calculation. This contrasts with the usual bound-state models.

In electron scattering we calculated the transverse form factors for 1^- and 2^- states because these form factors show the importance of magnetic effects. These results were compared with experiment, and the predicted levels are generally seen. The integrated form factors that can be derived from the energy spectra show reasonably good agreement with experiment. We have also seen that normalizing certain form factors by the ratio of experimental to theoretically predicted integrated photoabsorption cross sections improves the results, again pointing out the importance of $SU(4)$ to the nuclear physics of the giant resonances. The widths we predict for our states are in qualitative agreement with experiment. This model also justifies previous identification of narrow giant quadrupole states predicted by deForest. We have also investigated why the high-lying magnetic 1^- state does not become huge at large momentum transfers, although shell-model calculations, including this one, predict a vanishing of the giant dipole resonance for momentum transfers $q \approx 120$ MeV/c due to interference between the electric and magnetic parts of the 1^- transverse operator. This is not seen experimentally.

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Structure of the Nucleon-Nucleus Scattering Matrix in the Random-Phase Approximation*

J. N. Ginocchio, T. H. Schucan,† and H. A. Weidenmüller‡
Physics Department, Yale University, New Haven, Connecticut 06520

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The scattering matrix is derived for the scattering of nucleons by nuclei lacking one nucleon from being doubly magic. It is assumed that an average field has been determined through a Hartree-Fock procedure (HF). The residual interaction is treated in the random-phase approximation (RPA). In contrast to previous treatments, it is not assumed that this interaction is separable. The RPA ground state of the compound system is given by a correlated wave function $|\Psi_0\rangle$. It is assumed that states of the target and residual nucleus can be described as one-hole states in this correlated ground state $|\Psi_0\rangle$. It is found that the RPA equations allow for a proper definition of asymptotic states only if the full Hamiltonian (including the c.m. energy) is used in the HF procedure. A general, yet explicit, expression for the S matrix is obtained by applying to the channel-channel part of the residual interaction a method first proposed by Weinberg. The correlations contained in $|\Psi_0\rangle$ give rise to poles of the scattering matrix for real negative energies *below* the energy of the lowest bound state, i.e., the ground state $|\Psi_0\rangle$. In the energy region of physical interest, these poles have two effects on the scattering matrix. First, a constant background term is introduced. Second, the partial widths $\Gamma_{\lambda c}$ for decay of a compound state (λ) into an open channel (c) are complex. The sum of the partial widths, $\Sigma_{\lambda c}\Gamma_{\lambda c}$, is compared with the sum of the total widths, $\Sigma_\lambda \Gamma_\lambda$. It is found that the two sums differ by terms of second order in the admixture of correlations in the ground state. The influence of symmetry properties of the Hamiltonian on the RPA solutions is discussed. It is shown that the scattering matrix derived is that in the c.m. frame, and it is completely independent of the total momentum of the nucleus.

I. INTRODUCTION

THE random-phase approximation (RPA) is a useful tool for the understanding of collective properties of nuclear levels.[1] It serves primarily as a useful model in which the occurrence of collective modes of motion can be theoretically understood on a microscopic basis. Compared with the ordinary shell-model [or Tamm-Dancoff (TD)] treatment, the RPA offers an improved understanding of the relationship between symmetry properties of the Hamiltonian and the occurrence of collective modes.[2] The RPA also yields a semiquantitative account of the positions and electromagnetic properties of vibrational states.[3]

Since the RPA has turned out to be such a useful

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† Present address: Institut für Theoretische Physik der Universität Heidelberg, Heidelberg, West Germany.

‡ On leave of absence from Heidelberg University and from the Max Planck Institut für Kernphysik, Heidelberg, Germany.

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model for collective bound states, it is of interest to study the nucleon-nucleus scattering problem within the same approximation. Since the RPA is the only microscopic model of collective motion, one may thereby hope to gain an understanding of the properties of collective compound-nuclear resonances. The ordinary TD treatment of this problem has been studied rather widely.[4] The properties of the nucleon-nucleus scattering matrix in the TDA are well understood. The RPA differs from the Tamm-Dancoff approximation (TDA) in that the ground state of the nuclear system is assumed to be a *correlated* wave function. Furthermore, the excited states of the nuclear system are assumed to be given by applying a boson operator to this correlated ground state. We shall see in this paper that these two assumptions yield a scattering matrix that is different in structure from the one obtained from the TDA. We believe that the study of this modification sheds an interesting light on the question to what extent the analytical properties of the scattering matrix are influenced by the ground-state correlations.

The extension of the RPA to the nucleon-nucleus scattering problem was studied previously by various authors.[5–7] In these papers, explicit solutions were obtained only for the case of a separable two-body interaction. This assumption is not made in the present paper; it is hoped that one thereby exhibits those properties of the scattering matrix that are inherent in the RPA and thus separates them from other properties which are due to the assumption of a separable approximation.

This paper is concerned with the general solution of the RPA equations in the continuum and with the properties of this solution. Aside from the question raised above, we investigate whether there is collective enhancement of individual partial widths of nuclear resonances; we establish sum rules for these widths; we investigate the role of the linear-momentum operator in the RPA and the occurrence of spurious modes of excitation. We also give attention to the question why the RPA equations can be solved without any real difficulty, despite the fact that the RPA states contain 1p-1h, 2p-2h, ··· states, i.e., contain several particles in the continuum, so that one might expect to run into the complexities of the three- and the many-body scattering problems.

The present study is limited to even-even doubly magic compound nuclei, and we take into account only nucleon channels.

In Sec. II, we give the RPA equations and discuss the assumptions made in deriving these equations. General properties of the S matrix are discussed in Sec. III. The content of Secs. II and III partly parallels that of Refs. 5 and 6. In Sec. IV, we construct the S matrix explicitly for the case of no channel-channel coupling, and we derive sum rules for the partial widths. The channel-channel coupling is explicitly included in the treatment in Sec. V, which follows closely the algebraic treatment developed earlier for the TDA. Section VI deals with the c.m. problem. Section VII contains the conclusions.

II. RPA EQUATIONS

When applied to a spherical nucleus in which the nucleons interact via a two-nucleon potential V, the Hartree-Fock procedure determines a spherically symmetric single-particle potential v_0. The single-particle Hamiltonian $h_0=t+v_0$ (with t denoting the operator of kinetic energy) has a finite number of bound states and a continuum of scattering states. The energies $\epsilon_{nlj}<0$ of the bound states are labeled by the principal quantum number n, the orbital angular momentum l, and the total spin j of the nucleon. The bound-state wave functions $(1/r)w_{nlj}{}^b(r)\mathcal{Y}_{lj}{}^q$ are normalized to unity. Here,

$$\mathcal{Y}_{lj}{}^q=i^l\sum_\mu C(l\tfrac{1}{2}j;q-\mu,\mu)Y_l{}^{q-\mu}(\theta,\phi)\chi_{1/2}{}^\mu, \qquad (1)$$

where $Y_l{}^{q-\mu}$ is a spherical harmonic and $\chi_{1/2}{}^\mu$ a nonrelativistic spin wave function. The scattering states $(1/r)u_{lj}(r,k)\mathcal{Y}_{lj}{}^q$ correspond to the energy $\hbar^2k^2/2M\geq 0$ (with $M=$nucleon mass). They are normalized to a δ function in energy. The regular radial wave functions $u_{lj}(r,k)$ are chosen to be real. They are characterized by a real phase shift $\delta_{lj}(k)$.

The Hartree-Fock approximation to the ground state is given by the Slater determinant $|\,\Phi_0\rangle$ obtained by distributing the A nucleons over the lowest A bound states of h_0. In the following, we restrict ourselves to nuclei that are doubly magic. Then, $|\,\Phi_0\rangle$ is uniquely defined, except for a phase. We define the total Hamiltonian $H=T+V$, with

$$T=\sum_{j=1}^{A}t(j).$$

The expectation value of H with respect to $|\,\Phi_0\rangle$ gives the Hartree-Fock approximation E^{HF} to the total binding energy,

$$E^{\mathrm{HF}}=\langle\Phi_0\,|\,H\,|\,\Phi_0\rangle$$
$$=\tfrac{1}{2}\langle\Phi_0\,|\,T\,|\,\Phi_0\rangle+\tfrac{1}{2}\sum_{l=1}^{A}\epsilon_l. \qquad (2)$$

Here and in the following, the index l extends over the A bound states occupied in $|\,\Phi_0\rangle$.

We define the creation operator $\bar{a}_l{}^\dagger$ for a hole in $|\,\Phi_0\rangle$ by the removal of a nucleon in one of the eigenstates of h_0, so that $\bar{a}_l\,|\,\Phi_0\rangle=0$. Deferring until later in this section a discussion of the physical significance of those eigenfunctions of h_0 that are not occupied in $|\,\Phi_0\rangle$, we define the creation operator $a_p{}^\dagger$ for a particle in an eigenstate of h_0 not occupied in $|\,\Phi_0\rangle$, so that $a_p\,|\,\Phi_0\rangle=0$. The index p extends both over discrete and continuum

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states. The shell-model Hamiltonian

$$H_0=\sum_{j=1}^{A} h_0(j)$$

is then written using second quantization,

$$H_0=-\sum_{l}\epsilon_l\bar{a}_l^{\dagger}\bar{a}_l+\sum_{p}\epsilon_p a_p^{\dagger}a_p+\sum_{l;\text{occupied}}\epsilon_l$$
$$\equiv\sum_{l;\text{occupied}}\epsilon_l+:H_0:. \tag{3}$$

The dots indicate that we take the product of the creation and annihilation operators in normal order with respect to the Hartree-Fock ground state $|\Phi_0\rangle$. The total Hamiltonian takes the form

$$H=E^{\mathrm{HF}}+:H_0:+:V:. \tag{4}$$

Here, $:V:$ is the normal form of the interaction V. It is the residual interaction of the Hartree-Fock procedure. Figure 1 lists the (antisymmetrized) matrix elements of V contributing to $:V:$, together with their diagrammatic representation. The graphs on the far right-hand side of the figure stand for the antisymmetrized matrix elements.

The simplest modes of excitation of the system are the particle-hole states $a_p^{\dagger}\bar{a}_l^{\dagger}|\Phi_0\rangle$. The usual 1p-1h

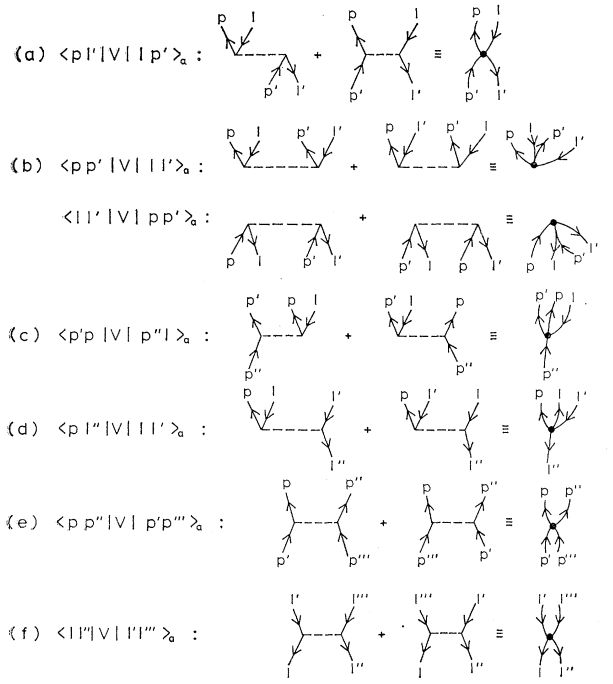

FIG. 1. Components of the residual interaction $:V:$ of the HF procedure. To the left is the antisymmetrized matrix element; the p's refer to particle states, the l's to hole states. In the center are the Feynman-Goldstone graphs corresponding to these matrix elements, both direct and exchange. The graph on the far right includes both direct and exchange matrix elements at the vertex. (a) "Forward-going" graphs included in both the TDA and RPA. (b) "Backward-going" graphs included only in the RPA, which give rise to correlations in the ground state. (c)–(f) Remaining parts of the residual interaction, which are ignored in both TDA and RPA.

shell model (or TDA) consists in diagonalizing H in the space of these states. The Hartree-Fock ground state $|\Phi_0\rangle$ is stationary with respect to such excitations, i.e., $\langle\Phi_0|Ha_p^{\dagger}\bar{a}_l^{\dagger}|\Phi_0\rangle=0$. Thus, $|\Phi_0\rangle$ remains the ground state of the TDA. Excited states are given as linear superpositions of 1p-1h states. The method can be extended to describe nucleon-nucleus scattering on targets approximated as hole states in $|\Phi_0\rangle$.

The TDA neglects all matrix elements of V except those shown under (a) in Fig. 1. It consists in summing over all 1p-1h intermediate states, as shown schematically in Fig. 2(a). The RPA goes one step further than the TDA. It includes the matrix elements of V shown in Fig. 1(b) as well as those shown in Fig. 1(a). Figure 2(b) demonstrates that the RPA includes "backward-going graphs" as well as those typical of the TDA. Inclusion of the matrix elements of Fig. 1(b) implies that the RPA ground state differs from the TDA ground state $|\Phi_0\rangle$. It contains 2p-2h, 4p-4h, $\cdots$ components built on the TDA ground state $|\Phi_0\rangle$. The excited states are described as 1p-1h excitations on this new *correlated* ground state.

An illustration of a graph not included in the RPA is given in Fig. 2(c). This graph is excluded because the RPA summation does not include the exchange of particle-hole lines, which do not begin from (or end at) the same vertex. In other words, the RPA summation keeps the bubbles intact. It has been shown[1] that retaining only bubble diagrams is equivalent to assuming that the particle-hole pairs $a_p^{\dagger}\bar{a}_l^{\dagger}$ are inert, i.e., that they behave like elementary particles. Since two creation operators for a particle-hole pair commute with each other, the RPA is equivalent to treating each 1p-1h pair as a boson. We therefore define boson operators $A^{\dagger}$, A as follows. We introduce the label $b\equiv(p,l)$ to denote a particle-hole pair where the particle is in a bound orbital; correspondingly, $c\equiv(p,l)$ stands for a particle-hole pair with the particle in a scattering eigenstate of h_0. Here and in the following, we pay no attention to angular momentum coupling, since we are only interested in the general structure of the RPA equations and their solutions. The boson operators are

$$(a_p^{\dagger}\bar{a}_l^{\dagger})\rightarrow A_b^{\dagger};\quad(\bar{a}_l a_p)\rightarrow A_b \qquad \text{if } p \text{ denotes a bound orbital,}$$
$$(a_p^{\dagger}\bar{a}_l^{\dagger})\rightarrow A_c^{\dagger}(E);\quad(\bar{a}_l a_p)\rightarrow A_c(E) \qquad \text{if } p \text{ denotes a scattering orbital.} \tag{5}$$

The energy E in the argument of $A_c^{\dagger}(E)$ is defined by

$$E=\hbar^2k^2/2M-\epsilon_l, \tag{6a}$$

where ϵ_l and $\hbar^2k^2/2M$ were defined above. For later use, we define

$$-\epsilon_l=\epsilon_c. \tag{6b}$$

Correspondingly, the energy (with respect to H_0) of

the operators $A_b^\dagger$, A_b is

$$E_b=\epsilon_p-\epsilon_l. \tag{6c}$$

The boson operators are assumed to obey the usual commutation relations

$$[A_b, A_{b'}^\dagger]=\delta_{bb'}, \qquad [A_c(E), A_{c'}^\dagger(E')]=\delta_{cc'}\delta(E-E'), \tag{7}$$

while all other commutators vanish.

