Nuclear charge form factor with intrinsic hyperspherical-coordinate wave functions

M. Sotona and J. Žofka
Institute of Nuclear Physics, Czechoslovak Academy of Sciences, Řež-Prague, Czechoslovakia
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A simple expression for the intrinsic form factor of magic nuclei in the lowest order approximation of the hyperspherical-coordinate method is derived, discussed, and applied to ^{16}O and ^{4}He. A good agreement with experimental values for $q^2 \lesssim 8$ fm^{-2} and the appearance of a dip in the form factor of ^{4}He are noticed.

[NUCLEAR STRUCTURE ^{4}He, ^{16}O; calculated intrinsic charge form factor. Lowest order hyperspherical-coordinate wave functions; comparison with experiment.]

Palumbo has recently suggested a procedure for calculating the nuclear charge form factor, if both an intrinsic operator

$$\hat{F}_{\rm intr} = \frac{1}{A}\sum_j \exp[i\vec{q}(\vec{r}_j - \vec{R})]$$

and intrinsic hyperspherical-coordinate (HSC) wave functions are used.[1] He has also derived an expression for the form factor in the lowest order $K = K_{\min}$ approximation of the HSC method. This expression is rather complicated and it is the aim of this work to bring the expression into a simpler and physically more lucid form and to apply it to ^{4}He and ^{16}O.

If the HSC wave function is $\Psi_{K_{\min}}(\rho, \Omega) = \chi(\rho)U(\Omega)$, the form factor is given by

$$F(q) = \int \chi^*(\rho)\mathcal{F}(q,\rho)\chi(\rho)d\rho\,. \tag{1}$$

Palumbo's result for $\mathcal{F}$ in this case (the last equation of Ref. 1) can be rewritten as follows:

$$\mathcal{F}(q,\rho) = \frac{\Gamma(\mathfrak{L}+\frac{3}{2})}{2\pi i \rho^{2\mathfrak{L}+1}}\int_c ds \exp\left(s\rho^2 + \frac{q^2}{4sA}\right) \times s^{-(\mathfrak{L}+3/2)}\langle\phi^{\sqrt{s}}_{\rm HO}|\hat{F}_{\rm lab}|\phi^{\sqrt{s}}_{\rm HO}\rangle\,, \tag{2}$$

where

$$\hat{F}_{\rm lab} = \exp(i\vec{q}\vec{R})\hat{F}_{\rm intr}\,.$$

In Eq. (2), $\mathfrak{L}$ is a hypermomentum,

$$\mathfrak{L} = K_{\min} + \tfrac{3}{2}(A-2) = \sum_{nl\ \text{occup.}} (2n+l) + \tfrac{3}{2}(A-2)\,,$$

n, l are quantum numbers of harmonic oscillator single-particle states of which the Slater determinant $|\phi^{\sqrt{s}}_{\rm HO}\rangle$ is composed with the spring constant $\alpha = \sqrt{s}$. Contour of the integration C is the same as in Ref. 2 (see Fig. 1).

Evaluation of the matrix element of the many-body operator $\hat{F}_{\rm intr}$ between HSC wave functions is then reduced to much simpler evaluation of the matrix element of the one-body operator $\hat{F}_{\rm lab}$ in the basis of the harmonic oscillator. For doubly magic nuclei, this quantity is concisely expressed in the Filippov representation[3]

$$\langle\phi^{\alpha}_{\rm HO}|\hat{F}_{\rm lab}|\phi^{\alpha}_{\rm HO}\rangle = \frac{4}{A}e^{-q^2/4\alpha^2}L^3_N\left(\frac{q^2}{2\alpha^2}\right), \tag{3}$$

where L^{β}_N is the Laguerre polynomial and $N = 2n' + l'$, where n' and l' are quantum numbers of the last filled oscillator shell. Insertion of Eq. (3) into (2) and the use of an integral representation of the Bessel function J_ν[4]

$$\left(\frac{Z}{2}\right)^{-\nu}J_\nu(Z) = \frac{1}{2\pi i}\int_{c'}\frac{\exp(x - Z^2/4x)}{x^{\nu+1}}dx \tag{4}$$

leads finally to

$$\mathcal{F}(q,\rho) = \frac{4}{A}\sum_{m=0}^{N}\frac{(-)^m(N+3)!}{(m+3)!(N-m)!\,m!}\left(\frac{q^2\rho^2}{2}\right)^m \times\frac{\Gamma(\mathfrak{L}+\frac{3}{2})}{\Gamma(\mathfrak{L}+m+\frac{3}{2})}\,{}_0F_1\left(\mathfrak{L}+m+\tfrac{3}{2}; -\frac{A-1}{A}\frac{q^2\rho^2}{4}\right). \tag{5}$$

