Treatment of the Three- and Four-Nucleon Systems by a Generalized Separable-Potential Model

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A formalism, proposed previously, which represents a generalization of the separable-potential model is applied to the four-nucleon system. It consists essentially in approximating the off-shell nucleon-deuteron scattering amplitude—as obtained in the separable-potential model—again by separable expressions. We arrive thereby at effective two-particle equations which are solvable by standard methods. The determination of the required separable terms is studied in detail. A first attempt to solve approximately our final integral equations is made. The results demonstrate the applicability of the method.

1. INTRODUCTION

IT is well known that, in the separable-potential model,[1] the two-particle problem is immediately reducible to an algebraic equation, while the three-particle problem can be reduced to an effective two-body theory.[2–6] Moreover, as we have pointed out previously,[7] the separable-potential approximation leads in the general n-particle case to effective $(n-1)$-body equations.[8] This observation suggested the possibility of approximating by separable terms not only the original two-particle interactions, but also the "potentials" obtained in these effective equations.[9,10] Repeated application of such a procedure allows us to reduce the dimension of the original problem, step by step, providing us finally with manageable equations.[7] In particular, after two steps we obtain algebraic equations for the three-particle problem, while use of the same separable terms leads in the four-particle case to effective two-particle equations. After angular momentum decomposition, these are one-dimensional equations.

We recall that the justification of the separable-potential approximation is given by the fact that the kernels of (genuine and effective) two-particle equations are of the Hilbert-Schmidt type, at least under suitable conditions on the two-particle interactions.[11] By calculating the Schmidt norms of the neglected nonseparable rest kernels, we are in a position to answer the question, decisive for practical applications of the method, as to whether only few separable terms are sufficient for a reasonable approximation to the potentials.[9]

It is the purpose of this paper to apply the twofold separable approximation scheme described above to the three-nucleon[12] and, in particular, to the four-nucleon problem. In contrast to the latter one, the three-nucleon problem is manageable after the first step and has been studied in this form by several authors.[2–5,13] The results obtained even with few separable terms for the nucleon-nucleon interaction are very satisfactory. Therefore, in this case our repeated separable approximation only serves to check the method by comparison with these results. Furthermore, it provides us with the separable potentials needed in the four-nucleon problem in order to approximate it by "two-body" equations. Some of the transition amplitudes that occur in these equations are just the (off-shell) amplitudes for deuteron-deuteron and nucleon-triton (He^3) elastic and rearrangement scattering. The others correspond to the scattering of unphysical "quasiparticles,"[7,9] introduced in order to improve the approximation.

In Sec. 2, the kinematics is fixed, and the separable

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[3] R. D. Amado, Phys. Rev. **132**, 485 (1963); R. Aaron, R. D. Amado, and Y. Y. Yam, *ibid.* **136**, B650 (1964).
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[7] P. Grassberger and W. Sandhas, Nucl. Phys. **B2**, 181 (1967) [compare also: Z. Physik **217**, 9 (1968)].
[8] A suitable (but not necessary) starting point for this reduction is the utilization of equations similar to those proposed by L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSSR **138**, 565 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1963)]; *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translations, Jerusalem, 1965).
[9] A generalization of the separable-potential model (quasiparticle method) allows us to take into account also the nonseparable parts of the interaction by means of perturbation theory. For the two-particle case, this method has been introduced by S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963). See also M. Scadron, S. Weinberg, and J. Wright, *ibid.* **135**, B202 (1964); J. V. Noble, *ibid.* **148**, 1527 (1966). A suitable extension to the three-body problem has been proposed by E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967), and to the n-body problem in Ref. 7.
[10] For the three-body problem compare also L. Rosenberg, Phys. Rev. **138**, B1343 (1965); **168**, 1756 (1968); R. Yaes, *ibid.* **170**, 1236 (1968); M. G. Fuda, *ibid.* **166**, 1064 (1968); F. Riordan, Nuovo Cimento **54**, 552 (1968).
[11] K. Meetz, J. Math. Phys. **3**, 690 (1962), and Refs. 6, 9, and 7.
[12] Compare in this context also the method of E. A. Harms and L. Laroze, Bull. Am. Phys. Soc. **14**, 21 (1969).
[13] A. C. Phillips. Phys. Rev. **142**, 984 (1966); **145**, 733 (1966).

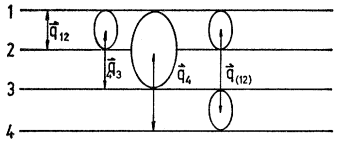

FIG. 1. Relative momenta used for the description of the four-nucleon system.

two-nucleon potentials used in the following are given. The latter are the ones conventionally employed in the separable-potential model of the two- and three-nucleon problems. Section 3 is devoted to the general problem of finding separable approximations with as few terms as possible. For this purpose the variational method of Wright and Scadron[14] is employed. (Several detailed questions, concerning the choice of the trial functions, are treated in the Appendixes A and B.)

In Sec. 4, the application to the effective two-particle formulation of the three-nucleon problem is given. Since the effective potentials are energy-dependent, the separable terms are chosen to be energy-dependent also, which allows us to adjust them at all energies. The obtained results provide us with the nucleon-triton "potentials" in the four-nucleon problem as presented in Sec. 5. Besides these nucleon-triton potentials, we also need there "potentials" describing the transition: nucleon+triton→deuteron+deuteron. These are constructed in Sec. 6. In Sec. 7, we present the results for the four-nucleon bound-state problem obtained by the Rayleigh-Ritz variational principle. For scattering energies, our effective two-body equations, although tractable by present-day computers, are still very cumbersome. Thus, we solved them only in a K-matrix Born approximation. The results are given in Sec. 8, both for the three- and for the four-nucleon problems. Some concluding remarks are contained in Sec. 9.

We should note that equations similar to ours have been proposed by Komarov and Popova[15] on the basis of more intuitive arguments. However, our final equations differ in several respects from theirs. Furthermore, numerical results for the four-nucleon problem have not yet been obtained in their framework.

Finally, we stress that our present results should be understood mainly as a first, crude, but, we believe, encouraging check of the applicability of our general method.

2. KINEMATICS AND THE TWO-NUCLEON INTERACTIONS

Instead of the particle momenta $\mathbf{k}_i$ themselves, we use relative momenta in the normalization of Lovelace[6] and Weyers.[16] In the four-nucleon system we have, besides the total momentum (m is the nucleon mass),

$$\begin{aligned}
\mathbf{q}_{12}&=[1/2(m)^{1/2}](\mathbf{k}_1-\mathbf{k}_2),\\
{}_4\mathbf{q}_3&=[1/2(3m)^{1/2}](\mathbf{k}_1+\mathbf{k}_2-2\mathbf{k}_3),\\
\mathbf{q}_{(12)}&=[1/2(2m)^{1/2}](\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4),\\
\mathbf{q}_4&=[1/2(6m)^{1/2}](\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-3\mathbf{k}_4),
\end{aligned}\qquad (2.1)$$

or the corresponding expressions for all other index combinations (see Fig. 1). Of course, only three of these momenta are linearly independent. Some of the relationships between them are given in Ref. 16. The normalization is chosen such that the c.m. kinetic energy is simply

$$H_0=\mathbf{q}_{12}^2+{}_4\mathbf{q}_3^2+\mathbf{q}_4^2=\mathbf{q}_{12}^2+\mathbf{q}_{34}^2+\mathbf{q}_{(12)}^2. \qquad (2.2)$$

Here, $\mathbf{q}_{12}$ evidently represents the relative momentum in the (1, 2) subsystem. The two momenta $\mathbf{q}_{12}$ and ${}_4\mathbf{q}_3$ are the ones relevant for the kinematical description of the (1, 2, 3) subsystem, being identical with the momenta $\mathbf{p}_3$ and $\mathbf{q}_3$ introduced by Lovelace in the three-particle problem. That is, the momenta used in the three-nucleon problem occur also in the basic set (2.1). Furthermore, if particles 3 and 4 are bound, the four-nucleon system is reduced to a formal three-particle system which is kinematically described by $\mathbf{q}_{12}$ and $\mathbf{q}_{(12)}$, corresponding again to Lovelace's three-particle momenta written down for this case. Thus, we see that the set (2.1) includes exactly the momenta necessary for treating the four-nucleon problem as a three-particle problem, as will be done in Sec. 5. All equations occurring in that section can, therefore, be taken over from Ref. 6, as far as kinematics is concerned.

For the nucleon-nucleon interaction, we take the usual version of the separable-potential model which includes neither hard-core nor tensor forces.[1,17] More explicitly, we have

$$V(\mathbf{p}',\mathbf{p})=-g_d(\mathbf{p}')\gamma_d{}^2 g_d(\mathbf{p})P_d-g_\phi(\mathbf{p}')\gamma_\phi{}^2 g_\phi(\mathbf{p})P_\phi, \qquad (2.3)$$

where P_d and P_ϕ are the isospin-spin projection operators onto the states with quantum numbers of the deuteron $d(I=0,\ S=1)$, and of the virtual singlet bound state $\phi(I=1,\ S=0)$.

The form factors $g_d(\mathbf{p})$ and $g_\phi(\mathbf{p})$ are chosen (as usual):

$$g_n(\mathbf{p})=1/(\mathbf{p}^2+\mu_n{}^2), \qquad n=d,\phi. \qquad (2.4)$$

The ansatz (2.3) leads to the following form of the two-

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[15] V. V. Komarov and A. M. Popova, Nucl. Phys. **69**, 253 (1965); **A90**, 635 (1967).
[16] J. Weyers, Phys. Rev. **145**, 1236 (1966).
[17] See also J. H. Naqui [Nucl. Phys. **58**, 289 (1964)] and the discussion of the various models, used up to now, in K. M. Watson and J. Nuttall [*Topics in Several Particle Dynamics* (Holden-Day, San Francisco, 1967)].

nucleon T matrix:

$$T(\mathbf{p}', \mathbf{p}, E) = -\sum_{n=d,\phi} g_n(\mathbf{p}') t_n(E) g_n(\mathbf{p}) P_n \quad (2.5)$$

with

$$t_n^{-1}(E) = \gamma_n^{-2} - \int d^3q [g_n(\mathbf{q})]^2/(\mathbf{q}^2 - E). \quad (2.6)$$

The parameters γ_d, γ_ϕ, μ_d, and μ_ϕ are determined by the binding energy of the deuteron, the singlet and triplet scattering lengths, and the singlet effective range. The following values are taken as input (for the singlet data we average over the nn and np data[13]):

$$\begin{aligned} E_d &= 2.226 \text{ MeV}, \\ a_d &= 5.396 \text{ fm}, \\ a_\phi &= -20.339 \text{ fm}, \\ r_\phi &= 2.844 \text{ fm}. \end{aligned} \quad (2.7)$$

We then obtain for γ_n and μ_n

$$\gamma_d = 10.26 \text{ (MeV)}^{3/4}, \quad \mu_d = 9.162 \text{ MeV},$$
$$\gamma_\phi = 5.888 \text{ (MeV)}^{3/4}, \quad \mu_\phi = 7.191 \text{ MeV}. \quad (2.8)$$

It is well known that neglect of tensor forces and short-range repulsion in the two-nucleon interaction leads to too strong a binding for the three-nucleon system. The same will be true for the four-nucleon problem.

