PHYSICAL REVIEW C VOLUME 1, NUMBER 2 FEBRUARY 1970

Off-Energy-Shell t Matrix for Local Potentials with Singular Core Interactions*

Y. E. Kim and A. Tubis
Department of Physics, Purdue University, Lafayette, Indiana 47907
(Received 24 October 1969)

As a preliminary to studying three-body problems involving local two-particle potentials with singular core interactions, we derive and numerically study an integral equation for the complete off-energy-shell t matrix for the associated two-particle scattering. The core interaction is described by the boundary-condition model (specified energy-independent logarithmic derivative of the radial wave function at the core radius). Numerical solutions of the integral equation for the special case of the Herzfeld potential (hard core with a square well outside) are found to agree well with the exact t matrix.

I. INTRODUCTION

THE nucleon-nucleon t matrix on the energy shell is now fairly well known over the elastic scattering region ($E_{\rm lab} \lesssim 400$ MeV),[1] and can be accounted for accurately in terms of a number of different potential models. Separable potentials,[2] local potentials with hard cores,[3] local potentials with core regions described by the boundary-condition model[4] (specified energy-independent logarithmic derivative of the radial wave function at the core radius), and local potentials with soft cores[5] have been used.

In order to reduce the ambiguity in the effective nucleon-nucleon interaction, one must study problems such as nuclear matter, p-p bremstrahlung, three-body bound, and scattering states, etc., which involve off-energy-shell t-matrix elements.

The development of the Faddeev equations,[6] whose solutions give the properties of three-body states in terms of off-energy-shell two-body t-matrix elements, has led to a great deal of research in three-body problems.[7]

The Faddeev equations reduce to two-variable integral equations in the case of local two-body interactions.[7] These integral equations are extremely difficult to solve with present computing facilities, but there has been considerable progress recently in finding efficient practical methods of solution.[8] Most of the Faddeev calculations have been based on separable two-body interactions[2] which lead to one-variable integral equations.[7]

Separable nucleon-nucleon interactions are unrealistic, especially for the long-range forces which are associated with single-meson exchanges. A number of "realistic" nucleon-nucleon interactions,[3–5] which give a good description of the t matrix on the energy shell, have been proposed. They incorporate theoretical estimates of the long-range (single-meson exchange) forces and a parametrized intermediate and short-range behavior. The latter is usually described in terms of hard cores,[3] the boundary condition model,[4] or soft cores.[5]

Thus, in order to use the Faddeev formalism to discriminate between various realistic nucleon-nucleon interactions, practical techniques must be developed for computing the complete off-shell t matrix for singular core interactions.

Van Leeuwen and Reiner[9] derived an exact expression for the complete off-shell t matrix for the case of a hard core with a chain of square wells outside. Brander[10] used the two-potential formalism of Gell-Mann and Goldberger[11] to derive a formal integral equation for the complete off-shell t matrix for the case of a hard core and an arbitrary outside potential. Kowalsky and Feldman[12] studied the half-off-shell t matrix for a hard-shell (infinite delta function) core interaction and an arbitrary outside potential. Laughlin and Scott[13] set up, and studied numerically, a formalism for determining the complete off-shell t matrix in the case of

* This work was supported by the U.S. Atomic Energy Commission.

[1] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **169**, 1128 (1968); **173**, 1272 (1968); G. Breit and R. D. Haracz, in *High Energy Physics*, edited by E. H. S. Burhop (Academic Press Inc., New York, 1967), Vol. I, p. 21. These works contain extensive references to the phase-shift analyses of the Livermore and Yale groups.

[2] See, e.g., F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964); T. Mongan, Phys. Rev. **178**, 1597 (1969).

[3] See, e.g., T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962); T. Hamada, Y. Nakamura, and R. Tamagaki, Progr. Theoret. Phys. (Kyoto) **33**, 769 (1965); see also the review article of G. Breit and R. D. Haracz in Ref. 1.

[4] H. Feshbach, E. L. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961); H. Feshbach and E. L. Lomon, Ann. Phys. (N.Y.) **29**, 19 (1964); E. L. Lomon and H. Feshbach, *ibid.* **48**, 94 (1968).

[5] R. V. Reid, Jr., Ann. Phys. (N.Y.) **50**, 411 (1968).

[6] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[7] See, e.g., C. Lovelace, Phys. Rev. **135**, B1225 (1964); A. Ahmadzadeh and J. A. Tjon, *ibid.* **139**, 1085 (1965); J. H. Hetherington and L. H. Schick, *ibid.* **137**, B935 (1965); R. Aaron, R. D. Amado, and Y. Yam, *ibid.* **136**, B650 (1964).

[8] J. S. Ball and D. Y. Wong, Phys. Rev. **169**, 1362 (1968); Y. E. Kim, J. Math. Phys. **10**, 1491 (1969).

[9] J. M. J. Van Leeuwen and A. S. Reiner, Physica **27**, 99 (1961).

[10] O. Brander, Arkiv Fysik **24**, 439 (1963).

