Accuracy of the distorted-wave impulse approximation for breakup reactions in a three-body model*

Stephen K. Young† and Edward F. Redish
University of Maryland, Department of Physics and Astronomy, College Park, Maryland 20742
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We have studied the reaction mechanisms of the $(p,2p)$ reaction on ^{4}He at energies of 65 and 100 MeV. The dynamics of the reaction is taken to be that of a three-particle system, the ^{4}He target being viewed as a proton bound to an inert triton. The particles interact via separable potentials chosen to fit bound state and scattering data of the two-particle subsystems. The protons are treated as identical bosons interacting with each other only in the relative S state while the proton-triton potential is limited to relative $L \leq 2$. The ^{4}He$(p,2p)$-^{3}H doubly differential cross section can be calculated exactly within this model by solving the Faddeev equations numerically using the method of deformed contours and Padé approximants. This can be compared with the distorted-wave impulse approximation (DWIA), calculated in this model by summing the appropriate subset of the multiple scattering series. This is done in momentum space and treats off-shell and finite range effects of the nucleon-nucleon T matrix as well as recoil effects exactly. We find that the DWIA is an adequate approximation at 100 MeV but is almost double the exact result at 65 MeV. Its use in the extraction of spectroscopic information below 100 MeV is therefore suspect. The shapes of the exact, distorted, and plane wave cross sections agree quite well with each other (and with the experimental shape), differing only in magnitude.

[NUCLEAR REACTIONS ^{4}He$(p,2p)^3$H, $E=65$, 100 MeV; calculated σ. Three-body model, reaction mechanisms.]

I. INTRODUCTION

Much of our present understanding of direct nuclear reactions (be it stripping, pickup, breakup, or elastic scattering) is based on a three-body framework. We operate on the premise that the reaction is essentially a two-body process, with two active nucleons engaging in violent reactions in the presence of distortion given rise to by the nuclear medium in which the reaction takes place. This involves viewing the target as nucleon plus core. The core provides the distortion while the nucleon reacts with the projectile. This picture gives rise to the distorted-wave Born approximation (DWBA) in pickup and stripping, to the optical potential in elastic scattering, and to the distorted-wave impulse approximation (DWIA) in breakup. In a lowest order approximation to the many body effects, the core is assumed inert and the quasi-three-body system contains pairwise interactions. This paper studies the multiple scattering aspect of breakup reactions and the commonly used DWIA at energies below 100 MeV. We use as the archetype of breakup the $(p,2p)$ reaction throughout this paper. A subsequent paper studies elastic scattering via an optical potential from a three-body model in view of the close ties between the optical potential used in breakup and elastic scattering. Stripping has been studied by three-body methods by Mitra,[1] by Aaron and Shanley,[2] and more recently by Bouldin and Levin[3] and others.[2,3]

In the past 15 years, there has been considerable interest in quasifree knockout reactions[4] of the type $(p,2p)$, (p,pn). These reactions are valuable as a probe of the nuclear inner shells and they compliment pickup reactions[5] (which are suitable for the outer shells) as the tools of studying nuclear structures. Moreover, they can potentially furnish off-shell information about the nucleon-nucleon t matrix.[6] However, all these attempts at extracting spectroscopic information and off-shell effects are futile unless we have a good understanding of the reaction mechanisms. At high energies (above 300 MeV) where single scattering is dominant one can invoke the plane wave impulse approximation (PWIA) and modify it to account for absorption effects via the semiclassical WKB method[7] in the spirit of the distorted wave treatment. As we go down in bombarding energies, matters become a lot more uncertain. At low energies, the multiple scattering series (MSS) is in general divergent.[8] This renders the DWIA suspect as to its adequacy in treating the multiple scattering effects. However, it has been used extensively[9] even at energies down to 50 MeV. The rationale for such an approximation is partly due to intuition, partly due to its tested success at higher energies, and last

but not least, it seems the best thing to do short of solving the scattering problem exactly. Some recent attempts have been made in justifying this last claim. Dodd and Greider[10] and Kazaks and Koshel[11] have shown that the DWIA is the first term of a "mathematically meaningful" series. (The name "DWIA" is used by some authors to imply on-shell approximations for the t matrix used in Eq. (1). Throughout this paper, we never use on-shell approximations for the nucleon-nucleon t matrix. In all calculations, the appropriate half-shell or fully-off-shell t matrix will be used.) They rearrange the three-particle Lippmann-Schwinger equation to get an integral equation with a compact kernel free of dangerous δ functions (hence the label "mathematically meaningful"). One must not, however, take this as a justification for the approximation, at least not until the higher order terms have been examined. Our understanding of the MSS has been considerably augmented by Sloan[8] in his study of n-d elastic scattering at low and medium energies. He reminds us that a compact kernel does not in any way guarantee a convergent MSS.

The experimental data fail to clarify this approximation either. At energies below 100 MeV, DWIA calculations do not fit the data well. Lim and McCarthy[9] have attempted to fit the $^{12}C(p,2p)$-^{11}B data at 50 MeV by Pugh *et al.*[12] While they fail to get good fit, they cannot pin down the cause of failure. This is due to the sensitive dependence of DWIA calculations on all the inputs in the calculation. The matrix element that we have to compute is

$$T_{fi} = \langle \chi_1^{(-)} \chi_2^{(-)} \mid t \mid \chi_1^{(+)} \phi \rangle, \tag{1}$$

where $\chi_i^{(\pm)} = 1, 2$ are distorted waves with outgoing and incoming waves, the indices 1, 2 stand for the two protons, t is the nucleon-nucleon t matrix, and ϕ is the overlap integral (or bound state wave function) between the target and residual nuclei. Apart from the approximate treatment of three-body final state,[9,13] there are more serious difficulties. First, the calculation is very sensitive to the shape of the distorting potentials.[14] As is well known, phenomenological fits to elastic scattering data do not uniquely specify the optical well. Recent attempts at calculating optical potentials via first principles corroborate this point. Lerner and Redish[15] have calculated the real optical well for $p - {}^{16}O$ elastic scattering at 65 MeV in the impulse approximation. The well shape so obtained differs appreciably from that of conventional ones. This real well, combined with a phenomenological imaginary potential, gives a good fit to the data. This shape ambiguity is a stumbling block in the way of a good DWIA calculation and points to the inadequacy of pure phenomenology. The second difficulty comes from our lack of knowledge of the overlap integral ϕ, apart from its tail behavior which we can infer from the bound state energy. The latter, however, is not sufficient for a reliable DWIA calculation. Thirdly, we need to know the nucleon-nucleon t matrix fully-off-shell. The calculation of Lim and McCarthy uses a pseudopotential model for this t matrix. While it fits on-shell data up to 300 MeV, one is not certain that it has the correct off-shell behavior and how sensitively the calculations depend on this off-shell ambiguity. Plagued by all these uncertainties, Lim and McCarthy fail to isolate the main cause of trouble, much less to establish the validity of the underlying distorted wave motif.

Our aim in this paper is to test the distorted wave premise at energies below 100 MeV using the reaction $^4He(p,2p)^3H$. We propose to do this within an exactly soluble three-body model so that we may compare our DWIA with the exact results. We try to make the model as realistic as we can while keeping the calculations down to manageable proportions. This involves using separable potentials and limiting two-body interactions to a few partial waves. The merit of such a model calculation is that the ingredients in the DWIA appear in the exact calculation in the same fashion. The only difference between the two is in their treatment of multiple scattering effects. Any discrepancies between them point to undue neglect of certain terms in the MSS and hence the inadequacy of the DWIA. If one can establish that the three-body aspects of the breakup problem are well represented by such a distorted wave treatment, one can hope to improve on the DWIA inputs.

This paper is organized as follows. Section II deals with the formalisms of the Faddeev theory[16,17] for the present quasi-three-body problem using separable potentials. In Sec. III we derive the DWIA from the Faddeev theory and point out its similarities with conventional approaches. Section IV discusses our particular three-body model. Calculational details are discussed in Sec. V, and in Sec. VI we discuss the results and conclusion of the study. Appendix A discusses the Born term or the driving term of the three-body integral equations for the unequal mass case. Its singularity structure is briefly studied. Appendix B gives the kinematics of the energy sharing geometry. Appendix C discusses the four-body problem in conjunction with the core assumption in the three-body model. The Padé approximant method is discussed in Appendix D.

II. FORMALISMS

In this section we shall establish our notation and formulate the three-body scattering problem using separable potentials. We employ standard cyclic notations for the three-body system consisting of particles α, β, and γ. Thus V_γ refers to the interaction between particles β and α or pair γ. We work in the center of mass (c.m.) frame such that a three-body state can be specified by two momenta $\vec{p}_\gamma$ and $\vec{q}_\gamma$. The former is the momentum of particle γ in the c.m. frame while the latter is the relative momentum of pair γ. We define the following masses:

$$\mu_\gamma = \frac{m_\alpha m_\beta}{m_\alpha + m_\beta}\ , \qquad N_\gamma = \frac{m_\gamma (m_\alpha + m_\beta)}{m_\alpha + m_\beta + m_\gamma}\ , \tag{2}$$

with which the $\vec{q}_\gamma$ can be written as

$$\vec{q}_\gamma = \frac{m_\beta \vec{p}_\alpha - m_\alpha \vec{p}_\beta}{m_\alpha + m_\beta} = \mu_\gamma \left(\frac{\vec{p}_\alpha}{m_\alpha} - \frac{\vec{p}_\beta}{m_\beta} \right)\ . \tag{3}$$

The masses μ_γ, N_γ are the reduced mass of pair γ and the mass of particle γ with pair γ, respectively. The c.m. frame imposes a condition on the $\vec{p}_\gamma$'s, namely

$$\vec{p}_\alpha + \vec{p}_\beta + \vec{p}_\gamma = 0. \tag{4}$$

This allows us to write

$$\vec{q}_\gamma = \vec{p}_\alpha + \frac{m_\alpha}{m_\alpha + m_\beta}\ \vec{p}_\gamma = -\vec{p}_\beta - \frac{m_\beta}{m_\alpha + m_\beta}\ \vec{p}_\gamma\ . \tag{5}$$

By permuting indices, we get analogous expressions for $\vec{q}_\alpha$ and $\vec{q}_\beta$. Furthermore, if pair γ is bound with energy E_B, the three-body c.m. energy E is given by

$$E = \frac{p_\gamma^{\,2}}{2N_\gamma} + E_B\ . \tag{6}$$

In scattering problems it is convenient to introduce a complex energy $S = E + i\epsilon$. The total Hamiltonian of the system is given by

$$H = H_0 + V = H_0 + \sum_\gamma V_\gamma\ , \tag{7}$$

where H_0 is the kinetic energy operator. Three-body forces are ignored. We can define a channel Hamiltonian H_γ and a residual interaction $\bar{V}_\gamma$ as

$$H = H_\gamma + \bar{V}_\gamma\ , \qquad H_\gamma = H_0 + V_\gamma\ , \tag{8}$$

and

$$\bar{V}_\gamma = V - V_\gamma\ .$$

The index 0 will stand for the breakup channel, and V_0 is defined to be zero. Channel states $|\Phi_{\gamma n}\rangle$ are represented by the kets $|\vec{p}_\gamma \phi_{\gamma n}\rangle$, where $|\phi_{\gamma n}\rangle$ is the nth bound state of pair γ. These channel states are eigenstate of H_γ, such that

$$H_\gamma |\Phi_{\gamma n}\rangle = \left(\frac{p_\gamma^{\,2}}{2N_\gamma} - \frac{\eta_{\gamma n}^{\,2}}{2\mu_\gamma} \right) |\Phi_{\gamma n}\rangle, \tag{9}$$

where $-\eta_{\gamma n}^{\,2}/2\mu_\gamma$ is the binding energy of the nth bound state in pair γ. Plane wave states are represented by the kets $|\vec{p}_\gamma \vec{q}_\gamma\rangle$.

We define the following Green's functions:

$$G_0 = (s - H_0)^{-1}\ , \qquad G_\gamma = (s - H_\gamma)^{-1}\ , \qquad \text{and} \qquad G = (s - H)^{-1}\ . \tag{10}$$

Further, we define three-body t matrices, which are two-body operators in the three-particle Hilbert space, such that

$$\langle \vec{p}'_\gamma \vec{q}'_\gamma | T_\gamma(s) | \vec{p}_\gamma \vec{q}_\gamma \rangle = \delta(\vec{p}'_\gamma - \vec{p}'_\gamma) \times \langle \vec{q}'_\gamma | t_\gamma (s - p'^{\,2}_\gamma / 2N_\gamma) | \vec{q}_\gamma \rangle\ . \tag{11}$$

We have used the lower case t for the t matrix in two-body space and capital T for its counterpart in three-body space. The T_γ's are defined by

$$T_\gamma = V_\gamma + V_\gamma G_0 T_\gamma = V_\gamma + V_\gamma G_\gamma V_\gamma\ . \tag{12}$$

The starting point of our calculation is the elegant formulation of the Faddeev equations by Alt, Grassberger, and Sandhas[17] (AGS). The transition operator $U_{\beta\alpha}$ for transition from channel α to channel β is expressed in a set of coupled integral equations (hereafter referred to as the AGS equation) with a particularly simple inhomogeneous term. The equations read

$$U_{\beta\alpha} = (1 - \delta_{\beta\alpha})(s - H_0) + \sum_{\gamma \neq \beta} T_\gamma G_0 U_{\gamma\alpha}\ . \tag{13}$$

The $U_{\beta\alpha}$'s are related directly to the S matrix via

$$S_{\beta n,\, \alpha m} = \delta_{\beta\alpha}\delta_{nm} - 2\pi i \delta(E_{\beta n} - E_{\alpha m}) \times \langle \Phi_{\beta n} | U_{\beta\alpha}(E_{\beta n} + i0) | \Phi_{\alpha m} \rangle\ . \tag{14}$$

By iterating (13) we get the Faddeev-Watson multiple scattering series (MSS)

$$U_{\beta\alpha} = (1 - \delta_{\beta\alpha})(s - H_0) + \sum_{\gamma \neq \beta \neq \alpha} T_\gamma + \sum_{\substack{\gamma \neq \beta \\ \gamma \neq \delta \\ \delta \neq \alpha}} T_\gamma G_0 T_\delta + \cdots\ . \tag{15}$$

The first term is the exchange term; the second is the single scattering or impulse term; the third is the double scattering term; and so on. Many of the approximations in reaction theory are contingent upon a convergent MSS. We shall look into this question in more detail later in the paper.