Therefore, to get a reliable measure of the approximations to be made, we have to compare our three-nucleon results with those obtained in exact calculations using the same simple ansatz for the two-nucleon forces.[4,13]

Finally, we note that more sophisticated separable potentials have been given[18] and used in the three-nucleon problem.[19] They can all be used in our treatment, but with increased numerical complexity.

3. DETERMINATION OF SEPARABLE "POTENTIALS"

According to the discussion in the Introduction, the use of the separable interaction (2.3) reduces the three-nucleon problem to effective two-particle equations, while the four-nucleon problem is reduced to a formal three-particle theory. Thus, our next step is to approximate by separable expressions the various "two-particle potentials" obtained thereby. In this section, we will, therefore, consider a general two-particle problem with a nonlocal energy-dependent potential.

The best approximation in the sense of the convergence of the quasi-Born series[20] is obtained by

$$\begin{aligned} V^{\text{sep}}(E) &= \sum_\nu V(E) \,|\, \psi_\nu(E)\rangle \eta_\nu(E) \langle \psi_\nu(E^*) \,|\, V(E) \\ &= \sum_\nu |\, \chi_\nu(E)\rangle \eta_\nu(E) \langle \chi_\nu(E^*) \,|, \end{aligned} \quad (3.1)$$

where the $|\,\psi_\nu(E)\rangle$ are Sturmian functions (representing the "ideal choice" of Weinberg), defined as solutions of

$$G_0(E) V(E) \,|\, \psi_\nu(E)\rangle = \eta_\nu(E) \,|\, \psi_\nu(E)\rangle \quad (3.2)$$

with the normalization[21]

$$\langle \chi_\nu(E^*) \,|\, G_0(E) \,|\, \chi_{\nu'}(E)\rangle = \delta_{\nu\nu'}. \quad (3.3)$$

Here, $G_0(E) = (H_0 - E)^{-1}$ is the free Green's function.

The above potential V^{sep} leads to the "two-particle" transition operator

$$T^{\text{sep}}(E) = -\sum_\nu |\, \chi_\nu(E)\rangle t_\nu(E) \langle \chi_\nu(E^*) \,| \quad (3.4)$$

with

$$t_\nu(E) = -\eta_\nu(E)/[1 + \eta_\nu(E)]. \quad (3.5)$$

To solve Eq. (3.2), at least approximately, we resort to variational principles, as described by Wright and Scadron,[14] for genuine two-particle problems. In fact, the solutions of (3.2) are obtained by extremizing the expressions

$$[\eta_\nu(\chi; E)] = \frac{\langle \chi_\nu(E^*) \,|\, G_0(E) V(E) G_0(E) \,|\, \chi_\nu(E)\rangle}{\langle \chi_\nu(E^*) \,|\, G_0(E) \,|\, \chi_\nu(E)\rangle} \quad (3.6)$$

or

$$[\eta_\nu'(\chi; E)] = \frac{\langle \chi_\nu(E^*) \,|\, G_0(E) \,|\, \chi_\nu(E)\rangle}{\langle \chi_\nu(E^*) \,|\, V^{-1}(E) \,|\, \chi_\nu(E)\rangle}. \quad (3.7)$$

Whenever the Lippmann-Schwinger equation, and therefore also Eq. (3.2), were uncoupled equations, we always took only one separable term, $\nu = 1$. In general, however, Eq. (3.2) is an $n \times n$ matrix equation describing coupled channels. Then, some technical modifications of the above method turned out to be useful. They are described in Appendix A, and lead to an approximation of V_{ik} by just n separable terms.

The important point, now, is the question of how to choose the trial form factors $|\,\chi_\nu(E)\rangle$. It is well known that the eigenfunctions of Eq. (3.2) become complex above threshold even for local and real potentials. Thus it is not very easy to make a reasonable guess. For real energies below threshold, and for the first (largest) eigenvalue, however, we were led by the following arguments: $V(\mathbf{p}', \mathbf{p}, E)$—as well as $G_0(\mathbf{p}; E)$ and $\chi_1(\mathbf{p}, E)$—

[18] F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964); R. D. Puff, *ibid.* **13**, 317 (1961); J. H. Naqui, Nucl. Phys. **A103**, 565 (1967); G. L. Strobel, *ibid.* **A116**, 465 (1968).

[19] V. S. Bhasin, G. L. Schrenk, and A. N. Mitra, Phys. Rev. **137**, B398 (1965); B. S. Bhakar and A. N. Mitra, Phys. Rev. Letters **14**, 143 (1965); G. L. Schrenk and A. N. Mitra, *ibid.* **19**, 530 (1967); A. G. Sitenko and V. F. Kharchenko, Yadern. Fiz. **1**, 994 (1965) [English transl.: Soviet J. Nucl. Phys. **1**, 708 (1965)]; A. G. Sitenko, V. F. Kharchenko, and N. M. Petrov, Phys. Letters **21**, 54 (1966); J. Borysowicz and J. Dabrowski, *ibid.* **24B**, 125 (1967); N. M. Petrov, S. A. Storozhenko, and V. F. Kharchenko, Yadern Fiz. **6**, 466 (1967) [English transl.: Soviet J. Nucl. Phys. **6**, 340 (1968)]; F. Tabakin, Phys. Rev. **137**, B75 (1965).

[20] S. Weinberg, Ref. 9.

[21] For simplicity, we have assumed $\eta_\nu \neq \eta_{\nu'}$ for $\nu \neq \nu'$.

is a (positive- or negative-) definite function. Therefore, it seems reasonable to approximate it in the integral equation (3.2), defining the form factors, by its value at some mean momentum $\bar{p}$. Then, we have from Eq. (3.2), after angular momentum decomposition,[22]

$$\chi_{l,1}(p;E)=[\eta_{l,1}(E)]^{-1}\int_0^\infty dp'p'^2V_l(p,p';E)$$
$$\times G_0(p';E)\chi_{l,1}(p';E)\sim V_l(p,\bar{p};E)[\eta_{l,1}(E)]^{-1}$$
$$\times\int_0^\infty dp'p'^2G_0(p';E)\chi_{l,1}(p';E). \quad (3.8)$$

According to (3.1), this leads to the separable potential

$$V_l^{\rm sep}(p',p;E)\propto V_l(p',\bar{p};E)V_l(\bar{p},p;E). \quad (3.9)$$

Here, the proportionality factor depends only on the energy. This form suggests the more general ansatz

$$V_l^{\rm sep}(p',p;E)=\xi(E)\,\frac{V_l(p',\alpha;E)V_l(\alpha,p;E)}{V_l(\alpha,\alpha;E)} \quad (3.10)$$

with an open parameter $\alpha(E)$ which has to be determined by the above variational principle, and with the normalization "constant" $\xi(E)$ which follows from the normalization condition (3.3), after having determined $\alpha(E)$. We would expect, however, that the value of $\xi(E)$ is not too far from unity, since then[23] $V_l^{\rm sep}(p',\alpha;E)\sim V_l(p',\alpha;E)$ and $V_l^{\rm sep}(\alpha,p;E)\sim V_l(\alpha,p;E)$. Actually, we made a further approximation in our calculation. Namely, we replaced $V_l(p,\alpha;E)$—which will, in general, be a rather unwieldy function—by a simpler function which coincides with it for $p\sim\alpha$. In all cases our final form factor has the correct threshold behavior $\propto p^l$ and the same asymptotic behavior for $p\to\infty$ as the exact solution of Eq. (3.2).

The relevance of the arguments of this section can be illustrated by inspection of the Yukawa potential. This is done in Appendix B.

[22] We choose

$$V(\mathbf{p}',\mathbf{p};E)=(1/4\pi)\Sigma_l(2l+1)P_l(\cos\theta)V_l(p',p;E).$$

[23] It is interesting to notice at this point that the method of H. P. Noyes [Phys. Rev. Letters **15**, 538 (1965)] and K. L. Kowalski [*ibid.* **15**, 798 (1965)] for solving the Lippmann-Schwinger equation for positive energies, is intimately connected with Eq. (3.10). For, if we take $\xi(E)=1$ and $\alpha=E^{1/2}$, we arrive at

$$V_l(p',p;E)$$
$$=[V_l(p',E^{1/2};E)V_l(E^{1/2},p;E)/V_l(E^{1/2},E^{1/2};E)]$$
$$+V_l'(p',p;E).$$

By use of this special decomposition of the potential into a separable and a nonseparable part, we are able, with the help of the quasiparticle method of Weinberg (Ref. 9), to reproduce all the results of Noyes and Kowalski.

4. THREE-NUCLEON PROBLEM

As a first application of the ideas developed in Sec. 3, we consider the three-nucleon problem. The results obtained will then serve as input for the calculations to be performed in the four-nucleon problem.

As already remarked in Sec. 2, we will restrict ourselves to the simplest model of only one separable term in each spin-isospin channel for the two-nucleon T matrix. Therefore, our calculations will suffer from the same defects as those of Refs. 4 and 13. Thus, it is clear that our results have to be measured against those of these authors, if one wants to judge the quality of our approximation.

We start by writing down the partial-wave projected scattering equations for orbital angular momentum equal to zero (we will, however, drop the index $L=0$; later on we shall also make some remarks on the P wave). The equations for the T matrix are, with correct symmetrization because of the identity of the nucleons,

$$T_{n,m}^{I,S}(q',q;E)=V_{n,m}^{I,S}(q',q;E)$$
$$-\sum_{r=d,\phi}\int_0^\infty dq''q''^2V_{n,r}^{I,S}(q',q'';E)$$
$$\times t_r(E-q''^2)T_{r,m}^{I,S}(q'',q;E). \quad (4.1)$$

Here, $I(S)$ denotes the total isospin (spin) of the three-nucleon system, both of which are conserved in our model. The indices n, m, and r stand for d (deuteron) and ϕ (virtual singlet state).

