Operator Reduction of Three-Body Scattering Equations*

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Operator multipliers of the Lippmann-Schwinger (LS) equation are used to obtain an uncoupled equation with calculable and unique solutions. The multiplier method is used as a basis for determining the relation between various formulations of the three-body problem. The formal but calculable solutions of the formulations of Faddeev, Lovelace, Rosenberg, Noble, and Newton are found to be identical in the sense that the different formulations are completely equivalent because they all use the same multiplier. They are also identical to the solution of the equation obtained by multiplying the L-S equation by an operator whose inverse exists. We also present an equation for the exact three-body bound-state wave function and put it into calculable form with the use of a multiplier. In a calculation that uses an incomplete set of basis states (such as in the shell model), we find on rigorous grounds that it is appropriate to use the t operator of the residual interaction rather than the residual interaction itself. To indicate the wide usefulness of the multiplier method, the exact distorted-wave formulation is obtained and put into calculable form with the use of a multiplier.

I. INTRODUCTION

THE Lippmann-Schwinger (LS) equations are found to be very useful in nonrelativistic two-body scattering theory and serve as a natural starting point in solving the three-body problem. However, it has been an annoying fact that the LS equations for the channel wave functions do not have a unique solution,[1] and, further, that the Born series solution for the LS operator equation does not converge.[2]

The work of Faddeev overcomes these difficulties in many-body scattering theory.[3–5] Weinberg[6] extended this work with a new formulation of the three-body problem as well as an excellent exposition of many-body scattering theory from the point of view of functional analysis, in which he points out that because of the noncompactness of the kernel, the three-body L-S equation is not amenable to a solution by any convergent calculational scheme. (Solutions of the LS equation cannot be calculated and in the three-body case the LS equations have then lost their utility. The Faddeev equations do not suffer from this property.) However, the formulation, which uses operator multiplier techniques, admits extra (spurious) bound-state solutions that are not solutions of the original Schrödinger equation (or the homogeneous LS equation).[7] It was subsequently found that the spurious states arise from the multiplier.[8]

A variety of specific three-body formulations other than those of Faddeev and Weinberg have also appeared in the literature: those of Lovelace,[9] Rosenberg,[10] Noble,[11] and Newton.[12] A method for treating the three-body problem using rather arbitary operators as multipliers of the LS equation was proposed by Sugar and Blankenbecler.[13] This method is applied in the formulation of Noble.

In this paper we also obtain a formulation of the three-body scattering problem by using multipliers. Our method yields one uncoupled equation with a compact kernel and the solution of our equation is identical to the formal solutions of the formulations of Faddeev, Lovelace, Rosenberg, Noble, and Newton. Our method shows the connections between the earlier formulations of three-body problems and establishes that they can be obtained from the one common multiplier procedure. Since our multiplier has no spurious solutions, we find that none of the formulations mentioned above, except for the Weinberg method, introduce spurious bound-state solutions.

The wide usefulness of the multiplier method is indicated when we consider the distorted-wave (DW) and shell-model formulations. We obtain the DW formulation with the use of a multiplier. The DW kernel is made compact and the exact solution calculable with the use of a multiplier. The new inhomogeneous term indicates that in most reactions the operator whose

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amplitude is to be evaluated between the distorted waves in distorted-wave Born-approximation (DWBA) calculation should be the t operator rather than the potential. We also obtain an equation for the exact three-body bound-state wave function. By exact we mean that the errors introduced by using an incomplete set of basis states, such as in shell-model calculations, are systematically corrected. Again, by using a multiplier, we find that the first term in a convergent solution is given by a shell-model (truncated-basis) calculation that uses the t operator of the residual interaction rather than the residual interaction itself.

In Sec. II we discuss the LS equations and point out the common origin of their deficiencies; that is, the solutions cannot be calculated by a convergent scheme because the kernel is not compact, and, further, the wave-function solution in a given channel is not unique because other channel-scattering wave functions satisfy the homogeneous LS equation.

In Sec. III we first discuss the conditions on the Sugar and Blankenbecler method and emphasize that the number of bound states should remain unchanged. Then by using a specific multiplier which does not introduce spurious states, we remove the noncompact part of the LS kernel. This multiplier is expressed in terms of a simpler multiplier that involves only one pair interaction. This method provides one uncoupled equation which yields calculable solutions for the total Green's function, the wave function, and the transition operators. The inhomogeneous term of the new equation provides an impulse-type approximation for the transition operators.

In Sec. IV we obtain formal, calculable solutions to the Faddeev, Lovelace, Rosenberg, and Noble formulations by uncoupling the equations with the use of operator algebra. The Newton formulation is rewritten for comparison with the others. All these solutions are seen to be identical to each other and identical to the solution of the multiplied equation obtained in Sec. III. We also find that the Lovelace equations can be obtained by multiplying the system of LS equations for the transition operators by a matrix of operators. The conditions on this matrix, in order to introduce no spurious states, are the same as the ones imposed on the operators that we used to multiply the individual LS equations. In Sec. V we treat the DW and shell-model formulations.

II. LIPPMANN-SCHWINGER EQUATIONS

It is known that the LS equation for the three-body problem is unsatisfactory in that neither does it possess a unique solution nor is it amenable to a solution by any convergent calculational scheme. In this section we introduce the notation, review how the unsatisfactory properties arise, and point out the common origin of these inadequacies.

Consider three spinless distinguishable particles with no internal structure. The Hamiltonian is given by

$$H=H_0+\sum_{\alpha=1}^{3} V_\alpha,$$

where the three-body kinetic energy operator is H_0 and V_α is the pair interaction between the two particles not labeled by α.

