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Nuclear muon-capture sum rules and mean nuclear excitation energies

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A discussion is given of non-energy-weighted and of energy-weighted sum rules in nuclear muon capture. It is argued that the mean nuclear excitation energy in muon capture does not vary appreciably as A and Z vary. A combined non-energy-weighted and energy-weighted sum rule which constitutes a three-parameter fit to the experimental data on total muon-capture rates is presented.

I. INTRODUCTION

Considerable progress has be achieved recently in the study of (a) the isospin structure of the matrix element of the operator which describes the total transition rate induced by an isovector current from an initial nuclear ground state to the various possible final nuclear excited states,[1] and (b) the effect of the internucleon isospace-exchange potentials on the photonuclear sum rules.[2] These developments can be applied to the treatment of the total rates of muon capture by nuclei and it is the aim of the present paper to outline such an application. The first section of the paper describes a closure approximation with respect to the final nuclear states obtained in the muon-capture process; here a suitable average is taken of the energy of the outgoing neutrino and the initial-state expectation value of the absolute square of the muon-capture current is decomposed into its isospin components. A similar investigation of the total muon-capture rate weighted by the mean nuclear excitation energy is carried out in the second section, while a description of the total muon-capture rate through a combination of a non-energy-weighted sum rule (NEWSR) and an energy-weighted sum rule (EWSR) is presented in the third section Some concluding remarks are set down in the last section.

II. NON-ENERGY-WEIGHTED SUM RULE (NEWSR)

The total muon-capture rate of the reaction from the ground state a to all energetically possible states b,

$$\mu^- + [Z,A]_{\text{g.s.}:a} \to \nu_\mu + [Z-1,A]_{\text{all}:b} \tag{1}$$

is given by[3-5]:

$$\Lambda_a = KZ_{\text{eff}}^{\,4} Z^{-1} \times \sum_{b\leq} \left|\frac{\vec{\nu}_{ba}}{m_\mu}\right|^2 \int \frac{d\hat{\nu}}{4\pi} |\langle b|J^{(-)}(|\vec{\nu}_{ba}|)|a\rangle|^2 , \tag{2a}$$

where $\sum_{b\leq}$ implies summation over all states of $[Z-1,A]$ with $(E_b - E_a) \leq (m_\mu - \epsilon_\mu)$ and where

$$\Lambda_a = \Lambda_{a;\,J_a-\frac12}\left(\frac{J_a}{2J_a+1}\right) + \Lambda_{a;\,J_a+\frac12}\left(\frac{J_a+1}{2J_a+1}\right);$$

$$K \equiv (G_V^{\,2} + 3G_A^{\,2} + G_P^{\,2} - 2G_PG_A)\frac{\alpha^3 m_\mu^{\,5}}{2\pi^2};$$

$$Z_{\text{eff}}^{\,4} \equiv Z^4\langle|\varphi_\mu(\vec{x})|^2\rangle;$$

$$|\vec{\nu}_{ba}| = (m_\mu - \epsilon_\mu) - (E_b - E_a);$$

$$J^{(\mp)}(|\vec{\nu}_{ba}|) \equiv \sum_{j=1}^{A} \tau_j^{(\mp)} e^{\mp i|\vec{\nu}_{ba}|\hat{\nu}\cdot\vec{x}_j} \tag{2b}$$

the notation being that of Refs. 3, 4, and 5. In Eqs. (2a) and (2b), for the sake of simplicity, we average the square of the muon wave function over the initial nuclear ground state $|a\rangle$, neglect relativistic components of the hadron weak current,[6] omit all meson-exchange effects,[7] and assume that[4]

$$|M_V|^2 = |M_A|^2 = |M_P|^2$$

$$\left(|M_{V,A,P}|^2 \equiv \sum_{b\leq}\left|\frac{\vec{\nu}_{ba}}{m_\mu}\right|^2 \int\frac{d\hat{\nu}}{4\pi}\; |\langle b|J^{(-)}_{V,A,P}(|\vec{\nu}_{ba}|)|a\rangle|^2\right.$$

with

$$J^{(\mp)}_{V,A,P}(|\vec{\nu}_{ba}|) \equiv \sum_{j=1}^{A} \tau^{(\mp)}_{j=1}\vartheta_{V,A,P;j}\, e^{\mp i|\vec{\nu}_{ba}|\hat{\nu}\cdot\vec{x}_j}; \qquad \vartheta_{V;j}=1_j, \qquad \vec{\vartheta}_{A;j}=\vec{\sigma}_j/\sqrt{3}, \qquad \vartheta_{P;j}=\vec{\sigma}_j\cdot\hat{\nu}\Big);$$

this last assumption is briefly examined below. Use of closure, i.e., replacement of $\sum_{b\le}$ by $\{\sum_{b\le}+\sum_{b>}\}$ $=\sum_{\text{all states } b \text{ of } [Z-1,A]}$ where $\sum_{b>}$ implies summation over all states of $[Z-1, A]$ with $(E_b - E_a) > (m_\mu - \epsilon_\mu)$, yields

$$\Lambda_a = KZ_{\text{eff}}{}^4 Z^{-1}\left(\frac{\nu_a}{m_\mu}\right)^2 \langle a|J^{(+-)}(\nu_a)|a\rangle;$$

$$J^{(+-)}(\nu_a) \equiv \int \frac{d\hat{\nu}}{4\pi} J^{(+)}(\nu_a)J^{(-)}(\nu_a) = \tfrac{1}{2}A + T^{(3)} + AK^{(+-)}(\nu_a);$$

$$\tfrac{1}{2}A + T^{(3)} = \sum_{j=1}^{A} \tau_j^{(+)}\tau_j^{(-)} = \sum_{j=1}^{A} \tfrac{1}{2}(1+\tau_j^{(3)});$$

$$K^{(+-)}(\nu_a) \equiv A^{-1}\int \frac{d\hat{\nu}}{4\pi} \sum_{k=1,\,j=1}^{A} (1-\delta_{kj})\tau_k^{(+)}\tau_j^{(-)} e^{i\nu_a\hat{\nu}\cdot(\vec{x}_k-\vec{x}_j)}$$

$$= A^{-1}\sum_{k=1,\,j=1}^{A}(1-\delta_{kj})\tfrac{1}{4}\left[\tfrac{2}{3}\vec{\tau}_k\cdot\vec{\tau}_j - \tfrac{1}{3}(3\tau_k^{(3)}\tau_j^{(3)} - \vec{\tau}_k\cdot\vec{\tau}_j)\right]\left[\frac{\sin(\nu_a|\vec{x}_k-\vec{x}_j|)}{(\nu_a|\vec{x}_k-\vec{x}_j|)}\right], \tag{3}$$

where ν_a is a suitable average of the $|\vec{\nu}_{ba}|$; thus

$$[\langle E_b\rangle - (E_b)_{\text{g.s.}}] = \{(m_\mu - \epsilon_\mu) - \nu_a - [(E_b)_{\text{g.s.}} - E_a]\} \tag{4}$$

is a mean nuclear excitation energy of $[Z-1, A]$ characteristic of the muon capture process in Eq. (1). The diagonal matrix element $\langle a|J^{(+-)}(\nu_a)|a\rangle$ can be decomposed into an isoscalar, an isovector, and an isotensor part, as follows:

$$\langle a|J^{(+-)}(\nu_a)|a\rangle \equiv \langle a|J_0^{(+-)}(\nu_a) + \left(\frac{T^{(3)}}{A}\right)J_1^{(+-)}(\nu_a) + \left[\frac{3(T^{(3)})^2 - (\vec{T})^2}{A^2}\right]J_2^{(+-)}(\nu_a)|a\rangle$$

