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Relativistic wave functions for pion-nucleon and pion-nucleus scattering*

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We use a modification of the method of Blankenbecler and Sugar to discuss various reductions of the Bethe-Salpeter equation to three-dimensional equations. Pion-nucleon and pion-nucleus scattering are treated, and the advantages of the use of different Green's functions are indicated. For the case of pion-nucleus scattering we propose a relativistic impulse approximation for use in construction of a relativistic optical model. The relations of the wave functions resulting from the solution of the three-dimensional equations to the Bethe-Salpeter wave function is indicated. Our method should be particularly useful in providing a framework in which phenomenological studies of the ***off-shell*** pion-nucleon T matrices may be made, since the relationship of our (relativistic) three-dimensional equations to a manifestly covariant theory has been made explicit.

[NUCLEAR REACTIONS Three-dimensional covariant equations for pion-nucleon and pion-nucleus systems; relativistic multiple scattering theory.]

1. INTRODUCTION

Recently there has been great interest in the description of the scattering of pions from nuclei. A great many calculations have been made including some based on Glauber theory,[1,2] Watson multiple scattering theory,[3] and field theory.[4] (In the latter case the attempt has been made to study an infinite system of nucleons using Chew-Low theory for the pion-nucleon dynamics in the medium.) These schemes are somewhat limited in their generality and we feel there is a need for a unified approach to the problem of the scattering of relativistic particles.

In this work we will attempt to present what we hope will form a general dynamical approach to pion-nucleon and pion-nucleus scattering *appropriate to the needs of workers in intermediate energy physics*. We see that the theoretical approaches given by other workers in this field may be obtained from our general approach. However, this correspondence requires that an additional number of kinematical or dynamical assumptions be made in the general theory.

2. REDUCTION OF THE BETHE-SALPETER EQUATION

In a previous paper[5] we discussed how the Blankenbecler-Sugar reduction of the Bethe-Salpeter equation could be used to obtain Lippmann-Schwinger equations for relativistic dynamics involving the scattering of a projectile from a nucleus. This procedure is based on the general relation for the Bethe-Salpeter invariant amplitude M,

$$M = K + KGM\,, \tag{2.1}$$

which may be rewritten as

$$M = U + UgM\,, \tag{2.2}$$

where

$$U = K + K(G-g)U\,. \tag{2.3}$$

The utility of replacing Eq. (2.1) by Eqs. (2.2) and (2.3) depends on the appropriate choice of g. In our previous work we discussed two choices of g which both led to Lippmann-Schwinger equations with energy dependent nonlocal potentials.[5] These two propagators were denoted as g_1 and g_2, where g_1 was essentially the Lomon-Partovi choice,[6] and g_2 was characterized by having one of the particles *always* constrained to the mass shell. While the formalism as expressed in terms of g_1 or g_2 could be used for pion-nucleus dynamics, it appears more appropriate to discuss the consequences of several other choices which will be defined in the following.

Again, we note that K is an *irreducible* amplitude for which we make the choice of kinematical variables as denoted in Fig. 1. Here $2W$ is the total center of mass energy, $L=(M_2^2-M_1^2)/2W$ is a factor whose origin is discussed by Partovi and Lomon,[6] and thus the four-vector $k_1=(k_1^0, \vec{k}_1) = (W+k^0-\frac{1}{2}L, \vec{k})$, etc., in this notation. [Note $k_1^2 = (k_1^0)^2 - \vec{k}_1^2$.]

Let us consider the pion-nucleon system in a specific channel so that we can avoid some complications of the isospin reduction. We may write

$$G(k|W)=\frac{i}{2\pi}\left[\frac{1}{(W+k-\frac{1}{2}L)^2-m_\pi{}^2+i\eta}\right]\times\left[\frac{1}{\gamma\cdot(W-k+\frac{1}{2}L)-m_N+i\eta}\right], \quad (2.4)$$

and define

$$g_3(k|W)=\int\frac{dW'}{W-W'+i\eta}\delta[(W'+k-\tfrac{1}{2}L)^2-m_\pi{}^2]\times\theta(W'+k^0-\tfrac{1}{2}L)\delta[(W'-k+\tfrac{1}{2}L)^2-m_N{}^2]\times\theta(W'-k^0+\tfrac{1}{2}L)[\gamma\cdot(W'-k+\tfrac{1}{2}L)+m_N] \quad (2.5)$$

$$=\left(\frac{m_N}{2\omega_{\vec{k}}E_{\vec{k}}}\right)\delta[k^0-\Delta_3(\vec{k})]\times\frac{1}{2W-(\omega_{\vec{k}}+E_{\vec{k}})+i\eta}\Lambda_+(\vec{k}). \quad (2.6)$$

In Eq. (2.6), $\omega_{\vec{k}}=(\vec{k}^2+m_\pi{}^2)^{1/2}$; $E_{\vec{k}}=(\vec{k}^2+m_N{}^2)^{1/2}$; $\Lambda_+(\vec{k})$ are the projection operators for positive energy nucleon spinors, and

$$\Delta_3(\vec{k})=\tfrac{1}{2}(L+\omega_{\vec{k}}-E_{\vec{k}}). \quad (2.7)$$

Following the procedure used previously,[6] with the definitions

$$\langle\vec{k}|\bar{M}_3(W)|\vec{k}'\rangle\equiv\langle\vec{k},k^0=\Delta_3(\vec{k})|M(W)|\vec{k}',k'^0=\Delta_3(\vec{k}')\rangle, \quad (2.8)$$

$$\langle\vec{k}|\bar{U}_3(W)|\vec{k}'\rangle\equiv\langle\vec{k},k^0=\Delta_3(\vec{k})|U_3(W)|\vec{k}',k'^0=\Delta_3(\vec{k}')\rangle, \quad (2.9)$$

and the restriction of Eqs. (2.8) and (2.9) to the space of positive energy nucleon spinors, we have

$$\langle\vec{k}|\bar{M}_3(W)|\vec{k}'\rangle=\langle\vec{k}|\bar{U}_3(W)|\vec{k}'\rangle+\int d\vec{p}\langle\vec{k}|\bar{U}_3(W)|\vec{p}\rangle\times\left[\frac{1}{2W-(\omega_{\vec{p}}+E_{\vec{p}})+i\eta}\right]R_{\pi N}(\vec{p})\times\langle\vec{p}|\bar{M}_3(W)|\vec{k}'\rangle, \quad (2.10)$$

where

$$R_{\pi N}(\vec{p})\equiv\frac{m_N}{2\omega_{\vec{p}}E_{\vec{p}}}. \quad (2.11)$$

Introducing

$$\langle\vec{k}|T_3(W)|\vec{k}'\rangle=R_{\pi N}{}^{1/2}(\vec{k})\langle\vec{k}|\bar{M}_3(W)|\vec{k}'\rangle R_{\pi N}{}^{1/2}(\vec{k}'), \quad (2.12)$$

$$\langle\vec{k}|V_3(W)|\vec{k}'\rangle=R_{\pi N}{}^{1/2}(\vec{k})\langle\vec{k}|\bar{U}_3(W)|\vec{k}'\rangle R_{\pi N}{}^{1/2}(\vec{k}'), \quad (2.13)$$

we see that T_3 satisfies a relativistic-wave equation of the form

$$\langle\vec{k}|T_3(W)|\vec{k}'\rangle=\langle\vec{k}|V_3(W)|\vec{k}'\rangle+\int d\vec{p}\langle\vec{k}|V_3(W)|\vec{p}\rangle\times[2W-(\omega_{\vec{p}}+E_{\vec{p}})+i\eta]^{-1}\times\langle\vec{p}|T_3(W)|\vec{k}'\rangle. \quad (2.14)$$

Indeed, Eq. (2.14) is of the form of the T-matrix equation which would be obtained by application of the Bakamjian-Thomas (BT) approach[7] to π-nucleon scattering (see Appendix A). Of course, we have some advantage over the BT method, since our "potential" $\langle\vec{k}|V_3(W)|\vec{k}'\rangle$ is ultimately defined in terms of the underlying field theory.