The "propagators" t_r are given by Eq. (2.6).[24] The "potentials" $V_{n,m}^{I,S}$ have the form[6]

$$V_{n,m}^{I,S}(q',q;E)=-\Lambda_{n,m}^{I,S}V_{n,m}(q',q;E) \quad (4.2)$$

with

$$V_{n,m}(q',q;E)=\frac{16}{3\sqrt{3}}\,2\pi\int_{-1}^{+1}d\cos\theta$$
$$\times\frac{g_n[3^{-1/2}(\mathbf{q}'+2\mathbf{q})]g_m[3^{-1/2}(\mathbf{q}+2\mathbf{q}')]}{\frac{1}{3}(4\mathbf{q}'^2+4\mathbf{q}^2+4\mathbf{q}\cdot\mathbf{q}')-E}, \quad (4.3)$$

θ being the angle between $\mathbf{q}$ and $\mathbf{q}'$.

The two-nucleon form factors g_n are defined in Eq. (2.4). The $\Lambda_{n,m}^{I,S}$ are the spin-isospin recoupling coefficients[25]:

$$\Lambda_{n,m}^{1/2,1/2}=\begin{pmatrix} & d & \phi \\ d & \frac{1}{4} & -\frac{3}{4} \\ \phi & -\frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

[24] Of course, t_d is not renormalized, i.e., its residue at the deuteron pole is not equal to 1; see Ref. 3. Correspondingly, also the $T_{dd}^{I,S}$ are not renormalized. In contrast, the normalization of t_ϕ is arbitrary.

[25] Note that the corresponding matrix of Ref. 6 is here multiplied by a factor (-2).

and

$$\Lambda_{n,m}{}^{1/2,3/2}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & 0\end{pmatrix}. \tag{4.4}$$

According to the discussion of Sec. 3, we must now approximate the "potentials" $V_{n,m}{}^{I,S}$ by separable ones. First, let us consider the quartet case ($I=\frac{1}{2}$, $S=\frac{3}{2}$), where only the element V_{dd} contributes. From (3.8) we would obtain

$$\chi_{qu}{}^{d}(q, E)\propto V_{dd}(q, \alpha; E). \tag{4.5}$$

With the correct normalization (3.3), and dropping terms proportional to $\cos\theta$ [compare Eq. (3.10) and the discussion following it], we get

$$\chi_{qu}{}^{d}(q; E)=\left(\frac{\xi_{qu}}{2\eta_{qu}}\right)^{1/2}\left(\frac{64\pi}{3\sqrt{3}}\right)^{1/2}\frac{[(5/3)\alpha^2+\mu_d{}^2][(8/3)\alpha^2-E]^{1/2}}{[\frac{1}{3}(q^2+4\alpha^2)+\mu_d{}^2][\frac{1}{3}(4q^2+\alpha^2)+\mu_d{}^2][\frac{4}{3}(q^2+\alpha^2)-E]}$$
$$=(\xi_{qu}/2\eta_{qu})^{1/2}\hat{\chi}_d(q; E), \tag{4.6}$$

and, correspondingly,

$$V_{d,d}{}^{1/2,3/2;\,\mathrm{sep}}(q', q; E)=\tfrac{1}{2}\xi_{qu}(E)\hat{\chi}_d(q'; E)\hat{\chi}_d(q; E). \tag{4.7}$$

The isoquartet state ($I=\frac{3}{2}$, $S=\frac{1}{2}$) can be treated in the same way, with the deuteron always replaced by ϕ. However, the results for it are not needed in the following, and so we left it out from our considerations. For the doublet potentials $V_{n,m}{}^{1/2,1/2}$, we proceed as described in Appendix A. As explained there, we approximate the potential in this case by two separable terms, with form factors $|\chi_\nu{}^n\rangle=C_\nu{}^n|\hat{\chi}_n\rangle$. Here, $|\hat{\chi}_d\rangle$ is given by Eq. (4.6), while $|\hat{\chi}_\phi\rangle$ is taken to be the same expression, with μ_d replaced by μ_ϕ. The coefficients $C_\nu{}^n(E)$, as well as the parameter α, are determined by the variational principle.[26] The results are the following:

(a) For quartet scattering, Eq. (3.6) leads to an eigenvalue $\eta_{qu}(E)$, which is shown in Fig. 2. The corresponding values of $\alpha(E)$ are depicted in Fig. 3.

(b) For the doublet channel, Eq. (A8) gives rise to two eigenvalues $\eta_1(E)$ and $\eta_2(E)$. They are also shown in Fig. 2 [the values for $\alpha(E)$ are the same as in the quartet case]. As can be seen from Fig. 2, only $\eta_1(E)$ goes through -1. The energy B_t corresponding to $\eta_1(-B_t)=-1$ is the triton binding energy. Our result, $B_t=10.3$ MeV, is indeed very close to the values obtained by Aaron *et al.*[4] and Phillips[13] by solving Eq. (4.1) exactly.

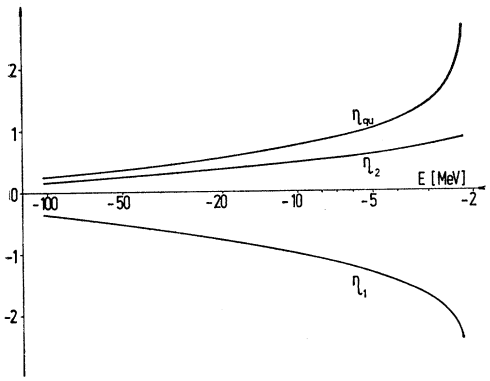


FIG. 2. The largest eigenvalue $\eta_{qu}(E)$ of the Lippmann-Schwinger kernel in the three-nucleon quartet state, and the two largest eigenvalues $\eta_1(E)$ and $\eta_2(E)$ in the three-nucleon doublet state.

[26] In principle, the parameter α should be different in the doublet and quartet cases, but it appeared from the calculation that the difference was completely negligible.

As we have discussed in Sec. 3, the factor $\xi_{qu}(E)$ in Eq. (4.7) is expected to be close to unity. Our results, shown in Fig. 4, verify this. An analogous consideration suggests in the doublet case (compare the definitions in Appendix A)

$$\xi_{n,m}(E)=\sum_{\nu=1}^{2}\eta_\nu(E)C_\nu{}^n(E)C_\nu{}^m(E)\sim-\Lambda_{n,m}{}^{1/2,1/2}. \tag{4.8}$$

At threshold ($E=-B_d$), we obtained indeed

$$\xi_{n,m}(-B_d)=\begin{pmatrix}-0.29 & 0.84\\ 0.84 & -0.30\end{pmatrix},$$

while at our value for the triton energy we got

$$\xi_{n,m}(-10.3)=\begin{pmatrix}-0.32 & 0.98\\ 0.98 & -0.34\end{pmatrix}.$$

Besides the triton binding energy, further important quantities, which can easily be compared with the "exact" results given in Refs. 4 and 13, are the scattering lengths for elastic nucleon-deuteron scattering. Their connection with the T matrices are [compare Eq. (8.3)]

$$a^{I,S}=\frac{\mu_d(\mu_d+B_d{}^{1/2})^3}{4\pi}\left(\frac{3B_d}{m}\right)^{1/2}T_{dd}{}^{I,S}(0, 0; -B_d), \tag{4.9}$$

where the somewhat unconventional coefficient arises from our normalization of the momenta and from the fact that we work with a not renormalized deuteron propagator.

Our results are given in Table I. They show large disagreement with the "exact" ones. This shortcoming, together with the very good result for the binding

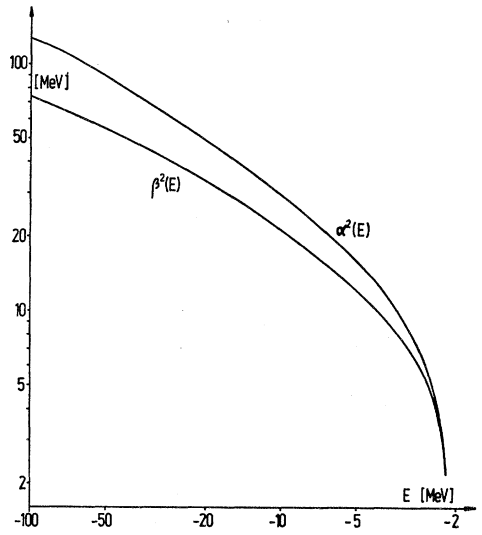


FIG. 3. The parameter $\alpha^2(E)$ in the triton form factors defined in Eq. (4.6), and the corresponding parameter $\beta^2(E)$ of the two-deuteron form factor [Eq. (6.6)].

energy, can be easily understood. First, it is well known that variational calculations yield errors in the binding energy which are of second order compared with the errors of the wave functions. Second, to represent the exact transition amplitude, we need (in principle) an infinite number of separable terms, while only one is required at the bound-state pole. To take into account the effect of the terms which were neglected by our separable approximation, we corrected Eq. (3.4) by adding the first quasi-Born approximation.[9]

Since we use the "ideal choice," this correction reduces to adding $V'=V-V^{\text{sep}}$ in Eq. (3.4). The result, also given in Table I, shows considerable improvement. In order to study the importance of V' from another point of view, we compare the Schmidt norms $\tau(E)$ of $t_d^{1/2}V_{dd}t_d^{1/2}$ and $\tau'(E)$ of $t_d^{1/2}V_{dd}'t_d^{1/2}$. (Hereby we also get upper bounds on the eigenvalues of these kernels.) For simplicity, we consider only the quartet case, since we do not expect that the coupling of the channels—as it occurs in the doublet case—alters the results essentially. The ratio $\tau'(E)/\tau(E)$ is shown in Fig. 5(b). From this it follows, e.g., that the magnitude of the second eigenvalue of $V_{dd}t_d$ is less than 20–30% of the

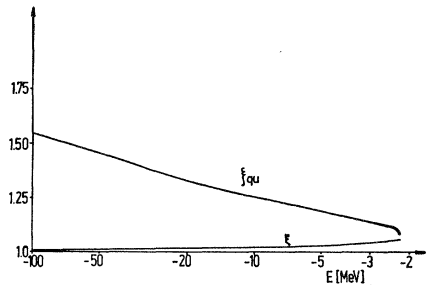


FIG. 4. The factor $\xi_{qu}(E)$ defined in Eq. (4.7), and the corresponding factor $\zeta(E)$ of Eq. (6.6).

