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Multiple-scattering theory and the relativistic optical model*

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It is shown how to construct Lippmann-Schwinger equations for the scattering of relativistic projectiles from nuclear targets. The method is based on the Blankenbecler-Sugar reduction of a covariant equation. Among other topics, we discuss the optical model, the relativistic impulse approximation, coupled channel equations, and inelastic scattering.

It is somewhat surprising that one does not currently have a well developed theory of the optical model appropriate to relativistic projectiles. The eikonal theory of Glauber[1] has, of course, had great success in interpretating the data in the relativistic domain; however, this theory rests on specialized assumptions, and a theory which does not presuppose the fixed-scatterer and eikonal approximations from the outset is clearly desirable. Further, while the multiple-scattering theory of Watson,[2] or Kerman, McManus, and Thaler[3] has been used for relativistic projectiles,[4] these applications are somewhat ambiguous in that these multiple-scattering theories are based completely on non-relativistic dynamics. In this work we hope to provide a basis for a systematic discussion of the scattering of a relativistic projectile from a nucleus and to indicate which approximation schemes may be useful. This work, further, forms a natural extension of the theory presented previously[5] for the scattering of non-relativistic projectiles from correlated nuclei and will reduce to that theory in the non-relativistic domain.

As in the case of previous theories of the scattering from a complex target, we will assume that the properties of the target and its various excited states are completely known. As we shall see, this means that we require a knowledge of a class of vertex functions describing the (virtual) breakup of the target and/or its excited states into their "constituents." We also assume that we may introduce a separate field for each nuclear bound state, that is, we treat each nuclear state as an elementary particle having its own propagator. This last assumption allows us to use Feynman diagrams to describe the dynamics of the scattering process. The further assumptions necessary for a viable theory will be indicated as we proceed. We now turn to a review of the Blankenbecler-Sugar[6] approach based on the discussion of Partovi and Lomon.[7]

The invariant amplitude, M, for the scattering of a projectile (particle 1) from a target nucleus (particle 2) satisfies the integral equation (see Fig. 1),

$$M = K + KGM, \tag{1}$$

where K is an (irreducible) kernel and G propagates the projectile and nucleus between interactions. Equation (1) is equivalent to the set of equations

$$M = U + UgM, \tag{2}$$

$$U = K + K(G - g)U, \tag{3}$$

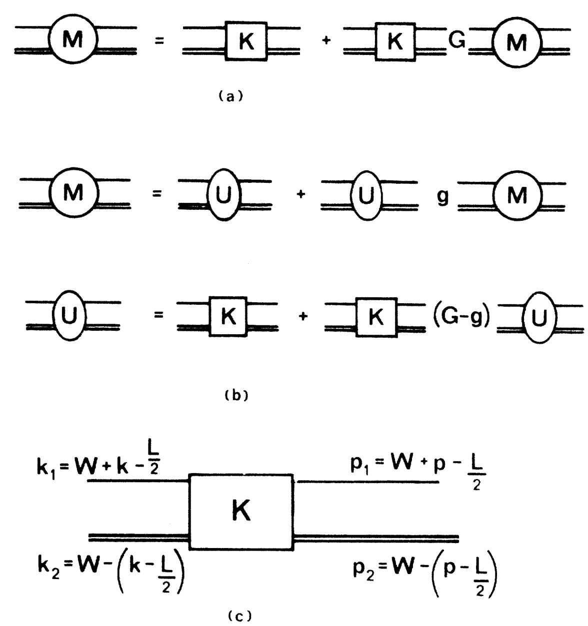

FIG. 1. (a) Schematic representation of the covariant equation for nucleon-nucleus scattering. The double line represents the nucleus. (b) A set of equations equivalent to that in (a). (c) Graphical representation of $K(k,p|W)$ where $2W$ is the total energy in the center-of-mass system, k and p are relative momenta, and the four-momentum L has only a fourth component $L^0 = (m_2^2 - m_1^2)/2W$. As discussed in Ref. 7, the use of L^0 serves to eliminate the *relative* time dependence in the incoming wave function. Similar diagrams may be drawn for $M(k,p|W)$ and $U(k,p|W)$.