Now it is apparent that simple approximations to V_3 (say of separable form) will not yield a T matrix of the Chew-Low form. To expose more of the physical content of the theory it is useful to obtain an equation that for the pion-nucleon system has some familiar properties with respect to the energy dependence of the on-shell T matrix. To this end we note that in Eq. (2.5) we can give up the restriction to positive energy meson states. If we define $g_4(k|W)$ by removing $\theta(W'+k^0-\frac{1}{2}L)$ from the definition of $g_3(k|W)$ we obtain

$$g_4(k|W)=\left(\frac{m_N}{2\omega_{\vec{k}}E_{\vec{k}}}\right)\left[\frac{\delta[k^0-\Delta_3(\vec{k})]}{2W-(\omega_{\vec{k}}+E_{\vec{k}})+i\eta}+\frac{\delta[k^0-\bar{\Delta}_3(\vec{k})]}{2W+\omega_{\vec{k}}-E_{\vec{k}}+i\eta}\right]\Lambda_+(\vec{k}), \quad (2.15)$$

with

$$\bar{\Delta}_3(\vec{k})=\tfrac{1}{2}(L-\omega_{\vec{k}}-E_{\vec{k}}). \quad (2.16)$$

Now since k^0 is restricted by either $\Delta_3(\vec{k})$ or $\bar{\Delta}_3(\vec{k})$, it is not possible to write a *single* equation starting with the two equations obtained using g_4,

$$M=U_4+U_4g_4M \quad (2.17)$$

and

$$U_4=K+K(G-g_4)U_4. \quad (2.18)$$

Therefore let us define, cf. Eq. (2.15),

$$g_4\equiv g_3+g_4^-\equiv g_4^++g_4^-\equiv\hat{g}_4^+\delta[k^0-\Delta_3(\vec{k})]+\hat{g}_4^-\delta[k^0-\bar{\Delta}_3(\vec{k})], \quad (2.19)$$

$$\langle\vec{k}|\bar{M}^{++}(W)|\vec{k}'\rangle=\langle\vec{k},k^0=\Delta_3(\vec{k})|M(W)|\vec{k}',k'^0=\Delta_3(\vec{k}')\rangle, \quad (2.20)$$

$$\langle\vec{k}|\bar{M}^{+-}(W)|\vec{k}'\rangle=\langle\vec{k},k^0=\Delta_3(\vec{k})|M(W)|\vec{k},k'^0=\bar{\Delta}_3(\vec{k}')\rangle, \quad (2.21)$$

and so forth.

We may write Eq. (2.17) as

$$\mathfrak{M} = \bar{\mathfrak{U}} + \bar{\mathfrak{U}}\,\bar{g}\,\mathfrak{M}\,, \tag{2.22}$$

where

$$\mathfrak{M} = \begin{pmatrix} \bar{M}^{++} & \bar{M}^{+-} \\ \bar{M}^{-+} & \bar{M}^{--} \end{pmatrix}, \tag{2.23}$$

$$\bar{g} = \begin{pmatrix} \hat{g}_4^+ & 0 \\ 0 & \hat{g}_4^- \end{pmatrix}, \quad \text{etc.} \tag{2.24}$$

Of course, we may obtain a single equation for M^{++}; however, that would take us back to Eq. (2.10) with a somewhat more explicit expression for $\bar{U}_3$. (Note that $g_4^+ = g_3$.)

We have gone to the trouble of constructing g_4, Eq. (2.15), and Eq. (2.22) since, as we shall see in the next section, *elementary* phenomenological approximations to U_4 yield the familiar expressions for the T matrix in the $(\frac{3}{2}, \frac{3}{2})$ channel. It is therefore possible that a more detailed study of Eq. (2.22) will yield useful parametrizations of the pion-nucleon T matrices, on and off the energy shell. We would, in this manner, hope to also include some kinematical features not contained in the static model.

It is important to note that we can keep *both* parts of the pion propagator *and* avoid coupled equations by introducing still another g, which we denote as g_5. As indicated in our previous work,[5] it is possible to introduce a propagator in which one of the particles is *always* on the mass shell. In this case, putting the *nucleon* on the mass shell, we have

$$\begin{aligned} g_5(k|W) &= \left[\int \frac{dW'}{W - W' + i\eta}\,\delta[(W' + k - \tfrac{1}{2}L)^2 - m_\pi{}^2]\right] \\ &\quad\times \delta[(W - k + \tfrac{1}{2}L)^2 - m_N{}^2]\,\theta(W - k^0 + \tfrac{1}{2}L) \\ &\quad\times [\gamma\cdot(W - k + \tfrac{1}{2}L) + m_N] \\ &= \left(\frac{m_N}{2\omega_{\vec{k}} E_{\vec{k}}}\right)\delta[k^0 - \Delta_5(\vec{k})] \\ &\quad\times \left[\frac{1}{2W - (\omega_{\vec{k}} + E_{\vec{k}}) + i\eta} \right. \\ &\quad\left. + \frac{1}{2W + \omega_{\vec{k}} - E_{\vec{k}} + i\eta}\right]\Lambda_+(\vec{k})\,, \end{aligned} \tag{2.25}$$

where

$$\Delta_5(\vec{k}) = (W + \tfrac{1}{2}L - E_{\vec{k}})\,. \tag{2.26}$$

Clearly, comparing $g_4(k|W)$ and $g_5(k|W)$ we see that the advantage of the latter is that it enables us to avoid the use of coupled equations. Since it is not clear as to which of these propagators leads to a more convergent approximation scheme we will discuss both the coupled equations obtained with g_4 and the single equation obtained if g_5 is used. [We note that for the physical values of $2W$, $2W \geq m_\pi + m_N$ we may drop the $i\eta$ in the second term of Eq. (2.25).]