The RPA to the Hamiltonian is obtained by dropping the interaction terms $(c-f)$ shown in Fig. 1 and by replacing the 1p-1h operators in the remaining interaction terms by the boson operators defined above. We obtain

$$H^B=E^{\rm HF}+H_0^B+V^B. \tag{8a}$$

Here, H_0^B is given by the RPA of the unperturbed Hamiltonian H_0,

$$H_0^B=\sum_b E_b A_b^\dagger A_b+\sum_c \int_{+\epsilon_c}^\infty dE\, E A_c^\dagger(E) A_c(E). \tag{8b}$$

Correspondingly, V^B is given by the RPA of the residual interaction,

$$\begin{aligned} V^B=&\sum_{bb'} \mathcal{V}_{bb'} A_b^\dagger A_{b'} \\ &+\left\{\sum_{b,c}\int_{\epsilon_c}^\infty dE\, \mathcal{V}_{bc}(E) A_b^\dagger A_c(E)+\text{H.c.}\right\} \\ &+\sum_{cc'}\int_{\epsilon_c}^\infty dE \int_{\epsilon_{c'}}^\infty dE' \mathcal{V}_{cc'}(E,E') A_c^\dagger(E) A_{c'}(E') \\ &+\tfrac{1}{2}\{\sum_{bb'} \mathcal{W}_{bb'} A_b^\dagger A_{b'}^\dagger+\text{H.c.}\} \\ &+\left\{\sum_{b,c}\int_{\epsilon_c}^\infty dE \mathcal{W}_{bc}(E) A_b^\dagger A_c^\dagger(E)+\text{H.c.}\right\} \\ &+\frac{1}{2}\left\{\sum_{c,c'}\int_{\epsilon_c}^\infty dE \int_{\epsilon_{c'}}^\infty dE' \mathcal{W}_{cc'}(E,E') \right. \\ &\left. \times A_c^\dagger(E) A_{c'}^\dagger(E')+\text{H.c.}\right\}. \end{aligned} \tag{8c}$$

The matrix elements denoted by $\mathcal{V}$ correspond to the 1p-1h matrix elements shown in Fig. 1(a). For instance,

$$\mathcal{V}_{bb'}=\langle l'p \mid V \mid p'l\rangle_a \qquad \text{if } p, p' \text{ bound.} \tag{9a}$$

Because of our choice of phases, all matrix elements are real. Hence,

$$\mathcal{V}_{b'b}=\mathcal{V}_{bb'}; \qquad \mathcal{V}_{cc'}(E,E')=\mathcal{V}_{c'c}(E',E); \qquad \mathcal{V}_{cb}(E)=\mathcal{V}_{bc}(E). \tag{9b}$$

The matrix elements denoted by $\mathcal{W}$ are those given in Fig. 1(b); e.g.,

$$\mathcal{W}_{bb'}=\langle pp' \mid V \mid ll'\rangle_a \qquad \text{if } p, p' \text{ bound.} \tag{9c}$$

These matrix elements are easily seen to have the

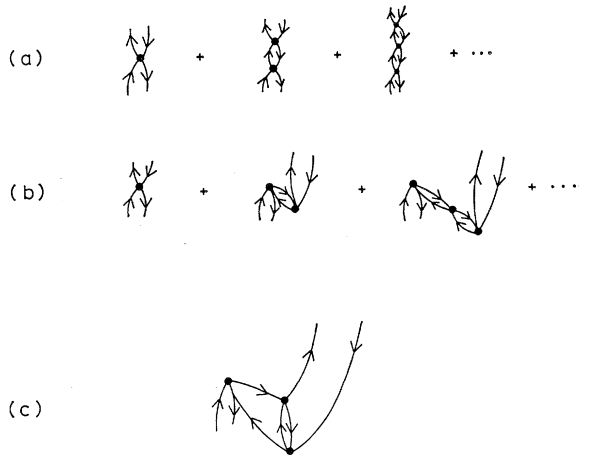

FIG. 2. (a) Intermediate states summed in the TDA and RPA. Intermediate p-h bubbles are kept intact. (b) Intermediate states also included in the RPA. p-h bubbles are kept intact. (c) Example of a diagram not included in TDA and RPA. One p-h bubble is broken.

symmetry properties

$$\mathcal{W}_{bb'}=\mathcal{W}_{b'b}; \qquad \mathcal{W}_{bc}(E)=\mathcal{W}_{cb}(E); \qquad \mathcal{W}_{cc'}(E,E')=\mathcal{W}_{c'c}(E',E). \tag{9d}$$

The factors $\frac{1}{2}$ appearing in front of the fourth and sixth term of the right-hand side of Eq. (8c) are to avoid double counting. The Hamiltonian (8) is a quadratic form in the boson operators. Indeed, the RPA can be thought of as a harmonic expansion of the Hamiltonian about the minimum found in the Hartree-Fock procedure. The eigenvalues and eigenvectors of (8) can be found by constructing the operators $Q_\mu^\dagger$, which are the normal modes of H^B, that is, which satisfy the equations

$$[H^B, Q_\mu^\dagger]=E_\mu Q_\mu^\dagger, \tag{10a}$$

$$[Q_\mu, Q_\mu^\dagger]=\delta_{\mu\mu'}. \tag{10b}$$

(At the present time, we do not concern ourselves with eigenbosons of the Hamiltonian that are related to symmetry properties of H^B, like the operator of total linear momentum, etc. Such bosons are dealt with in Sec. VI.) Let $\mid\Psi_0\rangle$ be the wave function of the (nondegenerate) ground state of H^B, $E^{\rm RPA}$ the corresponding eigenvalue,

$$H^B \mid \Psi_0\rangle=E^{\rm RPA} \mid \Psi_0\rangle. \tag{11}$$

In the following, we choose $E^{\rm RPA}$ as the zero of energy, $E^{\rm RPA}=0$. It follows from Eq. (10a) that excited states are given by

$$\mid\Psi_\mu\rangle=Q_\mu^\dagger \mid \Psi_0\rangle, \tag{12a}$$

and that their excitation energy is E_μ. Since $\mid\Psi_0\rangle$ is the ground state by definition, it also follows from Eq. (10a) that

$$Q_\mu \mid \Psi_0\rangle=0. \tag{12b}$$

Once the operators $Q_\mu^\dagger$ have been found, Eq. (12b) determines $|\Psi_0\rangle$, whereupon Eq. (12a) can be used to construct the states $|\Psi_\mu\rangle$.

The normal mode operators $Q_\mu^\dagger$ are written as linear combinations of the basic boson operators,

$$Q_\mu^\dagger = \sum_b \{Y_\mu(b)A_b^\dagger - \bar{Y}_\mu(b)A_b\}$$

$$+ \sum_c \int_{\epsilon_c}^\infty dE\{Z_\mu(c,E)A_c^\dagger(E) - \bar{Z}_\mu(c,E)A_c(E)\}. \quad (13)$$

The amplitudes $Y_\mu(b)$, $\bar{Y}_\mu(b)$, $Z_\mu(c,E)$, $\bar{Z}_\mu(c,E)$ are determined from the eigenvalue equation (10a) and from the orthonormalization condition (10b). These equations yield a finite number of discrete solutions, corresponding to the excited bound states of H^B. The Eqs. (10) also yield a continuum of scattering states. It is these states with which the present paper is mainly concerned. Such scattering states are fully defined only if proper boundary conditions are imposed in the asymptotic region, $r\to\infty$. We determine these boundary conditions in the following way.

We assume that the states of the target, and of the residual nucleus, are obtained by removing one particle from the correlated ground state $|\Psi_0\rangle$ of Eq. (11), more precisely, by operating with $\bar{a}_l^\dagger$ on $|\Psi_0\rangle$. It is thus postulated that a hole in the *correlated* ground state is an eigenstate of the Hamiltonian for the $(A-1)$-particle system. Aside from the basic RPA assumptions introduced above, this is the main assumption of the present approach. It implies, for instance, that the Hartree-Fock fields for the A-particle system and for the $(A-1)$-particle system are the same, within the accuracy in which the other assumptions made above are valid. Furthermore, the assumption implies that the ground-state correlations are unchanged when a hole is made. This is in keeping with the basic RPA assumptions. Finally, the assumption implies that we restrict ourselves to a subset of all the states of target and residual nucleus, since we do not consider, for instance, states composed of one particle and two holes. Hence, this assumption is rather stringent. It can only be removed, however, at the expense of considerable additional complications. In the light of the remarks made in the Introduction, we view the present paper essentially as a model for a theory with ground-state correlations and therefore do not want to consider such complications.

A channel is thus defined by the energy and other quantum numbers of the target or residual nucleus, and by the orbital angular momentum and spin of the nucleon in the continuum, i.e., by the set of quantum numbers labeled c in the relations (5). Since we have chosen the zero of energy by putting $E^{\mathrm{RPA}}=0$, the threshold energy for channel c is given by $\epsilon_c>0$, defined in Eq. (6b). The boundary condition on the continuum solutions of Eqs. (11) then is that the states $Q_\mu^\dagger|\Psi_0\rangle$, when taken in coordinate representation, have asymptotically an incoming wave only in channel c and outgoing waves in channel c and all channels $c'\neq c$. This boundary condition can be incorporated into Eqs. (11) by rewriting the latter in the form of a Lippmann-Schwinger equation. We take the commutator of Eq. (10a) with the boson operator A_b and use Eq. (13). This yields

$$[A_b,[H_0^B,Q_\mu^\dagger]]+[A_b,[V^B,Q_\mu^\dagger]]=+E_\mu Y_\mu(b). \quad (14a)$$

The first commutator on the left-hand side can easily be evaluated. We find

$$[A_b,[V^B,Q_\mu^\dagger]]=(E_\mu-E_b)Y_\mu(b). \quad (14b)$$

Corresponding equations are derived by taking the commutator of Eq. (10a) with $A_b^\dagger$, $A_c^\dagger(E)$, and $A_c(E)$, respectively. This gives

$$[A_b^\dagger,[V^B,Q_\mu^\dagger]]=(E_\mu+E_b)\bar{Y}_\mu(b), \quad (14c)$$

$$[A_c(E),[V^B,Q_\mu^\dagger]]=(E_\mu-E)Z_\mu(c,E), \quad (14d)$$

$$[A_c^\dagger(E),[V^B,Q_\mu^\dagger]]=(E+E_\mu)\bar{Z}_\mu(c,E). \quad (14e)$$

We solve these equations for the coefficients $Y_\mu(b)$, $\bar{Y}_\mu(b)$, $Z_\mu(c,E)$, $\bar{Z}_\mu(c,E)$. A problem arises when we divide Eq. (14d) by $E-E_\mu$; it is here that the boundary condition comes into play. We are presently only interested in scattering solutions. These are specified in terms of an incoming wave with energy E in channel c. To denote such solutions, we replace the index μ by the pair of indices c, E; for instance,

$$Z_\mu(c',E')\to Z_E^c(c',E').$$

We write

$$Q_\mu^\dagger\to Q_c^{(+)\dagger}(E)$$

to indicate the creation operator for a boson corresponding to the boundary condition just given. Equation (14d) takes the form

$$Z_E^c(c',E') = \exp(i\delta_c)\delta_{cc'}\delta(E-E')$$

$$+[1/(E^+-E')][A_{c'}(E'),[V^B,Q_c^{(+)\dagger}(E)]]. \quad (14f)$$

Since $Z_E^c(c',E')$ occurs only under the integral sign in Eq. (13), Eq. (14f) implies that we have an incoming wave only in channel c. The phase factor $\exp(i\delta_c)$ appears because the wave functions $u_{lj}(r,k)$ were chosen real. Here, $\delta_c\equiv\delta_{lj}$, where l, j are the quantum numbers of the continuum particle in channel c. Inserting all this into Eq. (13), we find the Lippmann-

Schwinger equation

$$Q_c^{(+)\dagger}(E)=\exp(i\delta_c)A_c^\dagger(E)+\sum_b\{[1/(E-E_b)][A_b,[V^B,Q_c^{(+)\dagger}(E)]]A_b^\dagger$$
$$-[1/(E+E_b)][A_b^\dagger,[V^B,Q_c^{(+)\dagger}(E)]]A_b\}+\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\{[1/(E^+-E')][A_{c'}(E'),[V^B,Q_c^{(+)\dagger}(E)]]$$
$$\times A_{c'}^\dagger(E')-[1/(E+E')][A_{c'}^\dagger(E'),[V^B,Q_c^{(+)\dagger}(E)]]A_{c'}(E')\}. \quad (15)$$

Equation (15) can equivalently be expressed as a Lippmann-Schwinger equation for the amplitudes $Y_E^c(b)$, $\bar{Y}_E^c(b)$, $Z_E^c(c',E')$, $\bar{Z}_E^c(c',E')$. The latter is obtained by taking the commutator of Eq. (15) with the boson operators and by working out the values of the commutators appearing in Eq. (15). We find

$$\begin{pmatrix}Y_E^c(b)\\ \bar{Y}_E^c(b)\end{pmatrix}=\begin{pmatrix}(E-E_b)^{-1} & 0\\ 0 & (E+E_b)^{-1}\end{pmatrix}\Bigg\{\sum_{b'}\begin{pmatrix}\mathcal{U}_{bb'} & \mathcal{W}_{bb'}\\ -\mathcal{W}_{bb'} & -\mathcal{U}_{bb'}\end{pmatrix}\begin{pmatrix}Y_E^c(b')\\ \bar{Y}_E^c(b')\end{pmatrix}$$
$$+\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\begin{pmatrix}\mathcal{U}_{bc'}(E') & \mathcal{W}_{bc'}(E')\\ -\mathcal{W}_{bc'}(E') & -\mathcal{U}_{bc'}(E')\end{pmatrix}\begin{pmatrix}Z_E^c(c',E')\\ \bar{Z}_E^c(c',E')\end{pmatrix}\Bigg\}, \quad (16a)$$

$$\begin{pmatrix}Z_E^c(c',E')\\ \bar{Z}_E^c(c',E')\end{pmatrix}=\begin{pmatrix}\exp(i\delta_c)\delta(E-E')\delta_{c'c}\\ 0\end{pmatrix}+\begin{pmatrix}(E^+-E')^{-1} & 0\\ 0 & (E+E')^{-1}\end{pmatrix}\Bigg\{\sum_{b'}\begin{pmatrix}\mathcal{U}_{c'b'}(E') & \mathcal{W}_{c'b'}(E')\\ -\mathcal{W}_{c'b'}(E') & -\mathcal{U}_{c'b'}(E')\end{pmatrix}$$
$$\times\begin{pmatrix}Y_E^c(b')\\ \bar{Y}_E^c(b')\end{pmatrix}+\sum_{c''}\int_{\epsilon_{c''}}^\infty dE''\begin{pmatrix}\mathcal{U}_{c'c''}(E',E'') & \mathcal{W}_{c'c''}(E',E'')\\ -\mathcal{W}_{c'c''}(E',E'') & -\mathcal{U}_{c'c''}(E',E'')\end{pmatrix}\begin{pmatrix}Z_E^c(c'',E'')\\ \bar{Z}_E^c(c'',E'')\end{pmatrix}\Bigg\}. \quad (16b)$$

Equations (16) constitute a coupled system of integral equations. The structure of the solution to these equations is exhibited in Secs. IV and V. The scattering state $|\Psi_E^{c(+)}\rangle$ corresponding to $Q_c^{(+)\dagger}(E)$ is given by

$$|\Psi_c^{(+)}(E)\rangle=Q_c^{(+)\dagger}(E)\,|\Psi_0\rangle. \quad (17)$$

From Eqs. (13) and (16), it is easily verified (see the Appendix) that

$$[Q_c^{(+)}(E),Q_{c'}^{(+)\dagger}(E')]=\delta_{cc'}\delta(E-E'), \quad (18a)$$
$$[Q_c^{(+)}(E),Q_{c'}^{(+)}(E')]=0=[Q_c^{(+)\dagger}(E),Q_{c'}^{(+)\dagger}(E')]. \quad (18b)$$

Equations (18) express the orthonormality properties of solutions of the Lippmann-Schwinger equation for the bosons. Instead of the operators $Q_c^{(+)\dagger}(E)$ defined by Eq. (15), we could have defined operators $Q_c^{(-)\dagger}(E)$ by replacing on the right-hand side of Eq. (15) the denominator (E^+-E') by (E^--E'), and $\exp(i\delta_c)$ by $\exp(-i\delta_c)$. The scattering states

$$|\Psi_c^{(-)}(E)\rangle=Q_c^{(-)\dagger}(E)\,|\Psi_0\rangle \quad (17')$$

have an outgoing wave only in channel c, and incoming waves in all channels. It can be shown that the operators $Q_c^{(-)}(E)$ also obey the commutation relations (18). The ground state $|\Psi_0\rangle$ obeys both relations:

$$Q_c^{(+)}(E)\,|\Psi_0\rangle=0=Q_c^{(-)}(E)\,|\Psi_0\rangle. \quad (19)$$

This is because the operators $Q_c^{(-)}(E)$ can be written as linear combinations of the operators $Q_c^{(+)}(E)$, and conversely.

