General boson expansion theory and Lie algebras B_n and D_n †

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A general boson expansion theory for even and odd fermion states has been developed in both Dyson and unitary representations. We can construct them in several different forms, one of which essentially reproduces the formula of Marshalek. The underlying algebra may or may not obey the associative law. A connection of the theory with the Lie algebras B_n and D_n has been investigated. As a byproduct, construction of fermion annihilation and creation operators solely in terms of boson operators has been found.

1. MARUMORI SPACE AND LIE ALGEBRAS B_n and D_n

The boson expansion method[1-5] is useful in providing a simple way to introduce collective variables in the nuclear (and other) many body problem. The general idea is to injectively map the many fermion Hilbert space into so-called ideal space which for even numbers of particles is a purely boson space with one boson representing each kind of pair excitation. However, the situation for odd number particle case is more complicated.[4,6,7]

The purpose of this article is first to generalize the method of Ref. 4 and of Marshalek in a rigorous way. Secondly, it will be shown that the boson expansion method is intimately related to the Lie algebras B_n and D_n. As a byproduct, we will give an example that fermion creation and annihilation operators can be expressed in terms of only boson operators in contrast to somewhat popularly held belief.

Let C_μ and its self-adjoint $\overline{C}_\mu \equiv C_\mu^\dagger$ ($\mu = 1, 2, \ldots, n$) be the standard annihilation and creation operators satisfying the commutation relation:

$$C_\mu C_\nu + C_\nu C_\mu = \overline{C}_\mu \overline{C}_\nu + \overline{C}_\nu \overline{C}_\mu = 0 ,$$

$$C_\mu \overline{C}_\nu + \overline{C}_\nu C_\mu = \delta_{\mu\nu} . \quad (1.1)$$

Then, the fermion space is the 2^n-dimensional Hilbert space spanned by completely antisymmetric states

$$|\mu_1, \mu_2, \ldots, \mu_q\rangle_F = \overline{C}_{\mu_1} \overline{C}_{\mu_2} \cdots \overline{C}_{\mu_q} |0\rangle_F , \quad (1.2)$$

with $q = 0, 1, 2, \ldots, n$, where $|0\rangle_F$ is the fermion vacuum state, satisfying

$$C_\mu |0\rangle_F = 0 . \quad (1.3)$$

Let V be a linear injective map of the fermion space into an ideal Hilbert space which in general contains an infinite-dimensional boson Fock space as its subspace. We set

$$|\mu_1, \mu_2, \ldots, \mu_q\rangle = V |\mu_1, \mu_2, \ldots, \mu_q\rangle_F . \quad (1.4)$$

The most interesting and important case is of course when V is isometric. This case will be discussed in Sec. 3. However, for discussions of Secs. 1 and 2, we *need not* require V to be isometric as is ordinarily assumed. As a matter of fact, its construction in the next section corresponds to a nonisometric V. We call the image of V be the (physical) Marumori space. If P is the projection operator in the ideal space *onto* the physical Marumori space, then V has its inverse U in that subspace, since V is assumed to be injective and hence one to one. Moreover, we have

$$UV = I, \quad PV = V, \quad VU = P, \quad (1.5)$$

where I is the identity operator in the original fermion space. If V is isometric, then we have an additional relation

$$U = V^\dagger . \quad (1.5')$$

Defining now linear operators A^μ_ν, $R^{\mu\nu}$, $R_{\mu\nu}$, A^μ, and A_μ in the ideal space by

$$A^\mu_\nu = -V\overline{C}_\mu C_\nu U, \quad R^{\mu\nu} = -R^{\nu\mu} = V\overline{C}_\mu \overline{C}_\nu U,$$

$$R_{\mu\nu} = -R_{\nu\mu} = -VC_\mu C_\nu U ,$$

$$A^\mu = V\overline{C}_\mu U, \quad A_\mu = VC_\mu U , \quad (1.6)$$

then these operators satisfy the condition

$$PQ = QP = Q , \quad (1.7)$$

where Q represents any of these operators. From

(1.6), we find

$$\begin{aligned}&(A^{\mu}_{\nu}+A^{\mu}A_{\nu})P=0\ ,\\&(R^{\mu\nu}-A^{\mu}A^{\nu})P=0\ ,\\&(R_{\mu\nu}+A_{\mu}A_{\nu})P=0\ ,\\&(A^{\mu}A_{\nu}+A_{\nu}A^{\mu}-\delta^{\mu}_{\nu})P=0\ .\end{aligned}\tag{1.8}$$

Although the presence of the projection operator P in (1.8) is actually not necessary because of (1.7), we placed it here by the following reason. All these operators are defined only in the finite-dimensional Marumori space. However, it is in practice more convenient to enlarge their domains of definition into a dense subset of the infinite-dimensional ideal space. Of course, this extension is in general not unique, but we shall use the same notations for these extended operators. Then, as we shall see in the next section, we can find a set of extended operators which satisfy commutation relations

$$\begin{aligned}&[A^{\mu}_{\nu},\ A^{\alpha}_{\beta}]=\delta^{\mu}_{\beta}A^{\alpha}_{\nu}-\delta^{\alpha}_{\nu}A^{\mu}_{\beta}\ ,\\&[A^{\mu}_{\nu},R^{\alpha\beta}]=-\delta^{\alpha}_{\nu}R^{\mu\beta}-\delta^{\beta}_{\nu}R^{\alpha\mu}\ ,\\&[A^{\mu}_{\nu},R_{\alpha\beta}]=\delta^{\mu}_{\alpha}R_{\nu\beta}+\delta^{\mu}_{\beta}R_{\alpha\nu}\ ,\\&[R_{\mu\nu},R^{\alpha\beta}]=\delta^{\alpha}_{\mu}A^{\beta}_{\nu}+\delta^{\beta}_{\nu}A^{\alpha}_{\mu}-\delta^{\alpha}_{\nu}A^{\beta}_{\mu}\\&\qquad\qquad-\delta^{\beta}_{\mu}A^{\alpha}_{\nu}+\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}-\delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\ ,\\&[R^{\mu\nu},\ R^{\alpha\beta}]=[R_{\mu\nu},R_{\alpha\beta}]=0\ ,\end{aligned}\tag{1.9}$$

as well as

$$\begin{aligned}&[A^{\mu}_{\nu},A_{\lambda}]=\delta^{\mu}_{\lambda}A_{\nu}\ ,\\&[A^{\mu}_{\nu},A^{\lambda}]=-\delta^{\lambda}_{\nu}A^{\mu}\ ,\\&[R^{\mu\nu},A_{\lambda}]=\delta^{\nu}_{\lambda}A^{\mu}-\delta^{\mu}_{\lambda}A^{\nu}\ ,\\&[R_{\mu\nu},A^{\lambda}]=\delta^{\lambda}_{\mu}A_{\nu}-\delta^{\lambda}_{\nu}A_{\mu}\ ,\\&[R^{\mu\nu},A^{\lambda}]=[R_{\mu\nu},A_{\lambda}]=0\ ,\end{aligned}\tag{1.10}$$

in a dense subset of the ideal space. However, the condition (1.8) will be preserved only if the projection operator P is now present there, i.e., it holds only in the Marumori space. In general, the range of these extended operators are also dense in the ideal space. Instead of (1.7), we have a weaker relation

$$QP=PQP \tag{1.7$'$}$$

so that (1.9) and (1.10) will be still valid in the finite-dimensional Marumori space. These relations are actually simple reflections of (1.1) and (1.6).