3. STATIC MODEL AND SEPARABLE INTERACTIONS

For our purposes we will *define* the static model on the basis of the approximations

$$(m_N/E_{\vec{k}}) \to 1\,;\quad R_{\pi N}(\vec{k}) \to (1/2\omega_{\vec{k}})\,;\quad (2W - E_{\vec{k}}) \to \omega\,;$$
$$\Delta_3(\vec{k}) \to 0\,;\quad \bar{\Delta}_3(\vec{k}) \to 0\,;\quad \Delta_5(\vec{k}) \to 0\,.$$

In this limit, therefore *both* $g_4(k|W)$ and $g_5(k|W)$ may be replaced by $g^s(k|W)$ where

$$g^s(k|W) = \frac{\delta(k^0)}{2\omega_{\vec{k}}}\left[\frac{1}{\omega - \omega_{\vec{k}} + i\eta} + \frac{1}{\omega + \omega_{\vec{k}} + i\eta}\right]\Lambda_+(\vec{k})\,. \tag{3.1}$$

We may then write for the static-model invariant matrix

$$M^s = U^s + U^s g^s M^s\,, \tag{3.2}$$

and defining

$$\langle \vec{k}|\bar{M}^s(\omega)|\vec{k}'\rangle = \langle \vec{k}, k^0 = 0|M^s(\omega)|\vec{k}', k'^0 = 0\rangle\,, \tag{3.3}$$

$$\langle \vec{k}|\bar{U}^s(\omega)|\vec{k}'\rangle = \langle \vec{k}, k^0 = 0|U^s(\omega)|\vec{k}', k'^0 = 0\rangle\,, \quad \text{etc.,} \tag{3.4}$$

we have, in the space of positive energy nucleon spinors,

$$\begin{aligned} \langle \vec{k}|\bar{M}^s(\omega)|\vec{k}'\rangle &= \langle \vec{k}|\bar{U}^s(\omega)|\vec{k}'\rangle \\ &\quad + \int d\vec{p}\,\langle \vec{k}|\bar{U}^s|\vec{p}\rangle \frac{1}{2\omega_{\vec{p}}} \\ &\quad \times \left[\frac{1}{\omega - \omega_{\vec{p}} + i\eta} + \frac{1}{\omega + \omega_{\vec{p}} + i\eta}\right] \\ &\quad \times \langle \vec{p}|\bar{M}^s(\omega)|\vec{k}'\rangle\,. \end{aligned} \tag{3.5}$$

Equation (3.5) is the dynamical equation for what we have termed the "static model." Now to indicate the possible utility of our approach for studying pion-nucleon dynamics (from a phenomenological point of view) we will explore the consequence of a simple separable approximation to the static approximation, Eq. (3.5).

Let us put in a separable form for $\bar{U}^s$, which is expected to be reasonable in the $(\frac{3}{2}, \frac{3}{2})$ channel *near the resonance*. We write (neglecting any ω dependence of the strength λ)

$$\langle \vec{k}|\bar{U}^s(\omega)|\vec{k}\rangle = \frac{-\lambda}{2\pi^2}\frac{v(k)}{\sqrt{\omega_{\vec{k}}}}\frac{v(k')}{\sqrt{\omega_{\vec{k}'}}}P_{33}\,, \tag{3.6}$$

where P_{33} contains the projection on angles and isospin for the $(\frac{3}{2}, \frac{3}{2})$ channel. The (static) invariant matrix is

$$\langle k|\bar{M}^s_{33}(\omega)|k'\rangle = \frac{-(\lambda/2\pi^2)v(k)v(k')/(\omega_{\vec{k}}\omega_{\vec{k}'})^{1/2}}{1+(2\lambda/\pi)\omega\int(p^2dp/\omega_{\vec{p}}^2)[v(p)]^2/(\omega+\omega_{\vec{p}})(\omega-\omega_{\vec{p}}+i\eta)}, \tag{3.7}$$

$$\simeq \frac{-(\lambda/2\pi^2)v(k)v(k')/(\omega_{\vec{k}}\omega_{\vec{k}'})^{1/2}}{1+(\lambda\omega/\pi)\int(d\omega_{\vec{p}}/\omega_{\vec{p}}^2)p[v(p)]^2/(\omega-\omega_{\vec{p}}+i\eta)}, \tag{3.8}$$

where we have used the approximation $(\omega+\omega_{\vec{p}})(\omega-\omega_{\vec{p}}+i\eta)\simeq 2\omega_{\vec{p}}(\omega-\omega_{\vec{p}}+i\eta)$ since $\omega > m_\pi$. We further note that the T matrix is related to the M matrix as in Eq. (2.12) so that in the static approximation

$$\langle k|T^s_{33}(\omega)|k'\rangle = \frac{-(\lambda/4\pi^2)[v(k)/\omega_{\vec{k}}][v(k')/\omega_{\vec{k}'}]}{1+(\lambda\omega/\pi)\int(d\omega_{\vec{p}}/\omega_{\vec{p}}^2)p|v(p)|^2/(\omega-\omega_{\vec{p}}+i\eta)}. \tag{3.9}$$

This expression is similar to the Chew-Low result without the crossing term. On shell $\omega=\omega_{\vec{k}}=\omega_{\vec{k}'}$

$$\langle k|T^s_{33}(\omega)|k\rangle_{\text{on-shell}} = \frac{-(\lambda/4\pi^2)[v(k)/\omega]^2}{1-(\lambda\omega/\pi)\int(d\omega_{\vec{p}}/\omega_{\vec{p}}^2)p[v(p)]^2/(\omega_{\vec{p}}-\omega-i\eta)} \tag{3.10}$$

$$= -\frac{1}{4\pi^2\omega k}e^{i\delta_{33}}\sin\delta_{33}, \tag{3.11}$$

which, with $v(k)=k$, is

$$\langle k|T^s_{33}(\omega)|k\rangle_{\text{on-shell}} = \frac{-(\lambda/4\pi^2)(k^2/\omega^2)}{1-(\lambda\omega/\pi)\int_{m_\pi}^{\Omega}d\omega_{\vec{p}}\,p^3/\omega_{\vec{p}}^2(\omega_{\vec{p}}-\omega-i\eta)}, \tag{3.12}$$

where Ω is a cutoff for the $\omega_{\vec{p}}$ integration.

We remark that attempts to parametrize the off-shell π-nucleon T matrices in terms of separable forms might lead to the simplest form factors if one makes this approximation in a *relativistic* equation such as Eq. (2.14) or a more elaborate parametrization may be tried using Eq. (2.22). In the latter equation one could try separable approximations for the submatrices of the $\mathfrak{U}$ matrix; i.e., $\bar{U}^{++}$, $\bar{U}^{+-}$, etc., could each be written in separable forms. The use of $g_5(k|W)$ without the "static" approximation clearly leads to a simple approximation if the corresponding potential ($\bar{U}_5$ or V_5) is replaced by a separable form. If we use $g_5(k|W)$ instead of $g^s(k|W)$ we are able to include some effects of the nucleon recoil.

Finally, it is clearly possible to include some ω dependence of the coupling strength λ, or to allow the potentials to be complex above production thresholds. (See Appendix C.)