TABLE I. Theoretical and experimental quantities for the low-energy three-nucleon system.

	B_t (MeV)	$a_{1/2}$ (F)	$a_{3/2}$ (F)
Aaron *et al.*[a] ($Z=0$)	11.01	−1.04	6.32
Phillips[b] ($V_4=0$)	11.1	−0.79	6.28
This work, separable approximation	10.3	5.19	3.73
First quasi-Born approximation	10.3	1.44	8.15
Experiment[c]	8.49	0.11±0.07	6.14±0.06

[a] Reference 4.
[b] Reference 13.
[c] W. T. H. Van Oers and Y. D. Seagrave, Phys. Letters **24B**, 562 (1967).

first one, for all energies below threshold. This shows that the second term in the eigenvalue expansion of $t_d^{1/2}V_{dd}t_d^{1/2}$ can be neglected as long as the first eigenvalue is not much larger than unity. The latter condition is violated only near threshold. This fact agrees with our bad result for the scattering lengths. But for all other energies, our separable approximation should be sufficient.

Up to now we have considered only S waves. We estimated the contribution of higher partial waves to show that we are allowed to neglect them in the following. For this purpose, we calculated the largest eigenvalue of the P-wave analog of Eq. (4.1), by the same variational method as applied to the S wave. The form factor was chosen, according to our above philosophy, as

$$\chi_n(q;E) \propto q/(q^2+4\alpha^2+3\mu_n^2)\,(4q^2+\alpha^2+3\mu_n^2)\,(4q^2+4\alpha^2-3E)^2.$$

The resulting eigenvalue was indeed about 1/2.5–1/5 of the eigenvalue in the S wave [see Fig. 5(a)].

5. FORMULATION OF FOUR-NUCLEON PROBLEM

Our treatment of the four-nucleon problem follows exactly the one described in Ref. 7 for the general

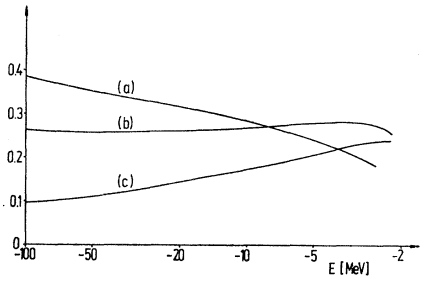


FIG. 5. (a) Ratio between the largest eigenvalue of the Lippmann-Schwinger kernel in the P wave and the one in the S wave (both in the three-nucleon quartet state). (b) Ratio between the Schmidt norms τ' of $t^{1/2}(V-V^{\text{sep}})t^{1/2}$ and τ of $t^{1/2}Vt^{1/2}$ in the three-nucleon quartet state with orbital angular momentum $L=0$. (c) The latter ratio for the kernel of Eq. (6.1) which describes the deuteron+deuteron subsystem.

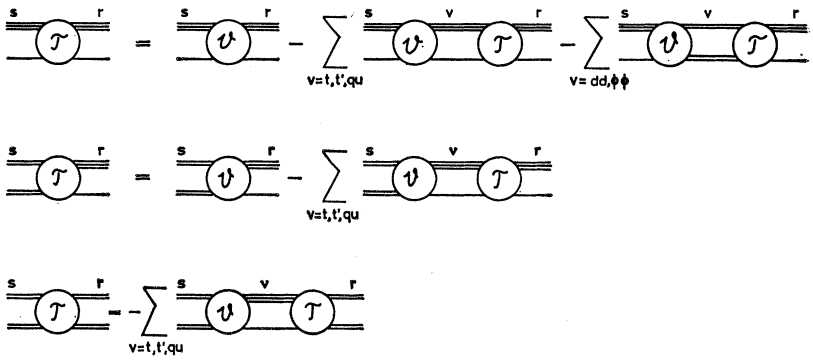


FIG. 6. The coupled Lippmann-Schwinger equations in the effective two-body formulation of the four-nucleon system. The nonexistence of the potentials $\mathcal{V}_{dd,dd}$, $\mathcal{V}_{\phi\phi,\phi\phi}$, and $\mathcal{V}_{dd,\phi\phi}$ stems from neglecting all nonseparable terms in the subsystems. The contributing channels dd and $\phi\phi$ correspond to either total spin or isospin being different from one.

four-body case, but with all nonseparable terms being discarded. After having taken properly into account the Pauli principle (see below), we are thus led, in the case $I=S=0$, to a four-channel "two-particle" problem with the following four channels:

(I) One free nucleon plus a three-nucleon bound state (assuming isospin invariance, we do not distinguish between the triton and the He^3), which we label by the index t.

(II) One nucleon plus the quasiparticle in the three-nucleon doublet subsystem described in Sec. 4. This channel we denote by t'.

(III) Two deuterons, a channel denoted by dd.

(IV) Two virtual two-nucleon S states, denoted by $\phi\phi$.

In the case $I=0$, $S=1$ four channels again contribute. But now channel (IV) consists of one free nucleon and the three-nucleon quartet state (labeled by qu), instead of two singlet states.

To take into account the Pauli principle, we use the method of Lovelace.[6] We start by assuming all particles to be formally distinguishable. For elastic and "inelastic" scattering (i.e., $t\to t$, $t\leftrightarrow t'$, $t\leftrightarrow qu$, $d\phi\to d\phi$, $dd\to dd$, $\phi\phi\to\phi\phi$, and $dd\leftrightarrow\phi\phi$), we then have to distinguish between direct amplitudes $\mathfrak{Z}^{\rm dir}$ and exchange amplitudes $\mathfrak{Z}^{\rm exch}$. The physical amplitudes are given by the following combinations, generalizing Ref. 6 (see also Ref. 27):

$$\mathfrak{Z}_{t,t}=\mathfrak{Z}_{t,t}{}^{\rm dir}+3\mathfrak{Z}_{t,t}{}^{\rm exch} \tag{5.1}$$

(similarly, for $\mathfrak{Z}_{t,t'}$, $\mathfrak{Z}_{t,qu}$ and $\mathfrak{Z}_{t',qu}$) and

$$\mathfrak{Z}_{dd,dd}=\mathfrak{Z}_{dd,dd}{}^{\rm dir}+2\mathfrak{Z}_{dd,dd}{}^{\rm exch} \tag{5.2}$$

(similarly, for $\mathfrak{Z}_{\phi\phi,\phi\phi}$, $\mathfrak{Z}_{dd,\phi\phi}$ and $\mathfrak{Z}_{d\phi,d\phi}$). For the nucleon transfer reactions (for instance, $pt\to dd$), the symmetrized amplitude is given in terms of the unsymmetrized transfer amplitudes $\mathfrak{Z}^{\rm tr}$ as follows:

$$\mathfrak{Z}_{dd,t}(\mathbf{q}',\mathbf{q};E)=\sqrt{3}\,(\mathfrak{Z}_{dd,t}{}^{\rm tr}(\mathbf{q}',\mathbf{q};E)\pm\mathfrak{Z}_{dd,t}{}^{\rm tr}(-\mathbf{q}',\mathbf{q};E)), \tag{5.3}$$

and analogously for the other transfer reactions.

The sign in Eq. (5.3) corresponds to whether the final state (with the two bosons $d+d$ or $\phi+\phi$) has even or odd internal wave functions. With analogous definitions for the "potentials," we are finally led to the equation (see Fig. 6) (we indicate again the total spin and isospin by upper indices)

$$\mathfrak{Z}_{s,r}{}^{I,S}=\mathcal{V}_{s,r}{}^{I,S}-\sum_{v}\mathcal{V}_{s,v}{}^{I,S}t_v\mathfrak{Z}_{v,r}{}^{I,S}. \tag{5.4}$$

The various potentials and propagators are the following ones (with the momenta defined in Sec. 2):

(a) For $t+p\to t+p$ (and, of course, for $He^3+n\to He^3+n$ and $t+p\leftrightarrow He^3+n$) we have according to Fig. 7(a)

$$\mathcal{V}_{t,t}{}^{I,S}(\mathbf{q}_2,\mathbf{q}_1;E)=-\frac{27}{16\sqrt{2}}\sum_{n=d,\phi}\Lambda_{t,t;n}{}^{I,S}\chi_t{}^n({}_2\mathbf{q}_1;E-\mathbf{q}_2{}^2)\times t_n(E-{}_1\mathbf{q}_2{}^2-\mathbf{q}_1{}^2)\chi_t{}^n({}_1\mathbf{q}_2;E-\mathbf{q}_1{}^2). \tag{5.5}$$

Here, $\Lambda_{t,t;n}{}^{I,S}$ is the product of spin- and isospin-recoupling coefficients [compare the analogous coefficients in Eq. (4.2)]:

$$\Lambda_{t,t;n}{}^{I,S}=\langle(\tfrac{1}{2},s_n)s_t,\tfrac{1}{2},S\mid(s_n,\tfrac{1}{2})s_t,\tfrac{1}{2},S\rangle\times\langle(\tfrac{1}{2},i_n)i_t,\tfrac{1}{2},I\mid(i_n,\tfrac{1}{2})i_t,\tfrac{1}{2},I\rangle. \tag{5.6}$$

The momenta ${}_1\mathbf{q}_2$ and ${}_2\mathbf{q}_1$ are linear combinations of $\mathbf{q}_1$ and $\mathbf{q}_2$, which can easily be obtained from Eq. (2.1):

$${}_1\mathbf{q}_2=(1/2\sqrt{2})(3\mathbf{q}_2+\mathbf{q}_1),\qquad {}_2\mathbf{q}_1=(1/2\sqrt{2})(3\mathbf{q}_1+\mathbf{q}_2). \tag{5.7}$$

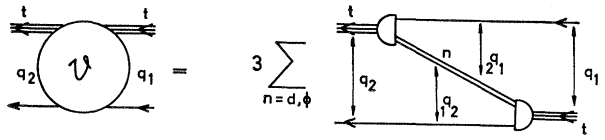
The factor[28] $27/16\sqrt{2}$ arises from our normalization of the momenta [compare Eq. (3.19) of Ref. 6].