Our algebra will become a Lie algebra if we use

$$K^{\mu}_{\nu}=A^{\mu}_{\nu}+\tfrac{1}{2}\delta^{\mu}_{\nu} \tag{1.11}$$

instead of A^{μ}_{ν}. In fact, (1.9) represents then the Lie algebra D_n of the Cartan classification. The n operators

$$H_{\mu}=K^{\mu}_{\mu}\quad(\mu=1,2,\ldots,n) \tag{1.12}$$

form the Cartan subalgebra of D_n, whose maximal eigenvalues define irreducible representations of D_n. If we define J_{ab} satisfying antisymmetric condition

$$J_{ab}=-J_{ba} \tag{1.13}$$

for a, $b=1,2,\ldots,2n$ by equations

$$\begin{aligned}&J_{\mu\nu}=\tfrac{1}{2}(R^{\mu\nu}-R_{\mu\nu}-K^{\mu}_{\nu}+K^{\nu}_{\mu})\ ,\\&J_{\mu+n,\ \nu+n}=\tfrac{1}{2}(-R^{\mu\nu}+R_{\mu\nu}-K^{\mu}_{\nu}+K^{\nu}_{\mu})\ ,\\&J_{\mu,\ \nu+n}=\tfrac{1}{2}i\,(R^{\mu\nu}+R_{\mu\nu}+K^{\mu}_{\nu}+K^{\nu}_{\mu})\ ,\\&J_{\mu+n,\ \nu}=\tfrac{1}{2}i\,(R^{\mu\nu}+R_{\mu\nu}-K^{\mu}_{\nu}-K^{\nu}_{\mu})\ ,\end{aligned}\tag{1.14}$$

for values of $\mu,\nu=1,2,\ldots,n$, then (1.9) is combined into a single relation

$$[J_{ab},J_{cd}]=\delta_{ad}J_{bc}+\delta_{bc}J_{ad}-\delta_{ac}J_{bd}-\delta_{bd}J_{ac} \tag{1.15}$$

for all $a,b,c,d=1,2,\ldots,2n$ so that J_{ab} is indeed familiar infinitesimal generator of the $2n$-dimensional orthogonal group $O(2n)$.

The larger Lie algebra generated by K^{μ}_{ν}, $R^{\mu\nu}$, $R_{\mu\nu}$, A^{μ}, and A_{μ} represents similarly the Lie algebra B_n in the physical Marumori space, which corresponds to the $2n+1$ dimensional orthogonal group $O(2n+1)$. Indeed, with identification

$$\begin{aligned}&J_{\mu,\ 2n+1}=\tfrac{1}{2}i\,(A_{\mu}+A^{\mu})\ ,\\&J_{\mu+n,\ 2n+1}=\tfrac{1}{2}(A_{\mu}-A^{\mu})\ ,\end{aligned}\tag{1.16}$$

we find the validity of (1.15) for all a,b,c,d including the new value $2n+1$ in the Marumori space. Note that $H_{\mu}=K^{\mu}_{\mu}$ ($\mu=1,2,\ldots,n$) are still the Cartan subalgebra of B_n.

The Casimir operator C of the Lie algebra D_n is given by

$$\begin{aligned}C&=\frac{1}{2}\sum_{\mu,\ \nu=1}^{n}(2K^{\mu}_{\nu}K^{\nu}_{\mu}+R^{\mu\nu}R_{\mu\nu}+R_{\mu\nu}R^{\mu\nu})\\&=\frac{1}{2}\sum_{a,\ b=1}^{2n}J_{ab}J_{ba}\ .\end{aligned}\tag{1.17}$$

Then, by means of (1.1) and (1.6), we can easily compute eigenvalue of C as

$$C=\tfrac{1}{2}n(n-\tfrac{1}{2})\ . \tag{1.18}$$

Similarly, the Casimir operator D of the larger algebra B_n is

$$D=C+\frac{1}{2}\sum_{\mu=1}^{n}(A_{\mu}A^{\mu}+A^{\mu}A_{\mu}) \tag{1.19}$$

whose eigenvalue is

$$D=\tfrac{1}{2}n(n+\tfrac{1}{2})\ . \tag{1.20}$$

In other words, the physical Marumori space corresponds to a representation of Lie algebras B_n and D_n with conditions (1.18) and (1.20). As we shall show in the next section, it is actually the irreducible spinor representation of B_n, which reduces to a direct sum of two inequivalent irreducible representations of the subalgebra D_n with the same dimensionality 2^{n-1} and with the same eigenvalue (1.18) for the Casimir operator C.

Finally, a subalgebra generated by A_ν^μ will form a Lie algebra of the n-dimensional unitary group $U(n)$, since its diagonal components A_μ^μ ($\mu=1,2,\ldots,n$) will assume only integral eigenvalues for its construction in the next section. If we subtract its trace from either A_ν^μ or K_ν^μ then they of course form the Lie algebra $A_{(n-1)}$ of the Cartan classification corresponding to the SU(n) group. These facts will play an important role in the Sec. 3 and in the Appendix.

2. DYSON REPRESENTATION FORMULA

Following the standard procedure,[1-5] we introduce boson annihilation and creation operators satisfying

$$\begin{aligned}&[B_{\mu\nu},B^{\alpha\beta}]=\delta_\mu^\alpha\delta_\nu^\beta-\delta_\nu^\alpha\delta_\mu^\beta\ ,\\&[B_{\mu\nu},B_{\alpha\beta}]=[B^{\mu\nu},B^{\alpha\beta}]=0\ ,\\&B_{\mu\nu}=-B_{\nu\mu},\quad B^{\mu\nu}=-B^{\nu\mu}\ ,\end{aligned} \tag{2.1}$$

where for simplicity we have set

$$B^{\mu\nu}=(B_{\mu\nu})^\dagger\ . \tag{2.2}$$

If we identify the resulting infinite-dimensional boson Fock space with the ideal space, then it is well known[4,5] that we can construct the physical Marumori space for even-number fermion states. However, if we wish to consider odd-number fermion states, this is not enough and it is necessary to enlarge the ideal space by introducing additional operators a_μ and its conjugate $\bar{a}_\mu\equiv(a_\mu)^\dagger$ which commute with $B_{\mu\nu}$ and $B^{\mu\nu}$. But we shall not yet specify commutation relations among a_μ and $\bar{a}_\nu$ except for

$$[\bar{a}_\mu a_\nu,\bar{a}_\alpha a_\beta]=\delta_\nu^\alpha\bar{a}_\mu a_\beta-\delta_\beta^\mu\,\bar{a}_\alpha a_\nu\ . \tag{2.3}$$

Next, let us set

$$B_\nu^\mu=-B^{\mu\lambda}B_{\nu\lambda}\ , \tag{2.4}$$

where the repeated index λ implies an automatical summation over λ from 1 to n hereafter. Let us consider a direct product algebra and define

$$\begin{aligned}&A_\nu^\mu=B_\nu^\mu-\bar{a}_\mu a_\nu\ ,\\&R^{\mu\nu}=B^{\mu\nu}+B^{\mu\lambda}B_\lambda^\nu-B^{\mu\lambda}\bar{a}_\nu a_\lambda+B^{\nu\lambda}\bar{a}_\mu a_\lambda,\\&R_{\mu\nu}=B_{\mu\nu}\ .\end{aligned} \tag{2.5}$$

Then, it is easy to check that these operators satisfy the commutation relation (1.9) corresponding to the Lie algebra D_n as well as the antisymmetric condition:

$$\begin{aligned}&R^{\mu\nu}=-R^{\nu\mu}\ ,\\&R_{\mu\nu}=-R_{\nu\mu}\ .\end{aligned} \tag{2.6}$$

These operators are defined in a dense subset of the boson Fock space. Also if we restrict ourselves to the Marumori space, then (2.5) agrees with those given in the Ref. 4. Note that our realization does not satisfy the Hermiticity condition

$$(R_{\mu\nu})^\dagger=R^{\mu\nu}\ , \tag{2.7}$$

although it satisfies the $U(n)$ condition

$$(A_\nu^\mu)^\dagger=A_\mu^\nu\ . \tag{2.8}$$

Hence, this representation is an analog of the formula discovered by Dyson[8] for the SU(2) group. Following Jansen *et al*,[4] we therefore call it the Dyson realization. However, as we shall prove in the next section, we can always find a transformation S in the Marumori space, which restores the Hermiticity condition (2.7).