The breakup operator $U_{0\alpha}$ can be obtained from (13) in terms of the rearrangement operators as

$$U_{0\alpha}=\sum_{\gamma} T_\gamma G_0 U_{\gamma\alpha}\,, \tag{16}$$

where we have used the fact that $(s-H_0)$ vanishes on-shell.

In the rest of this section we shall obtain explicit one-dimensional coupled integral equations for the case of separable potentials. The treatment parallels that of Sloan[18] where he treats the three-nucleon problem. Let the particles interact via pairwise spin-independent separable potentials of the form

$$v=-\sum_{LM}\frac{\hbar^2}{2\mu}\lambda_L\frac{4\pi}{2L+1}|g_{LM}\rangle\langle g_{LM}|\,, \tag{17}$$

where μ is the reduced mass of the interacting pair, L their relative angular momentum, and M its z projection. The strength parameter λ_L is determined from scattering data and $|g_{LM}\rangle$ is the form factor such that if $\vec{q}$ is the relative momentum of the pair,

$$\langle\vec{q}|g_{LM}\rangle=g_L(q)Y_L^M(\hat{q})\,. \tag{18}$$

The form factors $g_L(q)$ are chosen to be real and the Y_L^M's are the usual spherical harmonics. In momentum space, we have

$$\begin{aligned}\langle\vec{q}'|v|\vec{q}\rangle&=-\sum_{LM}\frac{\hbar^2}{2\mu}\lambda_L\frac{4\pi}{2L+1}g_L(q')g_L(q)Y_L^M(\hat{q}')\\&\quad\times Y_L^{M*}(\hat{q})\\&=-\sum_{L}\frac{\hbar^2}{2\mu}\lambda_L g_L(q')g_L(q)P_L(\hat{q}'\cdot\hat{q})\,.\end{aligned} \tag{19}$$

The two-body t matrix at energy z takes the form

$$t(z)=\sum_{LM}|g_{LM}\rangle F_L(z)\langle g_{LM}|\,, \tag{20}$$

which is a standard form derived from solving the two-body Lippmann-Schwinger equation in momentum-space. The F functions are given by

$$F_L(z)=\frac{-(\hbar^2/2\mu)[4\pi/(2L+1)]\lambda_L}{1-[4\pi/(2L+1)]\lambda_L\int g_L^2(q')q'^2dq'/[q'^2-(2\mu/\hbar^2)z]}\,. \tag{21}$$

The strength and bound state parameters are related by

$$\lambda_L=\frac{2L+1}{4\pi}\left[\int_0^\infty\frac{g_L^2(q')q'^2dq'}{\eta_L^2+q'^2}\right]^{-1}, \tag{22}$$

where $-(\hbar^2/2\mu)\eta_L^2$ is the binding energy of the bound state supported by the potential. The matrix element of t in momentum-space takes the form

$$\begin{aligned}\langle\vec{q}'|t(z)|\vec{q}\rangle&=\sum_{LM}g_L(q')g_L(q)F_L(z)Y_L^{M*}(\hat{q})Y_L^M(\hat{q}')\\&=\sum_L g_L(q')\frac{2L+1}{4\pi}F_L(z)g_L(q)P_L(\hat{q}'\cdot\hat{q})\,.\end{aligned} \tag{23}$$

With these two-body t matrices, the three-body t matrices $T_\gamma(s)$ can be written as

$$T_\gamma(s)=\sum_{L_\lambda M_\lambda}|\gamma L_\gamma M_\gamma\rangle\hat{\tau}_{L_\gamma}(s)\langle\gamma L_\gamma M_\gamma|\,, \tag{24}$$

where

$$\langle\vec{q}_\gamma|\gamma L_\gamma M_\gamma\rangle=g_{L_\gamma}(q_\gamma)Y_{L_\gamma}^{M_\gamma}(\hat{q}_\gamma)\,. \tag{25}$$

The quantum numbers L_γ, M_γ are the relative angular momentum of pair γ and its z projection and

$$\langle\vec{p}_\gamma'|\hat{\tau}_{L_\gamma}(s)|\vec{p}_\gamma\rangle=\delta(\vec{p}_\gamma-\vec{p}_\gamma')F_{L_\gamma}(s-\vec{p}_\gamma^{\,2}/2N_\gamma). \tag{26}$$

Written out explicitly, the matrix element of the three-particle t matrix is given by

$$\langle\vec{p}_\gamma'\vec{q}_\gamma'|T_\gamma(s)|\vec{p}_\gamma\vec{q}_\gamma\rangle=\delta(\vec{p}_\gamma-\vec{p}_\gamma')\sum_L g_L(q_\gamma')g_L(q_\gamma)\frac{2L_\gamma+1}{4\pi}F_{L_\gamma}\left(s-\frac{p_\gamma^{\,2}}{2N_\gamma}\right)P_{L_\gamma}(\hat{q}_\gamma\cdot\hat{q}_\gamma')\,. \tag{27}$$

Coupling in the angular momentum (l_γ and z projection m_γ) of the third particle relative to the pair (pair γ) to form total three-body momentum Γ and z projection μ, we get

$$T_\gamma(s)=\sum_{\substack{\Gamma\mu\\L_\gamma l_\gamma}}|\gamma L_\gamma l_\gamma\Gamma\mu\rangle\hat{\tau}_{L_\gamma}(s)\langle\gamma L_\gamma l_\gamma\Gamma\mu|\,, \tag{28}$$

where

$$|\gamma L_\gamma l_\gamma \Gamma\mu\rangle = \sum_{M_\gamma m_\gamma} \langle L_\gamma M_\gamma l_\gamma m_\gamma|\Gamma\mu\rangle |\gamma L_\gamma M_\gamma\rangle |l_\gamma m_\gamma\rangle . \tag{29}$$

We note that $|\gamma L_\gamma l_\gamma \Gamma\mu\rangle$ now acts on both $|\vec{p}_\gamma\rangle$ and $|\vec{q}_\gamma\rangle$ and that the ket $|l_\gamma m_\gamma\rangle$ satisfies the relation

$$\langle \hat{p}_\gamma | l_\gamma m_\gamma\rangle = Y^{m_\gamma}_{l_\gamma}(\hat{p}_\gamma) . \tag{30}$$

Now we are ready to get one-dimensional coupled integral equations from (13). Following AGS we multiply (13) on the right with $G_0|\alpha L_\alpha l_\alpha \Gamma\mu\rangle$ and on the left with $\langle\beta L_\beta l_\beta \Gamma\mu|G_0$. Defining

$$\begin{aligned}\hat{X}^\Gamma_{\beta\alpha} &= \langle\beta L_\beta l_\beta \Gamma\mu|G_0 U_{\beta\alpha} G_0|\alpha L_\alpha l_\alpha \Gamma\mu\rangle \\ \hat{Z}^\Gamma_{\beta\alpha} &= \langle\beta L_\beta l_\beta \Gamma\mu|G_0|\alpha L_\alpha l_\alpha \Gamma\mu\rangle(1-\delta_{\beta\alpha})\end{aligned} \tag{31}$$

gives

$$\hat{X}^\Gamma_{\beta\alpha} = \hat{Z}^\Gamma_{\beta\alpha} + \sum_{\gamma L_\gamma M_\gamma} \hat{Z}^\Gamma_{\beta\gamma}\hat{\tau}_{L_\gamma}(s)\hat{X}^\Gamma_{\gamma\alpha} . \tag{32}$$

We have used rotational invariance so that the matrix elements are μ-independent and Γ and μ are good quantum numbers. Taking matrix elements of the operator equation (32), we get

$$\langle p'_\beta|\hat{X}^\Gamma_{\beta\alpha}|p_\alpha\rangle = \langle p'_\beta|\hat{Z}^\Gamma_{\beta\alpha}|p_\alpha\rangle + \sum_{\gamma L_\gamma l_\gamma} \int_0^\infty p''^2_\gamma dp''_\gamma \langle p'_\beta|\hat{Z}^\Gamma_{\beta\gamma}|p''_\gamma\rangle F_{L_\gamma}(s - p''^2_\gamma/2N_\gamma)\langle p''_\gamma|\hat{X}^\Gamma_{\gamma\alpha}|p_\alpha\rangle , \tag{33}$$

which is a set of coupled one-dimensional integral equations with the $\vec{q}$ dependences projected out. It is possible to obtain coupled equations in a single vector variable because, in a separable potential model, the $\vec{q}$ dependences are completely specified by the form factors. Note that on the energy shell $G_0|\vec{p}_\alpha\rangle \times |g_{L_\alpha M_\alpha}\rangle$ is proportional to $|\vec{p}_\alpha|\phi_{L_\alpha M_\alpha}\rangle$. Therefore, the on-shell elastic and rearrangement amplitudes can be obtained via

$$\langle \vec{p}'_\beta \phi_{L_\beta M_\beta}|U_{\beta\alpha}|\vec{p}_\alpha \phi_{L_\alpha M_\alpha}\rangle = \frac{\hbar^2}{2\mu_\alpha}\frac{\hbar^2}{2\mu_\beta} C_{L_\beta} C_{L_\alpha} \sum_{\Gamma\mu} \sum_{\substack{l_\alpha m_\alpha \\ l_\beta m_\beta}} \langle L_\alpha M_\alpha l_\alpha m_\alpha|\Gamma\mu\rangle\langle L_\beta M_\beta l_\beta m_\beta|\Gamma\mu\rangle \times Y^{*m_\alpha}_{l_\alpha}(\hat{p}_\alpha) Y^{m_\beta}_{l_\beta}(\hat{p}_\beta)\langle p'_\beta|\hat{X}^\Gamma_{\beta\alpha}|p_\alpha\rangle , \tag{34}$$

once Eq. (33) has been solved for the $\langle p'_\beta|\hat{X}^\Gamma_{\beta\alpha}|p_\alpha\rangle$'s. The C's are the normalization constants for the bound state wave function ϕ,

$$\phi_{L_\alpha M_\alpha}(\vec{q}_\alpha) = \langle \vec{q}_\alpha | \phi_{L_\alpha M_\alpha}\rangle = C_{L_\alpha}\frac{g_{L_\alpha}(q_\alpha) Y^{M_\alpha}_{L_\alpha}(\hat{q}_\alpha)}{\eta_{L_\alpha}{}^2 + q_\alpha{}^2} . \tag{35}$$

We have explicitly exhibited the quantum numbers $\{L_\alpha, M_\alpha\}$ in ϕ.

For breakup, the final state consists of a plane wave with all three particles free. This state is labeled by $|\vec{p}'_1\vec{p}'_2\vec{p}'_3\rangle$ or alternatively by $|\vec{p}'_\gamma\vec{q}'_\gamma\rangle$ where $\vec{p}'_\gamma$ and $\vec{q}'_\gamma$ are related by

$$s = \frac{p'^2_\gamma}{2N_\gamma} + \frac{q'^2_\gamma}{2\mu_\gamma} . \tag{36}$$

From Eq. (16), and after a little algebraic manipulation, we get

$$\begin{aligned}\langle \vec{p}'_1\vec{p}'_2\vec{p}'_3 | U_{0\alpha}|\vec{p}_\alpha\phi_{L_\alpha M_\alpha}\rangle = -\frac{\hbar^2}{2\mu_\alpha} C_{L_\alpha} \sum_\gamma \sum_{\Gamma\mu} \sum_{\substack{l_\alpha m_\alpha \\ l_\gamma m_\gamma \\ L_\gamma M_\gamma}} &\langle L_\alpha M_\alpha l_\alpha m_\alpha|\Gamma\mu\rangle \\ &\times\langle L_\gamma M_\gamma l_\gamma m_\gamma|\Gamma\mu\rangle g_{L_\gamma}(q'_\gamma) F_{L_\gamma}\left(s - \frac{p'^2_\gamma}{2N_\gamma}\right) \\ &\times\langle p'_\gamma|\hat{X}^\Gamma_{\gamma\alpha}|p_\alpha\rangle Y^{*m_\alpha}_{l_\alpha}(\hat{p}_\alpha) Y^{M_\gamma}_{L_\gamma}(\hat{q}'_\gamma) Y^{m_\gamma}_{l_\gamma}(\hat{p}'_\gamma).\end{aligned} \tag{37}$$

Note that (37) expresses the breakup amplitude in terms of the half-off-shell elastic and rearrangement amplitudes without the need for solving additional equations or performing quadratures. This is because Alt, Grassberger, and Sandhas have cleverly utilized the off-shell freedom in the elastic and rearrangement amplitudes in defining their amplitudes $X_{\beta\alpha}$ [Eq. (31)].