where g is a new Green's function. The choice of g is somewhat arbitrary and a judicious choice is necessary for a good approximation scheme. Certain choices reduce Eq. (2) to a three-dimensional equation of the Lippmann-Schwinger form, with a non-local energy-dependent potential. For definiteness we consider the scattering of a spin-$\frac{1}{2}$ projectile from a spin-$\frac{1}{2}$ nucleus and note two possible choices for g. First we write Eqs. (1) and (2) as [see Fig. 1(c)]

$$M(p',p\,|\,W)=K(p',p\,|\,W)+\int d^4k K(p',k|\,W)G(k|\,W)M(k,p\,|\,W) \quad (4)$$

$$=U(p',p\,|\,W)+\int d^4k\, U(p',k|\,W)g(k|\,W)M(k,p\,|\,W), \quad (5)$$

where using the notation of Fig. 1(c),

$$G(k\,|\,W)\equiv\frac{i}{2\pi}\left[\frac{1}{\gamma\cdot(W+k-\frac{1}{2}L)-m_1+i\eta}\right]\times\left[\frac{1}{\gamma\cdot(W-k+\frac{1}{2}L)-m_2+i\eta}\right]. \quad (6)$$

Now the Partovi-Lomon choice for g is

$$g_1(k\,|\,W)=\int\frac{d\vec{q}'^2}{\vec{q}^2-\vec{q}'^2+i\eta}\delta[(W'+k-\tfrac{1}{2}L)^2-m_1^2]\times\theta(W'+k^0-\tfrac{1}{2}L)\delta[(W'-k+\tfrac{1}{2}L)^2-m_2^2]\times\theta(W'-k^0+\tfrac{1}{2}L)[\gamma\cdot(W'+k-\tfrac{1}{2}L)+m_1]\times[\gamma\cdot(W'-k+\tfrac{1}{2}L)+m_2] \quad (7)$$

with

$$2W=(\vec{q}^2+m_1^2)^{1/2}+(\vec{q}^2+m_2^2)^{1/2} \quad (8)$$

and

$$2W'=(\vec{q}'^2+m_1^2)^{1/2}+(\vec{q}'^2+m_2^2)^{1/2}. \quad (9)$$

Evaluating the integral in Eq. (7) we have

$$g_1(k|\,W)=\left(\frac{m_1+m_2}{E_1(\vec{k})+E_2(\vec{k})}\right)\times\frac{\delta[k^0-\Delta_1(\vec{k})]}{\vec{q}^2/2\mu-\vec{k}^2/2\mu+i\eta}\Lambda_+^{(1)}\Lambda_+^{(2)}, \quad (10)$$

where μ is the reduced mass and

$$E_1(\vec{k})=(\vec{k}^2+m_1^2)^{1/2},\quad E_2(\vec{k})=(\vec{k}^2+m_2^2)^{1/2},$$
$$\Delta_1(\vec{k})=\tfrac{1}{2}[L+E_1(\vec{k})-E_2(\vec{k})]. \quad (11)$$

$\Lambda_+^{(1)}$ is the positive energy projection operator for particle 1 etc.

Among the various alternate forms for g we also consider a form in which one of the particles is always on-shell,

$$g_2(k|\,W)=\left\{\int\frac{d\vec{q}'^2}{\vec{q}^2-\vec{q}'^2+i\eta}\delta[(W'+k-\tfrac{1}{2}L)^2-m_1^2]\theta(W'+k^0-\tfrac{1}{2}L)[\gamma\cdot(W'+k-\tfrac{1}{2}L)+m_1]\right\}\times\delta[(W-k+\tfrac{1}{2}L)^2-m_2^2]\theta(W-k^0+\tfrac{1}{2}L)(2m_2\Lambda_+^{(2)})$$
$$=\left(\frac{m_1+m_2}{E_1(\vec{k})+E_2(\vec{k})}\right)\frac{\delta[k^0-\Delta_2(\vec{k})]}{\vec{q}^2/2\mu-\vec{k}^2/2\mu+i\eta}\Lambda_+^{(1)}\Lambda_+^{(2)}, \quad (12)$$

with

$$\Delta_2(\vec{k})=W+\tfrac{1}{2}L-E_2(\vec{k}).$$

It is also useful to define the quantity,

$$R(m_1,m_2,\vec{k})\equiv\frac{(m_1+m_2)}{E_1(\vec{k})+E_2(\vec{k})} \quad (13)$$