We define the Green's functions as usual,

$$G(z)=(z-H)^{-1}, \tag{1}$$

$$G_0(z)=(z-H_0)^{-1}, \tag{2}$$

$$G_\alpha(z)=(z-H_0-V_\alpha)^{-1}, \tag{3}$$

where z is the energy of interest and is a complex number in the intermediate steps in scattering problems. The Green's functions satisfy the identities

$$G(z)=G_\alpha(z)+G_\alpha(z)v_\alpha G(z), \qquad \alpha=0,\,1,\,2,\,3 \tag{4}$$

$$G_\alpha(z)=G_0(z)+G_0(z)V_\alpha G_\alpha(z), \qquad \alpha=1,\,2,\,3 \tag{5}$$

where $V=\sum^3 V_\alpha$, $v_\alpha=V-V_\alpha$, and $V_0\equiv 0$.

Let $\psi_\alpha^+(E)$ represent a complete wave function with outgoing-wave boundary conditions. The unperturbed wave ϕ_α corresponds to the channel α, in which the particle labeled α does not interact, and particles β and γ are bound via the potential V_α. That is to say, ϕ_α is a continuum eigenfunction of H_0+V_α.

The LS equation for $\psi_\alpha^+(E)$ is

$$\psi_\alpha^+(E)=\phi_\alpha+\lim_{\epsilon\to 0} G_\alpha(E+i\epsilon)v_\alpha\psi_\alpha^+(E). \tag{6}$$

LS equations can also be written for the transition operators between the various channels

$$T_{\beta\alpha}^-=v_\alpha+\lim_{\epsilon\to 0} v_\beta G_\beta(E+i\epsilon)\,T_{\beta\alpha}^-. \tag{7}$$

This is the operator for the rearrangement $\alpha+(\beta+\gamma)\to\beta+(\alpha+\gamma)$. The other transition operators are written with the appropriate rearrangement of subscripts. In future notation the limit $\epsilon\to 0$ will be implicit.

It is well known that the LS kernels $G_\alpha v_\alpha$ or $v_\alpha G_\alpha$ of the integral equations (4), (6), or (7) are not compact.[6] This is because of the fact that the kernels contain terms in which one or another of the particles do not interact (these are called disconnected diagrams). Momentum conservation, which is required of the noninteracting particle, is represented by a δ function which cannot be factored out of the integral equation. In a coordinate representation there is no cutoff in the integration over the coordinate of the noninteracting (disconnected) particle.

Because the kernel is not compact, we face the first difficulty with the LS equations, namely, that there is no convergent calculational scheme for obtaining the solution. Specifically, the problem is that neither a Born

series nor a finite matrix representation of the resolvent of the kernel will converge.[6,14]

The second deficiency of the LS equations is that Eq. (6) does not have a unique solution whenever there is more than one channel. The other channel wave functions ψ_β^+ and ψ_γ^+ satisfy the homogeneous form of Eq. (6).[1] Thus, to any solution ψ_α^+ we may add arbitrary amounts of the other channel wave functions ψ_β^+ or ψ_γ^+. But this means that the desirable feature of integral equations—that of combining the differential equation and boundary conditions—is not present in Eq. (6). The inhomogeneous term ϕ_α, which is supposed to establish the boundary conditions, does not do so in all directions of configuration space.

Now a noncompact operator has a continuous spectrum in addition to a discrete one.[6] It is just these eigenvectors corresponding to the continuous eigenvalues of the kernel that introduce the nonuniqueness arising from other channel continuum states. The two deficiencies of the LS equations are then due to the one fact that the kernel is not compact.

III. USE OF MULTIPLIERS TO PRODUCE COMPACT KERNELS

We wish to modify the LS equation in such a way as to produce a compact kernel. We first examine a general method proposed by Sugar and Blankenbecler.[13]

A. Discussion of Multipliers

The LS equation can be written in the symbolic form

$$T=V+KT, \tag{8}$$

where T stands for the unknown, whether it is the transition operator, wave function, or Green's function. The inhomogeneous term V is that appropriate to the unknown T, and the noncompact kernel is K. The solution suggested by Sugar and Blankenbecler is

$$T=L[M(1-K)L]^{-1}MV. \tag{9}$$

The conditions they impose on the otherwise arbitrary operators L and M are that M^{-1} and $\det M(1-K)L$ exist. The existence of the determinant is necessary for the calculation of the solution by the Fredholm or other matrix inversion technique. It can be shown that the existence of the determinant is a sufficient condition for the new kernel to be compact. The existence of M^{-1} is necessary so that the proposed solution is a solution of Eq. (8). This is directly verified by putting the solutions given by Eq. (9) into Eq. (8).

However, if M is arbitrary the solution for the wave function as given in Eq. (9) with $L=1$ contains spurious solutions given by $M(z)\xi(z)=0$, where $\xi(z)$ is assumed nonzero. It may seem that these extraneous solutions are possible both in the scattering wave function, in which case $\xi(z)=[1-K(z)]\psi(z)-\phi(z)$, and in the bound-state wave function, in which case $\xi(z)=[1-K(z)]\psi(z)$. A necessary and sufficient condition to ensure that $M(z)$ annihilates no vectors is that $M^{-1}(z)$ exists for all z as prescribed by Sugar and Blankenbecler. However, we note that in order to produce a compact kernel, M^{-1} will annihilate (and M will be singular) on the continuous spectrum of K. By the same token, any continuum zeros of M must be annihilated by $1-K$. Therefore, if the new kernel is assumed compact, it is sufficient to require that $M^{-1}(z)$ exists at discrete values of z.