Before going into further detail, the Casimir operator C of D_n is easily computed from (1.17) and (2.5) to be

$$C=(\bar{a}_\mu a_\nu)(\bar{a}_\nu a_\mu)-n(\bar{a}_\mu a_\mu)+\tfrac{1}{2}n(n-\tfrac{1}{2})\ . \tag{2.9}$$

Note that C does not depend upon $B_{\mu\nu}$ and $B^{\mu\nu}$.

Next, in order to find correct expressions for operators A^μ and A_μ, let us make an ansatz[9]

$$\begin{aligned}&A^\mu=\bar{b}_\mu+\bar{b}_\lambda B_\lambda^\mu+B^{\mu\lambda}b_\lambda\ ,\\&A_\mu=b_\mu+B_{\mu\lambda}\bar{b}_\lambda\ ,\end{aligned} \tag{2.10}$$

where b_μ and $\bar{b}_\mu$ are some unspecified operators which are functions of only a_μ and $\bar{a}_\mu$ but not of $B_{\mu\nu}$ and $B^{\mu\nu}$. The commutation relations (1.10) turn out to be identically satisfied if we have

$$\begin{aligned}&\bar{b}_\lambda(\bar{a}_\mu a_\nu)=(\bar{a}_\mu a_\nu)b_\lambda=0\ ,\\&(\bar{a}_\mu a_\nu)\bar{b}_\lambda=\delta_{\nu\lambda}\bar{b}_\mu\ ,\\&b_\lambda(\bar{a}_\mu a_\nu)=\delta_{\lambda\mu}b_\nu\ .\end{aligned} \tag{2.11}$$

Actually, we can relax this condition into the following weaker form for the purpose of satisfying (1.10);

$$\begin{aligned}&[\bar{a}_\mu a_\nu,\bar{b}_\lambda]=\delta_{\nu\lambda}\bar{b}_\mu\ ,\\&[\bar{a}_\mu a_\nu,b_\lambda]=-\delta_{\lambda\mu}b_\nu\ ,\\&\bar{b}_\mu(\bar{a}_\nu a_\lambda)-\bar{b}_\nu(\bar{a}_\mu a_\lambda)=\delta_{\mu\lambda}\bar{b}_\rho(\bar{a}_\nu a_\rho)-\delta_{\nu\lambda}\bar{b}_\rho(\bar{a}_\mu a_\rho)\ .\end{aligned}$$

However, for the present aim the condition (2.11) is sufficient. Now, we have to specify commutation relation between a_μ and $\bar{a}_\nu$ as well as specific forms of b_μ and $\bar{b}_\mu$ so that all conditions (2.3) and (2.11) are fulfilled. There are several possibilities and we shall especially mention three interesting cases here.

Case I

We choose $b_\mu = a_\mu$ and $\bar{b}_\mu = \bar{a}_\mu$ and assume the following algebraic relations

$$\begin{aligned} &b_\mu \bar{b}_\nu = \delta_{\mu\nu} e, \quad ee = e, \\ &b_\mu e = e\bar{b}_\mu = 0, \quad eb_\mu = b_\mu, \quad \bar{b}_\mu e = \bar{b}_\mu . \end{aligned} \tag{2.12}$$

This algebra is due essentially to Marshalek.[7] From (2.12), we find

$$b_\mu b_\nu = \bar{b}_\mu \bar{b}_\nu = 0 \tag{2.13}$$

as well as (2.11), i.e.,

$$\begin{aligned} &\bar{b}_\lambda(\bar{b}_\mu b_\nu) = (\bar{b}_\mu b_\nu) b_\lambda = 0 , \\ &(\bar{b}_\mu b_\nu)\bar{b}_\lambda = \delta_{\nu\lambda}\bar{b}_\mu , \\ &b_\lambda(\bar{b}_\mu b_\nu) = \delta_{\lambda\mu} b_\nu . \end{aligned} \tag{2.14}$$

If we wish, we can identify e as

$$e = 1 - \bar{b}_\lambda b_\lambda . \tag{2.15}$$

The ideal Fock space for this algebra consists of the vacuum state $|0\rangle$ satisfying

$$b_\lambda |0\rangle = 0 \tag{2.16}$$

and one particle state defined by

$$|\lambda\rangle = \bar{b}_\lambda |0\rangle . \tag{2.17}$$

With the identification (2.15), the operator e is nothing but the projection operator for the vacuum state $|0\rangle$. Hereafter, we shall call b_μ and $\bar{b}_\mu$ the quasiparticle operators.

Case II

Let a_μ and $\bar{a}_\mu$ be ordinary Fermi or Bose operators satisfying

$$\begin{aligned} &a_\mu a_\nu \pm a_\nu a_\mu = \bar{a}_\mu \bar{a}_\nu \pm \bar{a}_\nu \bar{a}_\mu = 0 , \\ &a_\mu \bar{a}_\nu \pm \bar{a}_\nu a_\mu = \delta_{\mu\nu} . \end{aligned} \tag{2.18}$$

We then identify b_μ and $\bar{b}_\mu$ to be

$$\begin{aligned} &b_\mu = e a_\mu , \\ &\bar{b}_\mu = \bar{a}_\mu e , \end{aligned} \tag{2.19}$$

where e is now the projection operator for the vacuum state of either fermion or boson space given by

$$\begin{aligned} &e = \prod_{\lambda=1}^{n} e_\lambda , \\ &e_\lambda = \sum_{l=0}^{\infty} \frac{1}{l!}(-1)^l (\bar{a}_\lambda)^l (a_\lambda)^l . \end{aligned} \tag{2.20}$$

If we note an easily derivable relation

$$a_\mu e = e\bar{a}_\mu = 0 ,$$

it is simple to prove that b_μ and $\bar{b}_\mu$ defined by (2.19) and (2.20) satisfy all conditions (2.12)–(2.14) as well as (2.3) and (2.11). Hence, we may regard the present case as a realization of Marshalek algebra discussed in the case I. Although the ideal space is now spanned by the usual Fock space defined by (2.18) of either boson or fermion particles, the physical Marumori space utilizes only two distinct states given by (2.16) and (2.17) as we shall see shortly. We emphasize the fact that the lower sign case in (2.18) corresponds to the use of boson operators for a_μ and $\bar{a}_\mu$ so that our whole ideal space is purely bosonic.

So far, we have to utilize the vacuum projection operator e in our construction of A^μ and A_μ. However, we can dispense with its use, if we give up the associative law of the algebra.