and, for either function, $\Delta_1(\vec{k})$ or $\Delta_2(\vec{k})$, we define,

$$\overline{M}(\vec{p},\vec{p}'\,|\,W)\equiv M[\vec{p},p^0=\Delta(\vec{p});\vec{p}',p'^0=\Delta(\vec{p}')|\,W], \quad (14)$$

with the further restriction that $\overline{M}$ is only defined if matrix elements of Eq. (14) are taken between positive energy spinors. A similar definition is made relating $\overline{U}$ to U, so that $\overline{M}$ satisfies,

$$\overline{M}(\vec{p},\vec{p}'\,|\,W)=\overline{U}(\vec{p},\vec{p}'\,|\,W)+\int d\vec{k}\,\overline{U}(\vec{p},\vec{k}\,|\,W)\frac{1}{\vec{q}^2/2\mu-\vec{k}^2/2\mu+i\eta}\times R(m_1,m_2,\vec{k})\overline{M}(\vec{k},\vec{p}'\,|\,W). \quad (15)$$

Finally, defining a T matrix through,

$$T(\vec{p},\vec{p}'\,|\,W)=R(m_1,m_2,\vec{p})^{1/2}\overline{M}(\vec{p},\vec{p}'\,|\,W)R(m_1,m_2,\vec{p}')^{1/2}, \quad (16)$$

we find that

$$T(\vec{p}, \vec{p}' \mid W) = V(\vec{p}, \vec{p}' \mid W) + \int d\vec{k}\, V(\vec{p}, \vec{k} \mid W) \frac{1}{\vec{q}^2/2\mu - \vec{k}^2/2\mu + i\eta} \times T(\vec{k}, \vec{p}' \mid W), \quad (17)$$

if V is related to $\overline{U}$ as T is related to $\overline{M}$.

It is clear that in Eq. (17) we have the result that the scattering of a relativistic projectile from a nucleus may be described by a Lippmann-Schwinger equation with a non-local, energy-dependent potential as long as the relative energy $\vec{q}^2/2\mu$ is calculated using Eq. (8) to determine q.

Before proceeding to approximation schemes for the determination of $V(\vec{p}, \vec{p}' \mid W)$, we may note that Eq. (17) may be generalized to describe the coupling of the elastic channel to some given subset of inelastic *two-body channels*. To this end we may label the relevant channels by indices i, j, etc. The masses in channel i are then m_1^i and m_2^i, the propagators are $G^{(i)}$ and $g^{(i)}$, etc. Equation (17) becomes

$$T_{ij}(\vec{p}, \vec{p}' \mid W) = V_{ij}(\vec{p}, \vec{p}' \mid W) + \sum_l \int d\mathbf{k}\, V_{il}(\vec{p}, \vec{k} \mid W) \times \frac{1}{\vec{q}_l^{\,2}/2\mu_l - \vec{k}^2/2\mu_l + i\eta} \times T_{lj}(\vec{k}, \vec{p}' \mid W), \quad (18)$$

where $2W = (\vec{q}_l^{\,2} + m_1^{l2})^{1/2} + (\vec{q}_l^{\,2} + m_2^{l2})^{1/2}$ and μ_l is the reduced mass in channel l. Further, M_{ij} and U_{ij} are related to the invariant amplitude T_{ij} and V_{ij} through factors appropriate to the spins and masses of the particles in the channels. For example,

$$T_{ij}(\vec{p}, \vec{p}' \mid W) = R(m_1^i, m_2^i, \vec{p})^{1/2} \overline{M}_{ij}(\vec{p}, \vec{p}' \mid W) \times R(m_1^j, m_2^j, \vec{p}')^{1/2}, \quad (19)$$

where $R(m_1^i, m_2^i, \vec{p})$ is the appropriate numerical factor for the channel i [given by Eq. (13) for two spin-$\frac{1}{2}$ particles].