We now set $M=1$ and consider the solution with only the multiplier L present:

$$T=L[(1-K)L]^{-1}V. \tag{10}$$

By substituting this solution into Eq. (8), it is apparent that a solution of the form (10) is a solution of (8), and we are satisfied that (10) yields the desired solution of the inhomogeneous Eq. (8). We now ask whether the bound states, which satisfy the homogeneous form of Eq. (8) can be obtained from a solution of the form (10). If the criterion used for bound states is that they occur at energies corresponding to the poles of Eq. (10), there are still spurious solutions. Values of the energy z, for which a ϕ can be found to satisfy

$$[1-K(z)]L(z)\phi(z)=0, \tag{11}$$

yield poles in Eq. (10). Apart from the actual bound states

$$[1-K(E)]\psi(E)=0, \tag{12}$$

there are possible spurious states given by

$$L(z)\phi(z)=0. \tag{13}$$

We see that neither of the operators L and M is more or less arbitrary than the other. In order to produce no spurious poles, the inverses of both must exist at discrete values of energy.

B. Application of Multipliers

We now apply the method of multiplying the LS equation by an operator that makes the kernel compact. First, we define the two-body t operator:

$$t_\alpha=V_\alpha+V_\alpha G_0 t_\alpha, \tag{14}$$

$$t_\alpha=V_\alpha+V_\alpha G_\alpha V_\alpha. \tag{15}$$

These are two-body t operators in the three-body space, i.e., the kinetic-energy operator for all three bodies is included in the propagators. The t's can be calculated from Eq. (14) since the δ function in the momentum of the noninteracting particle can be factored from the whole equation. From Eqs. (14) and (15) we obtain the useful relations

$$(1-G_0V_\alpha)(1+G_0t_\alpha)=1, \tag{16a}$$

$$V_\alpha G_\alpha=t_\alpha G_0. \tag{16b}$$

[14] J. V. Corbett, J. Math. Phys. 9, 891 (1968).

In the LS equation (4), we use Eq. (5) to expand G_α to display the compact and noncompact parts of the kernel:

$$(1-G_0v_\alpha-G_0V_\alpha G_\alpha v_\alpha)G=G_\alpha. \tag{17}$$

The noncompact part is $G_0v_\alpha=G_0V_\beta+G_0V_\gamma$ since it is the sum of two terms each made up of the one interaction which leaves one particle unconnected. The other term in (17), $G_0V_\alpha G_\alpha v_\alpha=G_0t_\alpha G_0v_\alpha$, is already compact since it is made up of products of two different interactions and all particles are connected.[15] We see that if we use a multiplier $L=(1-G_0v_\alpha)^{-1}$ on Eq. (17), the noncompact term will cancel[11] and a solution of the form of Eq. (10) will result:

$$G=(1-G_0v_\alpha)^{-1}[1-G_0t_\alpha G_0v_\alpha(1-G_0v_\alpha)^{-1}]G_\alpha. \tag{18}$$

The inverse of this multiplier is

$$[1-G_0(z)v_\alpha]=G_0(z)(z-H_0-v_\alpha).$$

The factor $(z-H_0-v_\alpha)$ is everywhere bounded if $v_\alpha=V_\beta+V_\gamma$ is finite everywhere. The factor $G_0(z)$ is unbounded only on the continuous spectrum of H_0. Since the inverse of the multiplier is not unbounded at any discrete value of z, the multiplier $L=(1-G_0v_\alpha)^{-1}$ itself can annihilate no vector at a discrete energy and thus introduces no spurious bound states.

However, (18) still is not in calculable form because G_0v_α is not compact and therefore the factor $(1-G_0v_\alpha)^{-1}$ cannot be calculated. To overcome this we repeat the multiplier procedure. The multiplier satisfies operator identities of the type of Eqs. (4) and (5),

$$(1-G_0v_\alpha)^{-1}=(1-G_0V_\beta)^{-1}+(1-G_0V_\beta)^{-1}G_0V_\gamma(1-G_0v_\alpha)^{-1}, \tag{19}$$

which we rewrite using Eq. (16a):

$$[1-(1+G_0t_\beta)G_0V_\gamma](1-G_0v_\alpha)^{-1}=1+G_0t_\beta. \tag{20}$$

Equation (20) is an integral equation for the multiplier $(1-G_0v_\alpha)^{-1}$. The kernel of the equation is $(1+G_0t_\beta)G_0V_\gamma$ and it has the noncompact part G_0V_γ. To put the solution of (20) into calculable form (i.e., involving resolvents of compact operators) we use an L multiplier $L=1+G_0t_\gamma$. Using (14), the solution of Eq. (20), which is of the general form of Eq. (10), is

$$(1-G_0v_\alpha)^{-1}=(1+G_0t_\gamma)(1-G_0t_\beta G_0t_\gamma)^{-1}(1+G_0t_\beta). \tag{21}$$

We see that the multiplier $1+G_0t_\gamma$ has no zeros by noting that its inverse, which is obtained from Eq. (16a), has no discrete poles. Then this multiplier introduces no spurious solutions.

Of course, we could have expanded in terms of $(1-G_0V_\gamma)^{-1}$ instead and used the multiplier $1+G_0t_\beta$ to obtain another solution equivalent to Eq. (21), which has the labels β and γ interchanged. In either case, the solution involves the resolvent of a compact operator, $G_0t_\beta G_0t_\gamma$ or $G_0t_\gamma G_0t_\beta$, and we can calculate $(1-G_0v_\alpha)^{-1}$, which is the multiplier of the original LS equation.