Case III

We identify $b_\mu = a_\mu$ and $\bar{b}_\mu = \bar{a}_\mu$, but we assume

$$\begin{aligned} &a_\mu a_\nu = \bar{a}_\mu \bar{a}_\nu = 0 , \\ &a_\mu \bar{a}_\nu \pm \bar{a}_\nu a_\mu = \delta_{\mu\nu} \end{aligned} \tag{2.21}$$

as well as an additional conditions

$$(\bar{a}_\mu a_\nu)(\bar{a}_\alpha a_\beta) = \delta_{\nu\alpha}(\bar{a}_\mu a_\beta) \tag{2.22}$$

supplemented by (2.14) with replacing $b_\mu \to a_\mu$ and $\bar{b}_\mu \to \bar{a}_\mu$. Actually the choice (2.21) for the upper sign case is originally also due to Marshalek.[6] However, as we noted elsewhere,[9] any algebra satisfying (2.21) (with $n \geq 2$ for the upper sign case) *cannot* obey the associative law $x(yz) = (xy)z$. Therefore, our algebra is *not* associative. But the subalgebra generated by $\bar{a}_\mu a_\nu$ is associative. As the result, algebra among $R_{\mu\nu}$, $R^{\mu\nu}$, and A^μ_ν still obeys the associate law, although we have $A_\mu(A_\nu A_\lambda) \neq (A_\mu A_\nu)A_\lambda$ in general. Actually, there is an intimate connection between this nonassociative algebra and the Marshalek algebra, which we do not elaborate here upon.[9]

Since the nonassociative algebra is rather unfamiliar, we shall mostly deal with the associative cases (I) and (II), unless it is stated otherwise. Our real ideal space is the direct product space of two Fock Hilbert spaces, where the ideal vacuum state $|0\rangle$ satisfies

$$b_\mu|0\rangle = a_\mu|0\rangle = B_{\mu\nu}|0\rangle = 0 \ . \qquad (2.23)$$

Now, we shall construct the physical Marumori space as follows. For even-number particle states, we set

$$|\mu_1\nu_1, \mu_2\nu_2, \ldots, \mu_l\nu_l\rangle = R^{\mu_1\nu_1}R^{\mu_2\nu_2}\cdots R^{\mu_l\nu_l}|0\rangle \ , \qquad (2.24)$$

while the odd-number states are defined by

$$|\lambda, \mu_1\nu_1, \mu_2\nu_2 \ldots, \mu_l\nu_l\rangle = A^\lambda|\mu_1\nu_1, \mu_2\nu_2 \ldots \mu_l\nu_l\rangle \ . \qquad (2.25)$$

We can easily prove that all these states are completely antisymmetric for interchanges of any two indices involved. Actually, it is sufficient to verify this fact for the special cases $l=2$ in (2.24) and $l=1$ in (2.25) because we have $[R^{\mu\nu}, R^{\alpha\beta}] = [R^{\mu\nu}, A^\lambda] = 0$ and $R^{\mu\nu} = -R^{\nu\mu}$. Therefore, the subspace spanned by all states of the form (2.24) and (2.25) is a finite-dimensional space with the dimension 2^n, which we identify with the physical Marumori space. For even-number states, all terms involving $\bar{a}_\mu a_\nu$ in $R^{\mu\nu}$ do not contribute at all because of (2.23). Also, the odd-number states (2.25) contains exactly only one quasiparticle state. This formula coincides with the result of the Ref. 4. By the induction, we can show that we can combine both (2.24) and (2.25) into a single equation

$$|\lambda_1, \lambda_2, \ldots, \lambda_q\rangle = A^{\lambda_1}A^{\lambda_2}\cdots A^{\lambda_q}|0\rangle$$

$$(q = 2l, \text{ or } 2l+1) \qquad (2.26)$$

if we notice $A^\mu A^\nu|0\rangle = R^{\mu\nu}|0\rangle$. Actually (2.26) is also valid[9] for the nonassociative case III, if we interpret the product of the right side to imply operating A^λ successively from the right to the left.

Now, the linear injection map V of the previous section is simply defined by the one-to-one correspondence

$$|\lambda_1, \lambda_2, \ldots, \lambda_q\rangle = V|\lambda_1, \lambda_2, \ldots, \lambda_q\rangle_F \ . \qquad (2.27)$$

Since the states defined by (2.26) are not properly normalized (though they are orthogonal to each other), this injection map V is not isometric. This is essentially the reason behind the invalidity of the Hermiticity condition (2.7), which we shall remedy in the next section.

Let P be the projection operator *onto* the physical Marumori space, and let Q represent any of A^μ_ν, b_μ, $R^{\mu\nu}$, $R_{\mu\nu}$, A^μ, and A_μ. Then, we can show

$$QP = PQP. \qquad (2.28)$$

In other words, the Marumori space is invariant under operations of these operators. This is obvious for $Q = A^\mu$ in view of (2.26). The same argument is also applicable for $Q = R^{\mu\nu}$ because of (2.24) and (2.25) if we notice $[R^{\mu\nu}, A^\lambda] = 0$. With respect to $Q = A^\mu_\nu$, we utilize the commutation relation (1.9) together with $A^\mu_\nu|0\rangle = 0$ in order to prove the invariance of the Marumori space. Finally, the remaining two cases $Q = R_{\mu\nu}$, and A_λ can be demonstrated in a similar manner. Equation (2.28) implies that the commutation relations (1.9) and (1.10) are still valid for their restriction in the Marumori space. At this point, we should emphasize that we may not have $QP = PQ$ in general. As a matter of fact, this extra relation holds certainly for $Q = A^\mu_\nu$, A^λ, and $R^{\mu\nu}$, but not for two remaining cases $Q = R_{\mu\nu}$ and A_λ. This is because (2.28) leads to

$$PQ^\dagger = PQ^\dagger P \qquad (2.29)$$

if we use $P^\dagger = P$. Choosing $Q = R_{\mu\nu} = B_{\mu\nu}$, this gives

$$PB^{\mu\nu} = PB^{\mu\nu}P \neq B^{\mu\nu}P \ . \qquad (2.30)$$

The last inequality in (2.30) must be valid since $B^{\mu\nu}PR^{\alpha\beta}|0\rangle = B^{\mu\nu}B^{\alpha\beta}|0\rangle$, is *not* completely antisymmetric and it cannot belong to the Marumori space. Especially, this fact implies $PR_{\mu\nu} \neq R_{\mu\nu}P$ if we recall $R_{\mu\nu} = B_{\mu\nu}$. The validity of $QP = PQ$ for $Q = A^\mu_\nu$, A^λ, and $R^{\mu\nu}$ follows from (2.29) and (2.28).

Returning to the original discussion, we have finally to check the validity of (1.8). This requires some elaboration. For example we compute

$$A_\mu A_\nu = -B_{\mu\nu}e + B_{\mu\lambda}\bar{b}_\lambda b_\nu$$

which is not equal to $R_{\mu\nu}$ ($= B_{\mu\nu}$). However, as we shall show in the Appendix, we have fortunately many identities such as

$$\begin{aligned} &(B_{\mu\lambda}\bar{b}_\lambda b_\nu + B_{\mu\nu}\bar{b}_\lambda b_\lambda)P = 0 \ , \\ &(B^\mu_\lambda \bar{b}_\lambda b_\nu + B^\mu_\nu \bar{b}_\lambda b_\lambda)P = 0 \ , \\ &P(B^{\mu\lambda}\bar{b}_\nu b_\lambda + B^{\mu\nu}\bar{b}_\lambda b_\lambda) = 0 \ . \end{aligned} \qquad (2.31)$$

Then, we can easily derive now

$$A_\mu A_\nu P = -R_{\mu\nu}P \ .$$

Similarly, we can show

$$A^\mu A^\nu P = PA^\mu A^\nu P = PR^{\mu\nu}P = R^{\mu\nu}P$$

as well as

$$A^\mu A_\nu P = -A^\mu_\nu P \ ,$$

$$A_\nu A^\mu P = (\delta^\mu_\nu + A^\mu_\nu)P \ .$$

These reproduce (1.8). Note that the presence of the projection operator P is essential for their validity. From these, we find also

$$(A_\mu A_\nu + A_\nu A_\mu)P = 0 \ ,$$
$$(A^\mu A^\nu + A^\nu A^\mu)P = 0 \ ,$$
$$(A^\mu A_\nu + A_\nu A^\mu)P = \delta^\mu_\nu P \ . \tag{2.32}$$

This implies that the restriction of operators A^μ and A_μ in the Marumori space behave as ordinary fermion creation and annihilation operators. For the boson case of (II) corresponding to the lower sign in (2.18), this means that $2n$ fermion operators A^μ and A_μ can be constructed from $n(n+1)$ boson operators $B_{\mu\nu}$, $B^{\mu\nu}$, a_μ, and $\bar{a}_\mu$. Although this construction of fermions from pure bosons may appear rather surprising, a different example of this kind has been previously given by Kâlnay, MacCortonina, and Kademova[10] in a different context.