$$= \langle a\|J_0^{(+-)}(\nu_a)\|a\rangle + \left(\frac{Z-\frac{1}{2}A}{A}\right)\langle a\|J_1^{(+-)}(\nu_a)\|a\rangle + \left[\frac{3(Z-\frac{1}{2}A)^2 - T_a(T_a+1)}{A^2}\right]\langle a\|J_2^{(+-)}(\nu_a)\|a\rangle$$

$$= \tfrac{1}{2}A + A\langle a\|K_0^{(+-)}(\nu_a)\|a\rangle + \left(\frac{Z-\frac{1}{2}A}{A}\right)A + \left[\frac{3(Z-\frac{1}{2}A)^2 - T_a(T_a+1)}{A^2}\right]A\langle a\|K_2^{(+-)}(\nu_a)\|a\rangle, \tag{5a}$$

where

$$(\vec{T})^2|a\rangle = T_a(T_a+1)|a\rangle, \qquad T^{(3)}|a\rangle = T_a^{(3)}|a\rangle = (Z-\tfrac{1}{2}A)|a\rangle;$$

$$\langle a|L|a\rangle \equiv \langle \xi_a, T_a, T_a^{(3)} = (Z-\tfrac{1}{2}A)|L|\xi_a, T_a, T_a^{(3)} = (Z-\tfrac{1}{2}A)\rangle;$$

$$\langle a\|L_{0,1,2}\|a\rangle \equiv \langle \xi_a, T_a, T_a^{(3)} = -T_a|L_{0,1,2}|\xi_a, T_a, T_a^{(3)} = -T_a\rangle;$$

$$L = J^{(+-)}(\nu_a) \quad \text{or} \quad K^{(+-)}(\nu_a), \tag{5b}$$

and where the ξ_a are quantum numbers other than isospin which characterize the state $|a\rangle$. As an example, if $|a'\rangle$ and $|a''\rangle$ are members of the same isomultiplet,

$$\langle a'|J^{(+-)}(\nu_{a'})|a'\rangle - \langle a''|J^{(+-)}(\nu_{a''}\cong\nu_{a'})|a''\rangle = (Z'-Z'') + \left[\frac{3(Z'-\frac{1}{2}A)^2 - 3(Z''-\frac{1}{2}A)^2}{A^2}\right]A\langle a'\|K_2^{(+-)}(\nu_{a'})\|a'\rangle,$$

a relation particularly useful for $|a'\rangle = |{}^3\text{H}\rangle$, $|a''\rangle = |{}^3\text{He}\rangle$ where, in addition, $Z'-Z''=-1$, $(Z'-\frac{1}{2}A)^2 = (Z''-\frac{1}{2}A)^2$. Thus a knowledge of $(\Lambda_{^3\text{He}})_{\text{exper}}$, i.e., $(\Lambda_{^3\text{He}})_{\text{exper}} \cong 2200\ \text{sec}^{-1}$,[8] immediately permits the calculation of $(\Lambda_{^3\text{H}})_{\text{theor}}$, viz.:

$$(\Lambda_{^3\text{H}})_{\text{theor}} = \frac{(\Lambda_{^3\text{He}})_{\text{exper}}}{2^3} - K\left(\frac{\nu_{^3\text{He}}}{m_\mu}\right)^2 \cong 39\ \text{sec}^{-1} \cong 0.22(\Lambda_{^1\text{H}})_{\text{theor}}.$$

We now combine Eqs. (5a) and (5b) with Eq. (3) and assume, in addition, that, for all $|a\rangle$ of interest, $T_a=|T_a^{(3)}|=|Z-\frac{1}{2}A|$; this gives the non-energy-weighted sum rule for Λ_a

$$\Lambda_a = KZ_{\rm eff}{}^4\left(\frac{\nu_a}{m_\mu}\right)^2\left\{1+\left(\frac{A}{Z}\right)\langle a\|K_0^{(+-)}(\nu_a)\|a\rangle+\left[\frac{2(Z-\frac{1}{2}A)^2-|Z-\frac{1}{2}A|}{A^2}\right]\left(\frac{A}{Z}\right)\langle a\|K_2^{(+-)}(\nu_a)\|a\rangle\right\}$$

$$=KZ_{\rm eff}{}^4\left(\frac{\nu_a}{m_\mu}\right)^2\left\{1+\left(\frac{A}{2Z}\right)\langle a\|2K_0^{(+-)}(\nu_a)+K_2^{(+-)}(\nu_a)\|a\rangle-\left(\frac{A-Z}{2A}+\frac{|A-2Z|}{8ZA}\right)\langle a\|4K_2^{(+-)}(\nu_a)\|a\rangle\right\} \tag{6}$$

which is to be compared with the NEWSR for Λ_a deduced directly from Eq. (3) and approximately valid for $Z\gg 1$, viz. [see Eqs. (9)–(14) of Ref. 3]:

$$\Lambda_a \cong KZ_{\rm eff}{}^4\left(\frac{\nu_a}{m_\mu}\right)^2\left[1-\left(\frac{A}{2Z}\right)\delta_a'-\left(\frac{A-Z}{2A}\right)\delta_a\right];$$

$$\frac{\nu_a}{m_\mu}\cong\tfrac{3}{4},\quad \delta_a\cong 3,\quad \delta_a'\ll\delta_a\ . \tag{7}$$

The comparison shows that the term proportional to $(A-Z)/2A$, which involves only the isotensor part of $\langle a|K^{(+-)}(\nu_a)|a\rangle$,[9] arises from the difference between the space-symmetric and the space-antisymmetric nucleon-nucleon correlation functions [see Eqs. (7)–(9) of Ref. 3] and specifically yields

$$\langle a\|4K_2^{(+-)}(\nu_a)\|a\rangle\cong\delta_a\cong 3, \tag{8a}$$

$$-\langle a\|2K_0^{(+-)}(\nu_a)+K_2^{(+-)}(\nu_a)\|a\rangle\cong\delta_a'\ll\delta_a\cong 3\ , \tag{8b}$$

suggesting that a simple relationship, namely, $\langle a\|K_0^{(+-)}(\nu_a)\|a\rangle\cong-\frac{1}{2}\langle a\|K_2^{(+-)}(\nu_a)\|a\rangle$, may be generally valid. The NEWSR for Λ_a in Eq. (7) is in reasonably good agreement with experiment for $8\leq Z\leq 92$ (see Ref. 3); an even better agreement can be obtained with the NEWSR of Eq. (6) if

$$\langle a\|2K_0^{(+-)}(\nu_a)+K_2^{(+-)}(\nu_a)\|a\rangle,$$

$$\langle a\|4K_2^{(+-)}(\nu_a)\|a\rangle\ ,$$

and (ν_a/m_μ) are treated as constants independent of A and Z, but with numerical values adjusted for optimum fit to the data. Finally, remembering that there exist no systematic *a priori* calculations of (ν_a/m_μ) and that the prediction $\delta_a'\approx 0$ is based on an assumption regarding the short range character of effective nucleon-nucleon correlations in nuclei, we cannot exclude the possibility that, for example, $\langle a\|4K_2^{(+-)}(\nu_a)\|a\rangle$ is a constant independent of A and Z, while

$$\langle a\|2K_0^{(+-)}(\nu_a)+K_2^{(+-)}(\nu_a)\|a\rangle$$

and ν_a/m_μ slowly increase and slowly decrease, respectively, as A and Z increase.