Last, $\chi_t{}^n$ is just the form factor $\chi_1{}^n$, determined in Sec. 4:

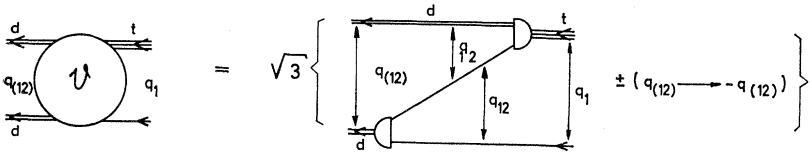
$$\chi_t{}^n(\mathbf{q};E)=C_1{}^n(E)(64\pi/3\sqrt{3})^{1/2}(\tfrac{5}{3}\alpha^2(E)+\mu_n{}^2)\times(\tfrac{8}{3}\alpha^2(E)-E)^{1/2}[(\tfrac{4}{3}\mathbf{q}^2+\tfrac{1}{3}\alpha^2(E)+\mu_n{}^2)\times(\tfrac{1}{3}\mathbf{q}^2+\tfrac{4}{3}\alpha^2(E)+\mu_n{}^2)(\tfrac{4}{3}\mathbf{q}^2+\tfrac{4}{3}\alpha^2(E)-E)]^{-1}. \tag{5.8}$$

[27] M. Golderger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

[28] There appears, in fact, an additional factor 3 from symmetrization. This cancels, however, against a factor $\frac{1}{3}$ since, when the nucleons are considered as distinguishable, the triton form factor is $(1/\sqrt{3})\chi_t{}^n$, instead of $\chi_t{}^n$.



(a)



(b)

FIG. 7. The potentials $\mathcal{V}_{t,t}$ [Fig. 7(a)] and $\mathcal{V}_{dd,t}$ [Fig. 7(b)] as defined in Eqs. (5.5) and (6.8), respectively.

The potentials for t replaced by t' or the quartet quasiparticle (qu) are completely analogous.

(b) The propagators for the states t, t', and qu are again obtained from the results of Sec. 3 [see Eq. (3.5)]:

$$t_t(E)=-\eta_1(E)/[1+\eta_1(E)],$$
$$t_{t'}(E)=-\eta_2(E)/[1+\eta_2(E)], \tag{5.9}$$

and analogously for t_{qu}.

Indeed, we did not use Eq. (5.9) as it stands, but multiplied by a constant factor of 0.9. This has the effect of shifting the three-nucleon binding energy from 10.3 to about 8 MeV, and should compensate for part of the short-range repulsion not included in our nucleon-nucleon interaction.

(c) The propagators for $d+d$ and for $\phi+\phi$, as well as the potentials describing transitions from $t+N$, $t'+N$, or $qu+N$ to these states, are not obtainable from the results of Sec. 4. Their determination is, therefore, left to Sec. 6.

6. TWO-DEUTERON (TWO-ϕ) INTERMEDIATE STATES

In Ref. 7, we have pointed out that in addition to the three-nucleon subsystems, we must also consider the subsystems with two pairs of mutually interacting nucleons.[29] This situation formally results from switching off all nucleon-nucleon potentials except, e.g., V_{12} and V_{34}. Similarly to the treatment of Sec. 4, we must now study the (off-shell) "transition amplitudes" $T_{(12)n,(34)m}$ and $T_{(12)n,(12)m}$ describing the processes shown in Fig. 8. As discussed in Ref. 7 (Sec. 4), separable approximations of these amplitudes provide us with the two-deuteron (two-ϕ) propagators and the form factors necessary for the calculation of the corresponding transition potentials. The amplitudes $T_{\beta n,\alpha m}$, with $\alpha=(12)$ or (34) and $\beta=(12)$ or (34), fulfill the integral equations analogous to (4.1),[29,30]

$$T_{\beta n,\alpha m}=V_{\beta n,\alpha m}-\sum_{\gamma=(12),(34)}\sum_{l=d,\phi}V_{\beta n,\gamma l}t_{\gamma l}T_{\gamma l,\alpha m}, \tag{6.1}$$

where (compare Fig. 9)

$$\begin{aligned}\langle \mathbf{q}_{34},\mathbf{q}_{(34)}' \mid V_{(12)n,(34)m}(E)\mid \mathbf{q}_{12},\mathbf{q}_{(34)}\rangle &=\delta^3(\mathbf{q}_{(34)}'-\mathbf{q}_{(34)})g_n(\mathbf{q}_{12})\\ &\quad\times[\mathbf{q}_{12}^2+\mathbf{q}_{34}^2+\mathbf{q}_{(34)}^2-E]^{-1}g_m(\mathbf{q}_{34})\\ &=\delta^3(\mathbf{q}_{(34)}'-\mathbf{q}_{(34)})V_{(12)n,(34)m}(\mathbf{q}_{34},\mathbf{q}_{12};E-\mathbf{q}_{(34)}^2)\end{aligned} \tag{6.2}$$

and

$$V_{\alpha n,\alpha m}=0.$$

The Pauli principle is included as in Sec. 5, but now the symmetrized $T_{n,m}$ and $V_{n,m}$ are given by

$$T_{n,m}=T_{(12)n,(34)m}+T_{(12)n,(12)m}, \tag{6.3}$$

$$V_{n,m}=V_{(12)n,(34)m}. \tag{6.4}$$

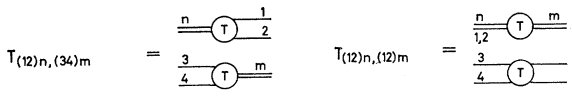


FIG. 8. The "transition amplitudes" $T_{(12)n,(34)m}$ and $T_{(12)n,(12)m}$.

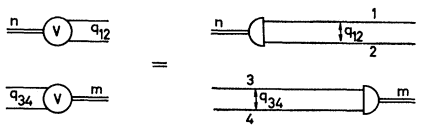


FIG. 9. The "potential" given by Eq. (6.2).

[29] For the moment we again consider the nucleons as distinguishable.

[30] In Ref. 7, we gave an explicit (formal) solution for $T_{\beta n,\alpha m}$. For our purpose, however, it is more practical to proceed in analogy to Sec. 4.

With this, Eq. (6.1) reduces to

$$T_{n,m}{}^{I,S}(E)=V_{n,m}{}^{I,S}(E) - \sum_l V_{n,l}{}^{I,S}(E)t_l(E)T_{l,m}{}^{I,S}(E). \quad (6.5)$$

The indices I and S on the potentials $V_{n,m}{}^{I,S}$ are superfluous and serve here only to fix the possible states n, m. [There are, of course, no recoupling coefficients as in the analogous three-nucleon equation (4.1)]. It is important to note that there is always only one possible intermediate state l for each spin and isospin: for $I=S=1$, only $V_{d\phi}$ and $V_{\phi d}$ are different from zero, while for all other combinations only the diagonal elements V_{dd} and $V_{\phi\phi}$ do not vanish.

Another important point is that $V_{nm}(E)$, as given by (6.2), is independent of the angle between $\mathbf{q}_{34}$ and $\mathbf{q}_{12}$. Thus, having only an S wave, a partial-wave decomposition is not necessary. Furthermore, the only nonseparable factor is $[\mathbf{q}_{12}{}^2+\mathbf{q}_{34}{}^2+\mathbf{q}_{(34)}{}^2-E]^{-1}$. For these reasons, a good separable approximation can now be found very easily. According to our general ansatz (A9) we write[31]

$$V_{n,m}{}^{I,S;\mathrm{sep}}(\mathbf{q}',\mathbf{q};E)=\xi_{nm}(E)[g_m(\mathbf{q}')/(\mathbf{q}'^2+\beta^2-E)] \times(2\beta^2-E)[g_n(\mathbf{q})/(\mathbf{q}^2+\beta^2-E)]. \quad (6.6)$$

In the variational determination of $\xi_{nm}(E)$ and $\beta(E)$, all elements of $\xi_{nm}(E)$ were equal, to a sufficient degree of accuracy, to a single function $\zeta(E)$: $\xi_{nm}(E)\sim\zeta(E)$ for all n, m. The values for $\zeta(E)$ are plotted together with the analogous quantity $\xi_{qu}(E)$ of the three-nucleon problem in Fig. 4, while $\beta^2(E)$ is shown, together with the function $\alpha^2(E)$, in Fig. 3. The eigenvalues $\eta_{dd}(E)$ and $\eta_{\phi\phi}(E)$ are shown in Fig. 10. The function $\zeta(E)$ is very close to unity, which, according to our discussion in Appendix B, is a first indication for the quality of our approximation. A stronger argument is that the resulting deuteron+deuteron-propagator has a pole at -4.35 MeV, which is very close to twice the deuteron energy ($2B_d=4.446$ MeV). Moreover, it appears [see Fig. 5(c)] that the approximation is excellent in the sense of the Schmidt norm.

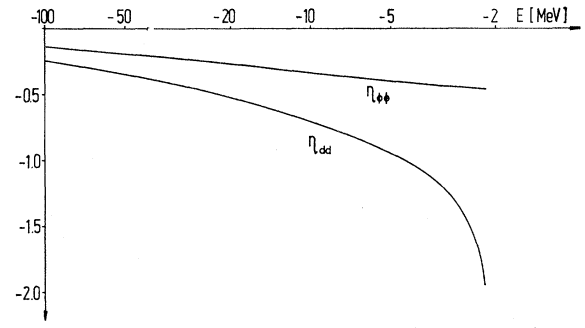


FIG. 10. The eigenvalues η_{dd} and $\eta_{\phi\phi}$ of the kernel of Eq. (6.1). The energy E at which $\eta_{dd}(E)=-1$ corresponds to twice the deuteron binding energy.

[31] Indeed, this goes over to (A9), if we write

$$\chi^{m;I,S}(\mathbf{q};E)=g_n(\mathbf{q})/(\mathbf{q}^2+\beta^2-E).$$

(Note here the different indices m and n). Due to the above discussion this defines $\chi^{m;I,S}$ uniquely, since, for fixed I and S, only one index n for each m leads to a nonvanishing potential. This fact establishes a one-to-one correspondence between n and m.