We conclude this section with two remarks. The Hartree-Fock procedure determines the A single-particle wave functions needed to form the uncorrelated Hartree-Fock ground state $|\Phi_0\rangle$. In the frame of the Hartree-Fock procedure, the states $a_p^\dagger|0\rangle$ have no distinct physical meaning. The HF procedure only defines the single-particle states that are occupied in the HF ground state $|\Phi_0\rangle$. Any unitary transformation in the space of single-particle states $a_p^\dagger|0\rangle$ would not change the Hartree-Fock results and would, therefore, lead to an equally acceptable starting point for the RPA. We therefore expect that our RPA results are invariant against such transformations. We now show that this is indeed the case. Clearly, the ansatz (13) is invariant in form under any transformation that leaves unchanged the states occupied in $|\Phi_0\rangle$ and only transforms the set of single-particle states not occupied in $|\Phi_0\rangle$ onto itself. Such a transformation would only modify the expansion coefficients $Y_E^c(b)$, $\bar{Y}_E^c(b)$, $Z_E^c(c',E')$, $\bar{Z}_E^c(c',E')$. The same remark applies to the Hamiltonian H^B given in Eqs. (8). Inasmuch as Eqs. (16) are a consequence of Eqs. (8) and (13), they must have the same property. The only additional fact used in deriving Eqs. (16) is that the states $\bar{a}_b^\dagger|\Psi_0\rangle$ are eigenstates of the $(A-1)$-particle Hamiltonian, with eigenvalues ϵ_l. Both statements refer to properties of states occupied in $|\Phi_0\rangle$ and are therefore unchanged under the transformation. We emphasize, in particular, that the energies ϵ_l are completely determined through the Hartree-Fock procedure. They determine, through Eq. (2), the *binding energy* E^{HF} of the nucleus in the Hartree-Fock approximation. We are therefore not

allowed to change the zero of energy of the quantities ϵ_l. This zero of energy determines, however, the threshold for the various channels and is the only additional piece of information used in the derivation of Eqs. (16).

We finally comment upon the possibility of using the RPA for the calculation of scattering states. In classical mechanics, the RPA has an analog. It is the expansion, up to second order in the displacements and their time derivatives, of the Hamiltonian around a static equilibrium position. This expansion is, of course, only valid for small displacements. One may wonder whether the RPA is applicable to a scattering problem, since infinite distances between particles and holes are involved. However, the expansion coefficients in quantum theory are not the displacements but rather the coefficients of the particle-hole admixtures in the correlated ground state. These coefficients are individually assumed to be small in the RPA. It is therefore as meaningful to use the RPA to calculate nucleon-nucleus scattering processes at low bombarding energies as to use the RPA for the determination of properties of nuclear levels in the same energy region.

III. DEFINITION AND GENERAL PROPERTIES OF THE SCATTERING MATRIX

In this section, we write down the formula for the scattering matrix and discuss unitarity, symmetry, and RPA symmetry of this matrix. By the latter, we mean a symmetry of the matrix that is due to the special approximations of the random-phase procedure; this symmetry has interesting consequences for the analytical properties of the S matrix. These consequences are discussed in the following sections.

Using the state vectors $|\Psi_c^{(+)}(E)\rangle$ and $|\Psi_c^{(-)}(E)\rangle$ introduced in Sec. II, we write the scattering matrix $S_{c'c}(E)$ in the form

$$\delta(E-E')S_{c'c}(E)=\langle\Psi_{c'}^{(-)}(E')\,|\,\Psi_c^{(+)}(E)\rangle. \quad (20)$$

For convenience, we have taken the factor $\delta(E-E')$ out of the ordinary definition of $\mathbf{S}(E)$. Using Eqs. (17) and (19) we find that the right-hand side of Eq. (20) equals the expectation value of $[Q_{c'}^{(-)}(E'), Q_c^{(+)\dagger}(E)]$ with respect to $|\Psi_0\rangle$. Since the commutator is only a c number, we obtain

$$\delta(E-E')S_{c'c}(E)=[Q_{c'}^{(-)}(E'), Q_c^{(+)\dagger}(E)]. \quad (21)$$

In the Appendix, we show that the right-hand side of Eq. (21) can be simplified, yielding

$$S_{c'c}(E)=\exp(i\delta_c+i\delta_{c'})[\delta_{cc'}-2i\pi T_{c'c}(E)], \quad (22a)$$

where

$$T_{c'c}(E)=\exp(-i\delta_c)[[A_{c'}(E), V^B], Q_c^{(+)\dagger}(E)]. \quad (22b)$$

As shown in the Appendix, the unitarity of the S matrix follows from the fact that the eigenmodes of H^B form a complete set of bosons. It is known that time-reversal invariance of the Hamiltonian implies symmetry of the scattering matrix. In the present context, this can be shown as follows. Since we assume H to be time-reversal invariant, we can choose all matrix elements appearing in Eq. (8) to be real. Since we have chosen our scattering states real, we have also $A_c^*(E)=A_c(E)$, $A_b^*(E)=A_b(E)$, etc. It then follows from Eq. (15) that

$$\{\exp(-i\delta_c)Q_c^{(+)\dagger}(E)\}^*=\{\exp(i\delta_c)Q_c^{(-)\dagger}(E)\}. \quad (23a)$$

We use this relation in the commutator appearing on the right-hand side of Eq. (21), taken for $E=E'$. We first notice that the commutator, being a c number, can be replaced by its transposed value (in Hilbert space, not in the space of channel wave functions):

$$\begin{aligned}[Q_{c'}^{(-)}(E), Q_c^{(+)\dagger}(E)]&=[Q_{c'}^{(-)}(E), Q_c^{(+)\dagger}(E)]^T\\&=[Q_c^{(+)*}(E), Q_c^{(-)T}(E)]\\&=[Q_c^{(-)}(E), Q_c^{(-)*\dagger}(E)]\\&=[Q_c^{(-)}(E), Q_c^{(+)\dagger}(E)]. \quad (23b)\end{aligned}$$

The equality of the first and the last terms in Eq. (23b) establishes the symmetry of the S matrix.

The scattering matrix obtained in the frame of the RPA has an additional symmetry property. We write Eq. (10a) for a continuum solution $Q_c^\dagger(E)$ in the form

$$[H^B, Q_c^{(\pm)\dagger}(E)]=EQ_c^{(\pm)\dagger}(E). \quad (24a)$$

Taking the transpose of this equation, we find, using the symmetry of H^B,

$$[H^B, Q_c^{(\pm)*}(E)]=[H^B, Q_c^{(\mp)}(E)]=-EQ_c^{(\mp)}(E). \quad (24b)$$

A comparison of Eqs. (24a) and (24b) suggests the definition

$$Q_c^{(\mp)\dagger}(-E)\equiv Q_c^{(\pm)}(E), \qquad E>0 \quad (25a)$$

$$Q_c^{(\pm)}(-E)\equiv Q_c^{(\mp)\dagger}(+E), \qquad E>0. \quad (25b)$$

We define the matrix $S_{c'c}(-E)$ by [see Eq. (21)]

$$\delta(E-E')S_{c'c}(-E)\equiv[Q_{c'}^{(-)}(-E'), Q_c^{(+)\dagger}(-E)]. \quad (26)$$

Using Eqs. (25), we obtain

$$S_{c'c}(-E)=S_{c'c}(+E). \quad (27)$$

We emphasize that $S_{c'c}(-E)$ as defined by Eq. (26) is different from the value obtained by continuing $S_{c'c}(+E)$ analytically on the physical sheet from $E>0$ to $E<0$. Therefore, the analytic continuation of $S_{c'c}$ studied in Secs. IV and V does not have the property (27). However, the property (24) of the solutions also has interesting consequences for this analytical continuation. Indeed, for the cases studied in this paper, it is found that singularities of $S_{c'c}(E)$ located on the cut physical E plane occur symmetrically with respect to $E=0$, the energy of the RPA ground state. We postpone a discussion and interpretation of this interesting result until Sec. IV.

IV. ANALYTIC PROPERTIES, POLE DECOMPOSITION, AND SUM RULES FOR THE S MATRIX WITHOUT CHANNEL-CHANNEL COUPLING

In this section, we simplify the basic equations (16) for the amplitudes Y, $\bar{Y}$, Z, $\bar{Z}$. We assume that those matrix elements of V connecting the channels directly with each other can be neglected, i.e., that the last term on the right-hand side of Eq. (16b) is zero:

$$\mathcal{V}_{c'c''}(E', E'') = 0; \qquad \mathcal{W}_{c'c''}(E', E'') = 0. \quad (28)$$

This assumption is introduced only for pedagogical reasons. Using Eqs. (28), we shall see that Eqs. (16) can easily be solved explicitly and that the resulting scattering matrix can easily be written down. This will make it possible to focus on those properties of $S_{c'c}(E)$ that are mentioned in the title of this section. We emphasize that, in many cases, it is not justified to use Eqs. (28). Then, the solution of Eqs. (16), although still feasible, is much more complicated, as shown in the following section. Once the solution has been constructed, however, the methods developed in the present section can be employed to derive, in the general case, results very similar in form to those obtained for the case when Eqs. (28) hold. The situation is thus very similar to that encountered in the TDA, where it is also found that the form of the scattering matrix obtained in the general case is not very different from the one valid if the first of Eqs. (28) applies.[4]

Evaluating the commutator in Eq. (22b) and using Eqs. (28), we find

$$T_{c'c}(E) = \exp(-i\delta_c) \sum_b \{\mathcal{V}_{c'b}(E) Y_E{}^c(b) + \mathcal{W}_{c'b}(E) \bar{Y}_E{}^c(b)\}. \quad (29)$$

We now solve Eqs. (16) for Y and $\bar{Y}$, again using Eqs. (28). Inserting the result into Eq. (29), we obtain

$$T_{c'c}(E) = \sum_{bb'} (\mathcal{V}_{c'b'}(E), \mathcal{W}_{c'b'}(E)) \times \begin{pmatrix} M_{b'b}{}^1 & M_{b'b}{}^2 \\ M_{b'b}{}^3 & M_{b'b}{}^4 \end{pmatrix} \begin{pmatrix} \mathcal{V}_{bc}(E) \\ -\mathcal{W}_{bc}(E) \end{pmatrix}. \quad (30a)$$

In addition to the summation over b and b', the right-hand side of Eq. (30a) also contains a 2×2 matrix product. The matrix M is defined by

$$\mathsf{M} \equiv \begin{pmatrix} M_{b'b}{}^1 & M_{b'b}{}^2 \\ M_{b'b}{}^3 & M_{b'b}{}^4 \end{pmatrix} = (\mathsf{D})^{-1}. \quad (30b)$$

The matrix D is a $2M\times 2M$ matrix, where M is the number of bound 1p-1h states. It can be written as

$$\mathsf{D} = 1E - \mathsf{N}, \quad (30c)$$

where the matrix N is given by

$$\mathsf{N} = \begin{pmatrix} N_{bb'}{}^1 & N_{bb'}{}^2 \\ -N_{b'b}{}^2 & -N_{bb'}{}^3 \end{pmatrix}. \quad (30d)$$

Here,

$$N_{bb'}{}^1 = E_b\delta_{bb'} + \mathcal{V}_{bb'} + \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{V}_{bc'}(E')\mathcal{V}_{c'b'}(E')}{E^+ - E'} - \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{W}_{bc'}(E')\mathcal{W}_{c'b'}(E')}{E + E'}, \quad (30e)$$

$$N_{bb'}{}^3 = E_b\delta_{bb'} + \mathcal{V}_{bb'} - \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{V}_{bc'}(E')\mathcal{V}_{c'b'}(E')}{E + E'} + \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{W}_{bc'}(E')\mathcal{W}_{c'b'}(E')}{E^+ - E'}, \quad (30f)$$

$$N_{bb'}{}^2 = \mathcal{W}_{bb'} + \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{V}_{bc'}(E')\mathcal{W}_{c'b'}(E')}{E^+ - E'} - \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{W}_{bc'}(E')\mathcal{V}_{c'b'}(E')}{E + E'}. \quad (30g)$$

It is obvious from Eqs. (30c)–(30g) that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathsf{D}$$

is symmetric. This shows that Eq. (30a) yields a symmetric T matrix, in agreement with the results obtained in Sec. III.

We now discuss the analytical properties of $T_{c'c}(E)$ as a function of the complex energy $\mathcal{E}$. The integral

$$\int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{V}_{bc'}(E')\mathcal{V}_{c'b'}(E')}{\mathcal{E} - E'} \quad (31)$$

defines, in the complex $\mathcal{E}$ plane, cut from $\epsilon_{c'}$ to ∞, a holomorphic function of $\mathcal{E}$. Similarly, the integral

$$\int_{\epsilon_{c'}}^{\infty} dE' \frac{\mathcal{V}_{bc'}(E')\mathcal{V}_{b'c'}(E')}{\mathcal{E} + E'} \quad (32)$$

defines, in the complex $\mathcal{E}$ plane cut from $-\epsilon_{c'}$ to $-\infty$, a holomorphic function of $\mathcal{E}$. All integrals occurring in Eq. (30c) for D are either of type (31) or of type (32). This shows that the matrix elements of D are holomorphic functions of $\mathcal{E}$ in the complex $\mathcal{E}$ plane, cut from ϵ_c to ∞, with $c = 1, \cdots, \Lambda$ (Λ is the number of channels), and also cut from $-\epsilon_c$ to $-\infty$, with $c = 1, \cdots, \Lambda$. The functions defined by Eq. (31) have a branch point at $\mathcal{E} = \epsilon_{c'}$. This leads to the cut from $\epsilon_{c'}$ to ∞. Conversely, the functions defined by Eq. (32) have a branch point at $\mathcal{E} = -\epsilon_{c'}$. This leads to the cut from $-\epsilon_{c'}$ to $-\infty$. The occurrence of branch points at $\mathcal{E} = \epsilon_c$, with $c = 1, \cdots, \Lambda$, is a well-known phenomenon. These branch points are kinematic singularities, caused by the existence of thresholds. However, the energies $-\epsilon_c$, $c = 1, \cdots, \Lambda$ do not correspond to thresholds. The branch points at these energies have a dynamical, rather than a kinematical, origin. They are caused by a specific symmetry

of the matrix $\mathbf{D}$ due to the RPA. It is easy to see that

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{D}(\mathcal{E})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=-\mathbf{D}(-\mathcal{E}). \tag{33}$$

This symmetry is typical of RPA equations. It shows that if an element of $\mathbf{D}$ has a singularity (a branch point) at $\mathcal{E}=\epsilon_c$, there must also be a matrix element of $\mathbf{D}$ with the same singularity at $\mathcal{E}=-\epsilon_c$, in accord with the statements made above. The symmetry (33) implies a further statement. Excited bound states of the compound system correspond to zeros of the function $\det(\mathbf{D})$ for $\mathcal{E}>0$. This can easily be seen by writing down Eq. (10a) for a $Q_\mu^\dagger$ creating a bound excited state. The corresponding homogeneous equation for the amplitudes Y, $\bar{Y}$, Z, $\bar{Z}$ has a nontrivial solution only if

$$\det[\mathbf{D}(\mathcal{E})]=0. \tag{34}$$

Conversely, by the same reasoning we can show that zeros of Eq. (34), with $\mathcal{E}>0$, correspond to bound states of the compound system. It is clear from Eq. (30) that every zero of Eq. (34), if located in the region of meromorphy of $T_{c'c}(\mathcal{E})$, causes a pole of $T_{c'c}(\mathcal{E})$ to occur. We thus retrieve the well-known statement that bound states of the system produce poles in the scattering matrix at the energies of the bound states. The symmetry (33) implies that the zeros of $\det(\mathbf{D})$ occur in pairs in the cut $\mathcal{E}$ plane, since

$$\det[\mathbf{D}(\mathcal{E})]=-\det[\mathbf{D}(-\mathcal{E})]. \tag{35}$$

Hence, to each pole of $T_{c'c}(\mathcal{E})$ at $\mathcal{E}=E_1>0$ caused by a bound state there corresponds a pole at $\mathcal{E}=-E_1$, if $(-E_1)$ is in the region of meromorphy of $T_{c'c}(\mathcal{E})$. This symmetry property is related to the symmetry (24) and originates in the fact that in the RPA, the boson creation and annihilation operators are treated on an equal footing, as displayed in Eqs. (8) and (13). Since, by definition, the RPA ground state has energy $E^{\mathrm{RPA}}=0$, it is clear that the poles at energies $\mathcal{E}<0$ are not associated with bound states of the system. They occur because, if there exists a creation operator $Q_1^\dagger$ for a bound state at energy $E_1>0$, there also exists a destruction operator of this bound-state boson, with energy $-E_1<0$. We thus see that the symmetric appearance about $\mathcal{E}=0$ of both branch points and poles of $\mathbf{D}^{-1}$ is caused by the symmetric treatment of creation and annihilation operators in the RPA.