III. ENERGY-WEIGHTED SUM RULE (EWSR)

We next consider an EWSR for Λ_a, viz.

$$(\langle E_b\rangle-E_a)=\frac{\sum_{b\leq}(E_b-E_a)|\vec{\nu}_{ba}/m_\mu|^2\int(d\hat{\nu}/4\pi)|\langle b|J^{(-)}(\vec{\nu}_{ba}|)|a\rangle|^2}{\sum_{b\leq}|\vec{\nu}_{ba}/m_\mu|^2\int(d\hat{\nu}/4\pi)|\langle b|J^{(-)}(|\vec{\nu}_{ba}|)a\rangle|^2}$$

$$=\Lambda_a{}^{-1}KZ_{\rm eff}{}^4Z^{-1}\sum_{b\leq}(E_b-E_a)\left|\frac{\vec{\nu}_{ba}}{m_\mu}\right|^2\int\frac{d\hat{\nu}}{4\pi}\,|\langle b|J^{(-)}(|\vec{\nu}_{ba}|)|a\rangle|^2$$

$$=\Lambda_a{}^{-1}KZ_{\rm eff}{}^4Z^{-1}\left(\frac{\nu_a}{m_\mu}\right)^2\sum_{b\leq}(E_b-E_a)\int\frac{d\hat{\nu}}{4\pi}\,|\langle b|J^{(-)}(\nu_a)|a\rangle|^2\ , \tag{9}$$

the ν_a in Eq. (9) being not very different from the ν_a in Eq. (3). Thus, using closure,

$$\Lambda_a=KZ_{\rm eff}{}^4Z^{-1}\left(\frac{\nu_a}{m_\mu}\right)^2\frac{(\nu_a{}^2/2m_N)\langle a|W^{(+-)}(\nu_a)|a\rangle}{\langle E_b\rangle-E_a}$$

$$=KZ_{\rm eff}{}^4Z^{-1}\left(\frac{\nu_a}{m_\mu}\right)^2\frac{(\nu_a{}^2/2m_N)\langle a|W^{(+-)}(\nu_a)|a\rangle}{(m_\mu-\epsilon_\mu)-\nu_a} \tag{10}$$

with

$$W^{(+-)}(\nu_a) \equiv \left(\frac{\nu_a^2}{2m_{\mathfrak{N}}}\right)^{-1} \int \frac{d\hat{\nu}}{4\pi}\, \tfrac{1}{2}\, \{[J^{(+)}(\nu_a), H]J^{(-)}(\nu_a) - J^{(+)}(\nu_a)[J^{(-)}(\nu_a), H]\}$$

$$H = \sum_{l=1}^{A}\left[\left(i^{-1}\frac{\partial}{\partial \vec{x}_l}\right)^2 \Big/ 2m_{\mathfrak{N}}\right] + \frac{1}{2}\sum_{l=1,\, m=1}^{A} (u_{lm} + \vec{\tau}_l \cdot \vec{\tau}_m v_{lm});$$

$$u_{lm} \equiv u(\vec{x}_l - \vec{x}_m, \vec{\sigma}_l, \vec{\sigma}_m), \quad v_{lm} \equiv v(\vec{x}_l - \vec{x}_m, \vec{\sigma}_l, \vec{\sigma}_m), \tag{11}$$

where H is the nuclear Hamiltonian, $\frac{1}{2}\sum_{l=1,m=1}^{A} u_{lm}$ is the internucleon *isospace-nonexchange* potential energy, and $\frac{1}{2}\sum_{l=1,m=1}^{A} \vec{\tau}_l \cdot \vec{\tau}_m v_{lm}$ is the internucleon *isospace-exchange* potential energy ($u_{ll} = v_{ll} = 0$). Evaluating the commutators, we obtain

$$W^{(+-)}(\nu_a) = \tfrac{1}{2}A + T^{(3)} + AX^{(+-)}(\nu_a) + AY^{(+-)}(\nu_a);$$

$$X^{(+-)}(\nu_a) \equiv \left[-A^{-1}\left(\frac{\nu_a^2}{2m_{\mathfrak{N}}}\right)\int \frac{d\hat{\nu}}{4\pi}\,\hat{\nu}\cdot \sum_{k=1,\,j=1}^{A} \tau_k^{(+)}\tau_j^{(-)} e^{i\nu_a\hat{\nu}\cdot(\vec{x}_k - \vec{x}_j)}\left(i^{-1}\frac{\partial}{\partial \vec{x}_k} + i^{-1}\frac{\partial}{\partial \vec{x}_j}\right)\right]\Big/\left(\frac{\nu_a^2}{2m_{\mathfrak{N}}}\right)$$

$$= -A^{-1}\sum_{k=1,\,j=1}^{A} \tfrac{1}{4}\,(\vec{\tau}_k \times \vec{\tau}_j)^{(3)}\left[\frac{\sin(\nu_a|\vec{x}_k - \vec{x}_j|) - (\nu_a|\vec{x}_k - \vec{x}_j|)\cos(\nu_a|\vec{x}_k - \vec{x}_j|)}{(\nu_a|\vec{x}_k - \vec{x}_j|)^2}\right]$$

$$\times\left(\frac{\vec{x}_k - \vec{x}_j}{|\vec{x}_k - \vec{x}_j|}\right)\cdot\left(\frac{i^{-1}\partial/\partial\vec{x}_k + i^{-1}\partial/\partial\vec{x}_j}{\nu_a}\right),$$

$$Y^{(+-)}(\nu_a) \equiv A^{-1}\left(\frac{\nu_a^2}{2m_{\mathfrak{N}}}\right)^{-1}\int\frac{d\hat{\nu}}{4\pi}\,\frac{1}{2}\left\{\frac{1}{2}\sum_{\substack{k=1,\,j=1\\ l=1,\,m=1}}^{A}\Big([\tau_k^{(+)}, \vec{\tau}_l\cdot\vec{\tau}_m]\tau_j^{(-)} e^{i\nu_a\hat{\nu}\cdot(\vec{x}_k - \vec{x}_j)} v_{lm}\right.$$

$$\left. - \tau_j^{(+)}[\tau_k^{(-)}, \vec{\tau}_l\cdot\vec{\tau}_m] e^{-i\nu_a\hat{\nu}\cdot(\vec{x}_k - \vec{x}_j)} v_{lm}\Big)\right\}$$

$$= A^{-1}\left(\frac{\nu_a^2}{2m_{\mathfrak{N}}}\right)^{-1}\left\{\sum_{k=1,\,j=1}^{A}(-\tfrac{1}{2})\,[\tfrac{4}{3}\vec{\tau}_k\cdot\vec{\tau}_j + \tfrac{1}{3}(3\tau_k^{(3)}\tau_j^{(3)} - \vec{\tau}_k\cdot\vec{\tau}_j) + (\tau_k^{(3)} + \tau_j^{(3)})]\left[1 - \frac{\sin(\nu_a|\vec{x}_k - \vec{x}_j|)}{(\nu_a|\vec{x}_k - \vec{x}_j|)}\right]v_{kj}\right.$$

$$+ \sum_{\substack{k=1,\,j=1,\\ m=1}}^{A}(1-\delta_{kj})(1-\delta_{mj})\tfrac{1}{2}[(\tau_k^{(3)}\tau_m^{(+)} - \tau_k^{(+)}\tau_m^{(3)})\tau_j^{(-)} - \tau_j^{(+)}(\tau_m^{(-)}\tau_k^{(3)} - \tau_m^{(3)}\tau_k^{(-)})]$$

$$\left.\times\left[\frac{\sin(\nu_a|\vec{x}_k - \vec{x}_j|)}{(\nu_a|\vec{x}_k - \vec{x}_j|)}\right]v_{km}\right\}. \tag{12}$$