We are now in a position to write down the potentials and propagators not yet obtained in Sec. 5 (see paragraph c). The two-deuteron propagator is

$$t_{dd}(E)=-\eta_{dd}(E)/[1+\eta_{dd}(E)]. \quad (6.7)$$

The $\phi+\phi$ and $\phi+d$ propagators have the same form, but with η_{dd} replaced by $\eta_{\phi\phi}$ and $\eta_{d\phi}$, respectively. The potentials for $t+N\to d+d$ (and also for the processes with ϕ instead of d, and t' or qu instead of t) are constructed similarly to Eq. (5.5) [compare Fig. 7(b)]:

$$\mathcal{U}_{t,dd}{}^{I,S}(\mathbf{q}_1,\mathbf{q}_{(12)};E)=-\sqrt{2}^{-1}\Lambda_{t,dd}{}^{I,S}(\tfrac{27}{8})^{1/2} \times[\chi_t{}^d({}_1\mathbf{q}_2;E-\mathbf{q}_1{}^2)t_d(E-\mathbf{q}_1{}^2-{}_1\mathbf{q}_2{}^2)\chi_{dd}{}^d(\mathbf{q}_{12};E-\mathbf{q}_{(12)}{}^2) +(-)^S\chi_t{}^d({}_1\mathbf{q}_2';E-\mathbf{q}_1{}^2)t_d(E-\mathbf{q}_1{}^2-{}_1\mathbf{q}_2'{}^2) \times\chi_{dd}{}^d(\mathbf{q}_{12}';E-\mathbf{q}_{(12)}{}^2)]. \quad (6.8)$$

Here, $\chi_{dd}{}^d$ is the form factor defined by Eq. (6.6):

$$\chi_{dd}{}^d(\mathbf{q};E)=\left(\frac{\zeta(E)}{\eta(E)}\right)^{1/2}\frac{(2\beta^2(E)-E)^{1/2}}{(\mathbf{q}^2+\beta^2(E)-E)(\mathbf{q}^2+\mu_d{}^2)}. \quad (6.9)$$

$\Lambda_{t,dd}{}^{I,S}$ is again the spin-isospin recoupling coefficient, similar to (5.6). The factor of $(\frac{27}{8})^{1/2}$ arises from the normalization of the momenta, and the $1/\sqrt{2}$ stems from symmetrization.[28] The momenta ${}_1\mathbf{q}_2$ and $\mathbf{q}_{12}$ are linear combinations of $\mathbf{q}_1$ and $\mathbf{q}_{(12)}$:

$$ {}_1\mathbf{q}_2=(1/\sqrt{2})(\mathbf{q}_1+\sqrt{3}\mathbf{q}_{(12)}),$$
$$\mathbf{q}_{12}=-(1/\sqrt{2})(\mathbf{q}_{(12)}+\sqrt{3}\mathbf{q}_1), \quad (6.10)$$

while ${}_1\mathbf{q}_2'$ and $\mathbf{q}_{12}'$ are the same combinations, with $\mathbf{q}_{(12)}$ replaced by $-\mathbf{q}_{(12)}$.

7. FOUR-NUCLEON BOUND-STATE PROBLEM

Our last task is to solve the Lippmann-Schwinger-type equation (5.4), decomposed into partial waves, on a computer. Since the bound-state problem is the easier one, we have devoted our main interest to it and are going to describe it first. Instead of solving the Lippmann-Schwinger equation by matrix inversion, we use again the variational principle as presented in Sec. 3 (Appendix A), and already applied in Secs. 4 and 6. We start with the discussion of the ground state of He^4, i.e., with the channel where isospin, spin, and orbital angular momentum are all zero. For simplicity, all trial functions needed in the variational treatment, i.e., the form factors corresponding to the vertices (α, nt), (α, nqu), (α, nt'), (α, dd), and $(\alpha, \phi\phi)$, are chosen identical. This is consistent with the correct behavior as $|\mathbf{q}|\to 0$, where they tend towards constants. At infinity, arguments similar to those leading to

(3.8) show that the form factors for (α, nt), (α, nt'), and (α, nqu) behave like q^{-11}, while the ones for (α, dd) and $(\alpha, \phi\phi)$ go like q^{-9}. This fact, and the explicit form of the potentials, suggested the ansatz

$$\chi(\mathbf{q}; E) \propto \{[\sigma(E)+3\mathbf{q}^2]^2[\sigma(E)+\mathbf{q}^2]^3\}^{-1} \qquad (7.1)$$

for the form factors. From the variational method, we found the binding energy of the α particle to be

$$B_\alpha = 50 \text{ MeV}. \qquad (7.2)$$

This result is obviously much too large (even worse, it represents only a lower limit to the exact solution). Such a failure can be understood by the fact that we have not taken into account the hard core and tensor forces in our nucleon-nucleon potentials (2.3). Moreover, this potential was constructed to fit the low-energy data while it is needed here at large negative energies. A better calculation should use more sophisticated potentials as proposed, e.g., in Refs. 18 and 19.

Nevertheless, we can draw some qualitative conclusions from our solution by looking at the matrix $N(E)$. According to its definition in Appendix A, this matrix is essentially the expectation value of the potential. Neglecting the t' would change our result for B_α by only a few percent. The major contribution is given by the $t+n$ channel, via deuteron and ϕ exchange. The channels $d+d$ and $\phi+\phi$ coupled to $t+n$ by one-nucleon exchange are of minor importance for the result.

A better test of the model is presumably to look at the first excited states of the four-nucleon system. Experimentally,[32] one finds several states with excitation energies between 20–30 MeV. The lowest one is a $J^P=0^+$ isosinglet state which should correspond to a radial excitation of the ground state. We have not looked for it because our program was not well-suited for that purpose. The next states, with "binding" energies 6.9 and 5.9 MeV, are again isosinglets with $J^P=0^-$ and 2^-, respectively. Together with a $J^P=1^-$ state found recently[33] at 4.4 MeV (i.e., only 0.05 MeV above the two-deuteron threshold), they should in our model correspond to states with $L=1$, $S=1$, $I=0$. Furthermore since we do not have any LS coupling, these states must be degenerate.

We took again the same form factor for all channels (where, now, the fourth channel contains a nucleon plus a quartet state, instead of two ϕ's):

$$\chi_m(\mathbf{q}; E) \propto Y_1^m(\mathbf{q}/q)q/\{[\rho(E)+q^2]^3[\rho(E)+3q^2]^3\}, \qquad (7.3)$$

which is easily seen to have the correct threshold and asymptotic behavior.

We found indeed a state at $\sim$12 MeV. This is already closer to the experimental values, although one still should be careful. In channels with total orbital angular momentum $L=1$, we expect that the neglect of three-nucleon P waves is much worse than in channels with $L=0$.

8. SCATTERING PROBLEM

The virtue of the above formulation is, of course, that it can be applied to scattering processes as well. However, the difficulties encountered in the actual calculation increase considerably there. Even the determination of the various subsystem form factors requires more effort.

Therefore, in this paper we resorted to a K-matrix Born approximation to get a first crude insight into the results to be expected. This approximation is obtained by replacing the triton-nucleon and the deuteron-deuteron propagators in Eq. (5.4) by their discontinuities along the two-particle cuts:

$$t_t(E-\mathbf{q}^2) \to i\pi R_t \delta(B_t+E-\mathbf{q}^2) \qquad (8.1)$$

and

$$t_{dd}(E-\mathbf{q}^2) \to i\pi R_{dd} \delta(2B_d+E-\mathbf{q}^2). \qquad (8.2)$$

Here, R_t and R_{dd} are the residues of $t_t(E)$ and $t_{dd}(E)$, and are given by $R_t=22.8$, $R_{dd}=9.2$. We note that these residues also enter the connection between our T matrices and the cross section [just as in the three-nucleon case, where the residue of t_d occurred in Eq. (4.9)]. If we take into account also the particular normalization of our momenta, we find the following (neglecting trivial complications due to isospin) for

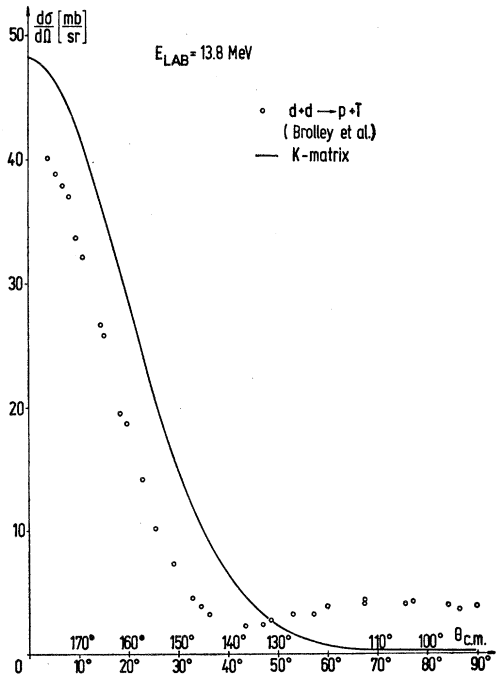


FIG. 11. Differential cross section for $d+d \to t+p$, at $E_{\text{lab}}=$ 13.8 MeV. Experimental points are from Ref. 35, the curve is the K-matrix approximation to Eq. (5.4).

[32] W. E. Meyerhof and T. A. Tombrello, Nucl. Phys. **A109**, 1 (1968).
[33] D. Fick and H. W. Franz, Phys. Letters **27B**, 541 (1968).

unpolarized particles:

$$\left(\frac{d\sigma_{n\to m}}{d\Omega}\right)_{\text{c.m.}} = \frac{1}{4(4\mu_n\mu_m)^{1/2}}\frac{q'}{q}(2\pi)^4 \mid (R_nR_m)^{1/2}\mathfrak{I}_{nm}(\mathbf{q}', \mathbf{q}) \mid^2, \quad (8.3)$$

where μ_n and μ_m are the reduced masses.