The preceding formalism is not particularly useful unless one has some idea as to a reasonable set of approximations for the calculation of the irreducible interaction K once a form for g is chosen. As an example, we consider the case of nucleon-nucleus scattering and define a decomposition of K into diagrams which are the relativistic analogs of those appearing in the non-relativistic multiple scattering theory. These diagrams are shown in Fig. 2 and are appropriate to a relativistic impulse approximation, containing no explicit meson propagators. In these figures the black dot refers to the (off-shell) nucleon-nucleon scattering amplitude and the open circles are various vertex functions. The diagrams on the right of Fig. 2(a) may be termed impulse, correlation, and exchange diagrams. These diagrams are further analyzed in Figs. 2(b), 2(c), 2(d), into diagrams of similar structure, but having different numbers of particles in the intermediate states. Our understanding of the target structure and the non-relativistic multiple scattering theory leads us to believe that this series can represent the most important (irreducible) scattering processes.

To show the utility of our approach we will discuss the first diagram in Fig. 2(b) in somewhat greater detail. This is by far the most important diagram at high energies for a target which may be described using Hartree-Fock or Brueckner-Hartree-Fock theory. (The other diagrams will be discussed in greater detail elsewhere.) Referring to Fig. 3, we note that if the energy of the projectile is quite large it may be reasonable to assume that the nucleon-nucleon scattering amplitude depends only weakly on the momentum κ. Indeed, let us assume that this scattering amplitude depends mainly on ϵ, the mean energy available in the center-of-mass system of the two nucleons, and upon the momentum transfer, q. With that approximation one may perform the integral over

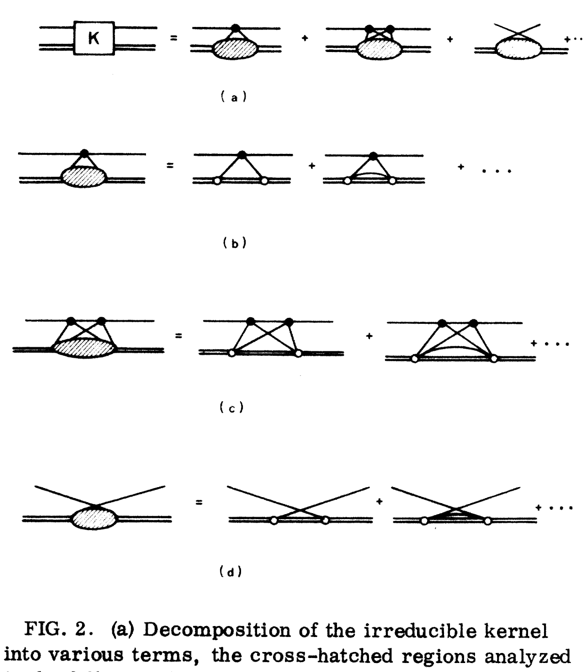

FIG. 2. (a) Decomposition of the irreducible kernel into various terms, the cross-hatched regions analyzed in the following figures. (b) Impulse terms decomposed according to the number of free propagators in the intermediate states. (c) Correlation terms decomposed as in (b). (d) Exchange scattering terms decomposed as in (b). These terms are small for high-energy projectiles.

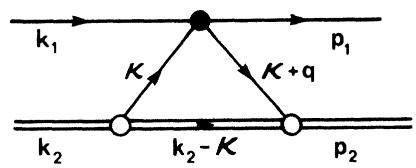

FIG. 3. The leading term in a relativistic impulse approximation. The single lines represent nucleons and the double lines represent nuclei. The open circles are vertex functions which are presumed to be known.

κ, and obtain a factorized result of the form,

$$K(\vec{k}, \vec{p} \mid W) \simeq \mathfrak{M}(\epsilon, q^2)\rho(q^2), \tag{20}$$

where $\mathfrak{M}$ is an on-shell approximation to the nucleon-nucleon scattering amplitude and $\rho(q^2)$ is a form factor. In the simplest approximation ($\overline{U} \simeq \overline{K}$), therefore,

$$\begin{aligned} V(\vec{p}, \vec{p}\,' \mid W) &\simeq R(m_1, m_2, \vec{p})^{1/2}\overline{K}(\vec{p}, \vec{p}\,' \mid W) \\ &\quad \times R(m_1, m_2, \vec{p}\,')^{1/2} \\ &\simeq \mathfrak{M}(\epsilon, \vec{q}^{\,2})\rho(\vec{q}^{\,2}) \end{aligned} \tag{21}$$