Thus, decomposing $\langle a|W^{(+-)}(\nu_a)|a\rangle$ into an isovector, an isoscalar, and an isotensor part, we get (remembering that $T_a = |T_a^{(3)}| = |Z - \frac{1}{2}A|$):

$$\langle a|W^{(+-)}(\nu_a)|a\rangle \equiv \left\langle a\left| W_0^{(+-)}(\nu_a) + \left(\frac{T^{(3)}}{A}\right)W_1^{(+-)}(\nu_a) + \left[\frac{3(T^{(3)})^2 - (\vec{T})^2}{A^2}\right]W_2^{(+-)}(\nu_a)\right|a\right\rangle$$

$$= \langle a\|W_0^{(+-)}(\nu_a)\|a\rangle + \left(\frac{Z - \frac{1}{2}A}{A}\right)\langle a\|W_1^{(+-)}(\nu_a)\|a\rangle + \left[\frac{2(Z - \frac{1}{2}A)^2 - |Z - \frac{1}{2}A|}{A^2}\right]\langle a\|W_2^{(+-)}(\nu_a)\|a\rangle$$

$$= \tfrac{1}{2}A + A\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle + \left(\frac{Z - \frac{1}{2}A}{A}\right)(A + A\langle a\|Y_1^{(+-)}(\nu_a)\|a\rangle)$$

$$+ \left[\frac{2(Z - \frac{1}{2}A)^2 - |Z - \frac{1}{2}A|}{A^2}\right]A\langle a\|Y_2^{(+-)}(\nu_a)\|a\rangle, \tag{13}$$

where we have omitted a term $\sim (Z - \frac{1}{2}A)^3$ which arises from the "three-nucleon correlation" term $\sum_{k=1,j=1,l=1}^{A}\cdots$ in Eq. (12) and where $\langle a|X^{(+-)}(\nu_a)|a\rangle = 0$ essentially because there is no correlation between $\hat{\nu}$ and the direction of motion of the proton which captures the muon [with neglect of the internucleon potential energy in H, one has, in a semiclassical picture

$$(\langle E_b\rangle - E_a) = \int\int \left\{ \frac{(\vec{\nu}_a - \vec{p}_{\rm prot})^2}{2m_{\mathfrak{N}}} - \frac{\vec{p}_{\rm prot}{}^2}{2m_{\mathfrak{N}}} \right\} \frac{d\hat{p}_{\rm prot}}{4\pi}\frac{d\hat{\nu}}{4\pi} = \frac{\nu_a{}^2}{2m_{\mathfrak{N}}}\left[1 - \int\int \left(\hat{\nu}\cdot\frac{2\vec{p}_{\rm prot}}{\nu_a}\right)\frac{d\hat{\nu}}{4\pi}\frac{d\vec{p}_{\rm prot}}{4\pi}\right]$$

and

$$\int\int\left(\hat{\nu}\cdot\frac{2\vec{p}_{\rm prot}}{\nu_a}\right)\frac{d\hat{\nu}}{4\pi}\frac{d\vec{p}_{\rm prot}}{4\pi}$$

which vanishes on the above mentioned no-correlation assumption, corresponds to $A\langle a|X^{(+-)}(\nu_a)|a\rangle]$.

We proceed to combine Eq. (13) with Eq. (10); this yields the EWSR for Λ_a:

$$\begin{aligned}\Lambda_a = {}& KZ_{\rm eff}{}^4\left(\frac{\nu_a}{m_\mu}\right)^2\left(\frac{\nu_a{}^2/2m_{\mathfrak{N}}}{(m_\mu-\epsilon_\mu)-\nu_a}\right)\left\{1+\left(\frac{A}{Z}\right)\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle + \left(\frac{Z-\frac{1}{2}A}{A}\right)\left(\frac{A}{Z}\right)\langle a\|Y_1^{(+-)}(\nu_a)\|a\rangle\right.\\ &\left. + \left[\frac{2(Z-\frac{1}{2}A)^2 - |Z-\frac{1}{2}A|}{A^2}\right]\left(\frac{A}{Z}\right)\langle a\|Y_2^{(+-)}(\nu_a)\|a\rangle\right\}\\ = {}& KZ_{\rm eff}{}^4\left(\frac{\nu_a}{m_\mu}\right)^2\left(\frac{\nu_a{}^2/2m_{\mathfrak{N}}}{(m_\mu-\epsilon_\mu)-\nu_a}\right)\left\{1+\left(\frac{A}{2Z}\right)\langle a\|2Y_0^{(+-)}(\nu_a)+Y_2^{(+-)}(\nu_a)\|a\rangle - \left(\frac{A-2Z}{2Z}\right)\langle a\|Y_1^{(+-)}(\nu_a)\|a\rangle\right.\\ &\left. - \left(\frac{A-Z}{2A}+\frac{|A-2Z|}{8ZA}\right)\langle a\|4Y_2^{(+-)}(\nu_a)\|a\rangle\right\};\\ \nu_a = {}& \{(m_\mu-\epsilon_\mu) - [\langle E_b\rangle - (E_b)_{\rm g.s.}] - [(E_b)_{\rm g.s.} - E_a]\}. \end{aligned} \tag{14}$$

An expression for Λ_a analogous to that in Eq. (14) but with an incorrect expression for the quantity in the curly bracket has been derived in Ref. 10. The error made in that reference arises from the unjustified replacement of the $W^{(+-)}(\nu_a)$ in Eqs. (10)–(13) by $\{W^{(+-)}(\nu_a)\}'$, where

$$\begin{aligned}\{W^{(+-)}(\nu_a)\}' &\equiv \left(\frac{\nu_a{}^2}{2m_{\mathfrak{N}}}\right)^{-1}\int\frac{d\hat{\nu}}{4\pi}\frac{1}{2}\{[[J^{(3)}(\nu_a),H],J^{(3)}(\nu_a)]\};\\ J^{(3)}(\nu_a) &\equiv \sum_{j=1}^{A}\frac{1}{2}(1+\tau_j^{(3)})e^{-i\nu_a\hat{\nu}\cdot\vec{x}_j}\end{aligned} \tag{15}$$

which, upon evaluation of the double commutator, becomes

$$\begin{aligned}\{W^{(+-)}(\nu_a)\}' &= \tfrac{1}{2}A + T^{(3)} + A\{Y^{(+-)}(\nu_a)\}';\\ \{Y^{(+-)}(\nu_a)\}' &\equiv A^{-1}\left(\frac{\nu_a{}^2}{m_{\mathfrak{N}}}\right)^{-1}\sum_{k=1,\,j=1}^{A}(-\tfrac{1}{4})\left[\tfrac{2}{3}\vec{\tau}_k\cdot\vec{\tau}_j - \tfrac{1}{3}(3\tau_k^{(3)}\tau_j^{(3)} - \vec{\tau}_k\cdot\vec{\tau}_j)\right]\left[1-\frac{\sin(\nu_a|\vec{x}_k-\vec{x}_j|)}{(\nu_a|\vec{x}_k-\vec{x}_j|)}\right]v_{kj},\end{aligned} \tag{16}$$

so that

$$\begin{aligned}\langle a|\{W^{(+-)}(\nu_a)\}'|a\rangle &\equiv \langle a|\{W_0^{(+-)}(\nu_a)\}' + \left(\frac{T^{(3)}}{A}\right)\{W_1^{(+-)}(\nu_a)\}' + \left[\frac{3(T^{(3)})^2 - (\vec{T})^2}{A^2}\right]\{W_2^{(+-)}(\nu_a)\}'|a\rangle\\ &= \langle a\|\{W_0^{(+-)}(\nu_a)\}'\|a\rangle + \left(\frac{Z-\frac{1}{2}A}{A}\right)\langle a\|\{W_1^{(+-)}(\nu_a)\}'\|a\rangle + \left[\frac{2(Z-\frac{1}{2}A)^2-|Z-\frac{1}{2}A|}{A^2}\right]\langle a\|\{W_2^{(+-)}(\nu_a)\}'\|a\rangle\\ &= \frac{A}{2} + A\langle a\|\{Y_0^{(+-)}(\nu_a)\}'\|a\rangle + \left(\frac{Z-\frac{1}{2}A}{A}\right)A + \left[\frac{2(Z-\frac{1}{2}A)^2-|Z-\frac{1}{2}A|}{A^2}\right]A\langle a\|\{Y_2^{(+-)}(\nu_a)\}'\|a\rangle.\end{aligned} \tag{17}$$