If the potentials are pure S waves the expressions (34) and (37) simplify, yielding

$$\langle \vec{p}'_\beta \phi_\beta | U_{\beta\alpha} | \vec{p}_\alpha \phi_\alpha \rangle = \frac{\hbar^2}{2\mu_\alpha} \frac{\hbar^2}{2\mu_\beta} C_\alpha C_\beta \sum_\Gamma \frac{2\Gamma+1}{4\pi} \langle p'_\beta | \hat{X}^\Gamma_{\beta\alpha} | p_\alpha \rangle P_\Gamma (\hat{p}'_\beta \cdot \hat{p}_\alpha) \tag{38}$$

and

$$\langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | U_{0\alpha} | \vec{p}_\alpha \phi_\alpha \rangle = -\frac{\hbar^2}{2\mu_\alpha} C_\alpha \sum_\gamma \frac{g_\gamma(q'_\gamma)}{\sqrt{4\pi}} F_\gamma (s - p'^2_\gamma / 2N_\gamma) \Big[\sum_\Gamma \frac{2\Gamma+1}{4\pi} \langle p'_\gamma | \hat{X}^\Gamma_{\gamma\alpha} | p_\alpha \rangle P_\Gamma (\hat{p}_\alpha \cdot \hat{p}'_\gamma) \Big] . \tag{39}$$

Thus, in order to get the breakup amplitude one first solves the system of equations (33) for the half-shell (since $p'_\beta \neq p_\alpha$) amplitudes $\langle p'_\beta | \hat{X}^\Gamma_{\beta\alpha} | p_\alpha \rangle$ and then obtains the breakup amplitude via (37).

III. DERIVATION OF THE DWIA IN THE FADDEEV FORMALISM

In this section, we shall formulate the DWIA from the Faddeev-AGS equation (13) and give the explicit expression for the breakup amplitude in the case of separable potentials. We make use of a set of formulas derived by Grassberger and Sandhas in Ref. 17. By an arbitrary splitting of T_γ into $(T^s_\gamma + T'_\gamma)$, one can split (13) into two sets of equations,

$$U'_{\beta\alpha} = (1 - \delta_{\beta\alpha})(s - H_0) + \sum_{\gamma \neq \beta} T'_\gamma G_0 U'_{\gamma\alpha} \tag{40}$$

and

$$U_{\beta\alpha} = U'_{\beta\alpha} + \sum_\gamma U'_{\beta\gamma} G_0 T^s_\gamma G_0 U_{\gamma\alpha} . \tag{41}$$

Consider the quasi-three-body problem of a nucleon incident on a nucleon-core bound state as in $(p, 2p)$. We label the nucleons as 1 and 2, the core as 3. The DWIA prescription tells us to turn on the nucleon-nucleon interaction (T_3) only once, while the nucleon-core interactions can occur an arbitrary number of times. Accordingly, we first solve the Faddeev three-body problem with the nucleon-nucleon force turned off. The equations are

$$U'_{\beta\alpha} = (1 - \delta_{\beta\alpha})(s - H_0) + \sum_{\substack{\gamma \neq \beta \\ \gamma \neq 3}} T_\gamma G_0 U'_{\gamma\alpha} , \tag{42}$$

obtained from (40) by letting

$$T'_\gamma = \begin{cases} T_\gamma & \gamma \neq 3 , \\ 0 & \gamma = 3 . \end{cases} \tag{43}$$

The exact operators $U_{\beta\alpha}$ can be obtained from $U'_{\beta\alpha}$ via (41) with the above splitting of the t matrices. This yields

$$U_{\beta\alpha} = U'_{\beta\alpha} + U'_{\beta 3} G_0 T_3 G_0 U_{3\alpha} . \tag{44}$$

Referring to the breakup operator in (16) we see that the DWIA demands the following approximations for the $U_{\beta\alpha}$'s, namely

$$U_{\beta\alpha} \approx \begin{cases} U'_{\beta\alpha} & \beta = 3 , \\ U'_{\beta 3} G_0 T_3 G_0 U'_{3\alpha} & \beta \neq 3 . \end{cases} \tag{45}$$

Upon substitution of (45) into (16) we get for the breakup operator in DWIA the following expression

$$U^{\mathrm{DWIA}}_{0\alpha} = T_3 G_0 U'_{3\alpha} + \sum_{\gamma = 1,2} T_\gamma G_0 U'_{\gamma 3} G_0 T_3 G_0 U'_{3\alpha} , \tag{46}$$

where the $U'_{\beta\alpha}$'s can be found by solving (42). Notice that we need the $U'_{\beta\alpha}$'s fully off-shell.

Next we shall write the exact breakup amplitude in the form of the "unified amplitude" of Ref. 11 to show the similarities of the two results. Using (16), (42), and (44) we get

$$\begin{aligned} U_{0\alpha} = U^{\mathrm{DWIA}}_{0\alpha} &+ (U'_{33} G_0 T_3 G_0 U'_{33} G_0 T_3 G_0 U_{3\alpha} \\ &+ T_3 G_0 U'_{33} G_0 T_3 G_0 U_{3\alpha}) \\ &+ \sum_{\gamma \neq 3} T_\gamma G_0 U'_{\gamma\alpha} . \end{aligned} \tag{47}$$

Borrowing the terminology of Ref. 11, we can name the second term in (47) the "resonance" type term, the third as the "recoil" type term. In this model the name "multistep" is perhaps more descriptive of the physical processes contained in the second term. We note that the nucleon-nucleon t matrix (T_3) appears once in the DWIA term, at least twice in the second, and not at all in the third. To see that this last term is indeed a recoil type term, consider

$$R = T_1 G_0 U'_{11} + T_2 G_0 U'_{21} . \tag{48}$$

On substituting for U'_{21} from (42) we get

$$R = [T_2 + (1 + T_2 G_0) T_1 G_0 U'_{11}] . \tag{49}$$

Defining the full Green's function for the problem with T_3 turned off as

$$G_M = (s - H_0 - V_1 - V_2)^{-1} , \tag{50}$$

we can write U'_{11} as[19]

$$U'_{11} = V_2 + V_2 G_M V_2 , \tag{51}$$

whereupon we get

$$T_1 G_0 U'_{11} = V_1 G_M V_2 \ , \tag{52}$$

yielding for R the following:

$$R = [\, T_2 + (1 + T_2 G_0) V_1 G_M V_2] = [\, 1 + (V_1 + V_2) G_M] \, V_2 \ , \tag{53}$$

which is of the form

$$R = \Omega^+_{12} \, V_2 \ . \tag{54}$$

The wave operator Ω^+_{12} is defined as

$$\Omega^+_{12} = 1 + (V_1 + V_2) G_M \ , \tag{55}$$

and distorts the plane waves in protons 1 and 2 into distorted waves. In this form, the "recoil" nature of R is apparent. In the infinite core case, the final state distorted waves in 1 and 2 factorize and the orthogonality of the distorted wave and bound state wave function in proton 2 will give zero for this term.

One of the merits of the three-body model is that we can compute the various terms in (47) exactly and assess their relative importance at various energies. This we shall set out to do, using $(p, p, {}^3\mathrm{H})$ as our model system as explained in the next section.

The system of one-dimensional integral equations we have to solve for the DWIA corresponding to using separable potentials in (42) is the same as in (33) with the F_{L_3} set equal to zero. We denote the solution by $\langle p'_\beta | \hat{X}^{\Gamma' }_{\beta\alpha} | p_\alpha \rangle$ where

$$\hat{X}^{\Gamma'}_{\beta\alpha} = \langle \beta L_\beta l_\beta \Gamma \mu \,|\, G_0 U'_{\beta\alpha} G_0 |\, \alpha L_\alpha l_\alpha \Gamma \mu \rangle \ . \tag{56}$$

With these amplitudes, we can then get the breakup amplitude in the DWIA as

$$\begin{aligned}
&\langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | \, U^{\mathrm{DWIA}}_{0\alpha} | \vec{p} \varphi_{L_\alpha M_\alpha} \rangle \\
&= -\hbar^2 / 2\mu_\alpha C_{L_\alpha} \Bigg\{ \sum_{\Gamma\mu} \sum_{\substack{l_\alpha m_\alpha \\ l_3 m_3 \\ L_3 M_3}} \langle L_\alpha M_\alpha l_\alpha m_\alpha | \, \Gamma\mu \rangle \langle L_3 M_3 l_3 m_3 | \, \Gamma\mu \rangle \\
&\qquad \times g_{L_3}(q'_3) F_{L_3}(s - p'^{\,2}_3 / 2N_3) \langle p'_3 | \hat{X}^{\Gamma'}_{3\alpha} | p_\alpha \rangle \, Y^{m_\alpha *}_{l_\alpha}(\hat{p}_\alpha) Y^{M_3}_{L_3}(\hat{q}'_3) Y^{m_3}_{l_3}(\hat{p}'_3) \\
&\quad + \sum_{\gamma=1,2} \sum_{\Gamma\mu} \sum_{\substack{l_\alpha m_\alpha \\ l_\gamma m_\gamma \\ L_\gamma M_\gamma}} \langle L_\alpha M_\alpha l_\alpha m_\alpha | \, \Gamma\mu \rangle \langle L_\gamma M_\gamma l_\gamma m_\gamma | \, \Gamma\mu \rangle \\
&\qquad \times g_{L_\gamma}(q'_\gamma) F_{L_\gamma}(s - p'^{\,2}_\gamma / 2N_\gamma) Y^{m_\alpha *}_{l_\alpha}(\hat{p}_\alpha) Y^{M_\gamma}_{L_\gamma}(\hat{q}_\gamma) Y^{m_\gamma}_{l_\gamma}(\hat{p}'_\gamma) \\
&\qquad \times \sum_{L_3 l_3} \int_0^\infty \langle p'_\gamma | \hat{X}'^{\Gamma}_{\gamma 3} | p''_3 \rangle \, F_{L_3}\!\left(s - \frac{p''^{\,2}_3}{2N_3}\right) \langle p''_3 | \hat{X}'^{\,\Gamma}_{3\alpha} | p_\alpha \rangle \, p''^{\,2}_3 dp''_3 \Bigg\} \ .
\end{aligned} \tag{57}$$

IV. MODEL

We propose to study the reaction $^4\mathrm{He}(p, 2p)^3\mathrm{H}$ at energies below 100 MeV. The ^{4}He target is viewed as a ^{3}H core plus a proton. The three particles are treated as spinless bosons interacting via pairwise spin-independent separable potentials. The use of separable potential is prompted by its success in the three-nucleon problem[20] and by the enormous numerical simplification it entails. Investigations by Harms[21] for the two-nucleon system show that separable potential calculations are more closely related to local potential calculations than we may think. He obtains a separable expansion of the potential in terms of the eigenfunctions of the homogeneous Lippmann-Schwinger equation. This separable expansion of the potential is then truncated and substituted into the Lippmann-Schwinger equation to obtain a t matrix which is both separable and unitary. This unitary pole expansion (UPE) gives t matrices in good agreement with those of the Reid potential at negative energies. A one term UPE or the unitary pole approximation (UPA) of the t matrix is in turn found to be a good representation of the UPE. Recent breakup calculations for $D(p, 2p)n$ (up to 150 MeV) by Wallace[22] using a phenomenologically scaled S-wave Yamaguchi force also indicate the relative insensitivity to details of the two-body input as long as two-body on-shell scattering data are well represented and the range is about right.

In the present model calculation, we use for the nucleon-nucleon force (v_{n-n}) an S-wave Yamaguchi

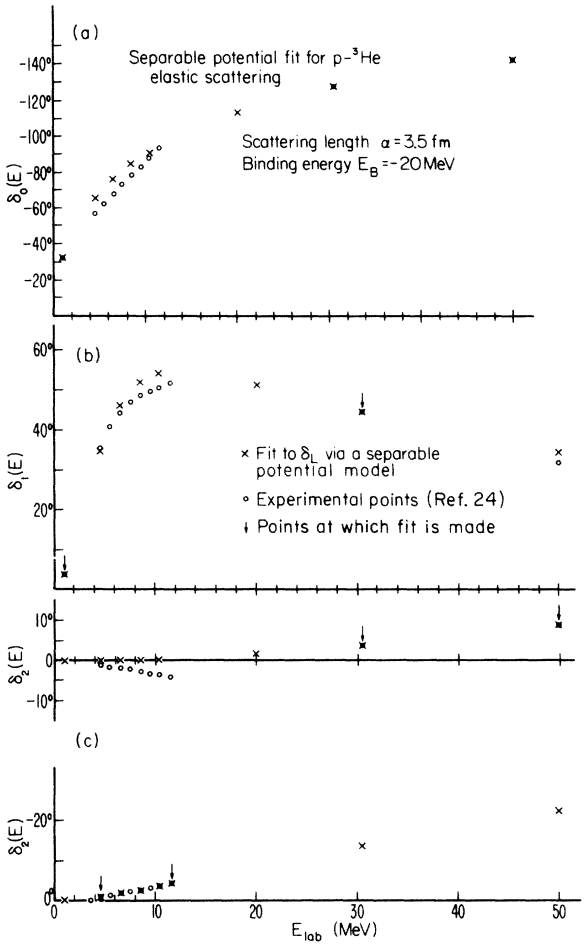

FIG. 1. p-^{3}He scattering data and separable potential fits for the interactions in the S, P, and D states.

potential[23] with parameters appropriate for the nucleon-nucleon triplet system. Explicitly,

$$\langle \vec{q}' \,|\, v_{n-n} \,|\, \vec{q} \rangle = -\frac{\hbar^2}{m} \lambda g(q') g(q) \,, \tag{58}$$

where the form factor and strength parameter are given by

$$g(q) = (q^2 + \beta^2)^{-1} \,,$$
$$\lambda = \beta \frac{(\alpha+\beta)^2}{\pi^2} \,. \tag{59}$$

The parameters α and β are chosen to fit the deuteron binding energy and the triplet scattering length. The potential gives a fairly good representation of the p-p on-shell scattering from 0–100 MeV.