4. WAVE FUNCTIONS

Assuming the reduction has been made to some single three-dimensional equation (e.g. using g_5), we may write [see, for example, Eq. (2.10)]

$$\langle\vec{p}'|\bar{M}(W)|\vec{p}\rangle = \langle\vec{p}'|\bar{U}(W)|\vec{p}\rangle + \int d\vec{p}''\langle\vec{p}'|\bar{U}(W)|\vec{p}''\rangle \mathcal{G}(\vec{p}''|W)R(\vec{p}'') \times\langle\vec{p}''|\bar{M}(W)|\vec{p}\rangle, \tag{4.1}$$

where $\mathcal{G}(\vec{p}''|W)=[2W-(\omega_{\vec{p}''}+E_{\vec{p}''})+i\eta]^{-1}$. Now we may define a wave function $|\bar{\psi}_{\vec{p},W}\rangle$ by[8]

$$\langle\vec{p}'|\bar{M}(W)|\vec{p}\rangle = \int\langle\vec{p}'|\bar{U}(W)|\vec{p}''\rangle d\vec{p}''\langle\vec{p}''|\bar{\psi}_{\vec{p},W}\rangle, \tag{4.2}$$

such that

$$\langle\vec{p}'|\bar{\psi}_{\vec{p},W}\rangle = \langle\vec{p}'|\vec{p}\rangle + \int d\vec{p}''\mathcal{G}(\vec{p}'|W)R(\vec{p}') \times\langle\vec{p}'|\bar{U}(W)|\vec{p}''\rangle\langle\vec{p}''|\bar{\psi}_{\vec{p},W}\rangle. \tag{4.3}$$

Equation (4.1) may be compared with the T-matrix equation [cf. Eq. (2.14)]

$$\langle\vec{p}'|T(W)|\vec{p}\rangle = \langle\vec{p}'|V(W)|\vec{p}\rangle + \int d\vec{p}''\langle\vec{p}'|V(W)|\vec{p}''\rangle\mathcal{G}(\vec{p}''|W) \times\langle\vec{p}''|T(W)|\vec{p}\rangle. \tag{4.4}$$

We may also write,[8] defining $\langle\vec{p}''|\psi_{\vec{p},W}\rangle$,

$$\langle\vec{p}'|T(W)|\vec{p}\rangle = \int d\vec{p}''\langle\vec{p}'|V(W)|\vec{p}''\rangle\langle\vec{p}''|\psi_{\vec{p},W}\rangle. \tag{4.5}$$

It is then not difficult to see that the relation

$$\langle\vec{p}'|\psi_{\vec{p},W}\rangle = \frac{R^{1/2}(\vec{p})}{R^{1/2}(\vec{p}')}\langle\vec{p}'|\bar{\psi}_{\vec{p},W}\rangle \tag{4.6}$$

is valid. We also note the wave equation for

$|\psi_{\vec{p},W}^{+}\rangle$,

$$\langle\vec{p}'|\psi_{\vec{p},W}^{+}\rangle=\langle\vec{p}'|\vec{p}\rangle+\int d\vec{p}''\mathcal{G}(\vec{p}'|W)\langle\vec{p}'|V(W)|\vec{p}''\rangle \times\langle\vec{p}''|\psi_{\vec{p},W}^{+}\rangle\,, \tag{4.7}$$

which follows from Eq. (4.4).

The wave function $|\bar{\psi}_{\vec{p},W}^{+}\rangle$ (and therefore $|\psi_{\vec{p},W}^{+}\rangle$) may be related *indirectly*, to the Bethe-Salpeter wave function, $|\psi_{\mathrm{p},W}^{\mathrm{BS}}\rangle\equiv|\psi_{\mathrm{p},p^0=\Delta(\vec{p}),W}^{\mathrm{BS}}\rangle$. Using Eq. (4.2) and noting that $\langle\vec{p}'|\bar{M}(W)|\vec{p}\rangle$ may also be written as

$$\begin{aligned}\langle\vec{p}'|\bar{M}(W)|\vec{p}\rangle&=\langle\vec{p}',p'^0=\Delta(\vec{p}')|M(W)|\vec{p},p^0=\Delta(\vec{p})\rangle\\&=\int\langle\vec{p}',p'^0=\Delta(\vec{p}')|U(W)|p''\rangle d^4p''\\&\quad\times\langle p''|\psi_{\mathrm{p},W}^{\mathrm{BS}}\rangle\,,\end{aligned} \tag{4.8}$$

we have the relation

$$\begin{aligned}&\int\langle\vec{p}',p'^0=\Delta(\vec{p}')|U(W)|p''\rangle d^4p''\langle p''|\psi_{\mathrm{p},W}^{\mathrm{BS}}\rangle\\&\quad=\int\langle\vec{p}',p'^0=\Delta(\vec{p}')|U(W)|\vec{p}'',p''^0=\Delta(\vec{p}'')\rangle d\vec{p}''\\&\qquad\times\langle\vec{p}''|\bar{\psi}_{\vec{p},W}^{+}\rangle\,.\end{aligned} \tag{4.9}$$

It is also clear from Eq. (4.9) that a knowledge of $|\bar{\psi}_{\vec{p},W}^{+}\rangle$ or $|\psi_{\vec{p},W}^{+}\rangle$ and the quantity $\langle\vec{p}'|\bar{U}(W)|\vec{p}\rangle$ is sufficient to calculate the on-shell invariant amplitude.

It is also of interest to consider the coupled problem as defined by Eq. (2.22) and the corresponding wave functions. We may write $\bar{\mathfrak{M}}=\bar{\mathfrak{U}}|\bar{\Psi}\rangle$ such that we obtain the (three-dimensional) equation,

$$|\bar{\Psi}\rangle=|\bar{\Phi}\rangle+\bar{g}_4\bar{\mathfrak{U}}|\bar{\Psi}\rangle\,. \tag{4.10}$$

Here $|\bar{\Psi}\rangle$ may be divided into "large" and "small" components:

$$|\bar{\Psi}\rangle=\begin{pmatrix}|\bar{\psi}^{+}\rangle\\|\bar{\psi}^{-}\rangle\end{pmatrix}. \tag{4.11}$$

Also, we have

$$|\bar{\Phi}_{\vec{p}}\rangle=\begin{pmatrix}|\vec{p}\rangle\\0\end{pmatrix}\equiv\begin{pmatrix}|\vec{p},p^0=\Delta(\vec{p})\rangle\\0\end{pmatrix}, \tag{4.12}$$

where p is the relative momentum in the incoming wave. Note, if $2W=(\vec{p}^2+m_\pi^{\,2})^{1/2}+(\vec{p}^2+m_N^{\,2})^{1/2}$, then $p^0=\Delta(\vec{p})=0$. The fact that $p^0=0$ for the incoming wave is a result of the subtraction of $\frac{1}{2}L$ from each zeroth component of momentum (see Fig. 1). In the case we wish to consider, the fully off-shell three-dimensional equation for the T matrix, Eq. (4.4), or for the wave function, Eq. (4.7), it is useful to define the quantity L as $L=E_{\vec{p}}-\omega_{\vec{p}}$, where p defined the relative momentum of the incoming wave. Of course, in the "on-shell" case, where $2W=(\vec{p}^2+m_\pi^{\,2})^{1/2}+(\vec{p}^2+m_N^{\,2})^{1/2}$, we regain the less general relation $L=(m_N^{\,2}-m_\pi^{\,2})/2W$. The more general definition of L, $L=E_{\vec{p}}-\omega_{\vec{p}}$, allows us to maintain the relation $p^0=0$ for the incoming wave in our equations.