Comparison of Eqs. (16) and (17) with Eqs. (12) and (13) shows that $\langle a|\{W^{(+-)}(\nu_a)\}'|a\rangle$ is *qualitatively* different from $\langle a|\{W^{(+-)}(\nu_a)\}|a\rangle$ since $\langle a\|\{Y_1^{(+-)}(\nu_a)\}'\|a\rangle$ vanishes while $\langle a\|Y_1^{(+-)}(\nu_a)\|a\rangle$ is definitely expected to be nonzero. In addition, the authors of Ref. 10 in effect replace the curly bracket on the right-hand side of Eq. (14) by 1, i.e., in effect neglect the terms proportional to $\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle$, $\langle a\|Y_1^{(+-)}(\nu_a)\|a\rangle$, and $\langle a\|Y_2^{(+-)}(\nu_a)\|a\rangle$ in the expression for Λ_a. This neglect, together with an attempt to fit the resultant (mutilated) expression for Λ_a to experiment, leads to the conclusion that ν_a must decrease (i.e., $[\langle E_b\rangle-(E_b)_{\text{g.s.}}]$ must increase) as A and Z increase. In fact, quite the contrary conclusion is reached regarding the variation of ν_a with A and Z by considering the EWSR of Eq. (14) for nuclei with $A=2Z$; for such nuclei Eq. (14) can be written as

$$\frac{\Lambda_{a;[Z,A=2Z]}}{Z_{\text{eff}}{}^4}=K\left(\frac{\nu_a}{m_\mu}\right)^2\left(\frac{\nu_a{}^2/2m_{\mathfrak{N}}}{m_\mu-\epsilon_\mu-\nu_a}\right)\times[1+2\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle], \tag{18}$$

with the left-hand side the same to within a few percent for $[Z,A=2Z]=[8,16]$, $[10,20]$, $[12,24]$, $[14,28]$, $[16,32]$, $[20,40]$, if one uses the appropriate experimental values of Λ_a.[11] Thus, assuming in addition that $\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle$ is a constant independent of A and Z, one sees that, at least up to $Z=20$, $A=40$, ν_a shows no tendency to decrease with increasing A and Z—also, the numerical value $\nu_a=\frac{3}{4}m_\mu$ [Eq. (7)] corresponds to $\langle a\|Y_0^{(+-)}(\nu_a)\|a\rangle=0.48$, which is quite reasonable. It is clear than an essentially identical conclusion can be reached on the basis of the NEWSR for Λ_a in Eq. (6).

IV. COMBINATION OF NEWSR AND EWSR

We now proceed to obtain a sum rule for Λ_a which does not contain in any explicit way the average neutrino energy ν_a; it will be recalled that the absence of any reasonably rigorous *a priori* calculation of ν_a as a function of A and Z constitutes a serious difficulty in the use of the NEWSR for Λ_a in Eq. (6) or Eq. (7). We have, from Eqs. (2a) and (2b)

$$\Lambda_a=KZ_{\text{eff}}{}^4Z^{-1}\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\sum_{b\leq}\left[1-\frac{(E_b-E_a)}{(m_\mu-\epsilon_\mu)}\right]^2\int\frac{d\hat{\nu}}{4\pi}\,|\langle b|J^{(-)}((m_\mu-\epsilon_\mu)-(E_b-E_a))|a\rangle|^2,$$

$$J^{(-)}((m_\mu-\epsilon_\mu)-(E_b-E_a))=\sum_{j=1}^{A}\tau_j^{(-)}e^{-i((m_\mu-\epsilon_\mu)-(E_b-E_a))\hat{\nu}\cdot\vec{x}_j}, \tag{19}$$

whence, neglecting terms of higher order than the first in $(E_b-E_a)/(m_\mu-\epsilon_\mu)$,

$$\Lambda_a=KZ_{\text{eff}}{}^4Z^{-1}\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\sum_{b\leq}\int\frac{d\hat{\nu}}{4\pi}\left(\left\{1-\frac{(E_b-E_a)}{(m_\mu-\epsilon_\mu)}\left[2+(m_\mu-\epsilon_\mu)\frac{\partial}{\partial(m_\mu-\epsilon_\mu)}\right]\right\}\left|\langle b|J^{(-)}(m_\mu-\epsilon_\mu)|a\rangle\right|^2\right) \tag{20}$$

so that, using closure, and remembering Eqs. (2a)–(3) and Eqs. (9)–(12),

$$\begin{aligned}\Lambda_a&=KZ_{\text{eff}}{}^4Z^{-1}\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\left\{\langle a|J^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle-\left[\frac{2}{m_\mu-\epsilon_\mu}+\frac{\partial}{\partial(m_\mu-\epsilon_\mu)}\right]\left[\frac{(m_\mu-\epsilon_\mu)^2}{2m_{\mathfrak{N}}}\langle a|W^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle\right]\right\}\\
&=KZ_{\text{eff}}{}^4Z^{-1}\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\left\{\langle a|J^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle-2\left(\frac{m_\mu-\epsilon_\mu}{m_{\mathfrak{N}}}\right)\langle a|W^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle\right.\\
&\qquad\left.-\left[\frac{(m_\mu-\epsilon_\mu)^2}{2m_{\mathfrak{N}}}\right]\left\langle a\left|\frac{\partial}{\partial(m_\mu-\epsilon_\mu)}W^{(+-)}(m_\mu-\epsilon_\mu)\right|a\right\rangle\right\}\\
&=KZ_{\text{eff}}{}^4\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\left\{\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_{\mathfrak{N}}}\right)\right]+\left(\frac{A}{Z}\right)\left[\langle a|K^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle-2\left(\frac{m_\mu-\epsilon_\mu}{m_{\mathfrak{N}}}\right)\langle a|Y^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle\right]\right.\\
&\qquad\left.-\left[\frac{(m_\mu-\epsilon_\mu)^2}{2m_{\mathfrak{N}}}\right]\left(\frac{A}{Z}\right)\left\langle a\left|\frac{\partial}{\partial(m_\mu-\epsilon_\mu)}Y^{(+-)}(m_\mu-\epsilon_\mu)\right|a\right\rangle\right\},\end{aligned} \tag{21}$$

where the last term can be neglected compared to the next to last term

$\Big\{$from Eq. (12), $$\left|\frac{2^{-1}\langle a|(m_\mu-\epsilon_\mu)\{[\partial/\partial(m_\mu-\epsilon_\mu)]Y^{(+-)}(m_\mu-\epsilon_\mu)\}|a\rangle}{2\langle a|Y^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle}\right|\approx\frac{1}{40}(m_\mu-\epsilon_\mu)^2\left(\frac{\langle a\|\vec{x}_k-\vec{x}_j|^4|a\rangle}{\langle a\|\vec{x}_k-\vec{x}_j|^2|a\rangle}\right)\ll1,$$

where k and j are nearest neighbors$\Big\}$.