For the p-^{3}H force (v_{n-c}) we use form factors of the form

$$g_L(q) = q^L/(q^2 + \bar{\beta}_L^2)^{(L+2)/2} \,. \tag{60}$$

The potential contains three two-body partial waves

$$\langle \vec{q}' \,|\, v_{n-c} \,|\, \vec{q} \rangle = -\sum_{L=0,1,2} \frac{\hbar^2}{2\mu} \lambda_L \, g_L(q') g_L(q) P_L(\hat{q}' \cdot \hat{q}), \tag{61}$$

where μ is the nucleon-core reduced mass. The particular form of the form factors used is motivated by its resemblance in coordinate space to a Yukawa dependence. The potential parameters are chosen to fit p-^{3}He elastic scattering data[24] from 0–50 MeV. There are two parameters per partial wave, namely η_L and $\bar{\beta}_L$. The strength parameters are given by

$$\lambda_0 = \frac{\bar{\beta}_0(\eta_0+\bar{\beta}_0)}{\pi^2} \,,$$
$$\lambda_1 = \frac{12}{\pi^2} \frac{\bar{\beta}_1(\eta_1+\bar{\beta}_1)^3}{3\eta_1+\bar{\beta}_1} \,, \tag{62}$$
$$\lambda_2 = \frac{40}{\pi^2} \frac{\bar{\beta}_2(\eta_2+\bar{\beta}_2)^4}{\bar{\beta}_2^2+4\bar{\beta}_2\eta_2+5\eta_2^2} \,.$$

The fits to p-^{3}He scattering data are shown in Fig. 1. Potential parameters are listed in Table I.

The two-body inputs into the Faddeev equations are in the form of t matrices. For the particular choice of form factor (60), the F functions are given as

$$F_0(Z) = -\frac{\hbar^2}{2\mu}\frac{4}{\pi} \frac{\bar{\beta}_0(\eta_0+\bar{\beta}_0)^2}{1-(\bar{\beta}_0+\eta_0)^2/(\bar{\beta}_0-i\kappa)^2} \,,$$
$$F_1(Z) = -\frac{\hbar^2}{2\mu}\frac{16}{\pi} \frac{\bar{\beta}_1(\bar{\beta}_1+\eta_1)/(\bar{\beta}_1+3\eta_1)}{1-[(\bar{\beta}_1-3i\kappa)(\bar{\beta}_1+\eta_1)^3]/[(\bar{\beta}_1-i\kappa)^3(\bar{\beta}_1+3\eta_1)]} \,, \tag{63}$$
$$F_2(Z) = -\frac{\hbar^2}{2\mu}\frac{32}{\pi} \frac{\bar{\beta}_2(\bar{\beta}_2+\eta_2)^4/(\bar{\beta}_2^2+4\bar{\beta}_2\eta_2+5\eta_2^2)}{1-[(\bar{\beta}_2^2-4i\bar{\beta}_2\kappa-5\kappa^2)(\bar{\beta}_2+\eta_2)^4]/[(\bar{\beta}_2-i\kappa)^4(\bar{\beta}_2^2+4\beta_2\eta_2+5\eta_2^2)]} \,,$$

where $\kappa = (2\mu z/\hbar^2)^{1/2}$ and z is the complex two-body energy. These F functions exhibit explicitly the poles at $\kappa = i\eta_L$. The Yamaguchi form is just a special case of (60) with $L=0$. Hence the nucleon-nucleon F function is given by $F_0(z)$ in (63) with $\mu = \frac{1}{2}m$, $\bar{\beta}_0 = \beta$, and $\eta_0 = \alpha$.

We have chosen a light target for study for the following reasons. First, we are aware of the failings of

the core assumption, namely the neglect of exchange between the active and core nucleons and the neglect of core breakup. By keeping the number of nucleons small, we may hope to track down what we are neglecting by comparing the three-body model with exact theory, like the four-body equations of Sloan.[25] This we shall discuss in Appendix C. Secondly, the fact that the ^{4}He target contains bound S-state nucleons enables us to check our calculations with the much simpler numerical calculation of the all S-wave model where only S-wave interactions are used. In fact we do find that the S-wave potentials give the bulk of the scattering.

V. CALCULATIONS

We have to solve a system of one-dimensional coupled integral equations given by (33) for the particular case of two identical nucleons plus a core. Performing explicit symmetrization between the two protons, the system of equations becomes

$$X^{\Gamma}_{\tau'\tau}(p',p) = Z^{\Gamma}_{\tau'\tau}(p',p) + \sum_{\tau''=\{L'',\,l''\}} \int_0^\infty Z^{\Gamma}_{\tau'\tau''}(p',p'')\bar{F}_{L''}\left(s-\frac{p''^2}{2N_1}\right)X^{\Gamma}_{\tau''\tau}(p'',p)p''^2dp''$$

$$+2\sum_{\tau''=\{L'',\,l''\}} \int_0^\infty \bar{\bar{Z}}^{\Gamma}_{\tau'\tau''}(p'p'')F_{L''}\left(s-\frac{p''^2}{2N_3}\right)\bar{X}^{\Gamma}_{\tau''\tau'}(p'',p)p''^2dp'' \,, \tag{64}$$

$$\bar{X}^{\Gamma}_{\tau'\tau}(p',p) = \bar{Z}^{\Gamma}_{\tau'\tau}(p',p) + \sum_{\tau''=\{L'',\,l''\}} \int_0^\infty \bar{Z}^{\Gamma}_{\tau'\tau''}(p',p'')\bar{F}_{L''}\left(s-\frac{p''^2}{2N_1}\right)X^{\Gamma}_{\tau''\tau}(p'',p)p''^2dp'' \,,$$

where

$$X^{\Gamma}_{\tau'\tau}(p',p) = \tfrac{1}{2}\sum_{\alpha,\beta=1,2}\langle p'_\beta|\hat{X}^{\Gamma}_{\beta\alpha}|p_\alpha\rangle \,,$$

$$\bar{X}^{\Gamma}_{\tau'\tau}(p',p) = \tfrac{1}{2}\sum_{\alpha=1,2}\langle p'_3|\hat{X}^{\Gamma}_{3\alpha}|p_\alpha\rangle \,,$$

$$Z^{\Gamma}_{\tau'\tau}(p',p) = \langle p'_2|\hat{Z}^{\Gamma}_{21}|p_1\rangle = \langle p'_1|\hat{Z}^{\Gamma}_{12}|p_2\rangle \,, \tag{65}$$

$$\bar{Z}^{\Gamma}_{\tau'\tau}(p',p) = \tfrac{1}{2}\sum_{\alpha=1,2}\langle p'_3|\hat{Z}^{\Gamma}_{3\alpha}|p_\alpha\rangle \,,$$

$$\bar{\bar{Z}}^{\Gamma}_{\tau'\tau}(p',p) = \tfrac{1}{2}\sum_{\alpha=1,2}\langle p'_\alpha|\hat{Z}^{\Gamma}_{\alpha 3}|p_3\rangle \,.$$

The label τ stands for $\{L,l\}$. The nucleon-nucleon F function is denoted by F_L while the nucleon-core F function is denoted by $\bar{F}_L$. The various combinations of $\bar{\tau}''$ that we can couple to via (64) are given in Table II.

Symbolically (64) can be written as

$$X = Z + Z\bar{F}X + 2\bar{\bar{Z}}F\bar{X} \,,$$
$$\bar{X} = \bar{Z} + \bar{Z}F\bar{X} \,, \tag{66}$$

and is shown diagrammatically in Fig. 2. The driving terms of the integral equations Z and $\bar{Z}$ describe pickup processes. Physically Z describes the process where the projectile nucleon gets picked up by the core and the target nucleon comes out. In other words the projectile and target nucleon trade places. The other term, $\bar{Z}$, describes the process whereby the projectile nucleon picks up the target nucleon and exits as a two-nucleon bound state. ($\bar{\bar{Z}}$ describes the time reversal of this process.) Explicit forms for the Born terms and their singularities will be discussed in Appendix A. It is easy to see that the on-shell Born terms are peaked in the backward direction in the c.m. frame.

After we have solved for X and $\bar{X}$, the breakup amplitudes can then be obtained from (37), which we may write symbolically as

$$B = \bar{g}\bar{F}X + gF\bar{X} \,, \tag{67}$$

where B stands for the breakup amplitude and $\bar{g}$ and g stand for the nucleon-core and the nucleon-nucleon form factors, respectively. Equation (67) is shown diagramatically in Fig. 3(a). Physically the first term describes the breakup process where the nucleon and the core interact last, and the second term describes the corresponding process where the nucleon-nucleon force operates last.

In the DWIA calculations, we first solve for

TABLE I. Potential parameters.

Nucleon-nucleon potential		
$\alpha_0 = 0.2317$ fm^{-1}		
$\beta_0 = 1.405$ fm^{-1}		
Nucleon-triton potentials		
l	η_l (fm^{-1})	$\bar{\beta}_l$ (fm^{-1})
0	0.8505	0.7249
1	−0.2529	0.9608
2	−0.9456	1.9160

$X'^{\Gamma}_{\tau'\tau}(p',p)$ and $\bar{X}^{\Gamma}_{\tau'\tau}(p',p)$ from (64) by setting $F_{L''}$ equal to zero. The two equations decouple, giving symbolically

$$X' = Z + Z\bar{F}X' ,$$
$$\bar{X}' = \bar{Z} + \bar{Z}\bar{F}X'. \tag{68}$$

However, to get the breakup amplitude in the DWIA according to (57), we need further the time reversal of $\bar{X}'$ which we denote symbolically by $\tilde{X}'$. This latter can be found by

$$\tilde{X}' = \bar{Z} + X'\bar{F}\bar{Z} . \tag{69}$$

With X', $\bar{X}'$, and $\tilde{X}'$ we can get the DWIA breakup amplitude as

$$B^{\mathrm{DWIA}} = 2\bar{g}\bar{F}\tilde{X}'F\bar{X}' + gF\bar{X}' \tag{70}$$

or

$$B^{\mathrm{DWIA}} = 2\bar{g}\bar{F}\bar{\bar{Z}}F\bar{X}' + 2\bar{g}\bar{F}X'\bar{F}\bar{\bar{Z}}F\bar{X}' + gF\bar{X}' . \tag{71}$$

This is shown diagrammatically in Fig. 3(b).

In the numerical calculations the Born terms are computed by numerical integration using a 10–20 point Gaussian quadrature rule, depending on the bombarding energy. The integral equations are solved numerically. We replace integrals by sums and map the domain of integration onto the interval $[-1, 1]$ via the mapping $p = \kappa_0(1+x)/(1-x)$. The integral equations are then solved by iteration, using the contour rotation method of Hetherington and Schick[26] and of Cahill and Sloan.[27] The angle of rotation ϕ is chosen such that no singularities are crossed. These three-

TABLE II. Various $\{L'',l''\}$ combinations relevant for our model problem.

	τ''		$(L'',l'') = \tau''$ classified by parity	
Γ	L''	l''	Even	Odd
	0	0	(0,0), (1,1), (2,2)	
0	1	1		
	2	2		
	0	1	(1,1), (2,2)	(0,1), (1,0), (1,2)
1	1	0,1,2		(2,1), (2,3)
	2	1,2,3		
	0	2	(0,2), (1,1), (1,3)	(1,2), (2,1), (2,3)
2	1	1,2,3	(2,0), (2,2), (2,4)	
	2	0,1,2,3,4		
	0	3	(1,3), (2,2), (2,4)	(0,3), (1,2), (1,4)
3	1	2,3,4		(2,1), (2,3), (2,5)
	2	1,2,3,4,5		
	0	4	(0,4), (1,3), (1,5)	(1,4), (2,3), (2,5)
4	1	3,4,5	(2,2), (2,4), (2,6)	
	2	2,3,4,5,6		
	0	5	(1,5), (2,4), (2,6)	(0,5), (1,4), (1,6)
5	1	4,5,6		(2,3), (2,5), (2,7)
	2	3,4,5,6,7		
	0	6	(0,6), (1,5), (1,7)	(1,6), (2,5), (2,7)
6	1	5,6,7	(2,4), (2,6), (2,8)	
	2	4,5,6,7,8		
	0	7	(1,7), (2,6), (2,8)	(0,7), (1,6), (1,8)
7	1	6,7,8		(2,5), (2,7), (2,9)
	2	5,6,7,8,9		
	0	8	(0,8), (1,7), (1,9)	(1,8), (2,7), (2,9)
8	1	7,8,9	(2,6), (2,8), (2,10)	
	2	6,7,8,9,10		
	0	9	(1,9), (2,8), (2,10)	(0,9), (1,8), (1,10)
9	1	8,9,10		(2,7), (2,9), (2,11)
	2	7,8,9,10,11		
	0	10	(0,10), (1,9), (1,11)	(1,10), (2,9), (2,11)
10	1	9,10,11	(2,8), (2,10), (2,12)	
	2	8,9,10,11,12		

body singularities are well known and have been studied extensively by Brayshaw,[28] who also justifies the contour rotation method. These singularities take the form of poles and branch cuts in the complex momentum plane arising from either the t matrices, the form factors, or the Born terms. Those arising from the Born terms are particularly troublesome. They take the form of moving cuts corresponding to particles being on-shell in the intermediate states. This vanishing of the Green's function is discussed in Appendix A. The angle of rotation ϕ into the lower half complex momentum plane has to be chosen so that no singularities are crossed. How this is done is also explained in Appendix A. The system of coupled equations (at most five in all for our model) is solved with a mesh of 32 Gaussian points. The number of equations is reduced due to the identity of the two nucleons in our spinless boson model. As a result only even-parity nucleon-core interactions come into the calculations. The iteration procedure in the solution of the integral equations converges in general for three-body angular momentum $\Gamma \geq 2$. In fact, for these partial wave amplitudes the Born terms give a good approximation. However for, $\Gamma = 0$ and 1 the series may diverge. This is particularly true for $\Gamma = 0$, where one would expect the most multiple scattering. Queen[29] argues that the particles "forget" their incident direction after many collisions and therefore come out isotropically. In cases where the MSS diverges, the method of Padé approximants[30] is used to sum the divergence. In all cases, the method works very well. Under most circumstances, a [3, 3] diagonal Padé will suffice. A theorem by Chisolm[31] guarantees convergence of the $[N, N]$ Padé series as $N \to \infty$ to the correct solution of an integral equation with compact kernel.