We briefly remark that (1.8) and (2.32) are also valid for the nonassociative case III, if we interpret the product such as $A_\mu A_\nu P$ to imply $A_\mu(A_\nu P)$. The third equation of (1.8) if for example to be read as

$$A_\mu(A_\nu P) = -R_{\mu\nu} P$$

which differs in general from $(A_\mu A_\nu)P$.

We can calculate eigenvalue of the Casimir operator C of the Lie algebra D_n from (2.9) to be (1.18). The Marumori space is invariant under actions of $R^{\mu\nu}$, $R_{\mu\nu}$, K^μ_ν, A^μ, and A_μ so that it defines a representation of the Lie algebras B_n and D_n. Let us first consider the algebra D_n. Two sectors corresponding to even- and odd-number particle states are then separately invariant under actions of $R^{\mu\nu}$, $R_{\mu\nu}$, and K^μ_ν. It is easy to check that they form two inequivalent irreducible representations of D_n. As is well known, any irreducible representation of D_n is specified by the highest weight state $|\phi\rangle$ satisfying

$$H_\mu|\phi\rangle = l_\mu|\phi\rangle \quad (\mu = 1, 2, \ldots, n) \tag{2.33}$$

such that $l_1 \geq l_2 \geq \cdots \geq l_n$. For the present case, the highest weight state in the even number sector is precisely the vacuum state $|0\rangle$ with $l_\mu = \frac{1}{2}$ for all $\mu = 1, 2, \ldots, n$. However, it is the one-particle state $|n\rangle = \bar{a}_n|0\rangle$ for the odd-number sector where we have $l_\mu = \frac{1}{2}$ for $n-1 \geq \mu \geq 1$ but $l_n = -\frac{1}{2}$. It is interesting to observe that both even and odd sectors have exactly the same dimensionality 2^{n-1} with the same eigenvalue for the Casimir operator C, although they belong to two different inequivalent irreducible representations of D_n. Obviously, both sectors are combined to give a single irreducible representations of the larger Lie algebra B_n with the dimensionality 2^n, corresponding to one of the fundamental spinor representation of the $O(2n+1)$ group. The vacuum state $|0\rangle$ is now the sole highest weight state of B_n.

Since the Marumori space is invariant under operations of $R^{\mu\nu}$, $R_{\mu\nu}$, and K^μ_ν, these operators are well defined also in the quotient of the ideal space over the Marumori space, where they define an infinite dimensional representation of the algebra D_n. An interesting question is whether any finite dimensional representation of D_n could exist in this quotient space. If the answer is positive, then we may have another realization of the boson expansion method in the quotient space. However, this problem is left for future investigation, although we can easily prove the answer to be negative for special cases $n=2$, and 3.

Last, we briefly mention that the Marumori states can be rewritten in a form[4]

$$|\lambda_1, \lambda_2, \ldots, \lambda_q\rangle = {}_F\langle 0|ST\bar{C}_{\lambda_1}\bar{C}_{\lambda_2}\cdots\bar{C}_{\lambda_q}|0\rangle_F \otimes |0\rangle \ ,$$
$$S = \exp\left(-\frac{1}{2}\sum_{\mu,\nu=1}^{n} B^{\mu\nu}C_\mu C_\nu\right) \ , \tag{2.34}$$
$$T = 1 + \sum_{\mu=1}^{n} \bar{b}_\mu C_\mu \ ,$$

where $\bar{b}_\mu$ and $B^{\mu\nu}$ are assumed to commute with C_ν and $\bar{C}_\nu$. Then, the completely anti-symmetric character of this state is transparent.

3. UNITARY REALIZATION

As we noted in the previous section, our formula does not satisfy the Hermiticity condition (2.7). This implies that the corresponding representation of the O($2n$) group is not unitary. However, we can always find a similarity transformation S to remedy this situation, since any finite dimensional representation of the groups O($2n$) and O($2n+1$) is known to be equivalent to an unitary representation. In this section, we shall find its explicit realization. The non-Hermiticity is also related to the nonisometric character of the injection map (2.27), which in turn is the result of the use of nonnormalized state vectors. Hence, the only thing we must do is to properly normalize our state vectors (2.26). We shall achieve this end as follows. First, it is convenient to set

$$M = -B^\lambda_\lambda = B^{\mu\nu}B_{\mu\nu} \ , \tag{3.1}$$

which is equal to 2 times the number operator for the boson field $B_{\mu\nu}$. The real total particle number N is related to M by

$$N = M + 1 - e \ . \tag{3.2}$$

In the Marumori space, M can assume any even integral eigenvalues from 0 to n, while it can choose any nonnegative even integral values in the whole ideal space. Note that N assumes eigenvalues $2l$ for (2.24) and $(2l+1)$ for (2.25), while the eigenvalue of M remains to be $2l$ for both states.

Now, let $F(M)$ be an arbitrary function of M and define

$$\begin{aligned} C_{\mu\nu}&=F(M)B_{\mu\nu}e+B_{\mu\nu}F(M)(1-e)\ ,\\ C^{\mu\nu}&=B^{\mu\nu}G(M)e+G(M)B^{\mu\nu}(1-e)\ ,\\ d_\mu&=F(M)b_\mu\ ,\\ \bar{d}_\mu&=G(M)\bar{b}_\mu\ , \end{aligned}\tag{3.3}$$

where $G(M)$ is the inverse of $F(M)$, i.e.,

$$G(M)F(M)=1\ .\tag{3.4}$$

When we notice

$$\begin{aligned} B_{\mu\nu}F(M)&=F(M+2)B_{\mu\nu}\ ,\\ B^{\mu\nu}F(M)&=F(M-2)B^{\mu\nu}\ ,\\ B^{\mu}_{\nu}F(M)&=F(M)B^{\mu}_{\nu}\ , \end{aligned}\tag{3.5}$$

then it is not difficult to check that the transformation

$$\begin{aligned} &B^{\mu\nu}\to C^{\mu\nu},\quad B_{\mu\nu}\to C_{\mu\nu}\ ,\\ &b_\mu\to d_\mu,\quad \bar{b}_\mu\to\bar{d}_\mu \end{aligned}$$

preserves all the algebraic relations specified in the previous section, excepting possibly the Hermiticity properties. This also holds for the nonassociative case III. At any rate, this fact suggests that we will have a transformation S such that

$$\begin{aligned} C_{\mu\nu}&=SB_{\mu\nu}S^{-1}\ ,\\ C^{\mu\nu}&=SB^{\mu\nu}S^{-1}\ ,\\ d_\mu&=Sb_\mu S^{-1}\ ,\\ \bar{d}_\mu&=S\bar{b}_\mu S^{-1}\ . \end{aligned}\tag{3.6}$$

Because of the reason which will become soon apparent, we are actually interested in the special case of

$$F(M)=(1+M)^{1/2}\ .\tag{3.7}$$

Then, the explicit form of S is found to be

$$\begin{aligned} S&=[f(\tfrac{1}{2}M)]^{-1/2}[e+(1+M)^{-1/2}(1-e)]\ ,\\ f(x)&=2^x\Gamma(x+\tfrac{1}{2})/\Gamma(\tfrac{1}{2})\ , \end{aligned}\tag{3.8}$$

where $\Gamma(x)$ stands for the gamma function. We may easily check

$$SP=PS=PSP\ ,\tag{3.9}$$

so that the transformation S does not change the antisymmetric character of the Marumori states.