In the RPA, the singularities of $\mathbf{D}^{-1}$ occurring for $\mathcal{E}<0$ are also singularities of the T matrix. These T-matrix singularities arise from the inclusion of the backward-going graphs shown in Fig. 2(b). Such graphs lead to a boson-boson interaction, i.e., an interaction between the unperturbed boson operators defined in Eq. (5), which is different from the interaction included in the TDA. This offers the following interpretation for these singularities. We remind the reader that branch points and poles (the latter not associated with bound states of the system) arise at negative energies in the analytically continued scattered amplitude already for the simple case of elastic scattering by a superposition of exponential potentials.[8] In high-energy physics, it is well known that singularities on the "left-hand cut" are associated with the forces between the particles. We may therefore say that the inclusion of backward-going graphs leads to an inter-boson force, which produces the singularities found at $\mathcal{E}<0$. These singularities have some influence on the scattering cross section in the physical region. In this way, the backward-going graphs produce a nonzero background scattering cross section outside the physical resonances, as shown in Eq. (48) below. The poles and branch points at $\mathcal{E}<0$ are rather far away from the region of physical scattering processes (we have chosen $E^{\mathrm{RPA}}=0$!). Hence, we expect the numerical effect of these singularities on the S matrix in the physical region to be rather small. It is quite likely, in fact, that this effect is smaller than the uncertainties inherent in the RPA, so that these singularities have more of an academic interest. Their occurrence does show, however, that the inclusion of backward-going graphs can lead to a behavior of the S matrix not encountered in the TDA. Here, we believe, lies the main interest of these singularities.

Can there be zeros of $\det[\mathbf{D}(\mathcal{E})]$ in the cut physical $\mathcal{E}$ plane other than those associated with bound states of the compound system and their mirror images about $\mathcal{E}=0$? We show that this is not the case. Suppose that $\det[\mathbf{D}(\mathcal{E})]$ had a zero at $\mathcal{E}=\mathcal{E}_1$ on the cut $\mathcal{E}$ plane, with $\mathrm{Im}\mathcal{E}_1\neq 0$. Then an operator $Q_1^\dagger$ can be constructed that creates an eigenboson of the Hamiltonian with this complex energy $\mathcal{E}_1$. It was shown by Thouless[2] that if the RPA was preceded by a Hartree-Fock minimization of the energy, then such bosons cannot exist, and all eigenvalues of H^B must be real.

The origin of the zeros of $\det[\mathbf{D}(\mathcal{E})]$ in the cut $\mathcal{E}$ plane can be elucidated in yet another way. It follows from the commutation relations (10a) that if $Q_1^\dagger$ is the creation operator of a bound boson with energy $E_1>0$, then the Hamiltonian has bound states at E_1, $2E_1$, $3E_1$, $\cdots$. This is due to the boson character of the operator $Q_1^\dagger$. However, the function $\det[\mathbf{D}(\mathcal{E})]$ has, in general, a zero only at $\mathcal{E}=E_1$, not at $\mathcal{E}=2E_1$, $\mathcal{E}=3E_1$, $\cdots$. This shows that the positive-energy zeros of $\det[\mathbf{D}]$ only give *some* bound states of the boson Hamiltonian H^B, i.e., those related to the presence of *one* boson. Correspondingly, the annihilation operator Q, for a bound boson with energy $E_1>0$, is itself a solution of Eq. (10a), with energy $-E_1<0$. There are other solutions of this equation, with energies $-2E_1$, $-3E_1$, $-4E_1$, $\cdots$. A negative-energy zero of $\det[\mathbf{D}(\mathcal{E})]$ occurs only at the energy $\mathcal{E}=-E_1$, not at the energies $\mathcal{E}=-2E_1$, $\mathcal{E}=-3E_1$, $\cdots$. Altogether, we see that the zeros of $\det[\mathbf{D}(\mathcal{E})]$, and hence the poles of the S matrix, are not associated with the bound states of H^B. Rather, they are associated with the one-boson solutions of

[8] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Co., New York, 1966).

Eq. (10a) with discrete energy, and no discrimination is made between positive-energy zeros corresponding to boson creation operators $Q^\dagger$ and negative-energy zeros corresponding to annihilation operators Q.

We now turn to the pole decomposition of the scattering matrix and establish sum rules for the partial and total widths similar to those[4] found in the TDA. We first observe that neither $T_{c'c}(\mathcal{E})$ nor $S_{c'c}(\mathcal{E})$ obeys a symmetry relation of type (33). This is not surprising, since this symmetry relation is a result of the RPA equations, but not of the boundary conditions imposed upon the scattering solutions. The matrix $\mathbf{D}(\mathcal{E})$ is quite independent of the particular boundary conditions chosen. This is not so with the S matrix, which therefore does not display the characteristic symmetry of the RPA equations. This also shows that $S_{c'c}(-E)\,(E>0)$ defined in Eq. (26) differs from the value obtained by continuing $S_{c'c}(E)$ analytically to $-E$ on the cut $\mathcal{E}$ plane.

A pole decomposition of $T_{c'c}(\mathcal{E})$ [or of $S_{c'c}(\mathcal{E})$] can be obtained by continuing these functions into the sheets adjacent to the positive real axis of the cut $\mathcal{E}$ plane. It is on these sheets that poles are found that correspond to the compound-nuclear resonances. Since the treatment of this problem follows very much the same lines as that of the general Humblet-Rosenfeld theory, we do not give it here. Rather, we focus on the pole decomposition obtained in a region of the Riemann surface in $\mathcal{E}$ (henceforth called the "region of interest") restricted in such a way that the integrals (31) and (32), as well as the single-particle phase shifts δ_c and matrix elements $\mathcal{V}_{bc}(E)$, $\mathcal{W}_{bc}(E)$ appearing in the expression (29) of the T matrix, may be regarded as constants. It is known[4] that this assumption makes it possible to obtain, in the frame of the TDA, sum rules for the total widths and the partial widths. The assumption implies that we are far from any threshold and from any single-particle resonances. The latter would cause a strong energy dependence of the quantities assumed constant above. Such resonances are dealt with in Sec. V. A glance at the matrix $\mathbf{D}(\mathcal{E})$ in Eqs. (30c)–(30g) shows that even if we assume all the integrals of type (31), (32) to be constant, zeros of det$\mathbf{D}$ will still appear symmetrically about the point $\mathcal{E}=0$. Those zeros corresponding to $\mathrm{Re}\mathcal{E}<0$ lie, of course, far outside the region where our approximation of constant integrals is valid. Therefore, they will not coincide with the true zeros of det$\mathbf{D}$, i.e., the zeros determined without using the approximation described above. Furthermore, the corresponding S-matrix poles will have little influence on the cross section in the energy region of interest. This influence will consist of a smooth background. For these reasons, we separate the T matrix into two parts. One part describes the influence of the close-lying poles, the other one is assumed to be constant. This operation is carried through in two steps. First, we introduce the eigenvectors and eigenvalues of the standard[3] bound-state RPA equations. This is done in order to establish the connection between the present and the standard treatment. Subsequently, we include the effect of the continuum and introduce the pole decomposition of the T matrix. Finally, we derive the sum rules.

We define the $2M\times 2M$ matrix $\mathbf{N}^{(0)}$ as follows:

$$\mathbf{N}^{(0)}=\begin{pmatrix}\delta_{bb'}E_b+\mathcal{V}_{bb'} & \mathcal{W}_{bb'}\\ -\mathcal{W}_{bb'} & -\delta_{bb'}E_b-\mathcal{V}_{bb'}\end{pmatrix}. \tag{36}$$

Comparison of Eq. (36) with Eqs. (30c)–(30g) shows that $\mathbf{N}^{(0)}$ differs from $\mathbf{N}$ by the omission of the integrals (31) and (32), which represent the influence of the various channels on our problem. The eigenvalue equation

$$\mathbf{N}^{(0)}\begin{pmatrix}X_b^\lambda\\ \bar{X}_b^\lambda\end{pmatrix}=\epsilon^\lambda\begin{pmatrix}X_b^\lambda\\ \bar{X}_b^\lambda\end{pmatrix} \tag{37}$$

is exactly the RPA equation that we would have obtained had we omitted all states with nucleons in continuum orbitals. It is the equation considered in practice[3] for the calculation of bound nuclear levels from the RPA. In the following, we assume that $\mathbf{N}^{(0)}$ does not have vanishing eigenvalues. This assumption is reasonable because of the following argument. It was shown by Thouless[2] that the *full* RPA equations yield vanishing eigenvalues for the three components of the linear momentum operator. By full RPA equations, we mean here the equations obtained by admitting a complete set of HF single-particle wave functions. The equations (37) are not identical to these full RPA equations, since they are obtained by omitting the continuum HF single-particle wave functions. Hence, the theorem of Thouless does not apply to these equations. If the effect of the continuum is small, we expect, according to Thouless's theorem, the occurrence in the vicinity of $\mathcal{E}=0$ of eigenvalues ϵ^λ of Eq. (37) associated with symmetry properties of the Hamiltonian and, therefore, with spurious modes of internal excitation. We would, however, consider it fortuitous if these eigenvalues were to vanish exactly. This point of view is supported by the results of numerical calculations.[3] We therefore assume that all eigenvalues of $\mathbf{N}^{(0)}$ differ from zero.

The eigenvalues and eigenvectors of Eq. (37) are real. This also follows from the work by Thouless,[2] who showed that if the RPA is preceded by a HF calculation, then the matrix of the full RPA equations is positive semidefinite if the metric matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

is used. The statement carries over immediately to the matrix $\mathbf{N}^{(0)}$, which is defined in a subspace of the full RPA equations.

The symmetry property of $\mathbf{N}^{(0)}$ displayed in Eq. (36) implies that to each eigenvalue ϵ^λ of Eq. (37), there is a corresponding eigenvalue $-\epsilon^\lambda$. The eigenvector asso-

ciated with $-\epsilon^\lambda$ is related to the quantities appearing in Eq. (37) by

$$\mathsf{N}^{(0)}\begin{pmatrix}\bar{X}_b{}^\lambda\\ X_b{}^\lambda\end{pmatrix}=-\epsilon^\lambda\begin{pmatrix}\bar{X}_b{}^\lambda\\ X_b{}^\lambda\end{pmatrix}. \tag{38}$$

Let ϵ_j, $j=1,\cdots,M$, be the M *positive* eigenvalues of Eq. (37), and let

$$\begin{pmatrix}X_b{}^j\\ \bar{X}_b{}^j\end{pmatrix},\quad j=1,\cdots,M$$

be the corresponding eigenvectors. (The quantities ϵ_j should not be confused with the threshold energies ϵ_c.) We define the matrices

$$\mathsf{O}=\begin{pmatrix}X_b & \bar{X}_b\\ \bar{X}_b & X_b\end{pmatrix},\qquad \mathsf{I}=\begin{pmatrix}\delta_{\beta\beta'} & 0\\ 0 & -\delta_{\beta\beta'}\end{pmatrix},$$

$$\mathfrak{e}=\begin{pmatrix}\delta_{jj'}\epsilon_j & 0\\ 0 & -\delta_{jj'}\epsilon_j\end{pmatrix}. \tag{39}$$

The eigenvectors can be chosen in such a way that (T stands for transposed)

$$\mathsf{O}^T\mathsf{I}\mathsf{N}^{(0)}\mathsf{O}=\mathfrak{e},\qquad \mathsf{O}^T\mathsf{I}\mathsf{O}=\mathsf{1}=\mathsf{O}\mathsf{O}^T\mathsf{I}. \tag{40}$$

The matrix D^{-1} appearing in Eq. (30b) can be written

$$\mathsf{D}^{-1}=\mathsf{O}\mathsf{O}^T\mathsf{I}\mathsf{D}^{-1}\mathsf{O}\mathsf{O}^T\mathsf{I}=\mathsf{O}[\mathsf{O}^T\mathsf{I}\mathsf{D}\mathsf{O}]^{-1}\mathsf{O}^T\mathsf{I}\equiv\mathsf{O}\bar{\mathsf{D}}^{-1}\mathsf{O}^T\mathsf{I}. \tag{41}$$

The last relation defines the symmetric matrix $\bar{\mathbf{D}}$. Inserting Eq. (41) into Eqs. (30), we find

$$T_{c'c}(E)=(\hat{V}_{c'b'}{}^T(E),\hat{W}_{c'b'}{}^T(E))(\bar{\mathbf{D}})^{-1}\begin{pmatrix}\hat{V}_{bc}(E)\\ \hat{W}_{bc}(E)\end{pmatrix}. \tag{42}$$

Equation (42) implies matrix multiplication in a $(2M)$-dimensional space. We have defined ($j=1,\cdots,M$)

$$\hat{V}_{cj}(E)=\sum_{b'}(\mathcal{U}_{cb'}(E)X_{b'}{}^j+\mathcal{W}_{cb'}(E)\bar{X}_{b'}{}^j), \tag{43a}$$

$$\hat{W}_{cj}(E)=\sum_{b'}(\mathcal{W}_{cb'}(E)X_{b'}{}^j+\mathcal{U}_{cb'}(E)\bar{X}_{b'}{}^j). \tag{43b}$$

According to Eq. (41), the matrix $\bar{\mathbf{D}}$ has the form

$$\bar{\mathbf{D}}=\begin{pmatrix}\delta_{jj'}E-\bar{N}_{jj'}{}^1 & -\bar{N}_{jj'}{}^2\\ -\bar{N}_{j'j}{}^2 & \delta_{jj'}E+\bar{N}_{jj'}{}^3\end{pmatrix}, \tag{43c}$$

where

$$\bar{N}_{jj'}{}^1=\epsilon_j\delta_{jj'}+\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{V}_{jc'}(E')\hat{V}_{c'j'}(E')}{E^+-E'}-\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{W}_{jc'}(E')\hat{W}_{c'j'}(E')}{E+E'}, \tag{43d}$$

$$\bar{N}_{jj'}{}^3=\epsilon_j\delta_{jj'}-\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{V}_{jc'}(E')\hat{V}_{c'j'}(E')}{E+E'}+\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{W}_{jc'}(E')\hat{W}_{c'j'}(E')}{E^+-E'}, \tag{43e}$$

$$\bar{N}_{jj'}{}^2=\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{V}_{jc'}(E')\hat{W}_{c'j'}(E')}{E^+-E'}-\sum_{c'}\int_{\epsilon_{c'}}^\infty dE'\,\frac{\hat{W}_{jc'}(E')\hat{V}_{c'j'}(E')}{E+E'}. \tag{43f}$$

Equations (43c)–(43f) show that the matrix elements $\mathcal{U}_{bb'}$, $\mathcal{W}_{bb'}$ appearing in Eq. (30c) have been removed by the transformation and that the only coupling terms left are the integrals of type (31), (32). This is one reason that we introduced the transformation leading from the energies E_b of the particle-hole states to the energies ϵ_j of the bound-state RPA equations. We wanted to display the connection with the usual, restricted RPA calculations.[3] We see that the modification of these calculations introduced by taking into account the continuum is quite analogous to that introduced similarly[4] in the TDA. The integrals appearing in Eq. (43c) can be decomposed into a real and an imaginary part. The former parts give rise to a shift matrix, the latter to a width matrix and thus to the particle-decay widths of the resonances. Typically,[4] integrals of type (31), (32) have absolute values that are smaller by a factor 10–30 than those of the matrix elements $\mathcal{U}_{bb'}$, $\mathcal{W}_{bb'}$. This is the other reason that the transformation leading to the matrix $\bar{\mathbf{D}}$ was introduced above. It is hoped that, in the region of the $\mathcal{E}$ plane presently under study, resonances are caused by RPA states with energies ϵ_j lying within that region. The influence of RPA states with energies ϵ_j outside the region of interest will be treated as a perturbation, as far as the sum rules are concerned. The implication of this procedure is the following. In the standard RPA treatment, ground-state correlations and collective effects are caused by the coupling terms $\mathcal{W}_{bb'}$. The procedure just outlined implies that collective effects not already included in the standard RPA treatment, i.e., in the transformation leading to the matrix $\bar{\mathbf{D}}$, can be obtained by using perturbation theory. Such additional collective effects might arise from the presence of the continuum. We have not been able to derive physically meaningful sum rules without using perturbation theory.