As we may suspect, when we turn off the nucleon-nucleon force in the three-body problem as dictated by the DWIA calculations, the series is in general convergent. However, in this calculation we need the amplitudes fully off-shell as shown in (70). An examination of the singularities of the problem indicates that we ran into no additional problems. These singularities are discussed in Appendix A.

VI. RESULTS AND CONCLUSIONS

The results of our model calculation are displayed in Figs. 4–8. The doubly differential cross section in the energy sharing geometry at 65 and 100 MeV are given in Fig. 4 (see Appendix B for kinematic details). In this figure we also show the cross sections obtained by truncating the multiple scattering series at one, two, and three terms. At both energies, the multiple scattering series in this model is convergent, as the largest

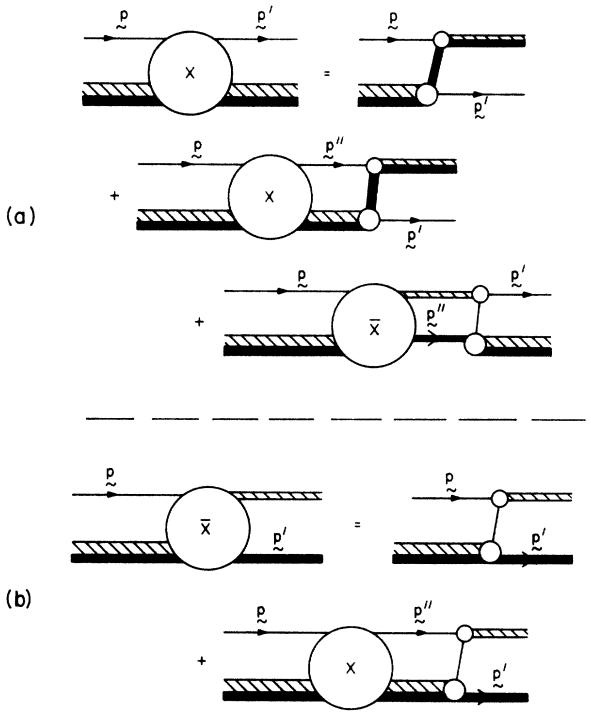

FIG. 2. Diagrammatic representation of the coupled equations.

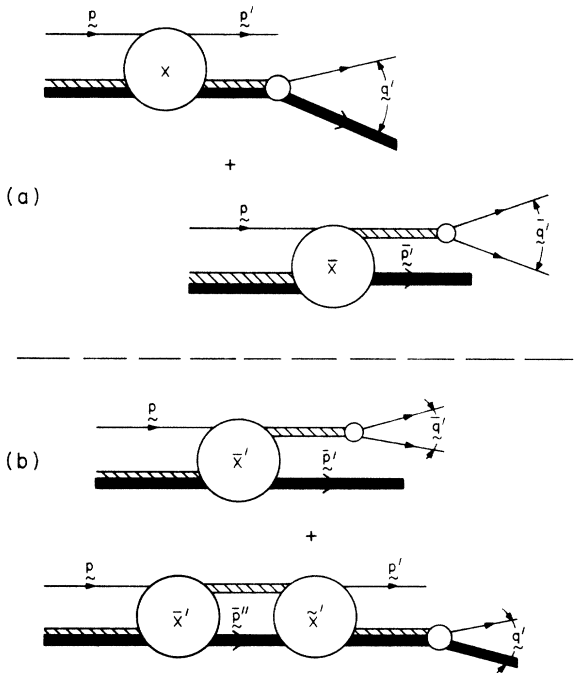

FIG. 3. Diagrammatic representation of the exact breakup amplitude and of the DWIA breakup amplitude.

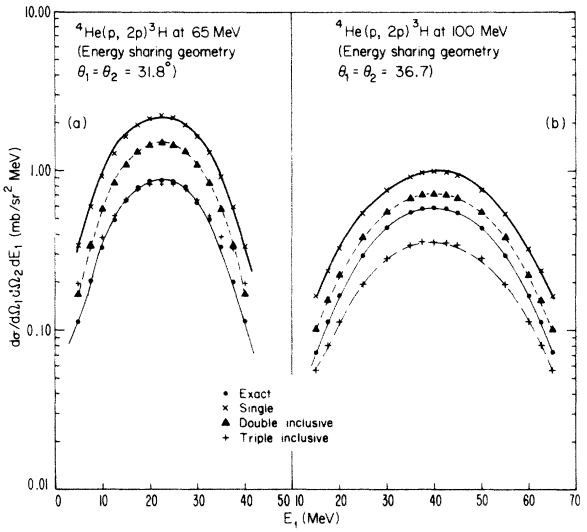

FIG. 4. ^{4}He$(p,2p)^3$H doubly differential cross section at 65 and 100 MeV via a MSS analysis.

eigenvalues of the Faddeev kernel are less than unity even for the three-body angular momentum $\Gamma = 0$. (In this partial wave the largest eigenvalues are $\lambda_0 = 0.6$ and 0.4 at 65 and 100 MeV, respectively.) However, the convergence rate is quite slow and the triple scattering estimate is not very good at either energy. The cross sections show the expected feature of a quasifree knockout peak centered at the equal energy point.

The amplitudes for breakup at 100 MeV are shown in Fig. 5, where we have also plotted the individual contributions to the total amplitude of the single, double, and triple scattering terms, as well as the exact result and the sum of all multiple scattering terms (all terms excluding the first). We make two observations. First, the multiple scattering terms interfere destructively with the single scattering term. This is well known experimentally, as the impulse approximation usually overestimates the cross section severely, especially at low energies. Second, the sum of the multiple scattering terms is nearly constant over the range of phase space probed by this geometry. In this case, the total amplitude may be written as the single scattering term plus a single complex constant characterizing all the multiple scattering effects. This is true even when the series diverges and must be summed by Padé methods. (This is discussed in more detail in Appendix D.)

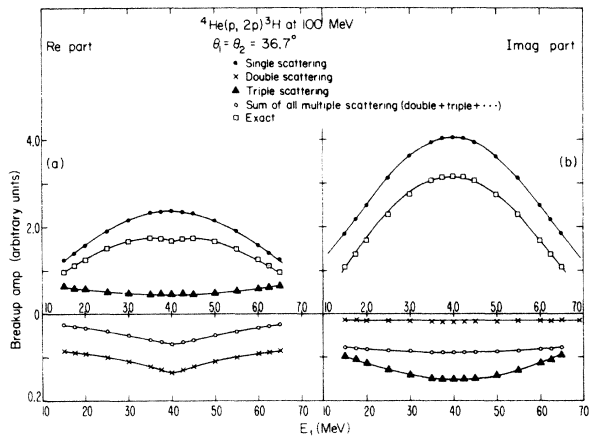

FIG. 5. The breakup amplitude at 100 MeV for ^{4}He-$(p,2p)^3$H with the contributions from various orders of multiple-scattering isolated.

Three-body calculations simplify considerably if only S-wave interactions are employed. We have therefore compared the result of the calculation including the interaction in the p-^{3}H D wave with a calculation in which the D-wave interaction has been turned off and only S-wave interactions retained. The results at 100 MeV are shown in Fig. 6. The S-wave interactions yield most of the scattering in this model, the inclusion of the D wave resulting in a 25% reduction of the cross section at the quasifree peak. At 65 MeV the reduction

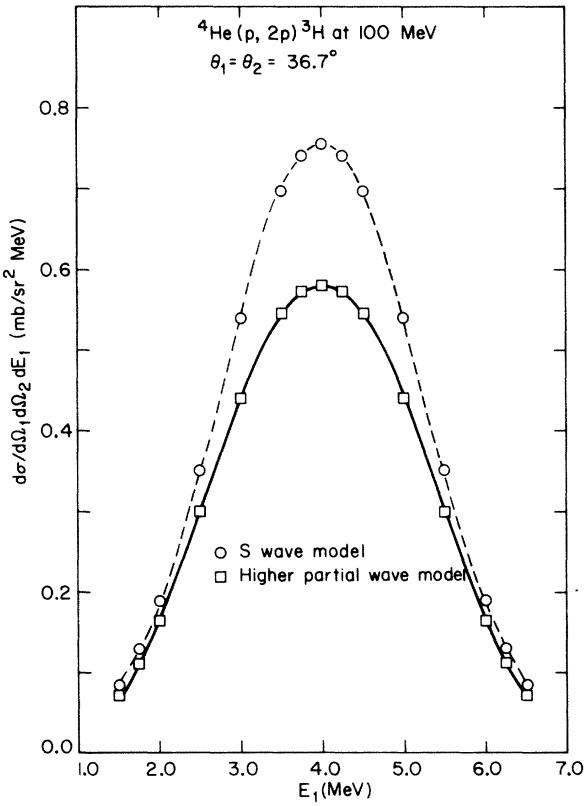

FIG. 6. Comparison between the S and higher partial wave models in the Faddeev calculation at 100 MeV.

produced by including the D wave is only 10%. To produce a truly realistic model, more p-^{3}H partial waves should be included, but the above result together with the fact that odd partial waves do not contribute in this model suggests that the effect of the higher partial waves may be less important than one would conclude from simple arguments. It is possible that perturbation theory would suffice for the inclusion of the higher partial waves, perhaps even for the D wave.

In Fig. 7 the results of the exact and DWIA calculations are compared. In (47), following the terminology of Ref. 11, the full amplitude is expressed as the sum of three terms, one of which is the DWIA. The other two are a resonance (or multistep) term and a recoil term. We can calculate the DWIA, the exact results, and the recoil term. From these the magnitude of the multistep term may be inferred. At 65 MeV the DWIA is about 1.7 times the exact result. It is, however, a significant improvement over the PWIA which is about 2.5 times the exact result at this energy. Inclusion of the recoil term produces about a 10% improvement. This indicates that at 65 MeV the multistep term is still quite important. At 100 MeV the DWIA has improved, being only about 20% larger than the exact result. Inclusion of the recoil term produces an improvement of about 5%.

We should point out that there is a difference between the DWIA as used here and as used in many practical calculations. Although the formal theories used to justify the DWIA (Dodd and Greider,[10] Kazaks and Koshel,[11] or the Faddeev formulation here employed) require the use of the particle-core interaction as a distorting potential in the incident channel, practical calculations often use the particle-target optical potential to produce the incident channel distortion.[32] The primary effect of this choice will be to increase the absorption by including the effect of knocking out the bound nucleon in directions other than those observed. This should reduce the DWIA somewhat and could produce slightly better agreement than is indicated in Fig. 7. We have not attempted to include this effect in our calculations as it would render them considerably more difficult.

Experimental data for the ^{4}He(p, $2p$) reaction have recently become available[33] and are compared to our results at 100 MeV in Fig. 8. It is seen that the shape of the curve can be obtained either with a PWIA, a DWIA, or with our Faddeev model. The normalization, however, is not given correctly by any of these simple models. The PWIA curve shown is that of Ref. 33. The DWIA shown is the one calculated in our model. A more realistic DWIA gives a similar normalization.[32] As displayed here, the normalization factors of 0.184, 0.308, and 0.380 required for the plane wave, distorted wave, and Faddeev calculations, respectively, to fit the data do not include the proper spectroscopic factor (~2) but take it equal to 1. A more realistic DWIA (Ref. 32) yields a normalization factor of about 0.35 even with a spectroscopic factor of 2. However, this includes very strong absorption in the incident channel.

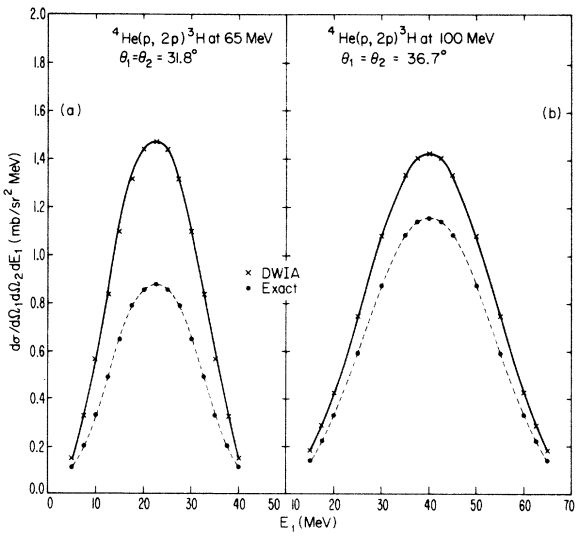

FIG. 7. DWIA versus exact calculations at 65 and 100 MeV for ^{4}He(p, $2p$)^{3}H.

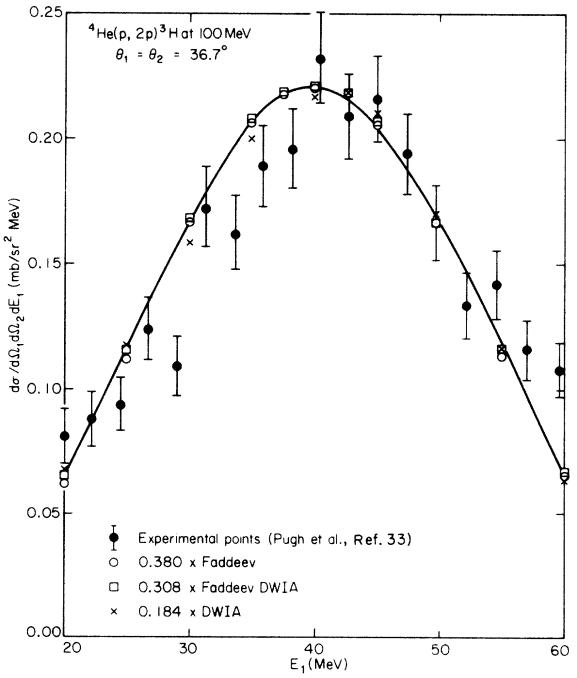

FIG. 8. Comparison between DWIA, PWIA, and Faddeev calculations and experiment.