Again we may write a more symmetrical form of Eq. (4.10) by introducing

$$|\Psi\rangle=\begin{pmatrix}|\psi^{+}\rangle\\|\psi^{-}\rangle\end{pmatrix}, \tag{4.13}$$

where for both $|\psi^{+}\rangle$ and $|\psi^{-}\rangle$ we define

$$\langle\vec{p}'|\psi_{\vec{p},W}^{+}\rangle\equiv\frac{R^{1/2}(\vec{p})}{R^{1/2}(\vec{p}')}\langle\vec{p}'|\bar{\psi}_{\vec{p},W}^{+}\rangle\,. \tag{4.14}$$

In that case we may write, with $|\Phi\rangle=|\bar{\Phi}\rangle$,

$$|\Psi\rangle=|\Phi\rangle+\mathcal{G}_4\mathcal{V}_4|\Psi\rangle\,, \tag{4.15}$$

where

$$\mathcal{G}_4(\vec{k}|W)=\begin{pmatrix}\dfrac{1}{2W-(\omega_{\vec{k}}+E_{\vec{k}})+i\eta}&0\\0&\dfrac{1}{2W+\omega_{\vec{k}}-E_{\vec{k}}+i\eta}\end{pmatrix}, \tag{4.16}$$

$$\mathcal{V}_4\equiv\begin{pmatrix}V_4^{++}&V_4^{+-}\\V_4^{-+}&V_4^{--}\end{pmatrix}, \tag{4.17}$$

and

$$\langle\vec{p}|V_4^{++}(W)|\vec{p}'\rangle=R^{1/2}(\vec{p})\langle\vec{p}|\bar{U}_4^{++}(W)|\vec{p}'\rangle R^{1/2}(\vec{p}')\,, \tag{4.18}$$

etc.

In the case that we work with the coupled equations, Eqs. (2.22), we may generalize Eq. (4.9) to

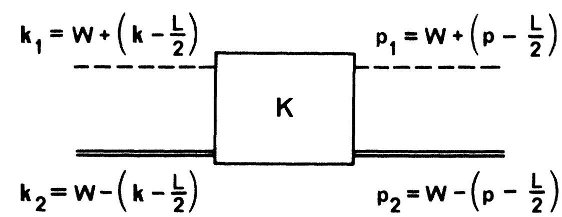

FIG. 1. The irreducible kernel for the Bethe-Salpeter equation, where W and L considered as four-vectors have only zeroth components. [See the work of Partovi and Lomon (Ref. 6) for a discussion of this representation.] The double line represents the heavy particle, a nucleon, or a nucleus. The dashed line is a pion.

the following set:

$$\int \langle \vec{p}', p'^0 = \Delta(\vec{p}') | U(W) | p'' \rangle d^4p'' \langle p'' | \psi^{BS}_{\vec{p}, W} \rangle$$
$$= \int \langle \vec{p}', p'^0 = \Delta(\vec{p}') | U(W) | \vec{p}'', p''^0 = \Delta(\vec{p}'') \rangle d\vec{p}'' \times \langle \vec{p}'' | \overline{\psi}^{+}_{\vec{p}, W} \rangle$$
$$+ \int \langle \vec{p}', p'^0 = \Delta(\vec{p}') | U(W) | \vec{p}'', p''^0 = \overline{\Delta}(\vec{p}'') \rangle d\vec{p}'' \times \langle \vec{p}'' | \overline{\psi}^{-}_{\vec{p}, W} \rangle , \qquad (4.19)$$

and

$$\int \langle \vec{p}', p'^0 = \overline{\Delta}(\vec{p}') | U(W) | p'' \rangle d^4p'' \langle p'' | \psi^{BS}_{\vec{p}, W} \rangle$$
$$= \int \langle \vec{p}', p'^0 = \overline{\Delta}(\vec{p}') | U(W) | \vec{p}'', p''^0 = \Delta(\vec{p}'') \rangle d\vec{p}'' \times \langle \vec{p}'' | \overline{\psi}^{+}_{\vec{p}, W} \rangle$$
$$+ \int \langle \vec{p}', p'^0 = \overline{\Delta}(\vec{p}') | U(W) | \vec{p}'', p''^0 = \overline{\Delta}(\vec{p}'') \rangle d\vec{p}'' \times \langle \vec{p}'' | \overline{\psi}^{-}_{\vec{p}, W} \rangle . \qquad (4.20)$$

These equations may be summarized in the following notation:

$$\overline{U\psi^{BS}}^{+} = \overline{U}^{++}\overline{\psi}^{+} + \overline{U}^{+-}\overline{\psi}^{-} \qquad (4.21)$$

and

$$\overline{U\psi^{BS}}^{-} = \overline{U}^{-+}\overline{\psi}^{+} + \overline{U}^{--}\overline{\psi}^{-} ; \qquad (4.22)$$

or more concisely,

$$\overline{U\psi^{BS}} = \overline{\mathfrak{U}}\,\overline{\Psi} . \qquad (4.23)$$

We note that the above discussion is general and could be used to discuss the wave functions for pion-*nucleus* scattering (see Sec. 5). To facilitate comparison of our methods to those based upon Klein-Gordon equation, we derive that equation using the static model introduced in Sec. 3. We recall Eq. (3.5) for the invariant amplitude $M^s(\omega)$ of the static model and now define

$$\langle \vec{k} | T^s(\omega) | \vec{k}' \rangle = \frac{1}{(2\omega_{\vec{k}})^{1/2}} \langle \vec{k} | \overline{M}^s(\omega) | \vec{k}' \rangle \frac{1}{(2\omega_{\vec{k}'})^{1/2}} , \qquad (4.24)$$

$$\langle \vec{k} | V^s(\omega) | \vec{k}' \rangle = \frac{1}{(2\omega_{\vec{k}})^{1/2}} \langle \vec{k} | \overline{U}^s(\omega) | \vec{k}' \rangle \frac{1}{(2\omega_{\vec{k}'})^{1/2}} , \qquad (4.25)$$

so that we may write

$$\langle \vec{k} | T^s(\omega) | \vec{k}' \rangle = \langle \vec{k} | V^s(\omega) | \vec{k}' \rangle + \int d\vec{p} \langle \vec{k} | V^s(\omega) | \vec{p} \rangle \left(\frac{2\omega}{\omega^2 - \omega_{\vec{p}}^2 + i\eta} \right) \times \langle \vec{p} | T^s(\omega) | \vec{k}' \rangle . \qquad (4.26)$$

If we define the static model wave functions $| \psi^{s(+)}_{\vec{k}, W} \rangle$ through the relation

$$\langle \vec{p} | T^s(\omega) | \vec{k} \rangle = \langle \vec{p} | V^s(\omega) | \psi^{s(+)}_{\vec{k}, W} \rangle , \qquad (4.27)$$

we find the equation for $| \psi^{s(+)}_{\vec{k}, W} \rangle$ to be

$$\langle \vec{p} | \psi^{s(+)}_{\vec{k}, W} \rangle = \langle \vec{p} | \vec{k} \rangle + \frac{1}{\omega^2 - \omega_{\vec{p}}^2 + i\eta} \langle \vec{p} | \mathfrak{V}(\omega) | \psi^{s(+)}_{\vec{k}, W} \rangle \qquad (4.28)$$

with the definition,

$$\langle \vec{p} | \mathfrak{V}(\omega) | \vec{p}' \rangle \equiv 2\omega \langle \vec{p} | V^s(\omega) | \vec{p}' \rangle . \qquad (4.29)$$

We recognize Eq. (4.28) as the Klein-Gordon equation with an energy-dependent, nonlocal potential $\mathfrak{V}$. In Appendix A we compare our "static model" result with a similar result based upon the use of a different g.