We examine the properties of the multiplier further. This will lead to a slight simplification of Eq. (18). We define $(1-G_0v_\alpha)^{-1}$ in terms of an unknown $\tau_{\beta\gamma}$:

$$(1-G_0v_\alpha)(1+G_0\tau_{\beta\gamma})=1.$$

This leads to an integral equation for $\tau_{\beta\gamma}$:

$$\tau_{\beta\gamma}=v_\alpha+v_\alpha G_0\tau_{\beta\gamma}. \tag{22}$$

We see that $\tau_{\beta\gamma}$ is the t operator for the two potentials V_β and V_γ. It is the t operator for all the interactions in which particle α participates. We can evaluate $\tau_{\beta\gamma}$ from Eq. (21) by using the operator identity

$$(1-AB)^{-1}A=A(1-BA)^{-1}. \tag{23}$$

We obtain

$$\tau_{\beta\gamma}=(1-t_\gamma G_0t_\beta G_0)^{-1}t_\gamma(1+G_0t_\beta)+(1-t_\beta G_0t_\gamma G_0)^{-1}t_\beta(1+G_0t_\gamma), \tag{24a}$$

or, alternatively,

$$\tau_{\beta\gamma}=t_\gamma(1-G_0t_\beta G_0t_\gamma)^{-1}(1+G_0t_\beta)+t_\beta(1-G_0t_\gamma G_0t_\beta)^{-1}(1+G_0t_\gamma). \tag{24b}$$

Now using (22) we rewrite (18) to obtain our final calculable expression for G:

$$G=(1+G_0\tau_{\beta\gamma})(1-G_0t_\alpha G_0\tau_{\beta\gamma})^{-1}G_\alpha. \tag{25}$$

We can apply the same multiplier $L=1+G_0\tau_{\beta\gamma}$ to the LS equation (6) for the wave function since the kernel is the same as in Eq. (4) or (17). We obtain the same form of solution:

$$\psi_\alpha^+(E)=(1+G_0\tau_{\beta\gamma})(1-G_0t_\alpha G_0\tau_{\beta\gamma})^{-1}\phi_\alpha. \tag{26}$$

A very similar multiplier, $1+\tau_{\beta\gamma}G_0$, can be used on the LS equation (7) for the transition operators. To eliminate the explicit appearance of potentials in the kernel, it should be used as an M-type multiplier since the kernel of Eq. (7) is $v_\alpha G_\alpha$ instead of $G_\alpha v_\alpha$. This will yield

$$T_{\alpha\beta}^-=(1-\tau_{\beta\gamma}G_0t_\alpha G_0)^{-1}(1+\tau_{\beta\gamma}G_0)v_\beta. \tag{27}$$

To summarize, we have used only multipliers that annihilate no vectors and thus we have not introduced spurious bound states. We have used the multiplier $(1-G_0v_\alpha)^{-1}$ to put the LS equation into a form with a compact kernel and the simpler multiplier $(1-G_0V_\beta)^{-1}$ to put the expression $(1-G_0v_\alpha)^{-1}$ into calculable form. The final expressions, Eqs. (25), (26), and (27), involve compact operators composed of two-body operators, e.g., t_β, and one can solve for t_β off the energy shell in a variety of ways for reasonable potentials.[16]

[15] The compactness is proved mathematically in Ref. 6, p. B255, and in C. Lovelace, *Strong Interactions in High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).

[16] Reference 6, p. B234, and M. Scadron, S. Weinberg, and J. Wright, Phys. Rev. **135**, B202 (1964).

C. Comments

It might be added that any L or M type of multiplier of the form $(1+G_0\tau_{\beta\gamma})(1+C)$, where C is compact, would also do to render the equations into a form with a compact kernel. To introduce no spurious solutions, the existence of $(1+C)^{-1}$ is required. An example[13] of a multiplier of this form is $(1-G_0V_\beta)^{-1}(1-G_0V_\gamma)^{-1}$.

At this stage it might not be untoward to present a physical interpretation of Eq. (27), since it has an especially simple one for elastic scattering:

$$T_{\alpha\alpha}{}^{-}=\tau_{\beta\gamma}+\tau_{\beta\gamma}G_0t_\alpha G_0T_{\alpha\alpha}{}^{-}. \tag{28}$$

This is the elastic scattering operator for the system $\alpha+(\beta+\gamma)\rightarrow\alpha+(\beta+\gamma)$. Equation (28) has a compact kernel and the Born series can be made convergent. We recall that the inhomogeneous term $\tau_{\beta\gamma}$ is the t operator for the sum of the two potentials $V_\beta+V_\gamma$. The first term then inolves the scattering to *all* orders of the particle α from both the particles β and γ, including multiple-scattering terms. In higher terms the rescattering within the bound system $\beta+\gamma$ appears to all orders of the interaction between particles β and γ. We note that if we neglect the compact operator $G_0t_\beta G_0t_\gamma$ in the "Born" term of the multiplied solution for $T_{\alpha\alpha}{}^{-}$, we arrive at the single-scattering impulse approximation[17]

$$T_{\alpha\alpha}{}^{-}\simeq t_\beta+t_\gamma. \tag{29}$$

In rearrangement scattering the "Born" term of, for example, $T_{\beta\alpha}{}^{-}$ is

$$(1+\tau_{\alpha\gamma}G_0)v_\alpha.$$

In the case of both rearrangement and elastic scattering the process of removing the disconnected diagrams in the LS kernel with the use of a multiplier results in an inhomogeneous term that is just like the original transition operator, except that it is defined with a free Green's function. The new inhomogeneous term is then an impulse-type approximation in which the binding potential in the exit channel is ignored. The intermediate states in this new Born term are then eigenfunctions of the exit-channel Hamiltonian.

The post form of the operator for the transition channel α goes to channel β is given by the LS equation

$$T_{\beta\alpha}{}^{+}=v_\beta+T_{\beta\alpha}{}^{+}G_\alpha{}^{-}v_\alpha.$$

The multiplier that makes the kernel compact is

$$(1+G_0\tau_{\beta\gamma}).$$

In this case the new inhomogeneous term is an impulse-type approximation in which the potential that produces binding in the entrance channel is ignored. The intermediate states in this new Born term are eigenfunctions of the entrance-channel Hamiltonian only.