If we set

$$|\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots,\tilde{\lambda}_q\rangle=S|\lambda_1,\lambda_2,\ldots,\lambda_q\rangle\ ,\tag{3.10}$$

then its effect is simply to normalize the state vectors. In fact, we find

$$\begin{aligned} &|\tilde{0}\rangle=|0\rangle\\ &|\tilde{\lambda}_1,\ldots,\tilde{\lambda}_q\rangle=[(q-1)!!]^{-1/2}|\lambda_1,\lambda_2\ldots\lambda_q\rangle\ ,\\ &q=\text{even}\geq 2\\ &|\tilde{\lambda}_1\ldots\tilde{\lambda}_q\rangle=(q!!)^{-1/2}|\lambda_1,\ldots,\lambda_q\rangle\ ,\\ &q=\text{odd}\geq 1\ . \end{aligned}\tag{3.11}$$

It is not difficult[4] to prove that the new states $|\tilde{\lambda}_1\cdots\tilde{\lambda}_q\rangle$ are now properly ortho-normalized. Therefore, if we set

$$\begin{aligned} &\tilde{V}=SV,\\ &|\tilde{\lambda}_1,\ldots,\tilde{\lambda}_q\rangle=\tilde{V}|\lambda_1\ldots\lambda_q\rangle_F=\tilde{A}^{\lambda_1}\cdots\tilde{A}^{\lambda_q}|\tilde{0}\rangle \end{aligned}\tag{3.12}$$

then the new injection map $\tilde{V}$ is isometric.

Hereafter, we shall consider the cases (I) and (III) of the Sec. 2. For the case II, we have to replace $\bar{a}_\mu a_\nu$ in $R^{\mu\nu}$ and A^{μ}_{ν} by $\bar{b}_\mu b_\nu$, which is justifiable in the Marumori space. Under this understanding, we find

$$\begin{aligned} \tilde{R}_{\mu\nu}&=(1+M)^{1/2}B_{\mu\nu}e+B_{\mu\nu}(1+M)^{1/2}(1-e)\ ,\\ \tilde{R}^{\mu\nu}&=B^{\mu\lambda}(B^{\nu}_{\lambda}+\delta^{\nu}_{\lambda})(1+M)^{-1/2}e+(1+M)^{-1/2}\\ &\quad\times[B^{\mu\lambda}(B^{\nu}_{\lambda}+\delta^{\nu}_{\lambda})(1-e)\\ &\quad-B^{\mu\lambda}\bar{b}_\nu b_\lambda+B^{\nu\lambda}\bar{b}_\mu b_\lambda]\ ,\\ \tilde{A}_\mu&=(1+M)^{1/2}b_\mu+B_{\mu\lambda}\bar{b}_\lambda\ ,\\ \tilde{A}^{\mu}&=(\bar{b}_\mu+\bar{b}_\lambda B^{\mu}_{\lambda})(1+M)^{-1/2}+B^{\mu\lambda}b_\lambda\ ,\\ \tilde{A}^{\mu}_{\nu}&=B^{\mu}_{\nu}-\bar{b}_\mu b_\nu\ , \end{aligned}\tag{3.13}$$

where operators $\tilde{Q}$ are defined by

$$\tilde{Q}=SQS^{-1}\ .\tag{3.14}$$

Now, as we shall prove in the Appendix, we have identities

$$\begin{aligned} &(B^{\lambda}_{\mu}B_{\lambda\nu}-MB_{\mu\nu})P=0\ ,\\ &P(B^{\mu\lambda}B^{\nu}_{\lambda}-B^{\mu\nu}M)=0\ ,\\ &(B^{\lambda}_{\mu}b_\lambda-Mb_\mu)P=0\ . \end{aligned}\tag{3.15}$$

Then, using (2.31), (3.15), and (3.5), we find

$$\begin{aligned} P\tilde{R}^{\mu\nu}&=P[B^{\mu\nu}(1+M)^{1/2}e+(1+M)^{1/2}B^{\mu\nu}(1-e)],\\ P\tilde{A}^{\mu}&=P[\bar{b}_\mu(1+M)^{1/2}+B^{\mu\lambda}b_\lambda]\ . \end{aligned}\tag{3.16}$$

Hence, comparing this with (3.13), we see that the Hermiticity conditions

$$\begin{aligned} (\tilde{R}_{\mu\nu}P)^\dagger&=P\tilde{R}^{\mu\nu}=\tilde{R}^{\mu\nu}P\ ,\\ (\tilde{A}_\mu P)^\dagger&=P\tilde{A}^{\mu}=\tilde{A}^{\mu}P \end{aligned}\tag{3.17}$$

are now satisfied in the Marumori space. However, our formulas are still different in form given by Marshalek[6,7] who introduces the square root operator D^μ_ν satisfying

$$D^\lambda_\nu D^\mu_\lambda = \delta^\mu_\nu + B^\mu_\nu \ . \tag{3.18}$$

Formally, we can rewrite the solution as

$$D^\mu_\nu = [(1+B)^{1/2}]^\mu_\nu \ . \tag{3.19}$$

Marshalek then expands the square root operator into a formal power series expansion in order to obtain his result. However, such an expansion diverges as is shown in the Appendix. We could of course use

$$D^\mu_\nu(Z) = [(1+ZB)^{1/2}]^\mu_\nu = \sum_{n=0}^{\infty} c_n (B^n)^\mu_\nu Z^n \ , \tag{3.20}$$

where the operator $(B^n)^\mu_\nu$ is defined in the Appendix and c_n is the numerical expansion coefficient. The expansion (3.20) converges in the Marumori space when the complex number Z is sufficiently small. Then, (3.19) can be obtained as an analytic continuation in Z for $Z \to 1$ of $D^\mu_\nu(Z)$. However, by this procedure, the Hermiticity condition

$$(D^\mu_\nu)^\dagger = D^\nu_\mu \tag{3.21}$$

is not obvious and must be checked. Fortunately, the situation is greatly simplified in the Marumori space. Indeed we can prove

$$D^\mu_\nu P = (1+M)^{-1/2}(\delta^\mu_\nu + B^\mu_\nu)P \ ,$$
$$P D^\mu_\nu = P(\delta^\mu_\nu + B^\mu_\nu)(1+M)^{-1/2} \tag{3.22}$$

as we shall prove in the Appendix. Note that the impossibility of the expansion (3.20) for $Z=1$ is related to the same difficulty for the expansion of $(1+M)^{-1/2}$ when we have $M \geq 1$. At any rate, if we note (2.31), (2.15), and (3.22) together with

$$B_{\mu\nu}P = PB_{\mu\nu}P, \quad PB^{\mu\nu} = PB^{\mu\nu}P \ ,$$

then we can finally rewrite (3.13) as

$$\tilde{R}_{\mu\nu}P = (D^\lambda_\mu B_{\lambda\nu} - [D^\alpha_\beta, B_{\mu\nu}]\bar{b}_\beta b_\alpha)P \ ,$$
$$P\tilde{R}^{\mu\nu} = P(B^{\mu\lambda}D^\nu_\lambda + [D^\alpha_\beta, B^{\mu\nu}]\bar{b}_\beta b_\alpha) \ ,$$
$$\tilde{A}_\mu P = (D^\lambda_\mu b_\lambda + B_{\mu\lambda}\bar{b}_\lambda)P \ ,$$
$$P\tilde{A}^\mu = P(\bar{b}_\lambda D^\mu_\lambda + B^{\mu\lambda}b_\lambda) \ . \tag{3.23}$$