We write the matrix $\bar{\mathbf{D}}$ in the form

$$\bar{\mathbf{D}}=\begin{pmatrix}\mathsf{A} & \mathsf{C}\\ \mathsf{C}^T & \mathsf{B}\end{pmatrix} \tag{43g}$$

and note that the quantities A, B, C are $M\times M$ matrices. The matrices A and B are symmetric. We also define

the $M\times M$ matrix

$$\mathbf{d}=\mathbf{A}-\mathbf{C}\mathbf{B}^{-1}\mathbf{C}^{T}. \tag{44a}$$

The inverse of $\bar{\mathbf{D}}$ takes the form

$$\bar{\mathbf{D}}^{-1}=\begin{pmatrix} \mathbf{d}^{-1} & -\mathbf{d}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{C}^{T}\mathbf{d}^{-1} & \mathbf{B}^{-1}+\mathbf{B}^{-1}\mathbf{C}^{T}\mathbf{d}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}. \tag{44b}$$

It follows from Eq. (43c) that $\det(\mathbf{B})$ has no zeros in the energy region of interest, its zeros all being located far to the left of this region. In the spirit of the remarks made above Eq. (36), we therefore assume that $\mathbf{B}^{-1}$ is constant in the region of interest. The T matrix (42) can be rewritten identically as follows. We introduce, for each channel c, the vector $\bar{\gamma}^{c}$ of M components $\gamma_b{}^{c}$, $b=1,\cdots,M$, with

$$\gamma_b{}^{c}=\hat{V}_{bc}-\sum_{b'b''}C_{bb'}(B^{-1})_{b'b''}\hat{W}_{b''c}. \tag{45a}$$

According to our assumptions, the quantities $\gamma_b{}^{c}$ are constant in the energy region of interest. Equation (42) takes the form

$$T_{c'c}(E)=(\bar{\gamma}^{c'})^{T}\mathbf{d}^{-1}\bar{\gamma}^{c}+\sum_{bb'}\hat{W}_{c'b}(B^{-1})_{bb'}\hat{W}_{b'c}. \tag{45b}$$

The last term on the right-hand side of Eq. (45b) represents a background term, which, under the assumptions made above, is constant in the energy region of interest. This term is caused by the ground-state correlations in $|\Psi_0\rangle$, as is evident from the occurrence of the matrix elements $\hat{W}_{bc}(E)$. Resonances, i.e., poles of $T_{c'c}(\mathcal{E})$, in the energy region of interest arise only from zeros of $\det(\mathbf{d})$, i.e., they are contained in the first term on the right-hand side of Eq. (45b). According to the defining equation (44a) and our assumptions, the complex, symmetric matrix $\mathbf{d}$ is the sum of the unit matrix multiplied by E and of a *constant* matrix. It can therefore be diagonalized by a complex, orthogonal matrix $\mathcal{O}$,

$$(\mathcal{O}\mathbf{d}\mathcal{O}^{T})_{bb'}=\delta_{bb'}(E-\mathcal{E}_b);$$

$$(\mathcal{O}\mathcal{O}^{T})_{bb'}=\delta_{bb'}=(\mathcal{O}^{T}\mathcal{O})_{bb'}. \tag{46}$$

Hence,

$$(\mathbf{d}^{-1})_{bb'}=\sum_{b''}(\mathcal{O}^{T})_{bb''}(E-\mathcal{E}_{b''})^{-1}\mathcal{O}_{b''b'}. \tag{47}$$

The quantities $\mathcal{E}_b$ ($b=1,\cdots,M$) are the complex resonance energies. Naturally, only those $\mathcal{E}_b$ are physically meaningful that lie in the energy region of interest. For these resonances, we expect that $\mathrm{Im}\mathcal{E}_b<0$, according to the statements made at the end of Sec. III. We now insert Eq. (47) into Eq. (45b) and obtain

$$T_{c'c}(E)=\frac{1}{2\pi}\sum_{b}\frac{\Gamma_{bc'}{}^{1/2}\Gamma_{bc}{}^{1/2}}{E-\mathcal{E}_b}+\sum_{b,b'}\hat{W}_{c'b}(B^{-1})_{bb'}\hat{W}_{b'c}. \tag{48a}$$

We have used

$$\Gamma_{bc}{}^{1/2}=(2\pi)^{1/2}\sum_{b'}\mathcal{O}_{bb'}\gamma_{b'}{}^{c}. \tag{48b}$$

The partial-width amplitudes $\Gamma_{bc}{}^{1/2}$ are complex quantities, both because the matrix $\mathcal{O}$ is complex and because the quantities $\gamma_b{}^{c}$ are defined in terms of complex matrices [see Eq. (45a)]. The matrix $\mathcal{O}$ occurs, in a similar form, also in the TDA version of Eqs. (48). The quantities $\gamma_b{}^{c}$ differ from their TDA analogs by the inclusion of backward-going graphs, both through the last term on the right-hand side of Eq. (45a) and through the definition of $\hat{V}_{bc}$ in Eq. (43a). The question arises whether collective enhancement of the partial-width amplitudes $\Gamma_{bc}{}^{1/2}$ is possible. Such enhancement, if it occurs, is expected to reside primarily in the quantities $\gamma_b{}^{c}$. However, we have not been able to devise a simple mechanism that would yield particularly large values for $\gamma_b{}^{c}$. This question deserves further attention and can probably only be clarified by explicit calculation of the quantities $\gamma_b{}^{c}$. We emphasize that the results obtained so far are independent of any perturbative treatment of far-lying resonances.

It is well known that for overlapping resonances, the sum of the partial widths $\sum_{c,\mathrm{open}}\Gamma_{bc}$ over all open channels may differ from the total width $(-2\,\mathrm{Im}\mathcal{E}_b)$. Only for isolated resonances are the two expressions equal, as a consequence of the unitarity of the S matrix. However, in the TDA it is possible to show that, on the average, i.e., by summing over *all* the resonant states b, the two expressions do become equal, even for overlapping resonances. Can we derive a similar result for the RPA? According to Eq. (48b), we have

$$\sum_{b}\sum_{c,\mathrm{open}}\Gamma_{bc}=2\pi\sum_{c,\mathrm{open}}\sum_{b}(\gamma_b{}^{c})^{2}. \tag{49}$$

We calculate the trace of the matrix $\mathbf{d}$ and find

$$2\,\mathrm{Im}\,\mathrm{Tr}\{\mathbf{d}\}=-2\sum_{b}\mathrm{Im}\mathcal{E}_b=2\,\mathrm{Im}\,\mathrm{Tr}\{\mathbf{A}-\mathbf{C}\mathbf{B}^{-1}\mathbf{C}^{T}\}. \tag{50}$$

A glance at Eq. (45a) shows that the expressions in Eqs. (49) and (50) differ. Indeed, the matrices $\mathbf{C}$ and $\mathbf{B}$ are both complex. The imaginary parts of $\gamma_b{}^{c}$ and of $\sum_{c,\mathrm{open}}\sum_{b}(\gamma_b{}^{c})^{2}$ do not vanish. Thus, the expression given by the right-hand side of Eq. (49) is complex, while Eq. (50) contains a purely real quantity. The two expressions become equal only in lowest (zeroth) order in $\mathbf{C}\mathbf{B}^{-1}$. In this case, we retrieve the TDA result. Since the elements of $\mathbf{C}\mathbf{B}^{-1}$ are expected to be of the order 10 keV/10 MeV (unless coherent summations come into play), this approximation is probably quite sufficient. We also notice that in lowest (zeroth) order in $\mathbf{B}^{-1}\,\mathrm{Im}\mathbf{B}$, the real part of expression (49) equals expression (50). It is interesting, however, that in the RPA, the expressions (49) and (50) are different, in general. This is because in the RPA there must always be some poles outside the energy region of interest, in contrast to the TDA. These poles produce the background in Eq. (48a). Unitarity of the S matrix alone is not sufficient to ensure the equality of the expressions (49) and (50).

The resonance energies $\mathcal{E}_b$ defined by Eq. (46) may not all lie in the energy region of interest. In this case,

even the restricted sum rule obtained in zeroth order in CB^{-1} is physically meaningless. However, a device can be found to remedy this situation. In Eq. (43d), one can define the matrix A to have fewer than M rows and columns, while the size of B is increased correspondingly, and the matrix C becomes a rectangular matrix. Thereby, one includes only part of the resonances in the matrix A and hopefully can select them in such a way that they all lie in the energy region of interest. One can then follow the steps leading from Eqs. (43) to Eq. (50) and obtain, in zeroth order in CB^{-1}, a sum rule for the resonances that lie in the region of interest.

V. CONTINUUM-CONTINUUM INTERACTION

A. General T Matrix

Neglect of the continuum-continuum interaction matrix elements (28) is not generally justifiable. If these matrix elements are not neglected, the second of the Lippmann-Schwinger equations (16) becomes an integral equation with a kernel of infinite rank. It can therefore no longer be solved algebraically. Rewriting Eqs. (16) in operator form we have

$$\mathsf{Y}_{E^c}=\mathsf{E}_B^{-1}\{\mathsf{V}_{BB}\mathsf{Y}_{E^c}+\mathsf{V}_{BC}\mathsf{Z}_{E^c}\},$$
$$\mathsf{Z}_{E^c}=\exp(i\delta_c)\,\mathsf{\Delta}_c+\mathsf{E}_C^{-1}\{\mathsf{V}_{CB}\mathsf{Y}_{E^c}+\mathsf{V}_{CC}\mathsf{Z}_{E^c}\}, \quad (16')$$

where the definition of the functions Y, Z, and $\mathsf{\Delta}$ and the operators E_B, E_C, V_{BB}, V_{BC}, V_{CB}, and V_{CC} is obtained by comparison.

Evaluating the commutator (22b) without the restriction (28), we find

$$[[A_{c'}(E),\,V^B],\,Q_c{}^{(+)\dagger}(E)]=\mathsf{\Delta}_{c'}{}^T\mathsf{V}_{CB}\mathsf{Y}_{E^c}+\mathsf{\Delta}_{c'}{}^T\mathsf{V}_{CC}\mathsf{Z}_{E^c}, \quad (51)$$

from which we get the T matrix by solving Eqs. (16′) for Y and Z. The solution for Y is

$$\mathsf{Y}_{E^c}=\exp(i\delta_c)\,\mathsf{D}^{-1}\mathsf{V}_{BC}(1+\mathsf{F})\,\mathsf{\Delta}_c. \quad (52)$$

The operator D is given by

$$\mathsf{D}=\mathsf{E}_B-\mathsf{V}_{BB}-\mathsf{V}_{BC}(1+\mathsf{F})\mathsf{E}_C^{-1}\mathsf{V}_{CB}. \quad (53)$$

The operator F is defined as resolvent of the operator

$$\mathsf{K}=\mathsf{E}_C^{-1}\mathsf{V}_{CC}=\begin{pmatrix}(E^+-E')^{-1} & 0\\ 0 & (E^++E')^{-1}\end{pmatrix}\times\begin{pmatrix}\mathcal{U}_{c'c''}(E',E'') & \mathcal{W}_{c'c''}(E',E'')\\ -\mathcal{W}_{c'c''}(E',E'') & -\mathcal{U}_{c'c''}(E',E'')\end{pmatrix}. \quad (54)$$

We therefore have

$$(1+\mathsf{F})\,(1-\mathsf{K})=1. \quad (55)$$

In the case of no channel-channel coupling, both operators F and K vanish, and the elements of D are given by Eq. (30c).

Inserting Eq. (52) and the corresponding solution for Z in the commutator (51), we find for the T matrix

$$T_{c'c}=\mathsf{\Delta}_{c'}{}^T\mathsf{V}_{CC}(1+\mathsf{F})\,\mathsf{\Delta}_c+\mathsf{\Delta}_{c'}{}^T\{1+\mathsf{V}_{CC}(1+\mathsf{F})\mathsf{E}_C^{-1}\}\times\mathsf{V}_{CB}\mathsf{D}^{-1}\mathsf{V}_{BC}(1+\mathsf{F})\,\mathsf{\Delta}_c. \quad (56)$$

As in the TD treatment, the transition matrix consists of a part involving only the interaction between the channels and a compound part involving the interaction with the bound states. In the case of no channel-channel coupling, the first part of Eq. (56) vanishes, and the second part is equal to Eq. (30a).

From the defining equation (55) of F, we find

$$\mathsf{F}^T=\mathsf{I}\mathsf{V}_{CC}(1+\mathsf{F})\mathsf{E}_C^{-1}\mathsf{I} \quad (57a)$$

and

$$\mathsf{D}^T=\mathsf{I}\mathsf{D}\mathsf{I}. \quad (57b)$$

Using these relations, we find that each of the two parts of the T matrix (56) is symmetric. It can also be seen that

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathsf{F}(E)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\mathsf{F}(-E) \quad (57c)$$

and that the matrix D defined in Eq. (53) fulfills the same RPA-symmetry relation (33) as the corresponding matrix defined in Sec. IV. It follows that the transition matrix (56) has the same structure and symmetries as Eq. (30) and that the discussion given in Sec. IV can be extended to include the channel-channel interaction.

B. Convergence of the Born Series

The transition matrix given by Eq. (56) can be calculated only if we can construct the resolvent F. A straightforward approach to solving the integral equation (55) is iteration. The resulting Born series

$$\mathsf{F}=\mathsf{K}+\mathsf{K}^2+\mathsf{K}^3+\cdots \quad (58)$$

converges, for a fixed value of E, if and only if all eigenvalues $\eta^{(\alpha)}(E)$ of $\mathsf{K}(E)$ obey the inequality

$$|\,\eta^{(\alpha)}(E)\,|<1. \quad (59)$$

The eigenvalues $\eta^{(\alpha)}(E)$ are defined by

$$\mathsf{E}_C^{-1}\mathsf{V}_{CC}\psi^{(\alpha)}(E)=\eta^{(\alpha)}(E)\psi^{(\alpha)}(E), \quad (60a)$$

where $\psi^{(\alpha)}(E)$ stands for the amplitudes

$$\psi^{(\alpha)}(E)=\begin{pmatrix}\psi_{E^{(\alpha)}}(c',E')\\ -\bar{\psi}_{E^{(\alpha)}}(c',E')\end{pmatrix}. \quad (60b)$$

It follows from the definitions (9) that the matrix elements of the residual interaction can be written as matrix elements between two single-particle states

$$\mathcal{U}_{c'c''}(E',E'')=\langle u_{p'}(k')\,|\,\mathcal{U}_{l'l''}\,|\,u_{p''}(k'')\rangle,$$
$$\mathcal{W}_{c'c''}(E',E'')=\langle u_{p'}(k')\,|\,\mathcal{W}_{l'l''}\,|\,u_{p''}(k'')\rangle. \quad (61)$$

The second of these equations is true because the radial wave functions have been chosen real. With the aid of these expressions, the eigenvalue equation (60) can be

written as a set of coupled integral equations

$$\sum_{c''}\int dE'\,\frac{|\,u_{p'}(k')\rangle\langle u_{p'}(k')\,|}{E-E'}\times\{\mathfrak{V}_{l'l''}\,|\,\phi_{c''}{}^{(\alpha)}(E)\rangle-\mathfrak{W}_{l'l''}\,|\,\bar{\phi}_{c''}{}^{(\alpha)}(E)\rangle\}=\eta^{(\alpha)}(E)\,|\,\phi_{c'}{}^{(\alpha)}(E)\rangle,$$

$$\sum_{c''}\int dE'\,\frac{|\,u_{\Gamma'}(k')\rangle\langle u_{p'}(k')\,|}{E+E'}\times\{\mathfrak{W}_{l'l''}\,|\,\phi_{c''}{}^{(\alpha)}(E)\rangle-\mathfrak{V}_{l'l''}\,|\,\bar{\phi}_{c''}{}^{(\alpha)}(E)\rangle\}=\eta^{(\alpha)}(E)\,|\,\bar{\phi}_{c'}{}^{(\alpha)}(E)\rangle, \tag{62a}$$

for the functions

$$|\,\phi_{c'}{}^{(\alpha)}(E)\rangle=\int dE'\psi_E{}^{(\alpha)}(c',E')\,|\,u_{p'}(k')\rangle,$$
$$|\,\bar{\phi}_{c'}{}^{(\alpha)}(E)\rangle=\int dE'\bar{\psi}_E{}^{(\alpha)}(c',E')\,|\,u_{p'}(k')\rangle. \tag{62b}$$

We assume that these functions, as well as the functions $\eta^{(\alpha)}(E)$, can be continued analytically to complex energies $\mathcal{E}$. Whenever one of the functions $\eta^{(\alpha)}(\mathcal{E})$ has a pole, there is a region in the complex-energy plane surrounding the pole for which $|\,\eta^{(\alpha)}(\mathcal{E})\,|\geq 1$. If one of these poles lies close to the real axis, the corresponding region overlaps the real axis, and the convergence condition (59) is violated.