IV. EXISTING THREE-BODY FORMULATIONS

In this section we examine the three-body formulations of Faddeev,[3] Lovelace,[9] Rosenberg,[10] Noble,[11] and Newton.[12] We show that the solutions of these formulations are identical to each other and identical to the solutions obtained in Sec. III with the use of the multiplier.

A. Faddeev and Lovelace Formulations

We first reproduce the Faddeev equations. The LS equation for the transition operator for all three particles free is

$$T_{00}=V+VG_0T_{00}. \tag{30}$$

Faddeev writes

$$T_{00}=T_1+T_2+T_3, \tag{31}$$

and obtains the following system of equations—the Faddeev equations:

$$T_1=t_1+t_1G_0(T_2+T_3), \tag{32}$$

$$T_2=t_2+t_2G_0(T_1+T_3), \tag{33}$$

$$T_3=t_3+t_3G_0(T_2+T_1). \tag{34}$$

The operators t_i are given by Eq. (14) and i is any of the Greek subscripts used before.

To obtain a formal solution for T_{00} we can uncouple Eqs. (32)–(34) by simple operator algebra. Equations (33) and (34) are used to obtain T_2 and T_3 in terms of T_1. Since $t_2G_0t_3G_0$ is a compact operator, we can formally solve for T_2 and T_3 in terms of T_1. When T_2 and T_3 are added we recognize that the two-potential operator τ_{23} of Eq. (24) appears as follows:

$$T_2+T_3=\tau_{23}(1+G_0T_1). \tag{35}$$

By using Eq. (35) in Eq. (32) we can solve for T_1 since $t_1G_0\tau_{23}G_0$ is compact:

$$T_1=(1-t_1G_0\tau_{23}G_0)^{-1}t_1(1+G_0\tau_{23}). \tag{36}$$

Then from (31), using (35) and (36), it is an easy matter to obtain

$$T_{00}=\tau_{23}+(1+\tau_{23}G_0)(1-t_1G_0\tau_{23}G_0)^{-1}t_1(1+G_0\tau_{23}). \tag{37}$$

With this operator one can simply obtain the total Green's function from the standard relation

$$G=G_0+G_0T_{00}G_0$$

to produce

$$G=(1+G_0\tau_{23})(1-G_0t_1G_0\tau_{23})G_1. \tag{38}$$

Solutions with the other permutations of subscripts are obtained if we solve for T_{00} in terms of T_2 or T_3. We see that this solution is identical to the solution, Eq. (25), obtained with the use of the multiplier.

[17] The single-scattering impulse approximation was used successfully by A. K. Kerman, H. McManus, and R. M. Thaler [Ann. Phys. (N.Y.) **8**, 551 (1959)] to calculate elastic nucleon-nucleus scattering at 100 MeV and higher.

The Lovelace equations involve coupling between the different channel transition operators and can be written in matrix form:

$$\begin{pmatrix} T_{\alpha\alpha}^{-} \\ T_{\beta\alpha}^{-} \\ T_{\gamma\alpha}^{-} \end{pmatrix} = \begin{pmatrix} 0 & t_\beta & t_\gamma \\ t_\alpha & 0 & t_\gamma \\ t_\alpha & t_\beta & 0 \end{pmatrix} G_0 \begin{pmatrix} T_{\alpha\alpha}^{-} \\ T_{\beta\alpha}^{-} \\ T_{\gamma\alpha}^{-} \end{pmatrix}.$$

If we uncouple and solve this system as was done for the Faddeev equations and again use only calculable resolvents, we obtain

$$T_{\gamma\alpha}^{-} = (1-\tau_{\alpha\beta}G_0 t_\gamma G_0)^{-1}(1+\tau_{\alpha\beta}G_0)v_\alpha. \tag{39}$$

Again, this solution is identical to the solution, Eq. (27), for transition operators that was obtained with the use of the multiplier.

Thus the formal solutions of the Faddeev and Lovelace equations involve only calculable inverses. They are identical to the solutions obtained by multiplying the LS equation. It is equivalent to say that the uncoupled Lovelace equations are identical to the equations obtained by multiplying the LS equations for the transition operators. They have the same inhomogeneous part and the same compact kernel.

Furthermore, we can obtain the system of Lovelace equations by multiplying the system of LS equations by a matrix of operators. This emphasizes the fact that multiplying the LS equation by an appropriate operator effects a cure to the noncompactness of the kernel in the identical way that the Faddeev or Lovelace equations do. The system of LS equations, in matrix form, is

$$\begin{pmatrix} 1-v_\alpha G_\alpha & 0 & 0 \\ 0 & 1-v_\beta G_\beta & 0 \\ 0 & 0 & 1-v_\gamma G_\gamma \end{pmatrix} \times \begin{pmatrix} T_{\alpha\alpha}^{-} \\ T_{\beta\alpha}^{-} \\ T_{\gamma\alpha}^{-} \end{pmatrix} = \begin{pmatrix} v_\alpha \\ v_\alpha \\ v_\alpha \end{pmatrix}. \tag{40}$$

With the use of Eq. (4) it is easy to verify that the multiplier M, where

$$M = \begin{pmatrix} 1+v_\alpha G & -V_\beta G & -V_\gamma G \\ -V_\alpha G & 1+v_\beta G & -V_\gamma G \\ -V_\alpha G & -V_\beta G & 1+v_\gamma G \end{pmatrix},$$

acting on Eq. (40) yields the system of Lovelace equations. In order to introduce no spurious solutions, we ask that M^{-1} exists. Again it is easy to verify that

$$M^{-1} = \begin{pmatrix} 1-v_\alpha G_0 & V_\beta G_0 & V_\beta G_0 \\ V_\alpha G_0 & 1-v_\beta G_0 & V_\gamma G_0 \\ V_\alpha G_0 & V_\beta G_0 & 1-v_\gamma G_0 \end{pmatrix}.$$

We see that every element of M^{-1} exists. In fact, the diagonal elements of M^{-1} are the inverses of the multipliers we use to put the individual LS equations into a form with a compact kernel.