If we forget the presence of the projection operator P, then this expression is equivalent to the unitary form[6,7] of Marshalek, which is an analog of the Holstein-Primakoff unitary realization[11] of the SU(2) group. However, the realization (3.16) can be simpler for some cases. For example, let us consider the fermion Hamiltonian

$$H = \sum_\mu \epsilon_\mu \bar{C}_\mu C_\mu + \sum_{\alpha,\beta,\mu,\nu} g^{\mu\nu}_{\alpha\beta}\bar{C}_\alpha \bar{C}_\beta C_\mu C_\nu \ . \tag{3.24}$$

Then, the corresponding Marumori Hamiltonian will be written as

$$\tilde{H} = -\sum_\mu \epsilon_\mu \tilde{A}^\mu_\mu - \sum_{\alpha,\beta,\mu,\nu} g^{\mu\nu}_{\alpha\beta}\tilde{R}^{\alpha\beta}\tilde{R}_{\mu\nu} \ . \tag{3.25}$$

Using (3.13) and (3.16), we can rewrite it as

$$P\tilde{H}P = PH'P \ ,$$
$$H' = \sum_\mu \epsilon_\mu \left(\sum_\lambda B^{\mu\lambda}B_{\mu\lambda} + \bar{b}_\mu b_\mu\right) - \sum_{\alpha,\beta,\mu,\nu} g^{\mu\nu}_{\alpha\beta}B^{\alpha\beta}B_{\mu\nu}(N-e) \ . \tag{3.26}$$

The same equation can be derived for the Dyson representation formula (2.5), if we notice identities (3.15) and (2.31). Since the particle number N is a constant of motion, (3.26) is essentially bilinear in the boson variable $B_{\mu\nu}$ and $B^{\mu\nu}$. We remark that this surprising simplification is due to the presence of the projection operator P in the left side of H'. Indeed, let us consider an eigenvalue problem

$$H\Psi = E\Psi \ ,$$
$$\Psi = \sum F(\mu_1 \ldots, \mu_q)\bar{C}_{\mu_1}\cdots\bar{C}_{\mu_q}|0\rangle_F \ , \tag{3.27}$$

where $F(\mu_1 \cdots \mu_q)$ is completely antisymmetric. Then we have an eigenvalue equation

$$(\epsilon_{\mu_1} + \epsilon_{\mu_2} + \cdots + \epsilon_{\mu_q} - E)F(\mu_1, \mu_2, \ldots, \mu_q) = \frac{1}{(q-2)!}\sum\sum_{\alpha,\beta,\theta}(-1)^\theta g^{\alpha\beta}_{\mu_1\mu_2}F(\alpha, \beta, \mu_3, \mu_4, \ldots, \mu_q). \tag{3.28}$$

Here the summation is over all $q!$ permutation θ among $\mu_1\mu_2 \cdots \mu_q$. On the other hand, the corresponding Marumori equation

$$PH'\Psi' = E\Psi' \ ,$$
$$\Psi' = \sum F(\mu_1, \ldots, \mu_q)|\tilde{\mu}_1 \ldots, \tilde{\mu}_q\rangle \tag{3.29}$$

leads to the exactly same equation (3.28), when we notice

$$PB^{\mu_1\nu_1}\cdots B^{\mu_l\nu_l}|0\rangle = [(2l-1)!!]^{-1/2}|\tilde{\mu}_1\tilde{\nu}_1, \ldots, \tilde{\mu}_l\tilde{\nu}_l\rangle \ ,$$
$$PB^{\mu_1\nu_1}\cdots B^{\mu_l\nu_l}\bar{b}_\lambda|0\rangle = [(2l+1)!!]^{-1/2}|\tilde{\lambda}, \tilde{\mu}_1\tilde{\nu}_1, \ldots, \tilde{\mu}_l\tilde{\nu}_l\rangle \ . \tag{3.30}$$

However, we have $PH' \neq H'P$ because of (2.30). This proves the desired equivalence between $\boldsymbol{H}$ and PH', but *not* between H and H'.

The author would like to express his gratitude to Professor E. R. Marshalek and Professor A. Klein for useful conversations.

APPENDIX

First, we remark that A^μ_ν satisfies the commutation relation

$$[A^\mu_\nu, A^\alpha_\beta] = \delta^\mu_\beta A^\alpha_\nu - \delta^\alpha_\nu A^\mu_\beta \ . \tag{A1}$$

Second, its diagonal components $F_\mu = A^\mu_\mu$ (no summation over μ) assume only integral eigenvalues in the Marumori space. Hence, (A1) defines the Lie algebra of the n-dimensional unitary group U(n), whose irreducible representations are specified by n integers satisfying

$$f_1 \geq f_2 \geq \cdots \geq f_n$$

which are maximal eigenvalues of F_μ. Let us call any irreducible representation of the U(n) a degenerate representation if there is an integer k satisfying

$$f_1 = f_2 = \cdots = f_k \geq f_{k+1} = f_{k+2} = \cdots = f_n .$$

As we shall prove elsewhere,[12] any degenerate representation satisfies an identity

$$(A^\lambda_\nu - f_n \delta^\lambda_\nu)[A^\mu_\lambda - (f_1 + n - k)\delta^\mu_\lambda]P = 0 \ , \tag{A2}$$

where P is the projection operator for the irreducible representation. Now, our Marumori space is an irreducible representation of the algebra B_n, which reduces to a direct sum of completely antisymmetric degenerate representations of the U(n) group. Indeed, the state vector (2.26) belongs to the completely antisymmetric representation of the U(n) with signature

$$f_1 = f_2 = \cdots = f_{n-q} = 0 \ ,$$
$$f_{n-q+1} = \cdots = f_n = -1. \tag{A3}$$

Therefore, we must have

$$(A^\lambda_\nu + \delta^\lambda_\nu)(A^\mu_\lambda + A^\alpha_\alpha \delta^\mu_\lambda)P = 0 \tag{A4}$$

if we notice $A^\alpha_\alpha |\lambda_1 \cdots \lambda_q\rangle = -q|\lambda_1 \cdots \lambda_q\rangle$, where P is now the projection operator for the Marumori space. Actually, we can prove a stronger result of

$$(A^\mu_\nu A^\alpha_\beta + A^\alpha_\nu A^\mu_\beta + \delta^\alpha_\nu A^\mu_\beta + \delta^\mu_\nu A^\alpha_\beta)P = 0 \tag{A5}$$

for any completely antisymmetric representation with a special form (A3). The proof of (A5) is essentially the same as has been given elsewhere.[12] This may be also directly checked from (2.26) together with

$$[A^\mu_\nu, A^\lambda] = -\delta^\lambda_\nu A^\mu$$

if we utilize the completely antisymmetric character of the Marumori space. Equations (A4) and (A5) are the basic tool of this Appendix. Before going into details, we remark that (A5) corresponds to an identity

$$(\bar{C}_\mu C_\nu)(\bar{C}_\alpha C_\beta) + (\bar{C}_\alpha C_\nu)(\bar{C}_\mu C_\beta) - \delta^\alpha_\nu(\bar{C}_\mu C_\beta) - \delta^\mu_\nu(\bar{C}_\alpha C_\beta) = 0$$

in the original fermion space through (1.6).