Inspection of Eq. (62a) shows that $\eta^{(\alpha)}(\mathcal{E})$ has a pole wherever there is a single-particle resonance. Indeed, such a resonance corresponds to a pole situated on one of the nonphysical sheets, of at least one of the expressions

$$\int dE'\,\frac{|\,u_{p'}(k')\rangle\langle u_{p'}(k')\,|}{E-E'}. \tag{63a}$$

Near the pole[8a] with energy $\mathcal{E}^{c_0}$, this integral can be approximated by[4]

$$\int dE'\,\frac{|\,u_{p'}(k')\rangle\langle u_{p'}(k')\,|}{E-E'}\approx\frac{|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\rangle\langle w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,|}{E-\mathcal{E}^{c_0}}\times\delta_{p'p_0}. \tag{63b}$$

The resonance energy $\mathcal{E}^{c_0}$ is given by

$$\mathcal{E}^{c_0}=\mathcal{E}_{p_0}+\epsilon_{c_0}, \tag{63c}$$

while the single-particle Gamow function $w_{p_0}{}^{\rm res}(\mathcal{E}_{\Gamma_0})$ is a solution of

$$(\mathcal{E}_{p_0}-k_0)\,|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\rangle=0, \tag{63d}$$

which behaves asymptotically like an outgoing wave. This condition can only be met at discrete (complex) values $\mathcal{E}_{p_0}$ of the energy. These are the energies of the single-particle resonances. In the following, we assume that values of $\eta^{(\alpha)}(E)$ for which $|\,\eta^{(\alpha)}(E)\,|>1$ on the real E axis occur only because there are single-particle resonances close to the real E axis. This is consistent with the main assumption made in this paper, namely, that the combined HF and RPA procedures give a good approximation to the actual nuclear wave function, and that the residual interactions to the HF are weak. For a more detailed discussion, we refer to Ref. 4.

If $\eta^{(\alpha)}(\mathcal{E})$ has a pole at $\mathcal{E}^{c_0}$, the coupled equations (62a) can be reduced to

$$(\mathcal{E}^{c_0}-h_0-\epsilon_{c'})\,|\,\phi_{c'}{}^{(\alpha)}(\mathcal{E}^{c_0})\rangle=0,$$
$$(\mathcal{E}^{c_0}+h_0+\epsilon_{c'})\,|\,\bar{\phi}_{c'}{}^{(\alpha)}(\mathcal{E}^{c_0})\rangle=0. \tag{64}$$

These equations do not contain the residual interaction, and their solution is

$$|\,\phi_{c'}{}^{(\alpha)}(\mathcal{E}^{c_0})\rangle=N_{c_0}{}^{(\alpha)}\,|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p0})\rangle\delta_{c'c_0},$$
$$|\,\bar{\phi}_{c'}{}^{(\alpha)}(\mathcal{E}^{c_0})\rangle=0. \tag{65}$$

In the vicinity of the energy $\mathcal{E}^{c_0}$, the background scattering function for elastic scattering from the HF potential v_0 is given by the one-level approximation[4]

$$S_{c_0c_0}{}^{(0)}(E)=\exp(2i\delta_{c_0})\approx\exp(2i\xi_0)[(E-\mathcal{E}^{c_0*})/(E-\mathcal{E}^{c_0})]. \tag{66}$$

Here, ξ_0 is the background potential scattering phase shift.

With this, we have established the connection between the poles of the functions $\eta^{(\alpha)}(\mathcal{E})$ and the narrow single-particle resonances. In order to simplify the presentation, we assume that only one of these functions has a pole in the energy region of interest. The generalization to the case where more of the functions have poles is straightforward. The number of $\eta^{(\alpha)}$ with $|\,\eta^{(\alpha)}(E)\,|>1$ is always finite, for each value of the energy E.[9]

The residue of the function $\eta(\mathcal{E})$ at the pole $\mathcal{E}^{c_0}$ is obtained from Eqs. (62) with the aid of the approximate relation (63b). Using this property and Eq. (65), we find

$$\eta(E)\approx N/(E-\mathcal{E}^{c_0}) \tag{67a}$$

with

$$N=\langle w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,|\,\mathcal{V}_{l_0l_0}\,|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\rangle. \tag{67b}$$

C. Separable Approximation

As we saw in Sec. V B, the series (58) cannot be used for the calculation of the resolvent $\mathbf{F}$, if narrow single-particle resonances exist. In this case, the scattering matrix can be obtained with a method proposed by Weinberg[10] and applied to nuclear reactions by Glöckle, Hüfner, and Weidenmüller.[9] The method consists in a modification of the continuum-continuum interaction operator $\mathbf{V}_{CC}$. An operator of finite rank $\mathbf{V}_S$ is subtracted from the full operator $\mathbf{V}_{CC}$ in such a way that the norm of the difference operator $\mathbf{E}_C{}^{-1}\{\mathbf{V}_{CC}-\mathbf{V}_S\}$ is sufficiently small to allow a perturbation calculation of the resolvent

$$\mathbf{F}_1=\mathbf{E}_C{}^{-1}\{\mathbf{V}_{CC}-\mathbf{V}_S\}+\mathbf{E}_C{}^{-1}\{\mathbf{V}_{CC}-\mathbf{V}_S\}\mathbf{E}_C{}^{-1}\{\mathbf{V}_{CC}-\mathbf{V}_S\}\cdots. \tag{68}$$

[8a] We assume that there is only one single-particle resonance associated with definite angular momentum and spin of the particle. Hence, the Kronecker symbol on the right-hand side of Eq. (63b).

[9] W. Glöckle, J. Hüfner, and H. A. Weidenmüller, Nucl. Phys. A90, 481 (1967).

[10] S. Weinberg, Phys. Rev. 130, 776 (1963); 131, 440 (1964).

Since the operator V_S is of finite rank, the resolvent of $\mathsf{K}_S=\mathsf{E}_C^{-1}\mathsf{V}_S$ can be constructed algebraically, and the full resolvent F is readily obtained.

It follows from the discussion in the last subsection that the ideal choice for the separable potential V_S is such that the eigenvalues of $\mathsf{K}-\mathsf{K}_S$ are identical to those of K, except for the finite number p of $\eta^{(\alpha)}(E)$ ($\alpha=1,\cdots,p$), which violate the convergence condition (59). The corresponding p eigenvectors of K will become null eigenvectors of $\mathsf{K}-\mathsf{K}_S$ if K_S is chosen ideally. As before, we assume $p=1$ for reasons of simplicity. The ideal choice $\mathsf{V}_S^{(0)}$ for the separable potential is then given by

$$\mathsf{V}_S^{(0)}=[1/N^{(0)}(E)]\mathsf{V}_{CC}\psi\psi^T\mathsf{V}_{CC} \tag{69a}$$

with

$$N^{(0)}(E)=\psi^T\mathsf{V}_{CC}\psi. \tag{69b}$$

With these definitions we have

$$\mathsf{V}_S^{(0)}\psi(E)=\mathsf{V}_{CC}\psi(E), \tag{69c}$$

and the operator $\mathsf{K}_S^{(0)}=\mathsf{E}_C^{-1}\mathsf{V}_S^{(0)}$ has the same poles and residues as the full operator K defined in Eq. (54). All eigenvalues of the difference operator $\mathsf{K}-\mathsf{K}_S^{(0)}$ have magnitude less than one. The resolvent of $\mathsf{K}-\mathsf{K}_S^{(0)}$ can thus be calculated with the Born series (68).

The prescription (69) for the construction of the ideal operator $\mathsf{V}_S^{(0)}$ can only be followed exactly if the eigenfunctions of K are known. This is usually not the case. It is, however, possible to replace these functions by approximate functions. If the latter functions are chosen in such a way that they agree, at $E=\mathcal{E}^{c0}$, with the exact eigenfunctions of K, it is still possible to split the kernel K into a separable part, which contains the effects of the poles, and a nonseparable part, which can be treated as perturbation. It can be seen from Eq. (65) that a possible choice for the approximate functions is given by

$$\begin{aligned}|\,W_{c'}(E)\,\rangle&=|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,\rangle\delta_{c'c_0},\\ |\,\bar{W}_{c'}(E)\,\rangle&=0.\end{aligned} \tag{70}$$

A different choice for $|\,W(E)\,\rangle$ is discussed in Ref. 4. Using the choice (70), we define

$$\mathsf{V}_{CC}\mathsf{W}=\begin{pmatrix}\langle u_{p'}(k')\,|\,\mathcal{U}_{l'l_0}\,|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,\rangle\\ \\ \langle u_{p'}(k')\,|\,\mathcal{W}_{l'l_0}\,|\,w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,\rangle\end{pmatrix}. \tag{71}$$

With this and Eq. (67b), we define the separable potential

$$\mathsf{V}_S=(1/N)\mathsf{V}_{CC}\mathsf{W}\mathsf{W}^T\mathsf{V}_{CC}. \tag{72}$$

It can be easily checked that the operator

$$\mathsf{K}_S=\mathsf{E}_C^{-1}\mathsf{V}_S \tag{73}$$

has, at $\mathcal{E}=\mathcal{E}^{c0}$, the same residue as K. In order to evaluate the transition matrix, we write the formal solution of the RPA equations (16′)

$$\mathsf{Z}_E{}^c=\exp(i\delta_c)\,\mathbf{\Delta}_c+\mathsf{K}_B\mathsf{Z}_E{}^c+\mathsf{K}_S\mathsf{Z}_E{}^c+(\mathsf{K}-\mathsf{K}_S)\mathsf{Z}_E{}^c \tag{74a}$$

with

$$\mathsf{K}_B=\mathsf{E}_C^{-1}\mathsf{V}_{CB}(\mathsf{E}_B-\mathsf{V}_{BB})^{-1}\mathsf{V}_{BC} \tag{74b}$$

and

$$\mathsf{K}_S=\mathsf{E}_C^{-1}\mathsf{V}_{CC}\mathsf{W}(1/N)\mathsf{W}^T\mathsf{V}_{CC}. \tag{74c}$$

The kernels K_B and K_S have finite ranks $2M$ and 1, respectively. The resolvent of $(\mathsf{K}_B+\mathsf{K}_S)$ can therefore be calculated algebraically. The resolvent of $\mathsf{K}-\mathsf{K}_S$, on the other hand, is obtained from the Born series (68) within our approximation. Carrying through this procedure, we find

$$\begin{aligned}T_{c'c}=&\,\mathbf{\Delta}_{c'}{}^T(\mathsf{V}_{CC}-\mathsf{V}_S)(1+\mathsf{F}_1)\mathbf{\Delta}_c\\ &+\mathbf{\Delta}_{c'}{}^T\{1+(\mathsf{V}_{CC}-\mathsf{V}_S)(1+\mathsf{F}_1)\mathsf{E}_C^{-1}\}\mathsf{V}_{CG}\mathsf{D}_G^{-1}\mathsf{V}_{GC}(1+\mathsf{F}_1)\mathbf{\Delta}_c,\end{aligned} \tag{75a}$$

where the index G indicates that one row or one column, or both, are added to the corresponding matrices occurring in Eq. (56). This is because we explicitly treated the single-particle resonance, approximated by a Gamow function. We have

$$\mathsf{V}_{GC}=\begin{pmatrix}\mathcal{U}_{bc'}(E') & \mathcal{W}_{bc'}(E')\\ -\mathcal{W}_{bc'}(E') & -\mathcal{U}_{bc'}(E')\\ \mathcal{U}_{2M+1,c'}(E') & \mathcal{W}_{2M+1,c'}(E')\end{pmatrix} \tag{75b}$$

and

$$\mathsf{V}_{CG}=\mathsf{I}\mathsf{V}_{GC}{}^T\mathsf{I}_G \tag{75c}$$

with

$$\begin{aligned}\mathcal{U}_{2M+1,c'}(E')&=\langle w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,|\,\mathcal{U}_{l_0l'}\,|\,u_{p'}(k')\,\rangle,\\ \mathcal{W}_{2M+1,c'}(E')&=\langle w_{p_0}{}^{\rm res}(\mathcal{E}_{p_0})\,|\,\mathcal{W}_{l_0l'}\,|\,u_{p'}(k')\,\rangle.\end{aligned} \tag{75d}$$

The metric matrix I_G is defined by

$$\mathsf{I}_G=\begin{pmatrix}\mathsf{I} & 0\\ 0 & 1\end{pmatrix}. \tag{75e}$$

For the $(2M+1)\times(2M+1)$ matrix D_G we obtain

$$\mathsf{D}_G=\begin{pmatrix}(E-E_b)\delta_{bb'}-\mathcal{U}_{bb'} & -\mathcal{W}_{bb'} & 0\\ +\mathcal{W}_{bb'} & (E+E_b)\delta_{bb'}+\mathcal{U}_{bb'} & 0\\ 0 & 0 & N\end{pmatrix}-\mathsf{V}_{GC}(1+\mathsf{F}_1)\mathsf{E}_C^{-1}\mathsf{V}_{CG}. \tag{76}$$

It can easily be checked that $\mathsf{D}_G\mathsf{I}_G$ is symmetric and therefore that both parts of the T matrix are symmetric, too.

The behavior of the scattering matrix in the vicinity of the resonance energy $\mathcal{E}^{c0}$ is best seen if the T matrix

(75a) is expressed as ratio of two determinants. Inserting this into Eq. (22a), we obtain

$$S_{c'c}(E)=\frac{\exp(i\delta_c+i\delta_{c'})}{\det \mathsf{D}_G}\begin{vmatrix} \mathsf{D}_G & 2i\pi \mathsf{V}_{GC}(1+\mathsf{F}_1)\,\mathbf{\Delta}_c \\ \mathbf{\Delta}_{c'}{}^T(1+\mathsf{F}_1{}^T)\mathsf{V}_{GC}{}^T\mathsf{I}_G & \delta_{cc'}-2i\pi\mathbf{\Delta}_{c'}{}^T(\mathsf{V}_{CC}-\mathsf{V}_S)(1+\mathsf{F}_1)\mathbf{\Delta}_{c'} \end{vmatrix}. \tag{77}$$

In order to illustrate the physical content of this expression, we confine ourselves, for simplicity, to one continuum of states, i.e., to the elastic scattering of nucleons in a state of fixed spin and parity. Near the resonance energy, we use the approximate relation (63b) to obtain

$$\det \mathsf{D}_G=\frac{N}{E-\mathcal{E}^{c0}}\begin{vmatrix} (E-E_b)\delta_{bb'}-\mathcal{U}_{bb'} & -\mathcal{W}_{bb'} & \mathcal{U}_{b,2M+1} \\ +\mathcal{W}_{bb'} & (E+E_b)\delta_{bb'}+\mathcal{U}_{bb'} & -\mathcal{W}_{b,2M+1} \\ \mathcal{U}_{2M+1,b'} & \mathcal{W}_{2M+1,b'} & E-\mathcal{E}^{c0}-N \end{vmatrix}, \tag{78a}$$

where the quantities

$$\begin{aligned}\mathcal{U}_{b,2M+1}&=\langle w_{p'}{}^b \mid \mathcal{U}_{l'l_0} \mid w_{p_0}{}^{\mathrm{res}}(\mathcal{E}_{p_0})\rangle,\\ \mathcal{W}_{b,2M+1}&=\langle w_{p'}{}^b \mid \mathcal{W}_{l'l_0} \mid w_{p_0}{}^{\mathrm{res}}(\mathcal{E}_{p_0})\rangle\end{aligned} \tag{78b}$$

have been defined in analogy to Eqs. (75d).