Thus, introducing Eq. (5a) and Eq. (13) into Eq. (21),

$$\Lambda_a = KZ_{\rm eff}^4 \left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2 \left[1-2\left(\frac{m_\mu-\epsilon_\mu}{2m_\pi}\right)\right]\left\{1+\left(\frac{A}{2Z}\right)\beta_{0;a}-\left(\frac{A-2Z}{2Z}\right)\beta_{1;a}-\left(\frac{A-Z}{2A}+\frac{|A-2Z|}{8ZA}\right)\beta_{2;a}\right\} \;;$$

$$\beta_{0;a} \equiv \left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1}\left(\left\langle a \Big\| \left\{2K_0^{(+-)}(m_\mu-\epsilon_\mu)+K_2^{(+-)}(m_\mu-\epsilon_\mu)\right) - 2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right.\right.$$

$$\left.\left.\times\left[2Y_0^{(+-)}(m_\mu-\epsilon_\mu)+Y_2^{(+-)}(m_\mu-\epsilon_\mu)\right]\Big\| a\right\rangle\right\},$$

$$\beta_{1;a} \equiv -\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1} 2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\left[\left\langle a\| Y_1^{(+-)}(m_\mu-\epsilon_\mu)\| a\right\rangle\right],$$

$$\beta_{2;a} \equiv \left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1}\left[\left\langle a\| 4K_2^{(+-)}(m_\mu-\epsilon_\mu)-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)4Y_2^{(+-)}(m_\mu-\epsilon_\mu)\| a\right\rangle\right], \quad (22)$$

which, with $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ treated as constants independent of A and Z, i.e., taken as the same for all initial nuclear ground states $|a\rangle$, constitutes a three-parameter fit to the experimental data on the total muon-capture rates.[11] With the values of $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$:

$$\beta_{0;a}=-0.030, \quad \beta_{1;a}=-0.25, \quad \beta_{2;a}=3.24, \quad (23)$$

the fit is characterized by the mean absolute deviation (see Table I),

$$\left|\frac{\Lambda_a^{\rm exper}-\Lambda_a^{\rm fit}}{\Lambda_a^{\rm exper}}\right| = 5.6\%, \quad (24)$$

where the right-hand side corresponds to an average over 57 elements: $[8]\leq[Z]\leq[92]$.

Viewed as a sum rule, the expression for Λ_a in Eq. (22) does not have any explicit dependence on ν_a and is for this reason preferable to the NEWSR sum rule for Λ_a in Eq. (6) and the EWSR sum rule for Λ_a in Eq. (14); in addition, the good quality of the fit given by Eqs. (23) and (24) is a compelling indication that $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ are, at least approximately, constants independent of A and Z.[12] Further, the sum rule for Λ_a in Eq. (22) must be viewed as a combined NEWSR and EWSR since not only the quantities $\langle a\|K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ [Eqs. (5a)–(6)], but also the quantities

$$\langle a\| Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)\| a\rangle$$

[Eqs. (12)–(14)] enter into the parameters $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$. As a general comment, we emphasize again that the $\langle a\|K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ and the $\langle a\|Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ depend on nucleon-nucleon correlations in the state $|a\rangle$ as conditioned by the internucleon forces and the exclusion principle [see Eqs. (3)–(5b) and Eqs. (11)–(13)]. Of course, the *a priori* calculation of the $\langle a\|K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ and the $\langle a\|Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$,[13] and so of the $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ [using Eqs. (3)–(5b), (11)–(13), and (22)] is a task of great difficulty but may nevertheless prove feasible in the present state of nuclear dynamics.[14]

TABLE I. Values of $\Lambda_a^{\rm fit}$, $\Lambda_a^{\rm exper}$, and $|(\Lambda_a^{\rm exper}-\Lambda_a^{\rm fit})/\Lambda_a^{\rm exper}|$ for a few representative elements. The number given at the right-hand side of Eq. (24) is the average over the 57 elements ($[8]\leq[Z]\leq[92]$) listed in Ref. 11.

Element	$\Lambda_a^{\rm fit}$ (10^6 sec^{-1})	$\Lambda_a^{\rm exper}$ (10^6 sec^{-1})	$\left\lvert\frac{\Lambda_a^{\rm exper}-\Lambda_a^{\rm fit}}{\Lambda_a^{\rm exper}}\right\rvert$ (%)
$_8$O	0.1151	0.0974 ± 0.0031	18.2
$_{16}$S	1.244	1.338 ± 0.007	7.0
$_{20}$Ca	2.46	2.45 ± 0.02	0.4
$_{24}$Cr	3.06	3.29 ± 0.04	7.1
$_{30}$Zn	5.48	5.74 ± 0.04	4.5
$_{42}$Mo	9.55	9.22 ± 0.06	3.6
$_{48}$Cd	10.66	10.62 ± 0.10	0.4
$_{54}$Ba	10.40	10.18 ± 0.10	2.2
$_{64}$Gd	11.89	12.09 ± 0.16	1.7
$_{73}$Ta	12.74	12.86 ± 0.13	0.9
$_{82}$Pb	13.05	13.02 ± 0.11	0.3
$_{92}$U	11.5	11.0 ± 0.5	4.9

We proceed to discuss the validity of the approximation $|M_V|^2=|M_A|^2=|M_P|^2$ used in Eqs. (2a) and (2b). This approximation becomes exact only in the limit of SU4 invariance of the internucleon forces but may still hold with reasonably good accuracy even if the internucleon forces break SU4 to an appreciable extent.[14a] Without the $|M_V|^2=|M_A|^2=|M_P|^2$ approximation, the combined NEWSR-EWSR sum rule for Λ_a in Eqs. (21)

and (22) is replaced by

$$\Lambda_a = KZ_{\rm eff}^4\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\left\{\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]+\left(\frac{A}{Z}\right)\left[\sum_{i=V,A,P} b_i\langle a|K_i^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\times\sum_{i=V,A,P} b_i\langle a|Y_i^{(+-)}(m_\mu-\epsilon_\mu)|a\rangle\right]\right\}$$

$$=KZ_{\rm eff}^4\left(1-\frac{\epsilon_\mu}{m_\mu}\right)^2\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]\left\{1+\left(\frac{A}{2Z}\right)\beta'_{0;a}-\left(\frac{A-2Z}{2Z}\right)\beta'_{1;a}-\left(\frac{A-Z}{2A}+\frac{|A-2Z|}{8ZA}\right)\beta'_{2;a}\right\};$$

$$\beta'_{0;a}\equiv\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1}\left(\sum_{i=V,A,P} b_i\left\langle a\|\{2[K_0^{(+-)}(m_\mu-\epsilon_\mu)]_i+[K_2^{(+-)}(m_\mu-\epsilon_\mu)]_i\}-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\{2[Y_0^{(+-)}(m_\mu-\epsilon_\mu)]_i+[Y_2^{(+-)}(m_\mu-\epsilon_\mu)]_i\}\|a\right\rangle\right),$$

$$\beta'_{1;a}\equiv-\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1}2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\left(\sum_{i=V,A,P} b_i\langle a\|[Y_1^{(+-)}(m_\mu-\epsilon_\mu)]_i\|a\rangle\right),$$