B. Rosenberg, Newton, and Noble Formulations

Noble[11] uses the same multiplier $(1-G_0v_\alpha)^{-1}$ as an M-type multiplier to solve for the total Green's function. The advantage of using it as an L-type multiplier as we have done is that only the two-body t operators appear in the result and there is no explicit dependence on the potentials. Noble's approach also differs from ours in that in order to put the multiplier into calculable form he breaks it up into a sum of two parts and obtains two coupled Faddeev-like equations for them:

$$(1-G_0v_\alpha)\,(1+G_0(X_\beta+X_\gamma)) \equiv 1, \tag{41}$$

where the X's satisfy

$$X_\beta = t_\beta + t_\beta G_0 X_\gamma, \tag{42}$$

$$X_\gamma = t_\gamma + t_\gamma G_0 X_\beta. \tag{43}$$

We uncouple these equations to produce compact kernels and solve them, to find that $X_\beta+X_\gamma=\tau_{\beta\gamma}$. Indeed we would expect this result from the definition of $\tau_{\beta\gamma}$ and the identical definition of the X's in Eq. (41).

Rosenberg[10] solves for the elastic transition operator $T_{\alpha\alpha}^{-}$ and obtains an equation identical to our Eq. (28) with the use of an operator identity [his Eq. (23)]. He evaluates the $\tau_{\beta\gamma}$ occurring in that equation exactly as does Noble, i.e., by Eqs. (42) and (43).

Using an operator identity of the type of Eq. (4) or (5), Newton[12] produces Eq. (21). Newton repeats the procedure for the resolvent of an operator made up of three terms treating the sum of two of the terms as one. Newton's equation (7) can be written, in our notation, using Eq. (23), to obtain

$$[1-G_0(V_1+V_2+V_3)]^{-1} = (1+G_0\tau_{12}) \times (1-G_0t_3G_0\tau_{12})^{-1}(1+G_0t_3). \tag{44}$$

From Eq. (4), with $\alpha=0$, we see that the above resolvent acting on G_0 yields G:

$$G = (1+G_0\tau_{12})\,(1-G_0t_3G_0\tau_{12})^{-1}G_3. \tag{45}$$

This last result is exactly our solution, Eq. (25), to the multiplied LS equation. Similarly, one can rewrite Newton's equation (8) to yield exactly Eq. (37) for T_{00}^{-}.

Multipliers are already explicitly present in the formulations of Faddeev, Rosenberg, and Noble. To obtain the Faddeev equations (32)–(34), one must use the three expressions $1+G_0t_i$ as multipliers. Each one is used separately to obtain one of the Faddeev equations. Similarly, to obtain the two coupled equations of Rosen-

berg or Noble, one must use two of the expression $1+G_0t_i$ as multipliers.

In summary, we have shown in this section that the apparently different formulations of Faddeev, Lovelace, Rosenberg, Noble, and Newton yield solutions identical to each other and identical to those obtained from the multiplied LS equations which has no spurious bound states. The method of making the kernel compact in the various formulations is in effect the same as the multiplier method. An intrinsically different solution would result if a more complicated multiplier of the form $(1+G_0\tau_{\beta\gamma})(1+C)$, where C is compact and $(1+C)^{-1}$ exists, were to be used. The different formulations have effectively used the simplest multiplier, simplest in the sense that it immediately removes the noncompact part of the L-S kernel. The multiplier method is a straightforward and general way to put scattering equations into calculable form. It is not clear which formulation is the most practical from a calculational point of view. It does seem that the uncoupled equation is more useful for making approximations.

For completeness we must add that a field-theoretic formulation of the three-body problem was introduced by Amado.[18] Rosenberg[19] has shown this approach to be equivalent to the Lovelace formulation with separable interactions.

V. DISTORTED-WAVE AND SHELL-MODEL FORMALISMS

In this section we discuss two methods often used as approximations in three-body problems. With each of these we use the multiplier method to obtain a calculable integral equation with compact kernel for the appropriate operator. In this manner we can explicitly determine what *should* be the first-order term in a calculationally convergent series.

A. Distorted-Wave Formalism

Both the undistorted-wave and the distorted-wave (DW) Born series have been shown to be divergent in general.[2,20,21] We will show how the multiplier method can be used to remedy the divergence by removing the noncompactness in the integral equation for the DW transition amplitude.

Greider and Dodd have derived the integral equation for the exact transition amplitude in the DW formalism[20]:

$$T_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^{+}+\omega_\beta^{-\dagger}(v_\beta-w_\beta)G_\beta T_{\beta\alpha}^{-}, \quad (46)$$

where w_α and w_β are the distorting potentials in the initial and final channels, respectively. The wave operators that will distort the unperturbed states ϕ_α and ϕ_β are ω_α^{+} and ω_β^{-}. They are defined by

$$\omega_\alpha^{+}=1+[1/(E-H_0-V_\alpha-w_\alpha+i\epsilon)]w_\alpha$$

and

$$\omega_\beta^{-\dagger}=1+w_\beta/(E-H_0-V_\beta-w_\beta+i\epsilon).$$

The presence of the distorted wave $\chi_\alpha^{+}=\omega_\alpha^{+}\phi_\alpha$ in the entrance channel α is calculationally convenient, but does not affect the kernel of Eq. (46). Therefore, it does not affect the convergence of the DW series for $T_{\beta\alpha}$; for clarity we temporarily set $w_\alpha=0$, or $\omega_\alpha^{+}=1$, in what follows.