with $\vec{q} = \vec{p}\,' - \vec{p}$, and

$$\overline{\rho}(\vec{q}^{\,2}) \equiv \rho(q^2) \mid_{q^0 = \Delta(\vec{p}\,') - \Delta(\vec{p})}, \tag{22}$$

$$\overline{\mathfrak{M}}(\epsilon, \vec{q}^{\,2}) \equiv \mathfrak{M}(\epsilon, q^2) \mid_{q^0 = \Delta(\vec{p}\,') - \Delta(\vec{p})}. \tag{23}$$

In Eq. (21) we have also used the fact that $R \simeq 1$ in the case of a heavy target. It is possible to show [most easily if we use g_2 of Eq. (12)] that $\overline{\rho}(\vec{q}^{\,2})$ is simply related to the Fourier transform of the target density. We also note that Eq. (21) may be rewritten in terms of the Lippmann-Schwinger two-body t matrix since a relation of the form of Eq. (16) would relate that t matrix to the invariant amplitude $\overline{\mathfrak{M}}$. In the non-relativistic limit $[m_1/E_1(\vec{p})] \simeq 1$, our result goes over to the standard result of multiple-scattering theory for the leading term of the optical model which is to be inserted into the Lippmann-Schwinger equation, i.e., $V(\vec{p}, \vec{p}\,' \mid W) \simeq t(\epsilon, \vec{q}^{\,2})\rho(\vec{q}^{\,2})$.

We may also discuss inelastic scattering in the impulse approximation. We have

$$\begin{aligned} T_{ij}(\vec{p}, \vec{p}\,' \mid W) &\simeq V_{ij}(\vec{p}, \vec{p}\,' \mid W) \\ &\simeq \overline{\mathfrak{M}}(\epsilon, \vec{q}^{\,2})\overline{\rho}_{ij}(\vec{q}^{\,2}) \end{aligned} \tag{24}$$

with

$$\overline{\rho}_{ij}(\vec{q}^{\,2}) \equiv \rho_{ij}(q^2) \mid_{q^0 = \Delta^{(i)}(\vec{p}\,') - \Delta^{(j)}(\vec{p})} \tag{25}$$

$$\overline{\mathfrak{M}}(\epsilon, \vec{q}^{\,2}) = \mathfrak{M}(\epsilon, q^2) \mid_{q^0 = \Delta^{(i)}(\vec{p}\,') - \Delta^{(j)}(\vec{p})}, \tag{26}$$

where $\rho_{ij}(q^2)$ is the inelastic form factor connecting channels i and j and $\Delta^{(i)}$ is the function appropriate to the g being used, defined in terms of the masses in channel i. It is worth noting that analysis of a great deal of experimental data based on the factorized form Eq. (24) has indicated that this form provides an excellent approximation.[8] Finally, it is fairly obvious how a relativistic *distorted wave* Born approximation for inelastic scattering may be formulated starting from equations such as Eq. (18).

The above considerations should be particularly useful in providing a theoretical framework for the use of the Schrödinger equation (with relativistic kinematics) for such process as nucleon-nucleus or pion-nucleus scattering at relativistic energies. Some of the difficulties of current formulations based upon *ad hoc* Schrödinger-like relativistic equations may be seen in Ref. 9.

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[1] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience, New York, 1959), Vol. I, p. 315; R. J. Glauber, in *High-Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland, Amsterdam, 1967), p. 311.
[2] K. M. Watson, Phys. Rev. 89, 575 (1953).
[3] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).
[4] R. Bertini *et al.*, Phys. Lett. 45B, 119 (1973).
[5] C. M. Shakin and R. M. Thaler, Phys. Rev. C 7, 494 (1973); R. R. Scheerbaum, C. M. Shakin, and R. M. Thaler, Ann. Phys. (N.Y.) 76, 333 (1973).
[6] R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
[7] M. H. Partovi and E. L. Lomon, Phys. Rev. 2, 1999 (1970).
[8] Y. Alexander and A. S. Rinat, report (to be published).
[9] B. W. Allardyce *et al.*, Nucl. Phys. A209, 1 (1973).