$$\beta'_{2;a}\equiv\left[1-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)\right]^{-1}\left(\sum_{i=V,A,P} b_i\left\langle a\|4[K_2^{(+-)}(m_\mu-\epsilon_\mu)]_i-2\left(\frac{m_\mu-\epsilon_\mu}{m_\pi}\right)4[Y_2^{(+-)}(m_\mu-\epsilon_\mu)]_i\|a\right\rangle\right);$$

$$b_V\equiv\frac{G_V^2}{G_V^2+3G_A^2+G_P^2-2G_PG_A},\quad b_A\equiv\frac{G_A^2}{G_V^2+3G_A^2+G_P^2-2G_PG_A},\quad b_P\equiv\frac{G_P^2-2G_PG_A}{G_V^2+3G_A^2+G_P^2-2G_PG_A},\tag{25}$$

where $\langle a\|[K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)]_V\|a\rangle$ and $\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)]_V\|a\rangle$ are our previous $\langle a\|K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ and $\langle a\|Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ and where $\langle a\|[K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)]_{A,P}\|a\rangle$ and $\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)]_{A,P}\|a\rangle$ are related to our previous $\langle a\|K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ and $\langle a\|Y_{0,1,2}^{(+-)}(m_\mu-\epsilon_\mu)\|a\rangle$ in the sense of the replacement of

$$\langle b|J^{(\mp)}(m_\mu-\epsilon_\mu)|a\rangle=\langle b|J_V^{(\mp)}(m_\mu-\epsilon_\mu)|a\rangle=\left\langle b|\sum_{j=1}^{A}\tau_j^{(\mp)}e^{\mp i(m_\mu-\epsilon_\mu)\hat{\nu}\cdot\vec{x}_j}|a\right\rangle$$

by

$$\langle b|\vec{J}_A^{(\mp)}(m_\mu-\epsilon_\mu)|a\rangle=\left\langle b|\sum_{j=1}^{A}\tau_j^{(\mp)}\frac{\vec{\sigma}_j}{\sqrt{3}}e^{\mp i(m_\mu-\epsilon_\mu)\hat{\nu}\cdot\vec{x}_j}|a\right\rangle$$

and

$$\langle b|J_P^{(\mp)}(m_\mu-\epsilon_\mu)|a\rangle=\left\langle b|\sum_{j=1}^{A}\tau_j^{(\mp)}\vec{\sigma}_j\cdot\hat{\nu}e^{\mp i(m_\mu-\epsilon_\mu)\hat{\nu}\cdot\vec{x}_j}|a\right\rangle.$$

Thus, if the $\beta'_{0;a}$, $\beta'_{1;a}$, and $\beta'_{2;a}$ are treated as constants independent of A and Z, a three-parameter fit to the experimental data on the total muon-capture rates[11] is provided by Eq. (25) [just as such a fit is provided by Eq. (22)] with the values of $\beta'_{0;a}$, $\beta'_{1;a}$, and $\beta'_{2;a}$ required for the fit being identical with the values of $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ given by Eq. (23). The general comments made above with regard to the $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ can also be made with regard to the $\beta'_{0;a}$, $\beta'_{1;a}$, and $\beta'_{2;a}$.

A few remarks should now be set down regarding the circumstances under which one may expect appreciable differences among the $|M_V|^2$, $|M_A|^2$, and $|M_P|^2$ and hence among the $\langle a\|[K_{0,2}^{(+-)}(m_\mu-\epsilon_\mu)]_i\|a\rangle$. Such differences are related to the magnitude of the quantities [see Eqs. (7a)–(8) in Ref. 3]

$$\alpha'\equiv\langle a|\{b_V[(\vec{T})^2-(T^{(3)})^2]+(b_A+b_P)[(\vec{Y}^{(1)})^2+(\vec{Y}^{(2)})^2]-(b_V+b_A+b_P)[(\vec{T})^2-(T^{(3)})^2]\}|a\rangle/Z$$

$$=\frac{b_A+b_P}{Z}\langle a|[(\vec{Y}^{(1)})^2+(\vec{Y}^{(2)})^2]-[(T^{(1)})^2+(T^{(2)})^2]|a\rangle,$$

$$T^{(1),(2),(3)}=\sum_{j=1}^{A}\tfrac{1}{2}\tau_j^{(1),(2),(3)},\qquad \vec{Y}^{(1),(2),(3)}=\sum_{j=1}^{A}\tfrac{1}{2}\tau_j^{(1),(2),(3)}(\vec{\sigma}_j/\sqrt{3}),\tag{26}$$

and

$$\alpha'' \equiv \langle a|\{b_V - \tfrac{1}{3}(b_A + b_P)\}(-2\vec{S}_{\rm prot}\cdot\vec{S}_{\rm neut}) - (b_V + b_A + b_P)(-2\vec{S}_{\rm prot}\cdot\vec{S}_{\rm neut})|a\rangle/Z$$

$$= \frac{8}{3}\frac{b_A + b_P}{Z}\langle a|\vec{S}_{\rm prot}\cdot\vec{S}_{\rm neut}|a\rangle ,$$

$$\vec{S}_{\rm prot} = \sum_{j=1}^{A}\left(\frac{1+\tau_j^{(3)}}{2}\right)\left(\frac{\vec{\sigma}_j}{2}\right), \quad \vec{S}_{\rm neut} = \sum_{j=1}^{A}\left(\frac{1-\tau_j^{(3)}}{2}\right)\left(\frac{\vec{\sigma}_j}{2}\right), \tag{27}$$

in the sense that large (small) α' and α'' compared to $1/Z$ imply large (small) relative differences among the $\langle a\|[K_{0,2}^{(+-)}(m_\mu - \epsilon_\mu)]_i\|a\rangle$. The quantities α' and α'' vanish in the ground state $|a\rangle$ of an even-even, or an even-odd, or an odd-even nucleus in the approximation in which $|a\rangle$ is a member of a SU4 multiplet, and tend to be significant in light even-even, even-odd, and odd-even nuclei with unfilled outer proton and outer neutron shells,[5] e.g., ${}^{12}_{6}$C where

$$\langle a|(\vec{Y}^{(1)})^2 + (\vec{Y}^{(2)})^2|a\rangle \neq 0 , \qquad \langle a|(T^{(1)})^2 + (T^{(2)})^2|a\rangle = 0 ,$$

and in light odd-odd nuclei with $Z = A - Z$, e.g., ${}^{14}_{7}$N where[14a]

$$\langle a|(\vec{Y}^{(1)})^2 + (\vec{Y}^{(1)})^2|a\rangle = 2, \quad \langle a|(T^{(1)})^2 + (T^{(2)})^2|a\rangle = 0 , \quad \langle a|\vec{S}_{\rm prot}\cdot\vec{S}_{\rm neut}|a\rangle = \tfrac{1}{4} .$$

In these light nuclei, monopole transitions which violate $|M_V|^2 = |M_A|^2 = |M_P|^2$ occur with appreciable probability; thus, in $\mu^- + ({}^{12}_{6}\mathrm{C})_{\mathrm{g.s.}:a} \to \nu_\mu + ({}^{12}_{5}\mathrm{B})_{\mathrm{g.s.}:b}$, we have

$$\langle b|\{J_V^{(-)}(m_\mu - \epsilon_\mu)\}_{l=0}|a\rangle = \left\langle b\left| \sum_{j=1}^{A}\tau_j^{(-)}\frac{\sin[(m_\mu - \epsilon_\mu)|\vec{x}_j|]}{[(m_\mu - \epsilon_\mu)|\vec{x}_j|]}\right|a\right\rangle = 0,$$

$$\langle b|\{\vec{J}_A^{(-)}(m_\mu - \epsilon_\mu)\}_{l=0}|a\rangle = \left\langle b\left| \sum_{j=1}^{A}\tau_j^{(-)}\frac{\vec{\sigma}_j}{\sqrt{3}}\left(\frac{\sin[(m_\mu - \epsilon_\mu)|\vec{x}_j|]}{[(m_\mu - \epsilon_\mu)|\vec{x}_j|]}\right)\right|a\right\rangle$$

$$\cong \left\langle b\left| \sum_{j=1}^{A}\tau_j^{(-)}\frac{\vec{\sigma}_j}{\sqrt{3}}\right|a\right\rangle \approx 1,$$

$$\langle b|\{J_P^{(-)}(m_\mu - \epsilon_\mu)\}_{l=0}|a\rangle = \sqrt{3}\,\hat{\nu}\cdot\langle b|\{\vec{J}_A^{(-)}(m_\mu - \epsilon_\mu)\}_{l=0}|a\rangle .$$