[11] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[12] K. L. Kowalski and D. Feldman, J. Math. Phys. **2**, 499 (1961); **4**, 507 (1963); Phys. Rev. **130**, 276 (1963).

[13] R. Laughlin and B. L. Scott, Phys. Rev. **171**, 1196 (1968).

a hard core and arbitrary outside potential. Fuda[14] used a separable expansion in Sturmian functions[15] to study the same problem. Lomon and his collaborators[16] investigated the half-off-shell t matrix for the boundary-condition model.

In this paper, we consider a two-body interaction with an outside local potential and a core region described by the boundary-condition model (hard-core models are, of course, a special case of this class of interactions). We then derive and study an explicit integral equation for the complete off-shell t matrix. In subsequent publications, we will study three-body systems involving two-body forces of the type considered here.

In Sec. II, we show that the boundary-condition model may be described in terms of a limiting procedure applied to a square repulsive well with a delta-function interaction at the edge of the well. In Sec. III, we derive an explicit integral equation for the complete off-shell t matrix. In Sec. IV, we consider the hard-core limit of the boundary-condition model and exhibit an exact expression for the t matrix in the case of the Herzfeld potential (hard core with a square well outside). In Sec. V, we numerically solve the integral equation for the t matrix for the case of the Herzfeld potential, and compare the results with the exact t-matrix expression. A summary and some concluding remarks are given in Sec. VI.

II. REPRESENTATION OF CORE INTERACTION IN BOUNDARY-CONDITION MODEL

In this paper, we will, for simplicity, consider uncoupled two-body partial-wave states of definite orbital angular momentum. The generalization of our analysis to coupled partial-wave states is straightforward.

The boundary-condition model (BCM) consists of a core interaction which gives rise to an energy-independent logarithmic derivative of the radial wave function at the core radius, and a local potential outside the core radius.

In studying the off-shell behavior of the t matrix for the BCM, it is convenient to represent the model as an appropriate limit of the local potential given in Fig. 1. A repulsive square barrier of strength V_0 extends from particle separation $r=0$ to $r=r_0$, a delta-function interaction $-V_1r_0\delta(r-r_0)$ borders the square repulsive barrier, and there is a local potential $\tilde{V}(r)$ for $r>r_0$. In this paper, we assume that $\tilde{V}(r)$ approaches zero faster than $1/r$ as $r\to\infty$. The case $\tilde{V}(r)\sim 1/r$ for $r\to\infty$ can easily be handled by using Coulomb functions

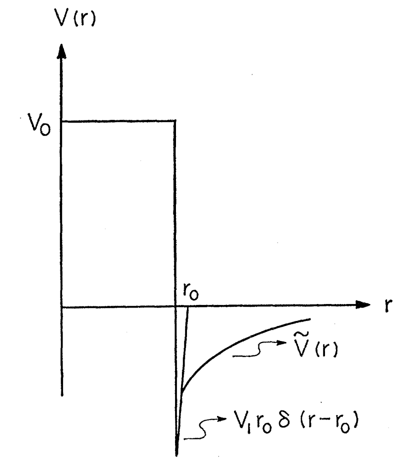

FIG. 1. Local potential which becomes the interaction of the BCM in the limit, $U_0=2\mu V_0/\hbar^2\to\infty$, $U_1=2\mu V_1/\hbar^2\to\infty$, $f_l=r_0(\sqrt{U_0}-r_0U_1)=$ finite const.

instead of Bessel functions to represent the asymptotic behavior of the wave functions.

The on-shell radial equation for the lth partial wave is

$$\left(\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right)u_l(r)=\frac{2\mu}{\hbar^2}V(r)u_l(r), \quad (2.1)$$

with the boundary condition

$$u_l(0)=0. \quad (2.2)$$

μ is the reduced mass, and the c.m. energy E is $\hbar^2k^2/2\mu$. For $0\leq r<r_0$, the solution of (2.1), with the $V(r)$ given in Fig. 1, has the general form

$$u_l(r\leq r_0)=A(i(U_0-k^2)^{1/2}r)j_l(i(U_0-k^2)^{1/2}r), \quad (2.3)$$

where

$$U_0=2\mu V_0/\hbar^2, \qquad U_0>k^2. \quad (2.4)$$

A is a constant, and j_l is the spherical Bessel function of order l. The delta function in $V(r)$ gives rise to a discontinuity in the derivative of $u_l(r)$,

$$r_0u_l'(r)\,|_{r_0^-}^{r_0^+}=r_0u_l'(r)\,|_{r_0^+}-r_0[i(U_0-k^2)^{1/2}\times rAj_l(i(U_0-k^2)^{1/2}r)]'\,|_{r=r_0^-}$$
$$=-U_1r_0Ai(U_0-k^2)^{1/2}r_0\times j_l(i(U_0-k^2)^{1/2}r_0), \quad (2.5)$$

where

$$U_1=2\mu V_1/\hbar^2. \quad (2.6)$$

The prime denotes differentiation with respect to r, and $r_0^{+(-)}$ signifies values of r infinitesimally larger (smaller) than r_0.