We rewrite Eq. (46) to yield

$$[1-\omega_\beta^{-\dagger}(v_\beta-w_\beta)G_\beta^{+}]T_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}v_\alpha. \quad (47)$$

It is an easy matter to show that Eq. (47) results when the LS equation is multiplied by $\omega_\beta^{-\dagger}$, if we use the relation

$$\omega_\beta^{-\dagger}=1+\omega_\beta^{-\dagger}w_\beta G_\beta^{+}.$$

The multiplier introduces no extraneous bound states since the only zeros of $\omega_\beta^{-\dagger}$ arise from the zeros of $z-H_0-V_\beta$. Since the potential V_β is a two-body potential, it cannot produce a three-body bound state.

We note that so far the distorting potential w_β is arbitrary.[20] In actual DW calculations in which the DW functions are taken from experimental data, the distorting potential w_β is taken to be either V_α or V_γ or something that approximates them.[21] If we choose $w_\beta=V_\alpha$, the new kernel in Eq. (47) becomes

$$\omega_\beta^{-\dagger}V_\gamma G_\beta=\omega_\beta^{-\dagger}V_\gamma G_0(1+V_\beta G_\beta).$$

We see that every term involves a product of at least two different interactions except for one term, $V_\gamma G_0$. This term is not compact. The use of the DW formalism has removed all but one of the noncompact terms of the LS kernel.

However, the DW equation can be made compact with the use of the multiplier $(1-V_\gamma G_0)^{-1}$. (This is essentially the procedure used by Dodd and Greider.[20]) This multiplier introduces no spurious solutions, and it can be used to multiply the LS equation before one multiplies by $\omega_\beta^{-\dagger}$. This yields an equation with a compact kernel whose inhomogeneous (corresponding to the Born) term is given by

$$\omega_\beta^{-\dagger}(1-V_\gamma G_0)^{-1}v_\alpha.$$

If the distorting potential had been chosen as $w_\beta=V_\gamma$, the multiplier $(1-V_\beta G_0)^{-1}$ would be necessary. We see that in order for the DW kernel to be compact, the potential *not used* in distorting the exit channel must be used in a multiplier of the form $(1-V_\gamma G_0)^{-1}$ or $(1-V_\beta G_0)^{-1}$. (In the post form of the transition operator $T_{\beta\alpha}^{+}$, a similar multiplier involving the potential not used in distorting the entrance channel is required.)

We now put back the distortion ω_α^{+} in channel α to obtain the inhomogeneous term for $T_{\beta\alpha}^{-}$, and set $w_\beta=V_\alpha$:

$$\omega_\beta^{-\dagger}(1-V_\gamma G_0)^{-1}(v_\alpha-w_\alpha)\omega_\alpha^{+}.$$

To further clarify the nature of this inhomogeneous term, we consider the specific process of a knockout reaction, like a (p,n) reaction on a heavy nucleus. The three-body model for this process requires that w_α

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[19] L. Rosenberg, Phys. Rev. **134**, B937 (1964).
[20] K. R. Grider and L. R. Dodd, Phys. Rev. **146**, 671 (1966).
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be set (approximately) equal to V_β,[21] and the inhomogeneous term becomes

$$\omega_\beta^{-\dagger}(1-V_\gamma G_0)^{-1}V_\gamma\omega_\alpha^{+}=\omega_\beta^{-\dagger}t_\gamma\omega_\alpha^{+}. \quad (48)$$

This term is then the first term in a series solution of the DW equation with a compact kernel. It is of importance in any DW calculation since we know that any exact solution of an integral equation is possible only if the kernel is compact. Furthermore, we know that a vector series and a matrix-element series are guaranteed to converge if the operator series is convergent.[14] The t_γ is the first term in the operator series and Eq. (48) is the first term in the matrix-element series. Use of t_γ rather than the usual Born term V_γ is appropriate in calculations based on a convergent formulation, and calculations using t_γ have been performed with some success.[22] From the definition of t_γ in Eq. (15), we see that the usual DWBA would be valid if $V_\gamma G_\gamma V_\gamma$ could be shown to be negligible compared to V_γ.

B. Shell-Model Method

Next we consider a three-body shell-model calculation. The three bodies are two extra-core nucleons and an inert core. We obtain an exact integral equation for the wave function, examine its kernel, and make the kernel compact with the use of a multiplier. We then use the lowest-order term of the equation with the compact kernel to obtain the expression that should be used in a standard shell-model calculation.

We assume that each extra-core nucleon n_1 and n_2 is bound to the core via the potential U_1 and U_2, respectively, and that the two nucleons interact with each other via the potential V. Further, we assume that we know the wave functions and the energy levels of the two nucleons outside the core when $V=0$; that is, we know the solutions of the Schrödinger equation (the basis states) for each nucleon separately bound to the core via the shell-model potential U_1 or U_2. In the usual shell-model calculation a finite number of these basis states are used, and a linear combination of the product wave functions is then diagonalized with respect to V to obtain the energy of the system.

We want to correct for the fact that only a finite number or a truncated subset of the complete set of basis states is considered. Since among the excluded states are the *continuum* states of each nucleon in the field of the core, we suspect and will show below that in an exact formulation disconnected diagrams (noncompact kernel) will appear. These diagrams can be dealt with by our multiplier method.