We also note that, in general, the differences among the $\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu - \epsilon_\mu)]_i\|a\rangle$ will be greater than the differences among the

$$\langle a\|[K_{0,2}^{(+-)}(m_\mu - \epsilon_\mu)]_i\|a\rangle$$

since, according to Eqs. (11)–(13), the spin dependent part of u_{lm} will contribute to

$$\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu - \epsilon_\mu)]_A\|a\rangle$$

and

$$\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu - \epsilon_\mu)]_P\|a\rangle$$

but not to

$$\langle a\|[Y_{0,1,2}^{(+-)}(m_\mu - \epsilon_\mu)]_V\|a\rangle .$$

Our combined NEWSR-EWSR expression for Λ_a in Eqs. (21) and (22) or in Eq. (25) bears some resemblance to the expression for Λ_a investigated by Bernabeu,[15] but differs from the latter in that $[\Lambda_a]_{\rm ours}$ results from an expansion in $(E_b - E_a)/(m_\mu - \epsilon_\mu)$ while $[\Lambda_a]_{\rm Bernabeu}$ results from an expansion in

$$\frac{(E_b - E_a) - (\langle E_b\rangle - E_a)}{(m_\mu - \epsilon_\mu)} = \frac{(E_b - E_a) - (m_\mu - \epsilon_\mu - \nu_a)}{(m_\mu - \epsilon_\mu)}$$

$$\cong \frac{(E_b - E_a) - \frac{1}{4}(m_\mu - \epsilon_\mu)}{m_\mu - \epsilon_\mu} ;$$

thus, as emphasized above, the Λ_a of Eqs. (21) and (22) or of Eq. (25) does not contain in any explicit way the average neutrino energy ν_a. We should also mention that if one expands the quantity

$$\frac{\sin[(m_\mu - \epsilon_\mu)|\vec{x}_k - \vec{x}_j|]}{(m_\mu - \epsilon_\mu)|\vec{x}_k - \vec{x}_j|}$$

entering into $K^{(+-)}(m_\mu - \epsilon_\mu)$ and $Y^{(+-)}(m_\mu - \epsilon_\mu)$ in a power series in $[(m_\mu - \epsilon_\mu)|\vec{x}_k - \vec{x}_j|]$ [which expansion corresponds after a suitable rearrangement of terms to a multipole expansion of the neutrino plane wave in $J^{(-)}(m_\mu - \epsilon_\mu)$] one again finds[5] (albeit somewhat model dependently) that, while the higher powers of $[(m_\mu - \epsilon_\mu)|\vec{x}_k - \vec{x}_j|]$ contribute relatively more to $\langle a|K^{(+-)}(m_\mu - \epsilon_\mu)|a\rangle$ and $\langle a|Y^{(+-)}(m_\mu - \epsilon_\mu)|a\rangle$ for large Z and A than for small Z and A, the mean nuclear excitation energy shows no general tendency to increase with increasing Z and A.[16]

V. CONCLUSION

Aside from the arguments which (a) exhibit a connection between the $(A-Z)/2A$ term in Λ_a and the isotensor part of the initial-state expectation value of the absolute square of the muon-capture current, and (b) which indicate that the mean nuclear excitation energy in muon capture does not vary appreciably as A and Z vary, the main result of the present paper is contained in the combined NEWSR-EWSR expression for Λ_a given in Eq. (25) [or Eq. (22)]; this expression provides a three-parameter fit to the experimental data on the total muon-capture rates [see Eqs. (23) and (24)]. The three parameters which enter the fit depend on the isoscalar, isovector, and isotensor parts of the expectation values over the initial nuclear ground states of certain nucleon-nucleon correlation functions. The *a priori* calculation of these parameters is, obviously, a task of great difficulty, but, if carried out successfully, would offer considerable insight into the interplay of the internucleon forces and the Pauli exclusion principle in determining the nucleon-nucleon correlations.

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[1]See, e.g., S. Fallieros, in *Proceedings of the International Conference on Photonuclear Reactions and Applications, Asilomar, 1973,* edited by B. L. Berman (Lawrence Livermore Laboratory, Univ. of California, Livermore, 1973).

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$$(\Delta E/E)_a \cong \{[3T_a(2T_a-1)K_{2;1^-}]/[T_a J_{2;1^-}]\}; \quad T_a >> 1,$$

where $K_{2;1^-}$ and $J_{2;1^-}$ are related to the $K_2^{(+-)}(\nu_a)$ and $J_2^{(+-)}(\nu_a)$ above by the replacement of $\tau_j^{(\mp)} e^{\mp i|\vec{\nu}_{ba}|\hat{\nu}\cdot\vec{x}_j}$ by $[(1+\tau_j^{(3)})/2]\,(\vec{x}_j - A^{-1}\sum_{m=1}^{A}\vec{x}_m)$.

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[12]The three parameters $\beta_{0;a}$, $\beta_{1;a}$, and $\beta_{2;a}$ are expected to exhibit larger fluctuations with varying Z and A for light nuclei than for heavy nuclei. Indeed, the mean deviation given in Eq. (24) is reduced to 5.0% if the nine elements with $[8] \le [Z] < [20]$ are omitted from the set of 57 elements over which $|(\Lambda_a^{\text{exper}} - \Lambda_a^{\text{fit}})/\Lambda_a^{\text{exper}}|$ is averaged.

[13]We note, since the v_{kj} appear explicitly in Eq. (12) but not in Eq. (3), that the ultimate dependence of the $\langle a\|Y_{0,1,2}^{(+-)}(m_\mu - \epsilon_\mu)\|a\rangle$ on the internucleon forces is both explicit and implicit while that of the $\langle a\|K_{0,2}^{(+-)}(m_\mu - \epsilon_\mu)\|a\rangle$ is only implicit.

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[16]This last result can be understood if one recalls that $(\langle E_b\rangle - E_a) = (\langle E_{b-\text{anal}}\rangle - E_a - E_C)$ where $E_{b-\text{anal}}$ is the state in $[Z, A]$ which is the isoanalog of the state $|b\rangle$ in $[Z-1, A]$ and E_C is the corresponding Coulomb-energy difference. In heavier nuclei, the larger relative contribution of higher multipole excitations is associated with transitions to higher isoanalog states and so with a tendency to greater $(\langle E_{b-\text{anal}}\rangle - E_a)$ and hence greater $(\langle E_b\rangle - E_a)$. This tendency, however, is compensated by an opposing tendency arising from the shrinkage of the nuclear level spacing $(\sim A^{-1/3})$ and the increase of E_C $(\sim Z)$ so that $(\langle E_b\rangle - E_a)$, and therefore $[\langle E_b\rangle - (E_b)_{\text{g.s.}}]$, actually remain more or less constant as A and Z increase.