In order to make Eq. (5) short, the Bessel function has been replaced by the generalized hypergeometric series[4]:

$${}_0F_1\left(\gamma+1; -\frac{x^2}{4}\right) = \frac{\Gamma(\gamma+1)}{(x/2)^\gamma}J_\gamma(x)\,. \tag{6}$$

For magic nuclei with $A \leq 16$, we obtain then

$$\mathcal{F}(q,\rho) = {}_0F_1\left(\mathfrak{L}+\tfrac{3}{2}; -\frac{A-1}{A}\frac{q^2\rho^2}{4}\right) - \frac{A-4}{A}\frac{2}{3(\mathfrak{L}+\frac{3}{2})}\frac{q^2\rho^2}{4}\,{}_0F_1\left(\mathfrak{L}+\tfrac{5}{2}; -\frac{A-1}{A}\frac{q^2\rho^2}{4}\right). \tag{7}$$

Equation (7) has been derived for ^{4}He ($\mathfrak{L} = 3$) and

for ^{16}O ($\mathfrak{L}=33$) only. Nevertheless it holds also for ^{12}C ($\mathfrak{L}=23$) provided its ground state is described by the HSC wave function with $K=K_{\min}$, corresponding to the oscillator configuration $(0s_{1/2})^4(0p_{3/2})^8$. However, the derivation of Eq. (7) for ^{12}C is more complicated, because simple Eq. (3) cannot be applied and the matrix element of $\hat{F}_{\text{lab}}$ has to be expressed as the sum of the single-particle contributions of $0s_{1/2}$ and $0p_{3/2}$ nucleons.

In Ref. 5, an expression for the HSC density distribution $n(r)$ of doubly magic nuclei in the approximation $K=K_{\min}$ is derived, taking into account only terms up to order $1/A$. We can thus obtain an approximate form factor by applying a Fourier transform to $n(r)$ of Ref. 5, which yields:

$$\mathfrak{F}^{\text{app}}(q,\rho)={}_0F_1\left(\mathfrak{L}+\tfrac{3}{2};-\frac{q^2\rho^2}{4}\right)-\frac{A-4}{A}\,\frac{2}{3(\mathfrak{L}+\frac{3}{2})}\,\frac{q^2\rho^2}{4}\,{}_0F_1\left(\mathfrak{L}+\tfrac{5}{2},-\frac{q^2\rho^2}{4}\right). \tag{8}$$

By simply dropping the factor $(A-1)/A$ we proceed from the exact form factors to approximate ones. However, it is seen that the introduction of an effective momentum $q_{\text{eff}}=[(A-1)/A]^{1/2}q$ in order to correct $\mathfrak{F}$ for neglected terms of order $1/A$ (which is sufficient for ^{4}He) does not suffice in the case of ^{16}O due to the coefficient $q^2\rho^2/4$ in the second term on the right hand side of Eq. (7).

In order to illustrate the above formulas, we have calculated the HSC form factors for ^{4}He and ^{16}O.

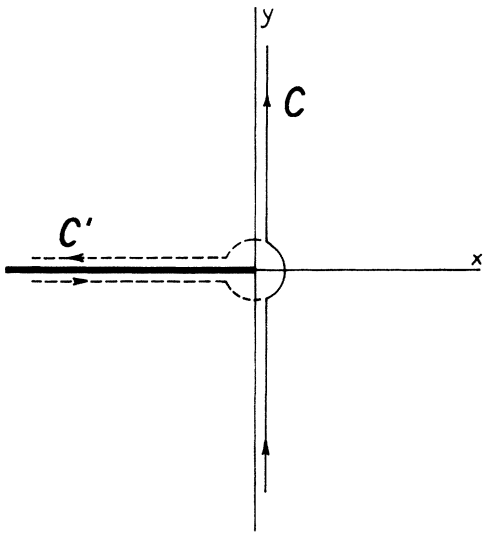

FIG. 1. Contours C and C' of the integration in Eqs. (2) and (4), respectively. The cut of the integrand is marked by a heavy line.

The HSC wave function $\chi(\rho)$ has been calculated[6] with the two-Gaussian interaction $B1$ of Brink and Boeker.[7] This interaction yields, for ^{4}He and ^{16}O, reasonable HSC binding energies (29.3 and 106.6 MeV) and rms radii (1.50 and 2.60 fm).[6] In Fig. 2, the HSC form factors F of ^{4}He, given by Eqs. (7) and (8) and corrected for the proton finite size, are compared with each other and also with experimental values.[8] The comparison with experimental values should not be regarded as a quantitative test of the potential used, because the model with $K=K_{\min}$ is primarily designed for heavier nuclei and for ^{4}He it may not work well enough. Such a comparison provides only a qualitative information on characteristic featues of the form factor which should be reproduced by any reasonable model. The use of the intrinsic form factor Eq. (7) changes considerably the behavior at small values of q, thus bringing the rms radius closer to the experimental value. It is remarkable that the simple HSC method, with its new degree of freedom in the hyperradius ρ, is rich enough to predict a dip. As compared with experiment, the position of the dip is shifted to higher values of q, which is likely due to an absence of shorter-range two-body correlations in this HSC method with $K=K_{\min}$ only. The intrinsic Slater determinant with harmonic oscillator single-particle wave functions, which is a very good approximation to the HSC wave function of heavier nuclei,[6] gives the form