5. PION-NUCLEUS SCATTERING

For simplicity we consider a pion scattering from a heavy nucleus of spin $\frac{1}{2}$. The preceding formalism may be taken over with the nucleon mass m_N replaced by the mass of the nucleus m_A. If we put $E_{\vec{k}} = (\vec{k}^2 + m_A{}^2)^{1/2}$, we may define $R_{\pi A}(\vec{k}) = (m_A / 2\omega_{\vec{k}} E_{\vec{k}}) \simeq (2\omega_{\vec{k}})^{-1}$. Since the use of a Green's function such as $g_5(k|W)$, which keeps the *nucleus* on the mass shell, is reasonable in this case we will not discuss coupled equations in this section. For the discussion of pion-nucleus scattering we may use a g, denoted as g_A, and defined by

$$g_A(k|W) = \frac{m_A}{2\omega_{\vec{k}} E_{\vec{k}}} \delta[k^0 - \Delta_A(\vec{k})] \Lambda_+(\vec{k}) \times \left[\frac{1}{2W - (\omega_{\vec{k}} + E_{\vec{k}}) + i\eta} + \frac{1}{2W + \omega_{\vec{k}} - E_{\vec{k}}} \right] , \qquad (5.1)$$

where $\Lambda_+(\vec{k})$ is now the projection operator for positive energy *nucleus* spinors and $\Delta_A(\vec{k}) = (W + \frac{1}{2}L - E_{\vec{k}}) \simeq 0$. Here $L = (m_A{}^2 - m_\pi{}^2)/2W$. [We might note, at this point, that if we go to the static limit, replacing $E_{\vec{k}}$ by the mass of the target nucleus in Eq. (5.1), we would have a propagator which allows us to use the Klein-Gordon equation for the meson.]

Since in this work we are not concerned with the details of the evaluation of the irreducible kernels K we will only indicate some of the more impor-

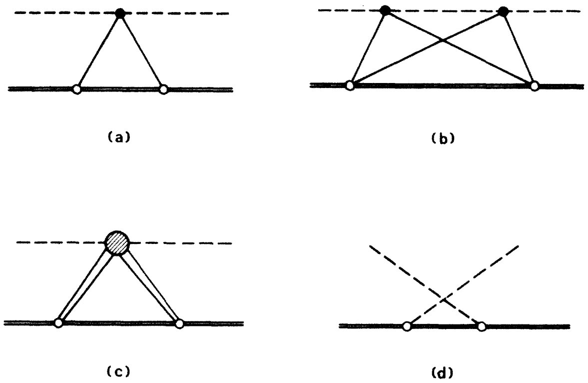

FIG. 2. Diagrams for the calculation of the invariant potential. (a) Relativistic impulse approximation, where the open circle is a presumably known vertex function and the filled circle is the pion-nucleon scattering amplitude (off-shell). (b) A diagram involving the scattering from a correlated pair. The iteration of (a) must be excluded from this diagram. (c) A diagram involving a $(\pi NN) \rightarrow (\pi NN)$ amplitude that is not expressible in the form indicated in (b). (d) A crossed diagram which is not contained in (a); that is, the pion is not absorbed and emitted by the same nucleon.

tant features which should be considered in an approach based upon a relativistic impulse approximation. The most important term to be included in the irreducible kernel is shown in Fig. 2(a). In this figure the filled circle represents an (off-shell) pion-nucleon scattering amplitude and the open circles are vertex functions describing the (virtual) breakup of the target (baryon number A) into a nucleon and a residual nucleus (baryon number $A-1$). As indicated previously,[5] we may use this diagram to make contact with the corresponding leading term in the Watson multiple scattering series. While the Feynman rules make precise the exact "off-shell" aspect of the pion-nucleon amplitude, it is often useful to consider the approximation in which the result of the evaluation of this diagram is factored into an on-shell pion-nucleon amplitude, depending upon the square of the four-momentum transfer q^2 and a form factor for the target.[5] (The evaluation of this diagram and other important diagrams will be discussed in a future publication.)

Away from the main peak in the elastic scattering cross section, the "correlation" diagram of Fig. 2(b) is expected to be important. This diagram involves vertex functions for the breakup of the target into two nucleons and a residual nucleus of baryon number $A-2$. The pion-nucleon amplitudes entering into the calculation of Fig. 2(b) will be much further "off-shell" than that of Fig. 2(a). However, for sufficiently high pion energies we may still hope to factor the result of the evaluation of Fig. 2(b) into a pair of pion-nucleon scattering amplitudes and a correlation function for the target. This term then can be seen to correspond to the double-scattering term of the Watson series for the optical potential. [It is necessary to point out that only the irreducible part of Fig. 2(b) should be calculated. The diagram as drawn contains the iteration of Fig. 1(a), which must be subtracted explicitly. This problem arises here because of the composite nature of the target.]

In general, the interaction of a pion with a pair of nucleons will not be fully represented by Fig. 2(b) so that one might wish to consider additional interaction terms. These are indicated by the large circle in Fig. 2(c) which denotes that part of the pion–two-nucleon scattering amplitude which is not already contained in Fig. 2(b).

In Fig. 2(d) we have indicated a crossed diagram which must be calculated such as not to include contributions already contained in Fig. 2(a). Other crossed diagrams may also be included; for example, one may obtain irreducible diagrams by interchanging the external pion lines in the iteration of Fig. 2(a) or from interchanging the external pion lines in Fig. 2(b).