By such a replacement, all other channels which do not correspond to physical particles are eliminated. Furthermore, from the δ functions we see that we

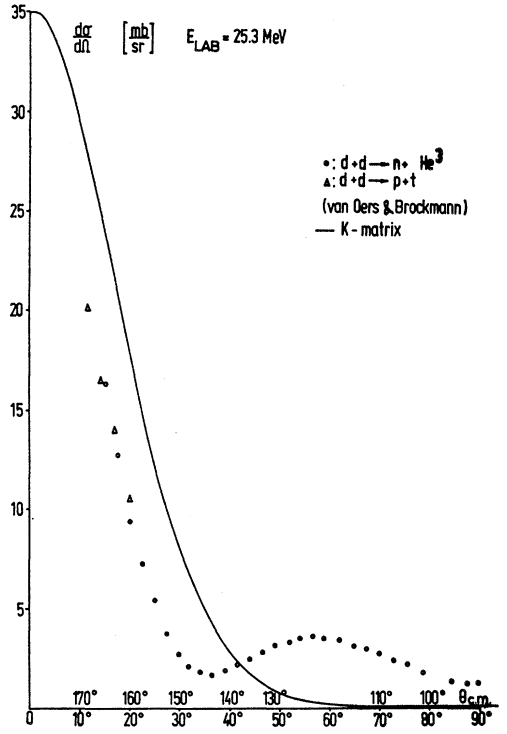


FIG. 12. The same as Fig. 11 but at $E_{\text{lab}}=25.3$ MeV. Experimental points from Ref. 36.

need the form factors only at the energy of their corresponding bound-states. That is, we need only $\chi_t{}^n(\mathbf{q}; E-\mathbf{q}^2=-B_t)$ and $\chi_{dd}{}^d(\mathbf{q}; E-\mathbf{q}^2=-2B_d)$, which we have already determined.

The replacements (8.1) and (8.2) lead to the same results as approximating the "two-body" K matrix by the potentials (5.5) and (6.8). By such an approximation we can, of course, not expect to get reasonable results for processes where the Born term vanishes. This is the case for elastic deuteron-deuteron scattering. A similarly bad situation holds for elastic triton-nucleon scattering where the Born term contributes only to backward scattering. In both of these cases, the inclusion of the second Born approximation to the K matrix[34] should yield considerable improvement. The only process where our transition potential may

[34] J. H. Sloan, Phys. Rev. **165**, 1587 (1968).

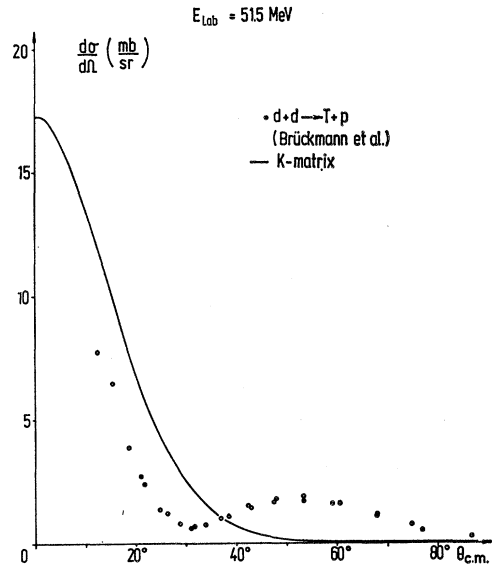


FIG. 13. The same as Fig. 11 but at $E_{\text{lab}}=51.5$ MeV. Experimental points from Ref. 37.

be expected to be a good starting point is $d+d\to t+p(\to\text{He}^3+n)$. Therefore, we can hope that there our K-matrix approach is reasonable. Indeed, this is verified by our results, as shown in Figs. 11–14.[35–38]

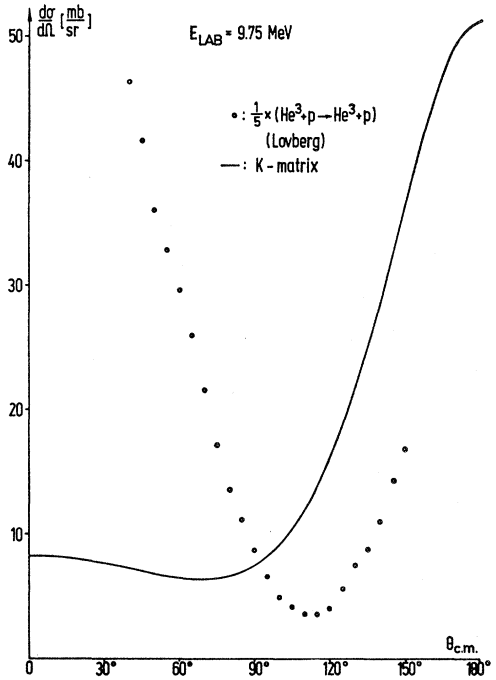


FIG. 14. Differential cross section for $\text{He}^3+p\to\text{He}^3+p$ at $E_{\text{lab}}=9.75$ MeV. Experimental points from Ref. 38.

[35] J. E. Brolley, Jr., T. M. Putnam, and L. Rosen, Phys. Rev. **107**, 820 (1957).

[36] W. T. H. Van Oers and K. W. Brockmann, Nucl. Phys. **48**, 625 (1963).

[37] H. Brückmann, E. L. Haase, W. Kluge, and L. Schänzler, Z. Physik **230**, 383 (1970).

[38] R. H. Lovberg, Phys. Rev. **103**, 1393 (1957).

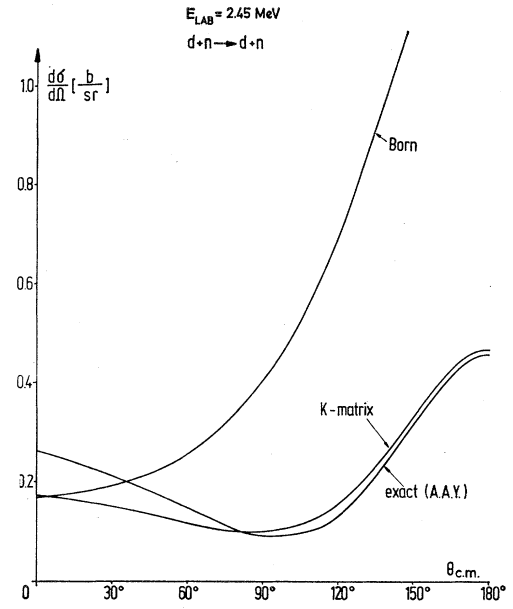


FIG. 15. Differential cross section for $d+n\rightarrow d+n$, at $E_{\rm lab}=2.45$ MeV. Shown are the exact solution of Eq. (4.1) taken from Ref. 4, the Born approximation to Eq. (4.1), and the K-matrix approximation obtained by replacing $t_d(E)$ by $i\pi R_d\delta(E+B_d)$.

Furthermore, we remark that the Born approximation to $d+d\rightarrow t+n$ is, in general, much too big, that is, it fails completely to describe the experiments.

To get some feeling for the validity of this K-matrix approach, we have applied it also to the three-nucleon problem as represented by Eq. (4.1). For n-d scattering, we can compare it with the exact solution of Aaron, Amado, and Yam.[4] This comparison is shown in Figs. 15 and 16. We find here again that the K matrix is a considerable improvement over the Born approximation. Furthermore, it is not too far from the exact values, especially near the backward direction where the Born term is maximal, and for low energies. These results support the arguments given above for the four-nucleon problem.

9. CONCLUSIONS

It was the main purpose of this paper to demonstrate in the four-nucleon system that the formalism proposed in Ref. 7 for the treatment of the few-particle problem is indeed a practical tool for calculations. Moreover, we wanted to test the validity of the various approximations involved. In particular, we found that the use of variational methods is essential to make the method really effective in this special problem.

Most effort was devoted to the problem of obtaining the integral equation (5.4) for triton-proton and deuteron-deuteron direct and exchange scattering, in a form amenable to modern computer facilities. The approximations necessary to arrive at it were extensively investigated. The crucial problem was to show that the effective potentials occurring in the three-nucleon Lippmann-Schwinger equation, derived with separable two-nucleon potentials, could again be approximated by only a few separable terms (it was just to keep the number of the latter as small as possible that we made use of variational methods). Such a result is not only decisive for our formulation of the four-nucleon problem, but may also be useful to simplify calculations in the three-nucleon system.

To solve the four-nucleon scattering equations, we have made rather drastic approximations, including a K-matrix approximation. Therefore, these results should be regarded only as a first step towards an exact solution. The bound-state problem has been studied more extensively.

Now, to judge the somewhat poor quality of our numerical results, two points should be kept in mind. First, these calculations are intended to provide a first glimpse at the four-nucleon problem. Therefore, rather crude approximations have been made. In fact, we were more concerned with the problem of establishing the effective two-particle Lippmann-Schwinger equations for this case than with their solution. The second and more important point is that in our formulation no open parameters exist except those of the nucleon-nucleon potential, and no uncontrollable approximations have been made apart from the final K-matrix approximation which, however, could be avoided by more computational effort. In the light of these facts,

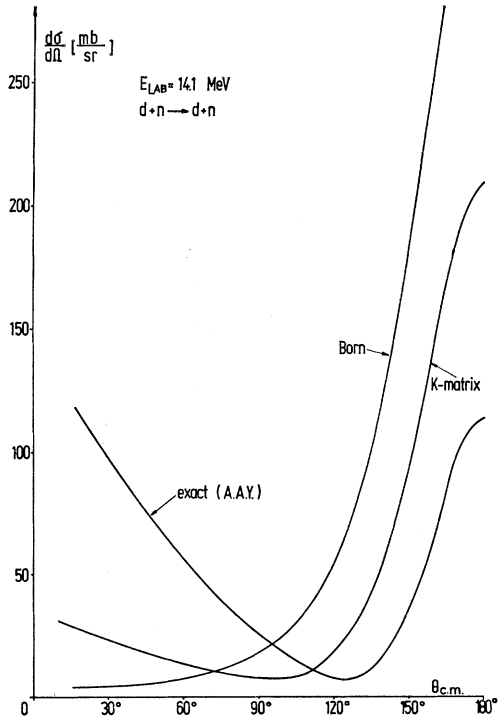


FIG. 16. The same as Fig. 15, but at $E_{\rm lab}=14.1$ MeV.

our results are encouraging and deserve further investigations along these lines.

ACKNOWLEDGMENTS

For encouraging interest we are indebted to Professor H. Rollnik. Generous grants of computing time on the IBM 7090 computer of the Institut für Instrumentelle Mathematik at Bonn are gratefully acknowledged.