Comparing Eq. (78a) with the background scattering phase shift (66), we see that the two denominators are the same. It can also be shown that the determinant in the numerator of Eq. (77) has a pole at $\mathcal{E}=\mathcal{E}^{c0}$*, which is canceled by the numerator of Eq. (66). As a result, the scattering matrix has no pole at the single-particle resonance energy $\mathcal{E}^{c0}$, but its poles are given by the zeros of the determinant (78a). In keeping with the discussion in the last section, we see that this determinant has $M+1$ zeros at positive energies and M zeros at negative energies. This asymmetry could be removed if the Weinberg state corresponding to negative single-particle energy would be included in the definition of the separable potential. This, however, is not practical, since this energy is far outside our region of interest. It is apparent from Eq. (78a) that the M zeros of the determinant at negative energies would not contribute to the cross section in the physical region at all, if the backward-going interaction $\mathcal{W}$ were switched off. From the similarity between Eq. (75a) and the corresponding result (30) of Sec. IV, it can be seen that pole decomposition, sum rules, and the separation of the poles at negative energies can be carried through by the same methods. The matrices $\mathcal{O}$, $\bar{\mathbf{D}}$, A, and $\mathbf{d}$ defined in Sec. IV have to be augmented by one row and one column, and a background part of the S matrix can be pulled out in a similar way.

VI. LINEAR MOMENTUM

The exact Hamiltonian H has a number of symmetry properties. It conserves particle number, linear momentum, angular momentum, and electric charge. The spherical HF procedure with separate orbits for neutrons and protons[11] conserves all of these quantum numbers except for linear momentum. There is a danger, then, in making an approximation to the residual interaction to obtain approximate excited states, that these excited states may have spurious components in them that correspond to the motion of the c.m. of the nucleus. Such a problem is certainly present in the TDA. However, the RPA separates out exactly the c.m. modes from the intrinsic modes of excitation. This fact follows from the translational and Galilean invariance of the interaction V. The former states that the linear momentum commutes with H,

$$[H, P_q]=0, \tag{79a}$$

and the latter leads to

$$[H, R_q]=[(P^2/2AM), R_q]=(\hbar/iAM)P_q, \tag{79b}$$

where P_q and R_q are the qth components ($q=1, 2, 3$) of the total linear-momentum operator and c.m. coordinate, respectively,

$$P_q=(\hbar/i)\sum_{j=1}^{A}\nabla_q(j), \qquad R_q=(1/A)\sum_{j=1}^{A}r_q(j), \tag{79c}$$

$$[P_q, R_{q'}]=(\hbar/i)\delta(q, q'). \tag{79d}$$

The RPA to P_q and R_q is derived by taking the p-h parts of these operators and replacing the p-h operators by bosons as prescribed in Eq. (5). Writing the p-h matrix elements of P_q as

$$\begin{aligned}\mathcal{P}_b{}^q&\equiv\langle p \mid P_q \mid l\rangle, && p \text{ bound}\\ \mathcal{P}_c{}^q(E)&\equiv\langle p \mid P_q \mid l\rangle, && p \text{ continuous}\end{aligned} \tag{80a}$$

the boson form of P_q is

$$P_q{}^B=\sum_b \mathcal{P}_b{}^q(A_b{}^\dagger-A_b)$$

$$+\sum_c\int_{\epsilon_c}^{\infty}dE\,\mathcal{P}_c{}^q(E)\,(A_c{}^\dagger(E)-A_c(E)). \tag{80b}$$

The linear combination $(A^\dagger-A)$ comes from the fact that P_q is an imaginary Hermitian operator. For $R_q{}^B$ we have

$$R_q{}^B=\sum_b \mathcal{R}_b{}^q(A_b{}^\dagger+A_b)$$

$$+\sum_c\int_{\epsilon_c}^{\infty}dE\,\mathcal{R}_c{}^q(E)\,(A_c{}^\dagger(E)+A_c(E)). \tag{80c}$$

[11] In principle, the HF energy could be lowered by mixing neutron and proton orbitals through the exchange part of the nuclear force. For even-even self-conjugate ($T_z=0$) nuclei, this will not happen. Furthermore, we assume it does not happen for spherical even-even nuclei, even if there is a neutron excess.

The linear combination $(A^\dagger+A)$ comes here because R_q is a real Hermitian operator. Since the boson commutation rules differ from those between p-h operators only by p-p and h-h terms, one finds that the commutation rules given in Eq. (79) are preserved in the RPA; that is,

$$[H^B, P_q{}^B]=0, \tag{81a}$$

$$[H^B, R_q{}^B]=(\hbar/iAM)P_q{}^B, \tag{81b}$$

$$[P_q{}^B, R_{q'}{}^B]=(\hbar/i)\delta_{q,q'}. \tag{81c}$$

Clearly, $P_q{}^B$ is an eigenmode of H^B, with zero eigenenergy. It commutes with both the bound and the scattering modes of excitation discussed in Sec. II. This is easily seen by commuting Eq. (10a) with $P_q{}^B$. Using Jacobi's identity for double commutators, with the fact that the commutator of two bosons is a c number, we have

$$[P_q{}^B, Q_\mu{}^\dagger]=0. \tag{82a}$$

Since $P_q{}^B$ is Hermitian, it follows that it also commutes with Q_μ. The operators $Q_\mu{}^\dagger$, Q_μ also commute with $R_q{}^B$. To see this, we commute again Eq. (10a) with $R_q{}^B$, use Jacobi's identity and Eqs. (81b) and (82a), and we find

$$[R_q{}^B, Q_\mu{}^\dagger]=[R_q{}^B, Q_\mu]=0. \tag{82b}$$

Thus the RPA excited modes are completely intrinsic modes of excitation. We have the freedom, as given in Eq. (12b), to choose the RPA ground state so that it is an eigenstate of the RPA linear-momentum operator, with eigenvalue zero[12]:

$$P_q{}^B \mid \Psi_0\rangle=0. \tag{83}$$

Because of Eq. (82a), the RPA excited states are also eigenstates of $P_q{}^B$, with eigenvalue zero. In particular, the scattering matrix derived in Sec. II is an intrinsic scattering matrix; that is, it is the scattering matrix in the c.m. frame.

The point may be raised that the RPA states are not eigenstates of the full linear-momentum operator P_q. However, Marshalek and Weneser[13] have emphasized that this is not the correct interpretation of the RPA. The exact dependence of the exact eigenstates of H on the linear momentum is known simply because of the translational and Galilean invariance of V. This tells us that an exact eigenstate (including scattering states), with total momentum $\mathbf{k}$ has the form (assuming box normalization)

$$\mid \mathrm{X}_{\mathrm{k},\mu}\rangle=N^{-1/2}\exp(i\mathbf{k}\cdot\mathbf{R})\mid \mathrm{X}_{0,\mu}\rangle, \tag{84a}$$

where $\mid \mathrm{X}_{0,\mu}\rangle$ is an intrinsic state; that is,

$$P_q \mid \mathrm{X}_{0,\mu}\rangle=0. \tag{84b}$$

The c.m. and the total momentum dependence of all operators are also known exactly and can be separated from the intrinsic part. Hence, we need only worry about the intrinsic structure of our states, since the only quantities of physical interest are the matrix elements of intrinsic operators. As seen above, the RPA produces intrinsic states to the RPA order; that is, the RPA to the linear-momentum operator gives zero when it operates on the intrinsic states, Eq. (83). The RPA order keeps terms in H quadratic in the bosons, but no higher. If we try to improve the approximation by including the cubic terms in the Hamiltonian as well, then the improved approximation to the linear-momentum operator will retain this property with respect to the new intrinsic states, so long as we keep *all* the cubic terms in the Hamiltonian. So it is with higher orders: As long as we are consistent and keep all boson terms to a given power, the resulting states will be intrinsic states.

Equation (81b) introduces an inhomogeneous set of RPA equations, unlike the homogeneous equations encountered in solving for the eigenmodes of H^B. Such a set of equations will occur only if there is an eigenboson with zero energy. Since we expect the ground state to be nondegenerate, such zero-energy modes occur only when an exact symmetry of H is broken in the HF approximation. From what was said at the beginning of this section, the $P_q{}^B$ are the only zero-energy bosons, and there is no other set of inhomogeneous equations than (81b). Knowing this, we may write H^B in its diagonal form. Denoting the bound eigenbosons by $Q_\mu{}^\dagger$, we have

$$H^B=E^{\mathrm{RPA}}+\sum_\mu E_\mu Q_\mu{}^\dagger Q_\mu+\sum_c\int_{\epsilon_c}^\infty dE\, EQ_c{}^{(+)\dagger}(E) \times Q_c{}^{(+)}(E)+(P^{B2}/2AM). \tag{85}$$

The c.m. energy is separated from the intrinsic energy dependence. In particular, the RPA ground-state energy E^{RPA} has the HF c.m. energy,

$$E_{\mathrm{c.m.}}{}^{\mathrm{HF}}=\langle\Phi_0 \mid P^2/2AM \mid \Phi_0\rangle,$$

subtracted out. The separation of the c.m. effects by the RPA that has been discussed in this section makes it unnecessary for us to choose, in the HF procedure, an intrinsic Hamiltonian,[14]

$$H_I=H-P^2/2AM. \tag{86}$$

A Lippmann-Schwinger equation cannot be written for the scattering states of such a Hamiltonian in the HF approximation. To see this, we note that the two-body part of $P^2/2AM$ gives a velocity-dependent interaction

$$\frac{1}{AM}\sum_{1=i<j}^{A} p(i)\cdot p(j).$$

[12] Because of this condition, $\mid \Psi_0\rangle$ is not normalizable. However, this does not affect the calculations of any intrinsic matrix elements, since the overlap $\langle\Psi_0 \mid \Psi_0\rangle$ always cancels out. The scattering matrix is, for example, given by

$$\delta(E-E')\,S_{c'c}(E)=\langle\Psi_0 \mid Q_{c'}{}^{(-)}(E')Q_c{}^{(+)\dagger}(E) \mid \Psi_0\rangle/\langle\Psi_0 \mid \Psi_0\rangle =[Q_{c'}{}^{(-)}(E'), Q_c{}^{(+)\dagger}(E)];$$

thus it agrees with Eq. (21).

[13] E. R. Marshalek and J. Weneser, Ann. Phys. (N.Y.) **53**, 569 (1969).

[14] A. K. Kerman, J. P. Svenne, and F. M. H. Villars, Phys. Rev. **147**, 710 (1966).

Such an interaction does not have a finite range, and thus the scattering solution is not asymptotically a free particle plus target nucleus. Physically this makes sense, since the outgoing nucleon has a constraint on its coordinate, namely,

$$r_q(A) = -\sum_{j=1}^{A-1} r_q(j),$$

and is not independent of the coordinates of the other nucleons.

It was stated in Sec. IV that, if the S matrix is continued analytically in energy, it has poles at energies corresponding to the eigenenergies E_μ of H^B. We saw above that H^B also has zero eigenenergies corresponding to the linear-momentum operators $P_q{}^B$. The question arises whether the S matrix has poles at zero energy. The answer is no. To show this, we expand the scattering solutions in terms of the complete set of bosons $\{Q_\mu{}^\dagger, Q_\mu, Q_c{}^{(+)\dagger}(E), Q_c{}^{(+)}(E), P_q{}^B, R_q{}^B\}$. Using similar arguments to those used in Sec. II for expanding the scattering solutions in terms of the complete set of bosons $\{A_b{}^\dagger, A_b, A_c{}^\dagger(E), A_c(E)\}$, we can derive the result

$$Q_c{}^{(+)\dagger}(E) = \exp(i\delta_c)\Big\{A_c{}^\dagger(E) + \sum_{c'}\int_{\epsilon_{c'}}^{\infty} dE'\Big(\frac{1}{E^+ - E'}[Q_{c'}{}^{(+)}(E'), [V^B, A_c{}^\dagger(E)]]Q_{c'}{}^{(+)\dagger}(E')$$
$$- \frac{1}{E+E'}[Q_{c'}{}^{(+)\dagger}(E'), [V^B, A_c{}^\dagger(E)]]Q_{c'}{}^{(+)}(E')\Big) + \sum_\mu\Big(\frac{1}{E-E_\mu}[Q_\mu, [V^B, A_c{}^\dagger(E)]]Q_\mu{}^\dagger$$
$$- \frac{1}{E+E_\mu}[Q_\mu{}^\dagger, [V^B, A_c(E)]]Q_\mu\Big) + \frac{i}{\hbar E}\sum_{q=1}^{3}\Big[[P_q{}^B, [V^B, A_c{}^\dagger(E)]]R_q{}^B$$
$$+\Big(\frac{\hbar}{iAME}[P_q{}^B, [V^B, A_c{}^\dagger(E)]] - [R_q{}^B, [V^B, A_c{}^\dagger(E)]]\Big)P_q{}^B\Big]\Big\}. \quad (87)$$

From the expression for the T matrix given in Eq. (22b), we see that the singularities encountered in the analytical continuation of the T matrix discussed in Sec. IV coincide with those of $\exp(-i\delta_c)Q_c{}^{(+)\dagger}(E)$, since $A_c(E)$ is well behaved. The second term in Eq. (87) gives rise to branch points at $E=\pm\epsilon_c$. The third term produces poles at $E=\pm E_\mu$. The last term appears to give a pole at $E=0$. Such a separation of pole terms is valid because, if a term in Eq. (87) gives a pole at a given energy, then it must be an eigenboson of H^B. To see this, we assume that the eigenboson equation (10a) can be analytically continued. Now let us suppose that $Q_c{}^{(+)\dagger}(E)$ has a pole at $E=\mathcal{E}$; we separate the pole term, call it $I^\dagger(E)$, from the regular part $J^\dagger(E)$,

$$Q_c{}^{(+)\dagger}(E) = \exp(-i\delta_c)\{[1/(E-\mathcal{E})]I^\dagger(E) + J^\dagger(E)\}. \quad (88a)$$

Premultiplying Eq. (10a) by $(E-\mathcal{E})$ and taking the limit $E\to\mathcal{E}$, we get

$$[H^B, I^\dagger(\mathcal{E})] = \mathcal{E}I^\dagger(\mathcal{E}). \quad (88b)$$

Hence, the pole part of $Q_c{}^\dagger(\mathcal{E})$ must be proportional to the eigenboson of H^B with eigenenergy $\mathcal{E}$.

This means that, to investigate the analytic behavior of $\exp(-i\delta_c)Q_c{}^\dagger(E)$ as $E\to 0$, we need only look at the coefficient of $P_q{}^B$ given in Eq. (87). We can simplify this coefficient by putting $V^B = H^B - H_0{}^B - E^{\mathrm{HF}}$ in the double commutators. By using the fact that the $A^\dagger$'s are eigenbosons for the unperturbed Hamiltonian

$$[H_0{}^B, A_c{}^\dagger(E)] = EA_c{}^\dagger(E), \quad (89a)$$

along with Jacobi's identity and Eqs. (80) and (81), we find that

$$[P_q{}^B, [V^B, A_c{}^\dagger(E)]] = E[A_c{}^\dagger(E), P_q{}^B] = E\mathcal{P}_c{}^q(E), \quad (89b)$$

$$[R_q{}^B, [V^B, A_c{}^\dagger(E)]]$$
$$= E[A_c{}^\dagger(E), R_q{}^B] + (\hbar/iAM)[A_c{}^\dagger(E), P_q{}^B]$$
$$= -E\mathcal{R}_c{}^q(E) + (\hbar/iAM)\mathcal{P}_c{}^q(E). \quad (89c)$$

The coefficient of $P_q{}^B$ in Eq. (87) then becomes just $(i/\hbar)\mathcal{R}_c{}^q(E)$. The problem reduces to investigating the analytic continuation of the single-particle matrix elements $\mathcal{R}_c{}^q(E)$. We need only look at the radial matrix element. This radial matrix element of $\mathcal{R}_c{}^q(E)$ is of the form

$$\frac{1}{A}\int_0^\infty dr\, u_{l'j'}(r,k) r w_{nlj}{}^b(r). \quad (90)$$

In this matrix element, u_{lj} and $w_{nlj}{}^b$ are the wave functions of the particle and hole, respectively, in the channel c. The energy is given by $E=\hbar^2k^2/2M-\epsilon_{nlj}$. Continuing the energy E to zero means continuing the particle energy $\hbar^2k^2/2M$ to take on the value of the hole energy ϵ_{nlj}. Thus, the question reduces to studying the behavior of the single-particle wave function $u_{l'j'}(r,k)$, as k is continued to $[(2M/\hbar^2)\epsilon_{nlj}]^{1/2}$. This wave function does have poles at energies corresponding to energies of single-particle bound states with the *same* orbital and total angular momentum that u_{lj} itself has. However, the hole wave function in the matrix element in Eq. (90) can never have the same orbital angular momentum as

the particle wave function in (90) (i.e., $l' \neq l$) because the c.m. operator changes the parity. Thus, $u_{nlj}(k, r)$ does not have a pole for k continued to $[(2M/\hbar^2)\epsilon_{nlj}]^{1/2}$, and consequently $\mathcal{R}_c^q(E)$ does not have a pole at $E=0$. Thus, the T matrix does not have a pole at $E=0$. This result is satisfactory because the zero corresponding to Eq. (81a) follows from the translational invariance of H^B (to the RPA order). The eigenmodes P_q^B are the generators of these translations. They do not generate internal excitations of the nucleus, but they generate translations of the nucleus as a whole and thus do not affect the scattering matrix in the c.m. frame.