We assume that some choice of diagrams has been made and one has calculated a sum of irreducible diagrams K_A and, in some approximation, the effective potential U_A, which satisfies

$$U_A = K_A + K_A(G - g_A)U_A \,. \tag{5.2}$$

Following the now standard procedure, we define

$$\langle \vec{k}' | V_A(W) | \vec{k} \rangle = R_{\pi A}{}^{1/2}(\vec{k}') \langle \vec{k}', k'^0 = \Delta_A(\vec{k}') | U_A(W) | \vec{k}, k^0 = \Delta_A(\vec{k}) \rangle R_{\pi A}{}^{1/2}(\vec{k}) \tag{5.3}$$

and

$$\langle \vec{k}' | T_A(W) | \vec{k} \rangle = R_{\pi A}{}^{1/2}(\vec{k}') \langle \vec{k}, k'^0 = \Delta_A(\vec{k}') \, | M(W) | \vec{k}, k^0 = \Delta_A(\vec{k}) \rangle R_{\pi A}{}^{1/2}(\vec{k}) \,. \tag{5.4}$$

We then have the equation

$$\langle \vec{k}' | T_A(W) | \vec{k} \rangle = \langle \vec{k}' | V_A(W) | \vec{k} \rangle + \int \langle \vec{k}' | V_A(W) | \vec{k}'' \rangle d\vec{k}'' \left[\frac{1}{2W - (\omega_{\vec{k}''} + E_{\vec{k}''}) + i\eta} + \frac{1}{2W + \omega_{\vec{k}''} - E_{\vec{k}''}} \right] \langle \vec{k}'' | T_A(W) | \vec{k} \rangle \tag{5.5}$$

which describes pion-nucleus scattering. Again, it is clear that we may obtain the Klein-Gordon equation by going to the static limit $(2W - E_{\vec{k}}) \to \omega$; $\Delta_A(\vec{k}) \to 0$. Finally, the above result may be generalized without difficulty to include inelastic *two-body channels*.[5]

6. CONCLUSIONS

In this work we have discussed the derivation of relativistic three-dimensional equations for pion-nucleon or pion-nucleus scattering. In the case of pion-nucleon scattering one may attempt to construct phenomenological potentials which, when inserted in these equations, reproduce the on-shell information. Of course, these potentials cannot be determined only from a knowledge of the on-shell T matrices. Therefore, we have all the standard problems associated with the freedom available in specifying the off-shell behavior of the scattering amplitude. In this connection we have shown how different off-shell extensions are related to the choice of the g used in deriving relativistic three-dimensional equations. (For example, see Appendix A.) In this case it is obviously desirable to have some fundamental theory which will allow one to calculate the potentials which enter our equations.

In the case of pion-nucleus scattering, we have provided a theory which we hope will replace the standard Watson multiple-scattering analysis.[3] The latter theory is essentially nonrelativistic and unsuited to the description of the scattering of a projectile which can be created or destroyed, as is the case with the pion.

We are still left with major questions as to the convergence of the theory. That is, it is not obvious which diagrams, beyond the most obvious, must be included in the calculation of the potentials for pion-nucleus scattering. In a future work we will discuss the calculation of the leading terms of the relativistic impulse approximation and try to answer some of these open questions.

APPENDIX A

Relativistic quantum mechanics has been considered as an alternative to quantum field theory and analytic S matrix theory. A prescription for constructing such a relativistic theory was proposed by Bakamjian and Thomas.[7,9] According to this scheme, the interaction between two particles is represented by a potential energy operator which is incorporated in the theory in such a way that the generators of a proper inhomogeneous Lorentz group satisfy the Lie algebra of this group. The potential is a rotationally invariant function of the internal c.m. dynamical variables. For the two-body case, Fong and Sucher[10] have proved that if the potential introduced in the Bakamjian-Thomas Hamiltonian vanishes sufficiently rapidly for large relative position vector, then the associated S matrix is covariant. The Bakamjian-Thomas Hamiltonian is therefore the most general form of interest from the viewpoint of relativistic scattering.

For π-N scattering, the Bakamjian-Thomas Hamiltonian in the c.m. system can be written as

$$H = (\vec{k}^2 + m_\pi^2)^{1/2} + (\vec{k}^2 + m_N^2)^{1/2} + V^{BT}(\vec{r}, \vec{k}, \vec{s}_N)\,. \tag{A1}$$

Here $\vec{r}$ and $\vec{k}$ are the nonrelativistic relative position and momentum operators, respectively, and $\vec{s}_N$ is the spin of the nucleon. If $2W$ is the total c.m. energy of the system, then the wave equation has the form

$$H\,|\psi\rangle = 2W\,|\psi\rangle \tag{A2}$$

and the corresponding Lippmann-Schwinger equation is

$$\langle\vec{k}|\,T(W)\,|\vec{k}'\rangle = \langle\vec{k}|\,V^{BT}|\,\vec{k}'\rangle + \int \langle\vec{k}|\,V^{BT}|\,\vec{p}\rangle \frac{d\vec{p}}{2W - \omega_{\vec{p}} - E_{\vec{p}} + i\eta} \times \langle\vec{p}|\,T(W)\,|\vec{k}'\rangle\,, \tag{A3}$$

where

$$\omega_{\vec{p}} = (m_\pi^2 + \vec{p}^2)^{1/2}\,, \quad E_{\vec{p}} = (m_N^2 + \vec{p}^2)^{1/2}\,.$$

Equation (A3) is in analogy to Eq. (2.14) of the text. The only difference is that the potential introduced in Eq. (2.14) can be determined, at least in principle, from the meson theory, while the potential introduced in Eq. (A3) is an unknown arbitrary function of $\vec{r}$, $\vec{k}$, and $\vec{s}_N$. Now, if we identify these two equations, we find that the potential in BT equation can be identified with the potential in Eq. (2.14). Thus we have provided a method to obtain the potential used in BT equation from field theory.

We must point out however that different approximate Green's functions may lead to similar wave equations having different off-shell extrapolations. To see this, we consider the Gross approximation.[11] We recall that in the treatment of Gross, the nucleon is put on its mass shell and no approximation is made for the pion propagator. We may therefore write the corresponding approximate Green's function as

$$g_G(k|W) = \delta[(W - k + \tfrac{1}{2}L)^2 - m_N^2]\,\theta(W - k^0 + \tfrac{1}{2}L) \times [\gamma\cdot(W - k + \tfrac{1}{2}L) + m_N] \times \frac{1}{(W + k - \tfrac{1}{2}L)^2 - m_\pi^2 + i\eta} \tag{A4}$$

which is obviously different from $g_5(k|W)$ of Eq. (2.25).

It is easy to show that with g_G we obtain for pion-nucleon scattering in the static limit

$$\langle\vec{k}|\bar{M}^s(\omega)|\vec{k}'\rangle = \langle\vec{k}|\bar{U}^s_G(\omega)|\vec{k}'\rangle + \int \langle\vec{k}|\bar{U}^s_G(\omega)|\vec{p}\rangle d\vec{p}\,\frac{1}{\omega^2-\omega_{\vec{p}}^{2}+i\eta} \times\langle\vec{p}|\bar{M}^s(\omega)|\vec{k}'\rangle\,, \tag{A5}$$

with $\langle\vec{k}|\bar{M}^s(\omega)|\vec{k}'\rangle$ and $\langle\vec{k}|\bar{U}^s_G(\omega)|\vec{k}'\rangle$ defined by equations analogous to Eqs. (3.3) and (3.4). We also note the difference between Eq. (A5) and Eq. (3.5). If we now define

$$\langle\vec{k}|\hat{T}^s(\omega)|\vec{k}'\rangle \equiv \frac{1}{\sqrt{2\omega}}\langle\vec{k}|\bar{M}^s(\omega)|\vec{k}'\rangle\frac{1}{\sqrt{2\omega}}, \tag{A6}$$

$$\langle\vec{k}|\hat{V}^s(\omega)|\vec{k}'\rangle \equiv \frac{1}{\sqrt{2\omega}}\langle\vec{k}|\bar{U}^s_G(\omega)|\vec{k}'\rangle\frac{1}{\sqrt{2\omega}}, \tag{A7}$$

we may write Eq. (A5) as

$$\langle\vec{k}|\hat{T}^s(\omega)|\vec{k}'\rangle = \langle\vec{k}|\hat{V}^s(\omega)|\vec{k}'\rangle + \int\langle\vec{k}|\hat{V}^s(\omega)|\vec{p}\rangle d\vec{p}\,\frac{2\omega}{\omega^2-\omega_{\vec{p}}^{2}+i\eta} \times\langle\vec{p}|\hat{T}^s(\omega)|\vec{k}'\rangle\,. \tag{A8}$$