This factor enters linearly in the PWIA and DWIA making them worse. It enters in a more complicated (and as yet unknown) manner in the Faddeev model. Note that the Faddeev calculation has no free parameters and that no effort was made to fit the data. The primary reason that our model produces too large a cross section is probably our failure to include absorption in the p-^{3}He interaction. A recent experiment[34] indicates that at 85 MeV the reaction cross section for $p+{}^3$He may be as large as 140 mb. This would require strong absorption in at least the first four partial waves. Although absorption can easily be included in this model by introducing a complex energy dependent coupling constant in the p-^{3}He interaction, we feel that an attempt to force a fit to data without a better understanding of the implications of the spectroscopic factor problem (i.e., there are two protons to be knocked out instead of just one) is unwarranted at this time.

The potential dependence of our conclusion on the accuracy of the DWIA was investigated by adjusting the p-^{3}He potential parameters to fit the bound state properties of the α particle rather than the scattering data. This results in new parameters: $\lambda_0 = 0.850\,538$ fm^{-1} and $\beta_0 = 1.40$ fm^{-1}. This change of parameters produces a fairly strong change in the p-^{4}He elastic scattering, shallowing and shifting the minimum. (The elastic results in this model will be discussed in a subsequent paper.) The results for breakup are similar to those obtained above. The DWIA is a reasonably good approximation at 100 MeV and not so good at 65 MeV.

In conclusion, we have investigated the accuracy of the DWIA for breakup in a solvable three-body model with parameters chosen to simulate the reaction $\alpha(p, 2p)$. The triton is treated as an inert core and absorption in the p-^{3}H channel is ignored. We find that the DWIA yields a reasonable shape for the cross section at both energies investigated (65 and 100 MeV) but that the magnitude is poorly reproduced at the lower energy. The distorted wave calculated correctly includes recoil and the full off-shell effects. This indicates that the DWIA is not correctly representing even the three-body aspects of the multiple scattering at energies below 100 MeV. Further investigations of multistep processes should be undertaken before low energy $(p, 2p)$ experiments can be analyzed for the extraction of nuclear spectroscopic factors.

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APPENDIX A: BORN TERM

The Born terms $\langle p'_\beta | Z^\Gamma_{\beta\alpha} | p_\alpha \rangle$ appearing in (33) drive the integral equations. For the present nuclear problem, there are two types of Born terms, $Z^\Gamma_{21}(p', L', l'; p, L, l)$ and $Z^\Gamma_{31}(p', L', l'; p, L, l)$, where we have adopted the notation in which the momenta and angular momenta are shown explicitly. (Z^Γ_{13} is related to Z^Γ_{31} via time reversal invariance.) The former describes the pickup of the core from the target by the projectile—the so called "heavy particle stripping." The latter describes the pickup of a nucleon from the target.

First we shall obtain explicit formulas for the Born terms. A discussion of their properties and singularities will follow.

1. Explicit formulas

Recall the definition of $\hat{Z}^\Gamma_{\beta\alpha}$ from (31). Inserting complete sets of $\tilde{q}_\alpha$, $\tilde{q}_\beta$, $\hat{p}_\alpha$, $\hat{p}'_\beta$ states and using (29), (30), and (25) we arrive at

$$\langle p'_\beta | \hat{Z}^\Gamma_{\beta\alpha} | p_\alpha \rangle = (1-\delta_{\beta\alpha}) \sum_{\substack{M_\alpha m_\alpha \\ M_\beta m_\beta}} \langle L_\beta M_\beta l_\beta m_\beta | \Gamma\mu \rangle \langle L_\alpha M_\alpha l_\alpha m_\alpha | \Gamma\mu \rangle \times \int d\hat{p}_\alpha d\hat{p}'_\beta \frac{g_{L_\alpha}(q_\alpha) Y^{M_\alpha}_{L_\alpha}(\hat{q}_\alpha) g_{L_\beta}(\hat{q}'_\beta) Y^{M^*_\beta}_{L_\beta}(\hat{q}'_\beta) Y^{m_\alpha}_{l_\alpha}(\hat{p}_\alpha) Y^{m^*_\beta}_{l_\beta}(\hat{p}'_\beta)}{E_{\beta\alpha}(s; p'_\beta, p_\alpha, \hat{p}'_\beta \cdot \hat{p}_\alpha)} , \tag{A1}$$

where, dropping the momentum subscripts,

$$E_{21}(s; p', p, x) = E_{12}(s; p', p, x) = s - \frac{\hbar^2}{m} \frac{A+1}{2A} \left(p^2 + p'^2 + \frac{2}{A+1} pp'x \right) ,$$

$$E_{31}(s; p', p, x) = E_{32}(s; p', p, x) = s - \frac{\hbar^2}{m} \left(p^2 + p'^2 \frac{A+1}{2A} + p'px \right) , \quad E_{13}(s; p', p, x) = E_{23}(s; p', p, x) = E_{31}(s; p, p', x) . \tag{A2}$$

In arriving at (A2) we have used (3) and the fact that G_0 is diagonal in the plane wave basis. We can simplify the Born term further. Rotational invariance dictates that the Born term be independent of μ, the z projection of the total conserved angular momentum Γ. Averaging over μ, we write

$$\langle p'_\beta|\hat{Z}^\Gamma_{\beta\alpha}|p_\alpha\rangle = \frac{1-\delta_{\beta\alpha}}{2\Gamma+1}\sum_\mu \sum_{M_\alpha M_\beta m_\alpha m_\beta} \langle L_\beta M_\beta l_\beta m_\beta|\Gamma\mu\rangle\langle L_\alpha M_\alpha l_\alpha m_\alpha|\Gamma\mu\rangle \times \int d\hat{p}_\alpha d\hat{p}'_\beta \frac{g_{L_\alpha}(q_\alpha)Y^{M_\alpha}_{L_\alpha}(\hat{q}_\alpha)g_{L_\beta}(q'_\beta)Y^{M^*_\beta}_{L_\beta}(\hat{q}'_\beta)Y^{m^*_\beta}_{l_\beta}(\hat{p}'_\beta)Y^{m_\alpha}_{l_\alpha}(\hat{p}_\alpha)}{E_{\beta\alpha}(s;p'_\beta,p_\alpha,\hat{p}_\alpha\cdot\hat{p}'_\beta)} . \tag{A3}$$

The integrand in (A3) above can be shown invariant under a rotation of coordinates in momentum space[18] so that it is a function of p_α, p'_β, and $\hat{p}_\alpha\cdot\hat{p}'_\beta$ only and the integration can be reduced to a single integration over $\hat{p}_\alpha\cdot\hat{p}'_\beta$. The end result is

$$\langle p'_\beta|\hat{Z}^\Gamma_{\beta\alpha}|p_\alpha\rangle = \frac{1-\delta_{\beta\alpha}}{2\Gamma+1}\,8\pi^2\left(\frac{2l_\gamma+1}{4\pi}\right)^{1/2}\sum_{\mu M_\beta}\langle L_\beta M_\beta l_\beta\,\mu\,-M_\beta|\Gamma\mu\rangle\langle L_\alpha\mu l_\alpha 0|\Gamma\mu\rangle \times\int_{-1}^{1}d(\cos\theta)\frac{g_{L_\beta}(q'_\beta)Y^{M^*_\beta}_{L_\beta}(\delta,0)g_{L_\alpha}(q_\alpha)Y^{\mu}_{L_\alpha}(\gamma,0)Y^{\mu-M^*_\beta}_{l_\beta}(\theta,0)}{E_{\beta\alpha}(s;p'_\beta,p_\alpha,\cos\theta)}, \tag{A4}$$

where

$$\theta=\sphericalangle(\hat{p}_\alpha,\hat{p}'_\beta)\,,\quad \gamma=\sphericalangle(\hat{p}_\alpha,\hat{q}_\alpha)\,,\quad \delta=\sphericalangle(\hat{p}_\alpha,\hat{q}'_\beta)\,. \tag{A5}$$

The notation $\sphericalangle(\hat{x},\hat{y})$ stands for the angle between $\hat{x}$ and $\hat{y}$.

2. Properties

A. Effect of symmetrization

In our model problem, the two nucleons (labeled 1 and 2) are considered identical bosons and differ from 3 (the core). This identity of 1 and 2 will put some restrictions on the Born term as we shall now see. Examine the symmetrized Born term

$$Z^\Gamma(p',L',l';p,L,l)=\tfrac{1}{2}[Z^\Gamma_{21}(p',L',l';p,L,l)+Z^\Gamma_{12}(p',L',l';p,L,l]$$

$$=\frac{1}{2}\frac{1}{2\Gamma+1}\sum_\mu\sum_{MmM'm'}\langle L'M'l'm'\,|\,\Gamma\mu\rangle\langle LMlm|\Gamma\mu\rangle \times\int d\hat{p}'d\hat{p}\,\frac{g_{L'}(q')g_L(q)Y^{m'}_{l'}(\hat{p}')Y^{m}_{l}(\hat{p})}{E_{21}(s,p',p,\hat{p}'\circ\hat{p})} \times[\,Y^{M'}_{L'}{}^*(\hat{q}')Y^{M}_{L}(\hat{q})+Y^{M'}_{L'}{}^*(-\hat{q}')Y^{M}_{L}(-\hat{q})]\,. \tag{A6}$$

Note the minus sign in the arguments of $Y^{M'}_{L'}{}^*$ and Y^M_L of the second integrand. It comes about because the arguments of $Y^{M'}_{L'}{}^*$ and Y^M_L in the calculation of Z^Γ_{21} are $\hat{q}'_2$ and $\hat{q}''_1$, respectively, while in calculating Z^Γ_{12} they are $\hat{q}'_1$ and $\hat{q}''_2$. It is clear from the kinematics relation in (5) that if we interchange indices 1 and 2 for the p's, $\vec{q}_1$ will go into $-\vec{q}_2$ and vice versa. Hence the minus signs. Recalling that $Y^M_L(-\hat{x})=(-)^L Y^M_L(\hat{x})$, we get for the symmetrized Born term

$$Z^\Gamma(p',L',l';p,L,l)=\tfrac{1}{2}[1+(-1)^{L'+L}]\frac{1}{2\Gamma+1}\sum_{\mu MmM'm'}\langle L'M'l'm'|\Gamma\mu\rangle\langle LMlm|\Gamma\mu\rangle \times\int d\hat{p}d\hat{p}'\,\frac{g_{L'}(q')Y^{M'}_{L'}{}^*(\hat{q}')g_L(q)Y^{M}_{L}(\hat{q})Y^{m}_{l}(\hat{p})Y^{m'}_{l'}{}^*(\hat{p}')}{E_{21}(s;p',p,\hat{p}\cdot\hat{p}')}. \tag{A7}$$

What it means physically is that, because of the identity of 1 and 2, the parity of the bound pair cannot be changed during a pickup (exchange) process. Similar results hold for $\bar{Z}^\Gamma$, the Z^Γ_{31} and Z^Γ_{32} symmetrized amplitude, and its time reversal counter part $\bar{\bar{Z}}^\Gamma$.

B. Parity conservation and time reversal invariance

In order to show the symmetry properties of the Born term, we further manipulate the formula (A4). First we define

$$I_{\beta\alpha}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \int_{-1}^{1} dx \frac{g_{l_1}(q'_\beta) Y_{l_1}^{m_1 *}(\delta, 0) g_{l_2}(q_\alpha) Y_{l_2}^{m_2}(\gamma, 0) Y_{l_3}^{m_3 *}(\theta, 0)}{E_{\beta\alpha}(s; p', p, x)} , \tag{A8}$$

with which shorthand notation the Born term reads

$$\begin{aligned} \langle p' | \hat{Z}^{\Gamma}_{\beta\alpha} | p \rangle = (1-\delta_{\beta\alpha}) \frac{8\pi^2}{2\Gamma+1} \left(\frac{2l+1}{4\pi}\right)^{1/2} \Bigg(& [1+(-1)^{L'+l'+L+l-2\Gamma}] \\ & \times \Bigg\{ \sum_{\mu, M'} \langle L\mu\, l0 | \Gamma\mu \rangle \Big[\langle L'M'l'\mu - M' | \Gamma\mu \rangle I_{\beta\alpha}\begin{pmatrix} L' & L & l' \\ M' & \mu & \mu - M' \end{pmatrix} \\ & + \langle L' - M' l' \mu + M' | \Gamma\mu \rangle I_{\beta\alpha}\begin{pmatrix} L' & L & l' \\ M' & \mu & \mu + M' \end{pmatrix} \Big] \\ & + \langle L0l0 | \Gamma 0 \rangle \sum_{M'} \langle L'M'l' - M' | \Gamma 0 \rangle I_{\beta\alpha}\begin{pmatrix} L' & L & l' \\ M' & 0 & -M' \end{pmatrix} \\ & + \sum_{\mu} \langle L'0l'\mu | \Gamma\mu \rangle \langle L\mu l0 | \Gamma\mu \rangle I_{\beta\alpha}\begin{pmatrix} L' & L & l' \\ 0 & \mu & \mu \end{pmatrix} \Bigg\} \\ & + \langle L0l0 | \Gamma 0 \rangle \langle L'0l'0 | \Gamma 0 \rangle I_{\beta\alpha}\begin{pmatrix} L' & L & l' \\ 0 & 0 & 0 \end{pmatrix} \Bigg), \end{aligned} \tag{A9}$$

TABLE III. Various momenta and quantitites related to the Green's function singularity of the Born term.