VII. CONCLUSIONS

Using single-particle wave functions determined from a HF potential, we have written down the RPA equations that determine the scattering of nucleons by nuclei lacking one particle from being doubly magic. It was assumed that states of the target and residual nuclei can be described as 1-h states in the correlated ground state $|\Psi_0\rangle$ of the RPA for the compound system. This assumption is probably one of the most severe of our approach. It can only be removed at the expense of considerable formal complications. We found that the RPA equations allow for the proper definition of asymptotic states only if the full Hamiltonian H is used in the HF procedure, rather than the intrinsic Hamiltonian given in Eq. (86).

The scattering eigenfunctions of the system determined by the RPA are linear combinations of n-particle–n-hole states, with $n=1, 3, 5 \cdots$, and thus contain states with 1, 3, 5, $\cdots$ nucleons in continuum orbitals of the HF Hamiltonian. In spite of this fact, the RPA equations are free from the mathematical complications of the quantum-mechanical three- and many-body scattering problem and have the structure of equations typically obtained in many-channel scattering. This is so because of the basic RPA assumption that excited states of the system (including the scattering states) are obtained by applying a boson operator $Q_\mu^\dagger$ to $|\Psi_0\rangle$. The operator $Q_\mu^\dagger$ is written as a linear combination of boson creation and destruction operators $A^\dagger$, A, with coefficients Y, $\bar{Y}$, Z, $\bar{Z}$. The RPA equations are linear equations for these coefficients and hence do not refer to more than one nucleon in a continuum orbital. They are therefore "connected" equations in the sense of Weinberg[10] and thus well behaved. In other words, the potentially dangerous many-particle–many-hole states all appear in the wave function of the correlated RPA ground state. There they cannot cause any difficulties, because the intrinsic part of $|\Psi_0\rangle$ is square-integrable.

Without resorting to separable two-nucleon potentials, we have given a general formula for the T matrix. For translationally and Galilean-invariant two-nucleon interactions, it was shown that this is the T matrix in the c.m. frame, in the RPA. All eigenfunctions constructed from the RPA are also eigenfunctions of the RPA to the linear-momentum operator. Failure of conservation of linear momentum can only be caused by a breakdown of the fundamental RPA assumptions, not by an incorrect treatment of the linear-momentum operator within the RPA. While $E=0$ is an eigenvalue of the RPA equations (with the linear-momentum operator as eigenboson), $E=0$ does not correspond to a singularity of the RPA T matrix.

From the general form of the T matrix, it was shown that the S matrix is unitary and symmetric. The form (56) for $\mathbf{T}$, while quite general, is not transparent, because it contains a yet-unknown resolvent operator $\mathbf{F}$. This is so because the RPA equations constitute a system of integral equations of infinite rank and therefore cannot be solved by algebraic means. In order to display the structure and energy dependence of $\mathbf{T}$ explicitly, we constructed the resolvent operator $\mathbf{F}$ following a procedure suggested by Weinberg.[10] $\mathbf{F}$ is the resolvent of the integral operator containing the channel-channel coupling $\mathbf{V}_{CC}$. While it was found that the Born series for $\mathbf{F}$ may diverge, and hence cannot always be used to calculate $\mathbf{F}$, the causes of the divergence were exhibited. It was seen that single-particle resonances in the HF field are the cause of such divergences. It was found possible, in full analogy to the TDA,[4,9] to introduce a separable potential $\mathbf{V}_S$ in such a way that the Born series for the operator containing $(\mathbf{V}_{CC}-\mathbf{V}_S)$ would always converge, while the treatment of $\mathbf{V}_S$ itself causes no problem, $\mathbf{V}_S$ being separable.

The resulting formulas for the T matrix make it possible to discuss the structure and, to some extent, the analytical properties of this matrix. For reasons of simplicity, this was only done for the case of no channel-channel coupling $\mathbf{V}_{CC}$. The extension of the treatment to include $\mathbf{V}_{CC}$ is straightforward. All S-matrix elements were found to contain the same function (det$\mathbf{D}$) in the denominator. Essentially, this function is the generalization of the Jost function to the present problem. Because of the RPA, the matrix $\mathbf{D}$ has the remarkable property that $\det\mathbf{D}(-\mathcal{E}) = \det\mathbf{D}(\mathcal{E})$. This leads to the statement that, on the physical sheet, poles and branch points of the S matrix caused by det$\mathbf{D}$ occur symmetrically about the point $E=0$. A detailed discussion of origin and physical significance of the left-hand singularities was given. They are due to the backward-going graphs.

The pole expansion of the T matrix yields, even in the case of no channel-channel coupling, a background term due to the left-hand poles. The presence of these poles also limits the possibility of deriving sum rules for the partial and total widths, in contrast to results obtained in the TDA. Indeed, the sum rules hold only up to and including first order in the matrix elements of the backward-going graphs. We remark that these matrix elements contribute to the correlations in the RPA ground state $|\Psi_0\rangle$, where they enter in first and higher order.

During preparation of this manuscript, the authors

received a manuscript by Hahne and Dover,[15] in which some of the problems studied in the present paper are investigated for the case of a separable two-nucleon interaction. In this case, an algebraic solution of the Lippmann-Schwinger equation can be obtained. While some of the problems discussed above (the relationship between the HF procedure and the RPA, the role of the linear-momentum operator) are not investigated in Ref. 15, the work there provides a very interesting and useful counterpart to the present investigation, because the motion of the poles in the complex plane and the values of the partial widths can be studied as functions of the parameters characterizing the separable potential.

APPENDIX

In this Appendix, we show that the scattering solutions given by Eqs. (15) and (17) agree with those derived by Dietrich and Hara.[5] We then show (i) that the scattering modes obey the boson commutation rules given in Eq. (18), (ii) that the scattering matrix is given by Eq. (22), and (iii) that the scattering matrix is unitary.

Let us define $U_c^{(+)\dagger}(E)$ to be

$$U_c^{(+)\dagger}(E) \equiv Q_c^{(+)\dagger}(E) - \exp(i\delta_c) A_c^\dagger(E). \quad \text{(A1)}$$

From Eq. (15), this is just

$$U_c^{(+)\dagger}(E) = \sum_b \left([A_b, [V^B, Q_c^{(+)\dagger}(E)]] A_b^\dagger \frac{1}{E-E_b} - [A_b^\dagger, [V^B, Q_c^{(+)\dagger}(E)]] \frac{A_b}{E+E_b} \right)$$
$$+ \sum_{c'} \int_{\epsilon_{c'}}^{\infty} dE' \left([A_{c'}(E'), [V^B, Q_c^{(+)\dagger}(E)]] A_{c'}^\dagger(E') \frac{1}{E^+-E'} - [A_{c'}^\dagger(E'), [V^B, Q_c^{(+)\dagger}(E)]] \frac{A_{c'}(E')}{E+E'} \right). \quad \text{(A2)}$$

In the above, we have put the energy denominators to the far right to illustrate better our argument. We then operate on the RPA ground state $|\Psi_0\rangle$ with $U_c^{(+)\dagger}(E)$. Consider the first of the energy denominators given in the above expression. Since $H^B|\Psi_0\rangle = 0$, we have

$$[1/(E-E_b)]|\Psi_0\rangle = [1/(E-E_b-H^B)]|\Psi_0\rangle, \quad \text{(A3)}$$

which is equivalent to

$$[1/(E-E_b)]|\Psi_0\rangle = [1/(E-E_b-H_0^B)]$$
$$\times \{1+(E^{\mathrm{HF}}+V^B)[1/(E-E_b-H^B)]\}|\Psi_0\rangle. \quad \text{(A4)}$$

Using the operator identity

$$A_b^\dagger [1/(E-E_b-H^B)] = [1/(E-H^B)] A_b^\dagger, \quad \text{(A5)}$$

we get

$$A_b^\dagger [1/(E-E_b)]|\Psi_0\rangle = [1/(E-H_0^B)] A_b^\dagger$$
$$\times \{1+[1/(E-E_b)](E^{\mathrm{HF}}+V^B)\}|\Psi_0\rangle. \quad \text{(A6)}$$

We can thus pull the $(E-H_0^B)^{-1}$ factor outside the summation sign in Eq. (A1). Doing this for all other energy denominators as well, and resumming (and integrating), we get

$$U_c^{(+)\dagger}(E)|\Psi_0\rangle = [1/(E^+-H_0^B)]\{[V^B, Q_c^{(+)\dagger}(E)]$$
$$+ U_c^{(+)\dagger}(E)(E^{\mathrm{HF}}+V^B)\}|\Psi_0\rangle. \quad \text{(A7)}$$

If we commute the last two operators in the curly brackets, the expression becomes

$$U_c^{(+)\dagger}(E)|\Psi_0\rangle$$
$$= [1/(E^+-H_0^B)]\{[V^B, Q_c^{(+)\dagger}(E)] + [U_c^{(+)\dagger}(E), V^B]$$
$$+ (E^{\mathrm{HF}}+V^B) U_c^{(+)\dagger}(E)\}|\Psi_0\rangle$$
$$= [1/(E^+-H_0^B)]\{\exp(i\delta_c)[V^B, A_c^\dagger(E)]$$
$$+ (E^{\mathrm{HF}}+V^B) U_c^{(+)\dagger}(E)\}|\Psi_0\rangle, \quad \text{(A8)}$$

where we have used the definition of $U_c^{(+)\dagger}(E)$ given in (A1). We then bring the second term on the right-hand side to the left-hand side, and we finally get

$$[1/(E^+-H_0^B)](E-H^B) U_c^{(+)\dagger}(E)|\Psi_0\rangle$$
$$= [1/(E^+-H_0^B)] \exp(i\delta_c)[V^B, A_c^\dagger(E)]|\Psi_0\rangle, \quad \text{(A9)}$$

from which we conclude that

$$U_c^{(+)\dagger}(E)|\Psi_0\rangle = [1/(E^+-H^B)]$$
$$\times \exp(i\delta_c)[V^B, A_c^\dagger(E)]|\Psi_0\rangle, \quad \text{(A10)}$$

using the fact that $U_c^{(+)\dagger}(E)$ has outgoing scattered waves only. This result agrees with that of Dietrich and Hara.[5]

The commutators between the creation and destruction operators for scattering modes can then be calculated. For example, we have

$$[Q_c^{(+)}(E'), Q_c^{(+)\dagger}(E)] = \exp(-i\delta_{c'})$$
$$\times [A_{c'}(E'), Q_c^{(+)\dagger}(E)]$$
$$+ [U_{c'}^{(+)}(E'), Q_c^{(+)\dagger}(E)]. \quad \text{(A11)}$$

[15] F. Hahne and C. B. Dover, Nucl. Phys. A135, 65 (1969).

If we use the expression in Eq. (15), the first term becomes

$$\exp(-i\delta_{c'})[A_{c'}(E'), Q_c^{(+)\dagger}(E)]=\delta_{cc'}\delta(E-E') + \exp(-i\delta_{c'})(E^+-E')^{-1}[A_{c'}(E'), [V^B, Q_c^{(+)\dagger}(E)]]. \quad (A12)$$

In order to evaluate the second term in (A11), we can sandwich it between the RPA ground states, since it is just a number:

$$[U_c^{(+)}(E'), Q_c^{(+)\dagger}(E)] = \langle\Psi_0 \mid [U_c^{(+)}(E'), Q_c^{(+)\dagger}(E)] \mid \Psi_0\rangle = \langle\Psi_0 \mid U_{c'}^{(+)}(E')Q_c^{(+)\dagger}(E) \mid \Psi_0\rangle, \quad (A13)$$

where we have used Eq. (12b). Using (A10) above, we get

$$[U_{c'}^{(+)}(E'), Q_c^{(+)\dagger}(E)]=-\exp(-i\delta_{c'})(E^+-E')^{-1} \times\langle\Psi_0 \mid [A_{c'}(E'), [V^B, Q_c^{(+)\dagger}(E)]] \mid \Psi_0\rangle. \quad (A14)$$

This term cancels with the second term in Eq. (A12), and we get

$$[Q_{c'}^{(+)}(E'), Q_c^{(+)\dagger}(E)]=\delta_{cc'}\delta(E-E'). \quad (A15)$$

It is easier to show that

$$[Q_{c'}^{(+)}(E'), Q_c^{(+)}(E)]=[Q_{c'}^{(+)\dagger}(E'), Q_c^{(+)\dagger}(E)]=0. \quad (A16)$$

Since the commutators are just numbers, we can evaluate them by sandwiching them between the RPA ground state as was done in (A13). Using (12b), we verify that these commutators vanish.

In a manner equivalent to deriving Eq. (A10), we can show that

$$U_c^{(-)\dagger}(E) \mid \Psi_0\rangle=[1/(E-H^B)] \times \exp(-i\delta_c)[V^B, A_c^\dagger(E)] \mid \Psi_0\rangle, \quad (A17)$$

where

$$U_c^{(-)\dagger}(E)\equiv Q_c^{(-)\dagger}(E)-\exp(-i\delta_c)A_c^\dagger(E). \quad (A18)$$

Using this relation, we can derive commutators like (A15) and (A16) with (+) superscripts replaced by (−) superscripts. Furthermore, we can derive the expression for the scattering matrix given in Eqs. (22). In a development similar to that used in Eqs. (A11)–(A14), we find

$$[Q_{c'}^{(-)}(E'), Q_c^{(+)\dagger}(E)]=\exp[i(\delta_{c'}+\delta_c)]\delta(E-E')\delta_{cc'} + \exp(i\delta_{c'})[A_{c'}(E'), [V^B, Q_c^{(+)\dagger}(E)]] \times\{(E^+-E')^{-1}-(E^--E')^{-1}\}. \quad (A19)$$

Hence, in this case only the real parts of the terms in curly brackets cancel, and the imaginary parts add. Using the fact that

$$\mathrm{Im}[1/(E^\pm-E')]=\mp\pi\delta(E-E'), \quad (A20)$$

we arrive at Eq. (22).

For reasons identical to those used in deriving Eqs. (A16), the other possible commutators vanish; namely,

$$[Q_{c'}^{(-)}(E'), Q_c^{(+)}(E)]=[Q_{c'}^{(-)\dagger}(E'), Q_c^{(+)\dagger}(E)]=0. \quad (A21)$$

All the scattering eigenbosons commute with bound eigenbosons simply because they are solutions to the eigenequations (10a) for different energies. Using Eq. (10a) and Jacobi's identity, we can easily see that

$$E_\mu[Q_c^{(+)}(E), Q_\mu^\dagger]=[Q_c^{(+)}(E), [H^B, Q_\mu^\dagger]] = E[Q_c^{(+)}(E), Q_\mu^\dagger]. \quad (A22)$$

Thus,

$$[Q_c^{(+)}(E), Q_\mu^\dagger]=0, \quad (A23)$$

if $E_\mu\neq E$.

Thus the set of bosons $\{Q_\mu^\dagger, Q_\mu, Q_c^{(+)\dagger}(E), Q_c^{(+)}(E), P_q^B, R_q^B\}$ form an orthonormal set of bosons. They also form a complete set in the sense that any boson expandable in the set $\{A_b^\dagger, A_b, A_c^\dagger(E), A_c(E)\}$ is expandable in this set. In particular, the scattering bosons $Q_c^{(-)}(E)$ are expandable in this set. These bosons commute with all the bosons in the former set of bosons, except for the eigenbosons $Q_c^{(+)\dagger}(E)$. The commutator with these bosons is just the scattering matrix, Eq. (20):

$$Q_c^{(-)}(E)=\sum_{c'} S_{cc'}(E)Q_{c'}^{(+)}(E). \quad (A24)$$

Commuting this equation with the conjugate boson $Q_{c''}^{(-)\dagger}(E'')$, we get

$$\delta(E-E'')\delta_{cc''}=\sum_{c'} S_{cc'}(E)[Q_{c'}^{(+)}(E), Q_{c''}^{(-)\dagger}(E'')] = \sum_{c'} S_{cc'}(E)\tilde{S}_{c'c''}^*(E)\delta(E-E''). \quad (A25)$$

Thus, the S matrix is unitary.