Then, we observe that the static-model wave function $|\hat{\psi}^{s(+)}_{\vec{k}',W}\rangle$, defined through

$$\langle\vec{k}|\hat{T}^s(\omega)|\vec{k}'\rangle = \langle\vec{k}|\hat{V}^s(\omega)|\hat{\psi}^{s(+)}_{\vec{k}',W}\rangle\,, \tag{A9}$$

satisfies the equation

$$\langle\vec{p}|\hat{\psi}^{s(+)}_{\vec{k},W}\rangle = \langle\vec{p}|\vec{k}\rangle + \frac{1}{\omega^2-\omega_{\vec{p}}^{2}+i\eta}\langle\vec{p}|2\omega\hat{V}^s(\omega)|\hat{\psi}^{s(+)}_{\vec{k},W}\rangle\,, \tag{A10}$$

which has the same structure as Eq. (4.28). Yet the *off-shell* extrapolation Eq. (A6) is very different from that in Eq. (4.24). This difference vanishes, of course, in the on-shell limit. This implies that $\mathfrak{V}(\omega)$ of Eq. (4.29) and

$$\langle\vec{p}|\hat{\mathfrak{V}}(\omega)|\vec{p}'\rangle \equiv 2\omega\langle\vec{p}|\hat{V}^s(\omega)|\vec{p}'\rangle \tag{A11}$$

are "elastically-equivalent" potentials, in that they yield the same on-shell T matrices.

A list of various possible choices for g, all of which lead to somewhat different three-dimensional relativistic equations, can be found in Ref. 12.

APPENDIX B

In this Appendix we will indicate the solution of our coupled equations for $|\bar{\Psi}\rangle$, Eq. (2.22). For simplicity we consider $\bar{U}^{++}=\bar{U}^{+-}=\bar{U}^{-+}=\bar{U}^{--}$. We may choose, then,

$$\langle\vec{k}|\bar{U}(\omega)|\vec{k}'\rangle = -\frac{\lambda}{2\pi^2}\frac{v(k)}{\sqrt{\omega_{\vec{k}}}}\frac{v(k')}{\sqrt{\omega_{\vec{k}'}}}P_{33} \tag{B1}$$

the form we used in the static model. It is then easy to obtain with $p'=|\vec{p}'|$, $p=|\vec{p}|$, etc.,

$$\langle p'|\bar{\psi}^+_p\rangle = \frac{\delta(p-p')}{pp'} - \left(\frac{m_N}{2\omega_{p'}E_{p'}}\right)\frac{1}{2W-\omega_{p'}-E_{p'}+i\eta} \times\left(\frac{\lambda}{2\pi^2}\right)\frac{v(p')v(p)}{(\omega_{p'}\omega_p)^{1/2}D(\omega)}, \tag{B2}$$

with

$$D(\omega) = \left[1+\frac{\lambda}{2\pi^2}\int d\vec{p}''\left(\frac{m_N}{2\omega_{p''}^{2}E_{p''}}\right)|v(p'')|^2 \times\left\{\frac{1}{2W-\omega_{p''}-E_{p''}+i\eta}+\frac{1}{2W+\omega_{p''}-E_{p''}+i\eta}\right\}\right]. \tag{B3}$$

Further

$$\langle p'|\bar{\psi}^-_p\rangle = -\left(\frac{m_N}{2\omega_{p'}E_{p'}}\right)\frac{1}{2W+\omega_{p'}-E_{p'}}\left(\frac{\lambda}{2\pi^2}\right)\frac{v(p')v(p)}{(\omega_{p'}\omega_p)^{1/2}D(\omega)}. \tag{B4}$$

Inspection of Eqs. (B2) and (B4) provides one with an idea of the relative magnitudes of the "large" and "small" components in the separable model. The T matrix for the static approximation may also be readily obtained.

APPENDIX C

In this Appendix we consider the case of an *energy dependent* potential and replace Eq. (3.6) by

$$\langle\vec{k}|\bar{U}^s(\omega)|\vec{k}'\rangle = -\frac{\tilde{\lambda}}{2\pi^2}\tilde{v}(k)\left(\frac{1}{\omega}\right)\tilde{v}(k')P_{33}\,. \tag{C1}$$

Then Eq. (3.7) is replaced by

$$\langle k|\bar{M}^s_{33}(\omega)|k'\rangle = \frac{-(\tilde{\lambda}/2\pi^2)\tilde{v}(k)\tilde{v}(k')/\omega}{1+\tilde{\lambda}I(\omega)} \tag{C2}$$

where

$$I(\omega) = \frac{2}{\pi}\int\frac{p^2dp}{\omega_{\vec{p}}}\frac{[\tilde{v}(p)]^2}{2\omega}\left[\frac{1}{\omega-\omega_{\vec{p}}+i\eta}+\frac{1}{\omega+\omega_{\vec{p}}}\right]. \tag{C3}$$

If we then define a "renormalized" constant λ_R by

$$\lambda_R = \tilde{\lambda}/[1+\tilde{\lambda} I(0)], \tag{C4}$$

we may obtain for the T matrix

$$\langle k | T^s_{33}(\omega) | k' \rangle = \frac{-(\lambda_R/4\pi^2)\tilde{v}(k)\tilde{v}(k')/(\omega^2 \omega^+_k \omega^+_{k'})^{1/2}}{1-(\lambda_R \omega/\pi)\int (p\,d\omega^+_p/\omega^{+2}_p)[\tilde{v}(p)]^2[1/(\omega^+_p-\omega-i\eta)-1/(\omega+\omega^+_p)]}, \tag{C5}$$

or

$$\langle k | T^s_{33}(\omega) | k \rangle_{\text{on-shell}} = \frac{-(\lambda_R/4\pi^2)[\tilde{v}(k)]^2/\omega^2}{1-(\lambda_R \omega/\pi)\int (p\,d\omega^+_p/\omega^{+2}_p)[\tilde{v}(p)]^2[1/(\omega^+_p-\omega-i\eta)-1/(\omega+\omega^+_p)]}. \tag{C6}$$

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