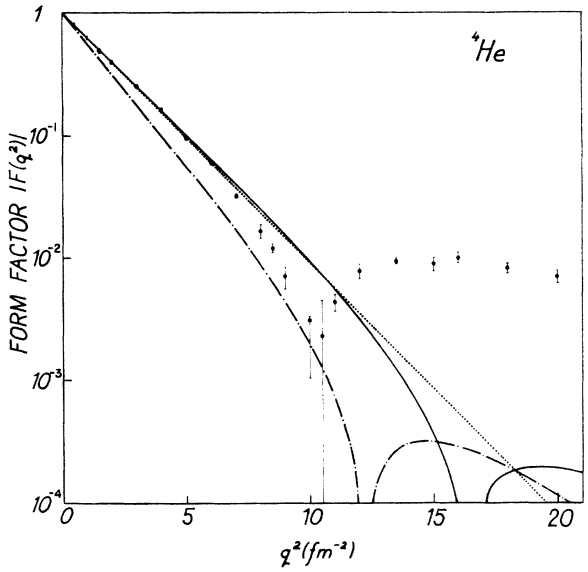

FIG. 2. HSC form factors of ^{4}He yielded by the exact formula (7) (solid line) and by the formula (9) corrected up to order $1/A$ (dashed-dotted line). The intrinsic form factor of the simple harmonic oscillator is marked by the dotted line. All values are corrected for the proton finite size. Experimental values are plotted by heavy dots with error bars (if possible).

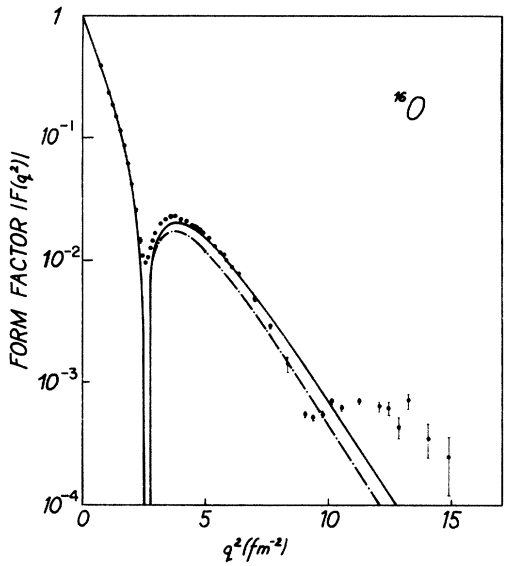

FIG. 3. HSC form factors of ^{16}O. For notation see Fig. 2.

factor

$$F^{110}(q) = \left(1 - \frac{A-4}{A}\frac{2}{3}\frac{q^2}{4\alpha^2}\right)\exp\left(-\frac{A-1}{A}\frac{q^2}{4\alpha^2}\right) \tag{9}$$

and for ^{4}He no dip occurs (dotted line in Fig. 2). In Eq. (9), the α giving the highest binding is chosen ($\alpha = 0.711$ fm^{-1} for ^{4}He).

In Fig. 3, the HSC form factors of ^{16}O, given by Eqs. (7) and (8) and corrected for the proton finite size, are compared with each other and also with experimental values.[9] Again, the use of the intrinsic form factor [Eq. (7)] improves the agreement with experimental values; this is most noticeable in the region of the second maximum. The simple HSC method is not able to predict the second dip and this could likely be improved by including correlations of shorter range, too. For ^{16}O, the oscillator model mentioned above approximates the HSC form factor F, based on Eq. (7), to a high degree of precision up to $q^2 \lesssim 15$ fm^{-2}.

In this comment, we have derived a simple expression for the intrinsic form factor of doubly magic nuclei, using an intrinsic operator and intrinsic hyperspherical-coordinate wave functions in the lowest order approximation $K = K_{\min}$. We have shown that the use of an effective momentum in order to correct for neglected terms of order $1/A$ may not be sufficient for nuclei heavier than ^{4}He. Finally, we have applied the formulas derived to ^{4}He and ^{16}O. We have found that the use of the intrinsic operator is important in the region of $q^2 \lesssim 8$ fm^{-2}, improving the agreement with experimental values, as compared to a treatment where terms of order $1/A$ are neglected. Moreover, the HSC method in the approximation $K = K_{\min}$ is itself able to produce a dip in the form factor of ^{4}He.

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