APPENDIX A

In the case where the Lippmann-Schwinger equation is a matrix equation, we modified the variational method described in Sec. 3 as follows: The separable ansatz (3.1) reads, with matrix indices,

$$V_{il}^{\rm sep}(E)=\sum_{\nu}|\,\chi_\nu^{\,i}(E)\rangle\eta_\nu(E)\langle\chi_\nu^{\,l}(E^*)\,|. \quad \text{(A1)}$$

For the form factors, we make the ansatz

$$|\,\chi_\nu^{\,l}(E)\rangle=C_\nu^{\,l}(E)\,|\,\hat{\chi}_\nu^{\,l}(E)\rangle \quad \text{(no summation)} \quad \text{(A2)}$$

with the $C_\nu^{\,l}(E)$ considered as variation parameters. The $|\,\hat{\chi}_\nu^{\,l}(E)\rangle$ are also varied independently but with only *one* parameter $\alpha(E)$ for each ν and all l. The ansatz (A2) provides us with enough flexibility without making the calculations too complicated. Inserting it into Eq. (3.6), we obtain

$$\sum_{i,l}(N_{il^\nu}-[\eta_\nu]M_{il^\nu})C_\nu^{\,i*}C_\nu^{\,l}=0 \quad \text{(A3)}$$

with

$$N_{il^\nu}=\langle\hat{\chi}_\nu^{\,i}\,|\,G_0V_{il}G_0\,|\,\hat{\chi}_\nu^{\,l}\rangle \quad \text{(A4)}$$

and

$$M_{il^\nu}=\langle\hat{\chi}_\nu^{\,i}\,|\,G_0\,|\,\hat{\chi}_\nu^{\,i}\rangle\delta_{il}. \quad \text{(A5)}$$

The condition

$$\delta[\eta_\nu]=0 \quad \text{(A6)}$$

yields, under variation of $C_\nu^{\,i*}$,

$$\sum_{l}(N_{il^\nu}-[\eta_\nu]M_{il^\nu})C_\nu^{\,l}=0, \quad \text{(A7)}$$

which has a solution if

$$\det(N^\nu-[\eta_\nu]M^\nu)=0. \quad \text{(A8)}$$

If N and M are $n\times n$ matrices, we have in general n different roots for $[\eta_\nu]$. Each one corresponds to a different (exact) solution of Eq. (A7), and to a different *approximate* solution of the Lippmann-Schwinger equation (3.2). By variation of $|\,\hat{\chi}_\nu^{\,l}\rangle$ [i.e., of the parameter $\alpha(E)$], one can only arrange *one* of them to become a (nearly) exact solution of Eq. (3.2). We took the largest of these $[\eta_\nu]$ and made it a maximum by variation. This yields a lower bound on the largest eigenvalue of VG_0 and on the binding energy of the ground state.

When constructing a separable approximation to the potentials, we found, fortunately, that in our application (*nd* scattering in the singlet state; here we deal with a 2×2 matrix) the second eigenvector of (A7) leads also to an approximate solution of (3.2), with similar accuracy.

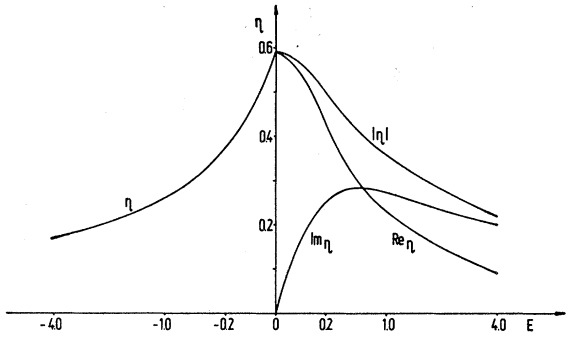


FIG. 17. The largest Yukawa eigenvalue $\eta(E+i0)$, as given by Eq. (3.6). Here, $\mu=m=\lambda=1$.

Therefore, we get in this case

$$V_{kl}^{\rm sep}(E)=\sum_{\nu=1,2}|\,\hat{\chi}^k(E)\rangle C_\nu^{\,k}(E)C_\nu^{\,l*}(E^*)\eta_\nu(E)\langle\hat{\chi}^l(E^*)\,|$$
$$=|\,\hat{\chi}^k(E)\rangle\xi_{kl}(E)\langle\hat{\chi}^l(E^*)\,| \quad \text{(A9)}$$

with

$$\xi_{kl}(E)=\sum_{\nu=1,2}C_\nu^{\,k}(E)C_\nu^{\,l*}(E^*)\eta_\nu(E). \quad \text{(A10)}$$

APPENDIX B

The quality of the separable ansatz, suggested by Eq. (3.10), can be illustrated rather directly for the s-wave part of the Yukawa potential $V(r)=\lambda(e^{-\mu r}/r)$:

$$V_0(p',p)=\frac{\lambda}{\pi}\int_{-1}^{+1}\frac{d\cos\theta}{p^2+p'^2-2pp'\cos\theta+\mu^2} \quad \text{(B1)}$$

$$=\frac{\lambda}{2\pi pp'}\ln\frac{\mu^2+(p+p')^2}{\mu^2+(p-p')^2}. \quad \text{(B2)}$$

For p and $p'\lesssim\mu$,

$$V_0(p',p)\cong\frac{2\lambda/\pi}{p^2+p'^2+\mu^2} \quad \text{(B3)}$$

represents a good approximation. With this, Eq. (3.10) simplifies to

$$V_0^{\rm sep}(p',p;E)$$
$$=\xi(E)\,\frac{2\lambda}{\pi}\,\frac{2\alpha^2+\mu^2}{[p'^2+\mu^2+\alpha^2(E)][p^2+\mu^2+\alpha^2(E)]}, \quad \text{(B4)}$$

corresponding to the form factor

$$\chi(p;E)\propto[p^2+\mu^2+\alpha^2(E)]^{-1}=[p^2+\beta^2(E)]^{-1}. \quad \text{(B5)}$$

This is, of course, the well-known Hulthén form factor (with energy-dependent range $1/\beta$) which is the form factor most widely used in separable-potential calculations. When we insert the ansatz (B4) in Eq. (3.7), we can do all integrations analytically.[39] The denomi-

[39] Wright and Scadron (Ref. 14) did the same using Eq. (3.6), which leads, however, to more complicated integrals.

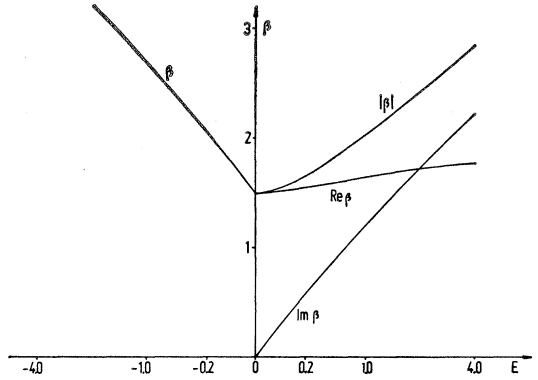


FIG. 18. The parameter $\beta(E+i0)$, as given by Eq. (B.7); again, $\mu=m=\lambda=1$.

nator is calculated in **x** space, making use of the locality of $V(r)$ to form $V^{-1}(r)$. We thus obtain

$$[\eta(E)]=\lambda(2\beta-\mu)^2/2\beta(\beta-iE^{1/2})^2. \quad \text{(B6)}$$

This becomes a maximum if

$$\beta(E)=\tfrac{3}{4}\mu-\tfrac{1}{2}(iE^{1/2})+\tfrac{1}{2}(9\mu^2/4-5i\mu E^{1/2}-E)^{1/2}. \quad \text{(B7)}$$

$\eta(E)$ and $\beta(E)$ are shown in Figs. 17 and 18. There we see that the imaginary parts both of β and of η soon become very important if one goes to positive energies. Looking at $E=0$, and using (B4) and (3.3), we find $\xi(0)=8/7$, which is indeed rather close to unity, as was suggested by the arguments leading to our ansatz (3.10). At this energy ($E=0$), we obtained the Schmidt norms (see also Ref. 9)

$$\tau(0)=|G_0^{1/2}(0)V_0(0)G_0^{1/2}(0)|=\lambda(2\ln 2-1)^{1/2}=0.6215\lambda$$

and

$$\tau'(0)=|G_0^{1/2}(0)V_0'(0)G_0^{1/2}(0)|$$
$$=\{\tau^2(0)-2\langle\chi|G_0(0)VG_0(0)|\chi\rangle+[\eta(0)]^2\}^{1/2}.$$

The term $\langle\chi|G_0V_0G_0|\chi\rangle$, calculated in configuration space,[40] is

$$\langle\chi|G_0(0)VG_0(0)|\chi\rangle=\frac{2\lambda\mu}{\beta(0)}[\eta(0)]\ln\frac{(\mu+\beta)^2}{(\mu+2\beta)\mu},$$

which gives

$$\tau'(0)=0.1795\lambda.$$

The work of Weinberg and collaborators[9,40] shows that this is sufficient to make V^{sep} a very good approximation to V, if the coupling constant λ is not too large. Since in the three-nucleon problem a similar weakening of the potentials is achieved by subtraction of separable terms (see the results given in Sec. 4), we may expect that these separable terms represent a good approximation for our problem.

[40] M. Scadron and S. Weinberg, Phys. Rev. **133**, B1589 (1964).

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New Statistical Theory of the Nuclear Surface. I*

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The Thomas-Fermi integral theory of the nuclear surface is shown to have singular solutions for the nuclear density. This failure is traced to its inadequate representation of the behavior of wave functions in a potential. A new theory based on approximate wave functions is developed and shown to be asymptotic to the Thomas-Fermi theory in the interior of the nucleus. An improved treatment of exchange forces is shown to be essential to obtaining realistic solutions.

I. INTRODUCTION

PREVIOUS statistical theories[1–3] of the nuclear surface have been formulated as variational problems in which the total energy W_T of a nucleus is minimized as a functional of its neutron and proton density distributions $\rho_n(\mathbf{r})$ and $\rho_p(\mathbf{r})$. The most thorough formulation of the problem is that of Bethe,[1] who writes W_T as an integral of the long-range direct force between nucleons, plus local-density approximations to the short-range force, the space-exchange integral, and the kinetic energy. Buchler *et al.*[2] approximate the long-range direct-force integral by a differential term, a fair approximation if the surface thickness is not too small; such a term includes the first-order correction of Weiszacker for the additional kinetic energy necessary to cause a varying density. In Sec. II, it is shown that the integral theory of Bethe can have discontinuous solutions. Such a solution is obtained numerically for the one-dimensional case. This singularity demonstrates that the surface thickness is due not only to the properties of the long-range direct nucleon-nucleon force, as argued by Wilets[3] and

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† National Science Foundation Graduate Fellow.
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