We can write the exact three-body bound-state wave function as

$$\psi=gV\psi,$$

where

$$g=(E-H_0-U_1-U_2)^{-1}.$$

The complete set of basis states consists of the eigenfunctions $|\,i\rangle$ of $H_0+U_1+U_2$:

$$(H_0+U_1+U_2)\,|\,i\rangle=\epsilon_i\,|\,i\rangle.$$

Let P denote the projection operator which projects out the small finite number of bound states used in a normal shell-model calculation,

$$P=\sum_i^N |\,i\rangle\langle i\,|\,, \quad (49)$$

where N is the number of basis states chosen. We then write

$$\psi=P\psi+Q\psi$$

or

$$\psi=P\psi+QgV\psi,$$

where

$$Q=1-P.$$

Since Q is a projection operator on the eigenstates of $H_0+U_1+U_2$ and g is the Green's function for the same operator, Q and g commute. Neither P nor Q, however, commutes with V. We wish to obtain an integral equation for ψ in which $P\psi$ is the lowest order, i.e., inhomogeneous term. We write

$$\psi=P\psi+gQV\psi$$

and expand g according to

$$g=G_0+G_0(U_1+U_2)g$$

to obtain the integral equation for ψ,

$$[1-G_0V+G_0PV-G_0(U_1+U_2)gQV]\psi=P\psi. \quad (50)$$

The inhomogeneous term $P\psi$ is the approximate wave function obtained by truncating the complete set of basis states. Now P as normally used in shell-model calculations is defined in Eq. (49) and is expressed in terms of states $|\,i\rangle$ which exponentially go to zero as the coordinates of particles 1 or 2 become large. Thus, PV involves at least two interactions and is a compact operator. From previous arguments, we know that

$$G_0(U_1+U_2)gQV=G_0\tau_{12}G_0QV$$

is compact. Then the only noncompact term in Eq. (50) is G_0V and we can remove it with the multiplier $(1-G_0V)^{-1}=1+G_0t$. If we are only interested in the lowest-order term of the multiplied solution to use in the shell-model calculation, it does not matter whether we use an L or M type of multiplier. For simplicity we use an L-type multiplier, for which the solution is

$$\psi=(1+G_0t)[1/(1+G_0Pt-G_0\tau_{12}G_0Qt)]P\psi. \quad (51)$$

This is a calculable expression for the solution

$$\psi=[1/(1-gQV)]PgV\psi. \quad (52)$$

If we consider only the lowest term, we have

$$\psi\simeq(1+G_0t)P\psi.$$

This is the first term in the solution of the equation with a compact kernel. We see that ψ should be calculated from $\psi\simeq PgV(1+G_0t)P\psi=PgtP\psi$ instead of from $\psi\simeq PgVP\psi$.

Alternatively, we can ask what is the form of the effective interaction operator V_{eff}, that will produce

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the exact wave function ψ from the truncated representation for g, i.e., what is V_{eff} if

$$(1-PgV_{\text{eff}})\psi=0$$

yields the exact wave function given in Eq. (51). Since we can write Eq. (52) as

$$\psi=PgV[1/(1-gQV)]\psi,$$

and using the calculable expression for $(1-gQV)^{-1}$ from Eq. (51), we obtain

$$\psi=PgV(1+G_0t)[1/(1+G_0Pt-G_0\tau_{12}G_0Qt)]\psi.$$

We then see that the effective operator which permits the use of a truncated representation for g, and yet yields the exact wave function, is

$$V_{\text{eff}}=t[1/(1+G_0Pt-G_0\tau_{12}G_0Qt)].$$

Neglecting the compact operator in the denominator of V_{eff}, we see that the wave function should again be approximately calculated from the lowest-order term:

$$\psi\simeq Pgt\psi.$$

We see that the t operator of the residual interaction rather than the residual interaction potential is the one that appears in a convergent formulation of three-body shell-model calculations if only a finite set of basis states, $P\psi$, is used in the diagonalization of the residual interaction. Calculations that use an approximation for the t matrix have recently been performed by Hodgson.[23] He finds that there is no definite improvement over a calculation that just employs the potential. Kuo and Brown[24] make use of the second Born term in the series for t, i.e., $t\simeq V+VG_0V$. They use core polarization states in the intermediate states of the term VG_0V. This implies that Kuo and Brown have included a structure in the core and have departed from a strict three-body shell-model interpretation of the problem. At any rate, they find the second Born term to be important.

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Nuclear Optical Model and Wave Properties: Barrier Penetration, Reflection, Absorption, and Resonance*

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A detailed study of the wave properties of the nuclear optical model is presented to elucidate the problem of barrier penetration by charged particles and to remove some of the mystique of optical-model calculations. The wave properties and the concomitant penetration are most straightforward for square wells, for which the resonance, reflection, and penetration are easily ascribed to separate factors. We show that the wave properties of more general diffuse-edge optical potentials achieve a similar simplicity by the construction of an equivalent square well (ESW) which has the same resonance, penetration, and absorption factors as the optical potential, but which differs in its reflection factor. A general construction of the ESW is given, and we apply it to the following problems: (1) the very narrow single-particle resonances of real optical potentials that occur at energies far below the Coulomb barrier, (2) the nuclear absorption cross sections in the presence of barriers, (3) the calculation of absorption cross sections at astrophysical energies (extreme barrier penetration) employing optical models fitted to data at higher energies, and (4) the value of the nuclear radius and sum-rule limits appropriate to the analysis of nuclear reactions. In some cases of extreme barrier penetration, the ESW fails to yield all the properties. For example, cases are described where the bulk of the absorption may attain in the distant "tail" of the imaginary term in the optical potential: The corresponding reaction rates can yield information about the behavior of the nucleus at distances much beyond the normal nuclear radius.

1. INTRODUCTION

THE behavior of most nuclear reactions at low energy—particularly those of interest for astrophysical systems—is dominated by Coulomb and angular-momentum barriers. Early treatments[1–3] of such reactions employed a simple picture: the "black-nucleus" or "black-box" picture. In this picture, a bombarding particle was viewed as passing through known Coulomb and angular-momentum barriers up to the nuclear radius. At the nuclear radius, it was ab-

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