	$Z^{\Gamma}(p',p'')$	$\bar{Z}^{\Gamma}(p',p'')$	$\bar{\bar{Z}}^{\Gamma}(p',p'')$
$\frac{a}{\hbar^2/m}$	$\frac{A+1}{2A}$	1	$\frac{A+1}{2A}$
$\frac{b}{\hbar^2/m}$	$\frac{1}{A}$	1	1
$\frac{c}{\hbar^2/m}$	$\frac{A+1}{2A}$	$\frac{A+1}{2A}$	1
p'_n	$\left(\frac{2A}{A+1}\frac{s}{\hbar^2/m}\right)^{1/2}$	$\left(\frac{2A}{A+1}\frac{s}{\hbar^2/m}\right)^{1/2}$	$\left(\frac{s}{\hbar^2/m}\right)^{1/2}$
p'_m	$\left(\frac{2(A+1)}{A+2}\frac{s}{\hbar^2/m}\right)^{1/2}$	$\left(\frac{4A}{A+2}\frac{s}{\hbar^2/m}\right)^{1/2}$	$\left(\frac{2(A+1)}{A+2}\frac{s}{\hbar^2/m}\right)^{1/2}$
p''_L	$\frac{1}{A+1}\left\{-p' + \left[\frac{2A(A+1)s}{\hbar^2/m} - A(A+2)p'^2\right]^{1/2}\right\}$	$\left(\frac{s}{\hbar^2/m} - \frac{A+2}{4A}p'^2\right)^{1/2} - \frac{p'^2}{2}$	$\frac{A}{A+1}\left[-p' + \left(\frac{A+1}{A}\frac{2s}{\hbar^2/m} - \frac{A+2}{A}p'^2\right)^{1/2}\right]$
p''_u	$\frac{1}{A+1}\left\{p' + \left[\frac{2A(A+1)s}{\hbar^2/m} - A(A+2)p'^2\right]^{1/2}\right\}$	$\left(\frac{s}{\hbar^2/m} - \frac{A+2}{4A}p'^2\right)^{1/2} + \frac{p'^2}{2}$	$\frac{A}{A+1}\left[p' + \left(\frac{A+1}{A}\frac{2s}{\hbar^2/m} - \frac{A+2}{A}p'^2\right)^{1/2}\right]$
p''_B	$\frac{1}{A+1}\left\{p' - \left[\frac{2A(A+1)s}{\hbar^2/m} - A(A+2)p'^2\right]^{1/2}\right\}$	$\frac{p'}{2} - \left(\frac{s}{\hbar^2/m} - \frac{A+2}{4A}p'^2\right)^{1/2}$	$\frac{A}{A+1}\left[p' - \left(\frac{A+1}{A}\frac{2s}{\hbar^2/m} - \frac{A+2}{A}p'^2\right)^{1/2}\right]$
p''_C	$\frac{1}{A+1}\left\{p' + \left[\frac{2A(A+1)s}{\hbar^2/m} - A(A+2)p'^2\right]^{1/2}\right\}$	$\frac{p'}{2} + \left(\frac{s}{\hbar^2/m} - \frac{A+2}{4A}p'^2\right)^{1/2}$	$\frac{A}{A+1}\left[p' + \left(\frac{A+1}{A}\frac{2s}{\hbar^2/m} - \frac{A+2}{A}p'^2\right)^{1/2}\right]$
$\bar{x}$	$(A+1)\frac{\bar{p}''}{p'}$	$2\frac{\bar{p}''}{p'}$	$\frac{A+1}{A}\frac{\bar{p}''}{p'}$
$\bar{p}''$	$\left(p'^2 - \frac{2A}{A+1}\frac{s}{\hbar^2/m}\right)^{1/2}$	$\left(\frac{A+1}{2A}p'^2 - \frac{s}{\hbar^2/m}\right)^{1/2}$	$\left[\frac{2A}{A+1}\left(p'^2 - \frac{s}{\hbar^2/m}\right)\right]^{1/2}$

where the conservation of parity is manifest. The Born term vanishes unless parity $[(-)^{L+l}]$ is conserved.

The condition for time reversal invariance reads

$$Z^{\Gamma}_{\beta\alpha}(p',\ L',\ l';p,\ L,\ l) = Z^{\Gamma}_{\alpha\beta}(p,\ L,\ l;\ p',\ L',\ l'). \tag{A10}$$

In order to see this, we examine (A1). The energy denominators in the Green's function satisfy explicitly the condition

$$E_{\beta\alpha}(s;p',p,\hat{p}'\cdot\hat{p}) = E_{\alpha\beta}(s;\ p,p',\hat{p}\cdot\hat{p}')\ . \tag{A11}$$

As for the Y_l^M's in the integrand, we have for the $Z^{\Gamma}_{\beta\alpha}$ case, $Y_L^M(\hat{q})Y_l^m(\hat{p})Y_{L'}^{M'*}(\hat{q}')Y_{l'}^{m'*}(\hat{p}')$ while for $Z^{\Gamma}_{\alpha\beta}$, we have $Y_L^{M*}(\hat{q})Y_l^m(\hat{p})Y_{L'}^{M'}(\hat{q}')Y_{l'}^{m'}(\hat{p}')$. The difference in the complex conjugates goes away via $Y_L^{M*}(\hat{x}) = (-)^M Y_L^{-M}(\hat{x})$. The combined phase factor is unity since $M'+m' = M+m = \mu$, the z projection of Γ. A change of summation index from M', m', M, m, μ to $-M'$, $-m'$, $-M$, $-m$, $-\mu$ gives the desired relation (A10).

3. Singularities

It is of prime importance to understand the singularity structure of the Born term. Otherwise we may get into serious computational problems as we analytically continue the AGS equations into the complex momentum plane. These equations are ultimately solved via a contour rotation method and the contour we choose cannot run into any singularity of the Faddeev kernel. This latter criterion was conjectured by Aaron and Amado[35] and shown rigorously by Brayshaw.[28] The singularities of the Faddeev kernel can come either from the Born term or the t matrices. Those from the latter (bound state poles and the unitarity cut) are well known

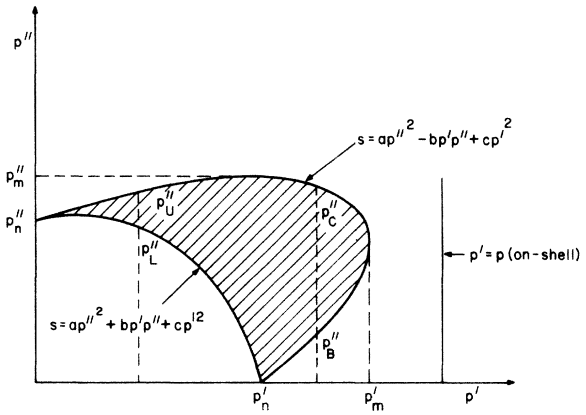

FIG. 9. The Born term singularity from the propagator. $p'_n = (s/c)^{1/2}$, $p'_m = [s/(c - b^2/4a)]^{1/2}$, $p''_n = (s/a)^{1/2}$, $p''_m = [s/(a - b^2/4c)]^{1/2}$.

while those from the Born term have been discussed by Amado and by Brayshaw for the equal mass problem. The Born term singularities can be divided into two classes. First, we have cuts arising from potentials (or the form factors of the potentials). Second, we have cuts coming from the vanishing of the Green's function denominator—the particle being on-shell, in the intermediate states. This Green's function singularity is by far the more troublesome one. In this section, we shall summarize the Born term singularities for this unequal mass problem. The analysis follows much the same line as in the three-nucleon problem and details will be omitted. A criterion for choosing the phase angle ϕ for the contour rotation method is given for the particular Mongan type form factor $q^L/(q^2+\beta^2)^{(L+2)/2}$ used in the present model calculations.

The Green's function appearing in $Z^{\Gamma}_{\beta\alpha}(p',\ L',\ l',\ p'',\ L'',\ l'')$ has the general form

$$G(p',p,x;s) = (s - ap^2 - bpp'x - cp'^2)^{-1}\ , \tag{A12}$$

where a, b, and c are positive constants given in Table III for our present problem of two equal mass particles (1, 2) and a third particle (3) with mass Am and $x = \hat{p}'\cdot\hat{p}''$.

The Green's function denominator vanishes when

$$s = ap^2 + bpp'x + cp'^2\ , \tag{A13}$$

where $-1 \leq x \leq 1$. In Fig. 9, we have sketched in the p', p'' plane the curves corresponding to (A13) for the extreme value of x. These curves define the boundary for the region where the Green's function denominator vanishes and hence gives an ill-defined Born term. This Green's function singularity takes the form of moving cuts (as a function of momentum) with branch points at $\pm p''_L$, $\pm p''_u$ for the region $0 \leq p' \leq p'_n$ and at $\pm p''_C$, $\mp p''_B$ for the region $p'_n \leq p' \leq p'_m$. These branch cuts are shown in Fig. 10 in which we have drawn in the integration contours (I and II) along which the integral equation is solved. These contours have been dsicussed in detail elsewhere[26,27] and will not be repeated here.

The phase angle ϕ has to be chosen so that no singularities are crossed. The rotated contour avoids the moving Green's function cut discussed above, plus the poles and cut of the t matrix. However, ϕ has to be such that we cross no potential singularities. For the particular type of form factors we use here, we get the condition

$$\tan\phi < \inf\left\{\frac{\beta}{p'}, \frac{\bar{\beta}}{p'}, \frac{2\alpha}{p'}, \frac{A+1}{A}\frac{\eta}{p'}, \frac{\pi}{4}\right\}, \tag{A14}$$

where (α, β), $(\eta, \bar{\beta})$ are the nucleon-nucleon and

nucleon-core potential parameters, respectively, and A is the number of nucleons in the core. As we notice, the angle ϕ is severely limited as we go up in energy, an obvious disadvantage of the method. New techniques have been developed in the last few years including (a) solution of the Faddeev equations on the real axis[36] and (b) a variational method.[37] The latter looks especially promising.

APPENDIX B: KINEMATICS OF THE ENERGY SHARING GEOMETRY IN $(p, 2p)$

In the energy sharing geometry, the protons (1, 2) are detected at equal lab angles θ from the incident beam direction. The angle θ is chosen such that if the two protons emerge with equal energies in the lab, the residual nucleus (mass Am) is at rest (in the lab). The doubly differential breakup cross section $(d\sigma/d\Omega_1 d\Omega_2 dE_1)$ is measured as a function of E_1, the energy of one of the emerging protons. As E_1 ranges from 0 to $(E_0 - E_B)$, E_2 decreases from $(E_0 - E_B)$ to 0. Here E_0 is the lab bombarding energy and $-E_B$ is the binding energy of the struck proton. The breakup cross section is symmetric around $\frac{1}{2}(E_0 - E_B)$.

From momentum and energy conservation we have

$$\vec{k}_0 = \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \tag{B1}$$

and

$$\frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2mA} + E_B , \tag{B2}$$

where $\vec{k}_0$ is the lab momentum of the incident proton, $\vec{k}_1$ and $\vec{k}_2$ are the lab momenta of protons 1 and 2, and $\vec{k}_3$ is the recoil momentum of the residual nucleus. From (B1) and (B2) we can infer the values of θ. The result can be easily seen to be

$$\theta = \cos^{-1}\left\{ \frac{k_0}{2[(E_0 - E_B)m/\hbar^2]^{1/2}} \right\} . \tag{B3}$$

Given E_0 and E_1 (hence k_0 and k_1) and θ, we can determine k_2, k_3, and $\bar{\theta}$ using (B1) and (B2). $\bar{\theta}$ is the angle which the recoil momentum of the residual nucleus makes with the incident direction.

The results are

$$k_2 = \frac{1}{A+1}\left\{ k_0\cos\theta - k_1\cos 2\theta + \left[(k_0\cos\theta - k_1\cos 2\theta)^2 + \frac{2A(A+1)m}{\hbar^2}\left(\frac{A-1}{A}E_0 - E_B - \frac{A+1}{A}E_1 + \frac{2}{A}(E_1E_0)^{1/2}\cos\theta \right) \right]^{1/2} \right\} , \tag{B4}$$

$$k_3 = (k_0^2 + k_1^2 + k_2^2 - 2k_0k_1\cos\theta - 2k_0k_2\cos\theta + 2k_1k_2\cos 2\theta)^{1/2} , \tag{B5}$$

$$\bar{\theta} = \cos^{-1}\left(\frac{k_0 - (k_1 + k_2)\cos\theta}{k_3} \right) . \tag{B6}$$

Having obtained these lab kinematics, we can transform into the c.m. frame in which our three-body calculations are carried out. Let p_i' be the c.m. counterpart of k_i $(i = 1, 2, 3)$ and p be the c.m. momentum of incident proton. We find

$$p = \frac{A+1}{A+2}k_0 , \tag{B7}$$

$$p_1' = \left(k_1^2 - \frac{2k_0k_1}{A+2}\cos\theta + \frac{k_0^2}{(A+2)^2} \right)^{1/2} , \tag{B8}$$

$$p_2' = \left(k_2^2 - \frac{2k_0k_2}{A+2}\cos\theta + \frac{k_0^2}{(A+2)^2} \right)^{1/2} , \tag{B9}$$

$$p_3' = \left(k_3^2 - \frac{2A}{A+2}k_0k_3\cos\bar{\theta} + \frac{A^2k_0^2}{(A+2)^2} \right)^{1/2} . \tag{B10}$$

The angles ϕ_i that p_i' makes with the z axis are

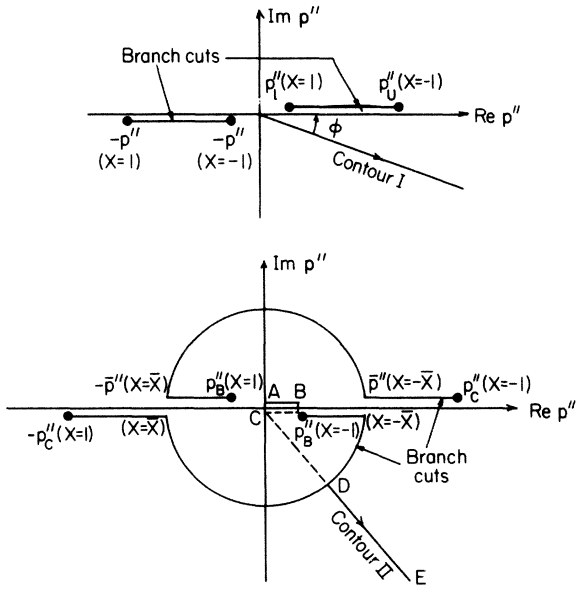

FIG. 10. The moving cuts in the Born term and the integration contours.