Next, let us take a commutator of (A5) with A_λ. If we notice

$$[A^\mu_\nu, A_\lambda] = \delta^\mu_\lambda A_\nu$$

and (2.28), i.e.,

$$A_\lambda P = PA_\lambda P \ ,$$

then we can easily derive

$$(A^\mu_\lambda A_\nu + A^\mu_\nu A_\lambda)P = 0 \tag{A6}$$

which is an analog of

$$(\bar{C}_\mu C_\lambda)C_\nu + (\bar{C}_\mu C_\nu)C_\lambda = 0 \ .$$

Multiplying e to (A6) and noting

$$e\bar{b}_\mu = 0, \quad eb_\mu = b_\mu \ ,$$

we find

$$(B^\mu_\lambda b_\nu + B^\mu_\nu b_\lambda)P = 0 \ . \tag{A7}$$

Setting $\mu = \nu$, this reproduces the last equation of (3.15). Actually, in our derivation, we are considering the cases (I) and (III) of the Sec. 2. For the case II, we can derive the same if we notice

$$(\bar{a}_\mu a_\nu)P = (\bar{b}_\mu b_\nu)P \ .$$

From (A7), we find

$$(B^\mu_\lambda \bar{b}_\alpha b_\nu + B^\mu_\nu \bar{b}_\alpha b_\lambda)P = 0 \tag{A8}$$

which reproduces the second equation in (2.31). Because of (A8), we can rewrite (A5) as

$$(B^\mu_\nu B^\alpha_\beta + B^\alpha_\nu B^\mu_\beta + \delta^\alpha_\nu B^\mu_\beta + \delta^\mu_\nu B^\alpha_\beta)P = 0 \ . \tag{A9}$$

This implies that the Marumori states are also completely antisymmetric with respect to the bosonic part of the U(n) group, whose generator is now B^μ_ν instead of A^μ_ν. This antisymmetricity is also evident from (2.24) and (2.25).

Next, let us take a commutator of (A8) with $B_{\tau\beta}$. Noting $B_{\tau\beta}P = PB_{\tau\beta}P$ and

$$[B^\mu_\nu, B_{\tau\beta}] = \delta^\mu_\tau B_{\nu\beta} + \delta^\mu_\beta B_{\tau\nu} \tag{A10}$$

then this gives us

$$[B_{\lambda\nu}(\bar{b}_\alpha b_\beta) + B_{\lambda\beta}(\bar{b}_\alpha b_\nu)]P = 0 \tag{A11}$$

which reprodues the first equation of (2.31). The last equation in (2.31) is simply its Hermitian conjugate. Similarly, commuting $B_{\lambda\tau}$ with (A9), we obtain

$$(B^{\lambda}_{\mu} B_{\alpha\nu} + B^{\lambda}_{\nu} B_{\alpha\mu})P = 0 \ . \qquad \text{(A12)}$$

Together with its Hermitian conjugate, this proves the first two equations of (3.15).

If we set $\mu = \beta$ in (A9), we find

$$[B^{\lambda}_{\nu} B^{\mu}_{\lambda} + (1 - M)B^{\mu}_{\nu} - M\delta^{\mu}_{\nu}]P = 0 \ , \qquad \text{(A13)}$$

where we have set

$$M = -B^{\lambda}_{\lambda} \ . \qquad \text{(A14)}$$

Now, let us define a tensor operator $(B^n)^{\mu}_{\nu}$ recursively by

$$(B^0)^{\mu}_{\nu} = \delta^{\mu}_{\nu}, \quad (B^{n+1})^{\mu}_{\nu} = (B^n)^{\lambda}_{\nu} B^{\mu}_{\lambda} \ . \qquad \text{(A15)}$$

Suppose that $f(z)$ is an holomorphic function of z near the origin $z = 0$ with

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \text{(A16)}$$

then we can define a tensor operator by

$$[f(B)]^{\mu}_{\nu} = \sum_{n=0}^{\infty} c_n (B^n)^{\mu}_{\nu} \ . \qquad \text{(A17)}$$

However, in view of (A13), we can reduce all $(B^n)^{\mu}_{\nu}$ with $n \geq 2$ as linear combinations of B^{μ}_{ν} and δ^{μ}_{ν} in the Marumori space. The result is

$$[f(B)]^{\mu}_{\nu} P = \left\{ \frac{1}{1+M} f(M)[B^{\mu}_{\nu} + \delta^{\mu}_{\nu}] - \frac{1}{1+M} f(-1)[B^{\mu}_{\nu} - M\delta^{\mu}_{\nu}] \right\} P \ . \qquad \text{(A18)}$$

Note that this formula can be regarded as a definition of $[f(B)]^{\mu}_{\nu}$ even when the power series expansion (A16) for $f(z)$ diverges at $z = -1$ and/or $z = M$. This is essentially an analog of the symbolic calculus[13] used in the functional analysis. Indeed, for the special choice $f(z) = (1 + z)^{1/2}$, the power series fails for $z = -1$ and $z = M$ with $M \geq 1$, so that the formal method utilized by Marshalek is, strictly speaking not justifiable. At any rate, our formula reproduces the first equation of (3.22), while the second one is simply its Hermitian conjugate.

We can find many other identities by similar method. For example, taking commutators of (A5) successively with $R_{\mu\nu}$ and $R^{\mu\nu}$, we can derive

$$(R^{\lambda\mu} R_{\alpha\beta} + A^{\mu}_{\alpha} A^{\lambda}_{\beta} + \delta^{\lambda}_{\alpha} A^{\mu}_{\beta})P = 0 \ . \qquad \text{(A19)}$$

This equation is an analog of an identity

$$(\bar{C}_{\mu} \bar{C}_{\lambda})(C_{\alpha} C_{\beta}) + (\bar{C}_{\mu} C_{\alpha})(\bar{C}_{\lambda} C_{\beta}) - \delta^{\lambda}_{\alpha}(\bar{C}_{\mu} C_{\beta}) = 0$$

in the original fermion space. Equation (A19) assures us of the fact that we can always obtain the same result whenever we express the four fermion Hamiltonian (3.24) in terms of boson variables in different forms. For example, we may reexpress (3.25) also as

$$\tilde{H} = -\sum_{\mu} \epsilon_{\mu} \tilde{A}^{\mu}_{\mu} - \sum_{\mu,\nu,\alpha,\beta} g^{\mu\nu}_{\alpha\beta} (\tilde{A}^{\alpha}_{\mu} \tilde{A}^{\beta}_{\nu} + \delta^{\beta}_{\mu} \tilde{A}^{\alpha}_{\nu}) \qquad \text{(A20)}$$

in the Marumori space.

Also, combining (A11) and (A12), we find

$$(A^{\lambda}_{\mu} R_{\alpha\nu} + A^{\lambda}_{\nu} R_{\alpha\mu})P = 0 \qquad \text{(A21)}$$

which corresponds to an identity

$$(\bar{C}_{\lambda} C_{\mu})(C_{\alpha} C_{\nu}) + (\bar{C}_{\lambda} C_{\nu})(C_{\alpha} C_{\mu}) = 0 \ .$$

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