found to be

$$\phi_1 = \cos^{-1}\left[\frac{k_1\cos\theta - k_0/(A+2)}{p_1'}\right], \tag{B11}$$

$$\phi_2 = \cos^{-1}\left[\frac{k_2\cos\theta - k_0/(A+2)}{p_2'}\right], \tag{B12}$$

$$\phi_3 = \cos^{-1}\left[\frac{k_3\cos\theta - Ak_0/(A+2)}{p_3'}\right]. \tag{B13}$$

Finally we need the q_i''s corresponding to the relative momentum of pair i in the c.m. frame. This we get using the on-shell condition

$$s = \frac{p_i'^2}{2N_i} + \frac{q_i'^2}{2\mu_i}, \tag{B14}$$

where s is the c.m. energy. The results are

$$q_1' = \left[\frac{2A}{A+1}\left(\frac{ms}{\hbar^2} - \frac{A+2}{2(A+1)}p_1'^2\right)\right]^{1/2}, \tag{B15}$$

$$q_2' = \left[\frac{2A}{A+1}\left(\frac{ms}{\hbar^2} - \frac{A+2}{2(A+1)}p_2'^2\right)\right]^{1/2}, \tag{B16}$$

$$q_3' = \left(\frac{ms}{\hbar^2} - \frac{A+2}{4A}p_3'^2\right)^{1/2}. \tag{B17}$$

APPENDIX C: CORE APPROXIMATION EXAMINED IN AN EXACT FOUR-BODY FRAMEWORK

Consider the four-particle problem of particle 1 incident on a bound state of 2, 3, and 4. We shall arrange the exact four-body Sloan equations[25] such that the "core approximation" of treating the target as core (4, 3) plus nucleon (2) is manifest. Such an approximation reduces the exact four-body problem into an approximate three-body one—the minimal problem we have to handle if we want to treat three-body final states. A comparison between the core-approximated and the exact result will reveal the nature of the approximation and serve as a guide to the core approximation in a general N-body problem. As we shall see it also lends itself to a systematic correction scheme to the core approximation.

We start with the Sloan equations which are four-body analogs to the AGS equations. These Sloan equations only connect two-body channels—many-body channels are expressed only in terms of two-body ones. As Sloan has pointed out, these two-body channels are crucial since any iteration of the internal connected scatterings in two different two-body channels become connected. (This is not true for the three and four particle channels.) The Sloan equations have simple inhomogeneous terms and require as input sums of internal connected scatterings in the seven two-body channels—two- and three-body t matrices off the energy shell. We shall briefly review the channels in the four-body problem. These channels are of four kinds:

(a) Channels (i)—the ith particle is free and the other three particles are bound. There are four channels of this kind, namely 1, 2, 3, and 4. (b) Channels $(ij)(kl)$—the (ij) pair is bound and the (kl) pair is bound. There are three channels, namely (12)(34), (13)(24), and (14)(23). Sets (a) and (b) are the seven two-body channels and they play a central role in the four-body problem. (c) Channels (ij)—(ij) is bound and the other two particles are free. These include (12), (13), (14), (23), (24), and (34)—six in all. (d) Channel 0 where all four particles are free. There are 14 channels in all.

The Sloan equations are

$$U_{\beta\alpha} = (1-\delta_{\beta\alpha})(s - H_0 - V^{\beta\alpha}) + \sum_\sigma W_{\beta\sigma}G_0U_{\sigma\alpha} \tag{C1}$$

The indices α, β run over the 14 channels while σ runs over the seven two-body channels. The U's are the usual transition operators. H_0 is the kinetic energy operator and G_0 is the free Green's function. $W_{\beta\sigma}$ is the sum of all connected internal scatterings in the two-body channel σ that must not end in an interaction internal to channel β, but it can start with any pair interaction. For example, in W_{41} only V_{23} is internal to channels 1 and 4. Accordingly, W_{41} cannot end in V_{23}. It can end in V_{24} or V_{34}. (In particular, $W_{0\sigma}$ has no restriction on the final interaction and represents all internal connected scatterings in channel σ. Moreover, $W_{\sigma\sigma} = 0$.) $V^{\beta\alpha}$ is the pair interaction common to both channels α and β. (In the three-body problem, this is zero.) The fact that interaction can be common to two different channels is an essential complication of the $N \geq 4$ problem.

In conjunction with the core approximation let us assume that 3 and 4 are bound in an inert core. Of the seven two-body channels listed under (a) and (b) above, only three of them are allowed. They are 1, 2, (12), (34). We refer to them as channels i, j, and k while the forbidden channels are referred to as a, b, c, and d. We can write (C1) in two sets:

$$U_{ij} = (1-\delta_{ij})(s - H_0 - V^{ij}) + \sum_k W_{ik}G_0U_{kj} + \sum_a W_{ia}G_0U_{aj}, \tag{C2}$$

$$U_{aj} = \sum_k W_{ak}G_0U_{kj} + \sum_b W_{ab}G_0U_{bj}. \tag{C3}$$

Denoting $H_0 + V^{ij}$ as $\tilde{H}_0$ and eliminating V_{aj} from (C2) via (C3), we get formally

$$U_{ij} = (1-\delta_{ij})(s - \tilde{H}_0) + \sum_k \tilde{T}_{ik}G_0U_{kj}, \tag{C4}$$

where

$$\tilde{T}_{ik} = W_{ik} + \sum_{a,b} W_{ia} G_0 (1 - W G_0)^{-1}_{ab} W_{bk} . \quad \text{(C5)}$$

We note that

$$\begin{aligned} V^{1,2} &= V_{34} , \\ V^{1,(12)(34)} &= V_{34} , \\ V^{2,(12)(34)} &= V_{34} . \end{aligned} \quad \text{(C6)}$$

This tells us that $\tilde{H}_0$ is simply the free Hamiltonian for the case where we have (34) always bound as a core. Moreover, we note that

$$\begin{aligned} W_{i1} &= T_{2c} \quad i \neq 1 , \\ W_{i2} &= T_{1c} \quad i \neq 2 , \\ W_{i,(12)(34)} &= T_{12} \quad i \neq (12)(34) . \end{aligned} \quad \text{(C7)}$$

where T_{1c} and T_{2c} stand for the one-core and two-core interaction, respectively. From (C4)–(C7), the core approximation is apparent. In the core approximation of the present four-body problem, we have the AGS equation

$$U_{ij} = (1 - \delta_{ij})(s - H_0) + \sum_{k \neq i} T_k G_0 U_{kj} , \quad \text{(C8)}$$

where T_k are the nucleon-nucleon and nucleon-core t matrices that we input into the problem. By comparing (C4) with (C8) we see clearly that the core approximation consists of the approximation

$$\tilde{T}_{ik} \approx W_{ik} \quad \text{(C9)}$$

in (C5).

Formally expanding (C5) we can examine what kind of diagrams are omitted by the core approximation. Doing so, we get

$$\tilde{T}_{ik} = W_{ik} + \sum_{a} W_{ia} G_0 W_{ak} + \sum_{a,b} W_{ia} G_0 W_{ab} G_0 W_{bk} \cdots , \quad \text{(C10)}$$

where we notice that the core approximation ignores those terms which involve the polarization of the core, and all exchange processes of 1 and 2 with the core nucleons 3 and 4. To illustrate, consider the first order correction to $\tilde{T}_{21}$,

$$\Delta \tilde{T}_{21} = \sum_{a} W_{2a} G_0 W_{a1} . \quad \text{(C11)}$$

Diagrammatically some of the correction terms are shown in Fig. 11. (Time runs from left to right.) Diagrams (a) and (b) represent exchange of the projectile nucleon (1) with the core nucleons (3, 4). Diagrams (c) and (d) represent pickup of one of the core nucleons by the projectile.

We see that these exchange and breakup effects can be partially included in a phenomenological manner by choosing a nucleon-core interaction that produces the correct reaction cross section (this will entail a complex potential) and which gives the correct pickup behavior (in the backward direction) in nucleon-core elastic scattering. In any case (C10) provides guidance for systematically correcting the core approximation.

APPENDIX D: PADÉ APPROXIMANT

Consider the integral equation

$$X = B + \lambda K X , \quad \text{(D1)}$$

where B is the Born term and λ is the complex coupling constant. It has the formal iterated solution

$$X = B + (\lambda K) B + (\lambda K)^2 B + \cdots . \quad \text{(D2)}$$

The above series may or may not converge, depending on the magnitude of the largest eigenvalue of the kernel K. If it exceeds unity, the series diverges. In such cases, we may use the method of Padé approximants[38] to extract the solution X from the terms in various orders of λ.

The $[n, n]$ Padé approximant for X is the ratio of two nth degree polynomials in λ so constructed that its Taylor expansion agrees with the iterated solutions for X up to order λ^{2n}. Thus it has the general form

$$[n, n] = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_N\lambda^N}{1 + b_1\lambda + b_2\lambda^2 + \cdots + b_N\lambda^N} . \quad \text{(D3)}$$

It is clear that if $X(\lambda)$ is meromorphic in λ, (D3) can be a very good representation. However, (D3) can also approximate cuts.[38]

The convergence properties of the Padé approximant are not well known. However, a theorem by Chisolm[31] justifies our use of it in the three-body problem. Chisolm shows that the exact solution of

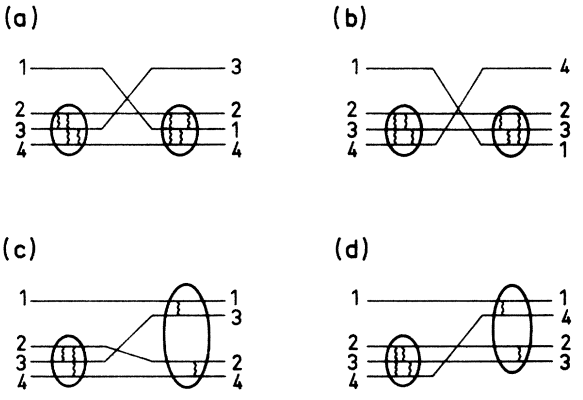

FIG. 11. Some contributions violating the inert core assumption.

a nonhomogeneous linear integral equation with a kernel K of finite rank is given by forming the $[n, n]$ Padé from the first $[2n+1]$ terms of the perturbation series solution. It follows that if the kernel is compact, the solution is $\lim_{n\to\infty}[n, n]$. Baker[38] has given explicit formula for $[n, n]$ given knowledge of the perturbation (D2). It is, denoting the term of order λ^n in (D2) as g_n,

$$[n,n]=\frac{\begin{vmatrix} g_1 & g_2\cdots & g_{n+1} \\ \vdots & & \\ g_n & g_{n+1}\cdots & g_{2n} \\ g_0\lambda^n & (g_0\lambda^{n-1}+g_1\lambda^n) & \sum_0^n g_i\lambda^i \end{vmatrix}}{\begin{vmatrix} g_1 & g_2\cdots & g_{n+1} \\ \vdots & & \\ g_n & g_{n+1}\cdots & g_{2n} \\ \lambda^n & \lambda^{n-1} & 1 \end{vmatrix}} . \tag{D4}$$

The physical solution is the case $\lambda = 1$. In particular, for our present study, (D2) is the Faddeev-Watson MSS with g_0 as the impulse approximation, g_1 the double scattering contribution, g_2 the triple scattering contribution, and so on.

Expanding the determinants in (D4) along the bottom row and setting $\lambda = 1$ for simplicity, we get

$$[n,n]=\frac{g_0M_{n1}+(g_0+g_1)M_{n2}+\cdots+(g_0+g_1+\cdots+g_n)M_{n,n+1}}{M_{n1}+M_{n2}+\cdots+M_{n,n+1}}, \tag{D5}$$

where

$$M_{nj}=(-1)^{n+j}=\begin{vmatrix} g_1 & g_2\cdots & g_{j-1} & g_{j+1} & \cdots & g_{n+1} \\ g_2 & & & & & \cdot \\ \vdots & & & & & \cdot \\ g_n & g_{n+1}\cdots & g_{n+j-2} & g_{n+j} & \cdots & g_{2n} \end{vmatrix} . \tag{D6}$$

Notice that (D6) is independent of g_0. From (D5) it is clear that

$$[n,n]=g_0+\frac{g_1M_{n2}+(g_1+g_2)M_{n3}+\cdots+(g_1+g_2+\cdots+g_n)M_{n,n+1}}{M_{n1}+M_{n2}+\cdots+M_{n,n+1}} . \tag{D7}$$

We observe an interesting feature in (D7) above. The $[n, n]$ Padé approximant for the scattering amplitude can be expressed as the sum of the impulse approximation g_0 and a correction term coming only from the multiple scattering contributions. This enables us to parametrize the scattering amplitude X as follows:

$$\begin{aligned} X &= \lim_{n\to\infty}[n,n]=g_0+\lim_{n\to\infty} f_n(g_1,g_2,\cdots g_{2n}) \\ &=g_0+f . \end{aligned} \tag{D8}$$

The function f is dependent on energy and on the momenta of the emerging protons in the breakup process. Depending on the geometry of the experiment, the function f may or may not have strong variations. For example, in the energy sharing geometry the function f does not vary much with the variation in energy of the emerging proton. In the cases where the variation of f is small, we can parametrize the multiple scattering effects for the whole angular correlation curve by two real constants, the real and imaginary part of f. Such an attempt has been made by Rogers *et al.*,[39] in analyzing their $d(p, 2p)n$ data.

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†Present address: Department of Physics, University of California, Los Angeles, Los Angeles, California 90024.

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