Now consider (2.5) in the limit

$$U_0,\,U_1\to\infty, \quad (2.7)$$

with

$$f_l=r_0(\sqrt{U_0}-r_0U_1) \quad (2.8)$$

equal to a finite constant. From the asymptotic proper-

[14] M. G. Fuda, Phys. Rev. **178**, 1682 (1969).

[15] K. Meetz, J. Math. Phys. **3**, 690 (1961); M. Rotenberg, Ann. Phys. (N.Y.) **19**, 262 (1962); S. Weinberg, Phys. Rev. **131**, 440 (1963).

[16] E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) **23**, 439 (1963); M. M. Hoenig and E. L. Lomon, *ibid.* **36**, 363 (1966).

ties of j_l, it follows that

$$\lim_{\substack{U_{0,1}\to\infty \\ f_l=r_0(\sqrt{U_0}-r_0U_1)}} \frac{r_0u_l'(r)\,|_{r=r_0^+}}{u_l(r_0)}=f_l, \tag{2.9}$$

independent of k. The potential of Fig. 1 with conditions (2.7) and (2.8) is thus equivalent to the BCM. Note that for $r\leq r_0$, and finite U_0,

$$u_l(r)=u_l(r_0)\,\frac{j_l(i(U_0-k^2)^{1/2}\,r)}{j_l(i(U_0-k^2)^{1/2}\,r_0)}. \tag{2.10}$$

For very large U_0, (2.10) gives

$$u_l(r)\sim u_l(r_0)\,\exp[-(\sqrt{U_0})(r_0-r)], \tag{2.11}$$

so that $u_l(r<r_0)=0$ in the BCM. This result has been derived in a more general way by Hoenig and Lomon.[16]

III. INTEGRAL EQUATION FOR OFF-SHELL t MATRIX

In this section, we first review briefly the integral-equation formalism for scattering by nonsingular potentials, and then generalize the formalism to the case of the BCM by an appropriate limit procedure.

We assume a nonsingular, central, Hermitian potential V which is local and generally (orbital) angular-momentum-dependent. Thus,

$$\langle \mathbf{r}'\,|\,V\,|\,\mathbf{r}\rangle=\frac{\delta(r-r')}{r^2}\,\frac{1}{4\pi}\sum_l V_l(r)\,(2l+1)P_l(\hat{r}\cdot\hat{r}'), \tag{3.1}$$

where

$$r=|\,\mathbf{r}\,|,\quad \text{etc.},$$
$$\hat{r}=\mathbf{r}/|\,\mathbf{r}\,|,\quad \text{etc.}$$

$V_l(r)$ is a real function of r, and P_l is the Legendre polynomial of order l.

The t operator is defined by the formal integral equation,

$$t(q)=V+V\,\frac{1}{(\hbar^2q^2/2\mu)-[(\mathbf{P}_{\mathrm{op}})^2/2\mu]}\,t(q), \tag{3.2}$$

where $\mathbf{P}_{\mathrm{op}}$ is the relative-momentum operator and

$$q=(2\mu E)^{1/2}/\hbar, \tag{3.3}$$

where E is the (generally complex) energy parameter. The eigenvectors of $\mathbf{P}_{\mathrm{op}}$ are the plane-wave states $|\,\mathbf{k}\rangle$ which satisfy

$$\hbar^{-1}\mathbf{P}_{\mathrm{op}}\,|\,\mathbf{k}\rangle=\mathbf{k}\,|\,\mathbf{k}\rangle, \tag{3.4}$$

$$\langle \mathbf{k}'\,|\,\mathbf{k}\rangle=\delta^{(3)}(\mathbf{k}'-\mathbf{k}), \tag{3.5}$$

$$\langle \mathbf{r}\,|\,\mathbf{k}\rangle=[1/(2\pi)^{3/2}]\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{3.6}$$

$$\int d^{(3)}\mathbf{k}\,|\,\mathbf{k}\rangle\langle\mathbf{k}\,|=1. \tag{3.7}$$

The general off-shell t-matrix element $\langle\mathbf{k}'\,|\,t(q)\,|\,\mathbf{k}\rangle$ may be expressed as

$$\langle\mathbf{k}'\,|\,t(q)\,|\,\mathbf{k}\rangle=\langle\mathbf{k}'\,|\,V\,|\,\Psi_{q,\mathbf{k}}\rangle, \tag{3.8}$$

where $|\,\Psi_{q,\mathbf{k}}\rangle$ satisfies the off-shell Lippmann-Schwinger equation

$$|\,\Psi_{q,\mathbf{k}}\rangle=|\,\mathbf{k}\rangle+\frac{1}{(\hbar^2q^2/2\mu)-[(\mathbf{P}_{\mathrm{op}})^2/2\mu]}\,V\,|\,\Psi_{q,\mathbf{k}}\rangle. \tag{3.9}$$

The on-shell t-matrix element $\langle\mathbf{k}'\,|\,t(k)\,|\,\mathbf{k}\rangle$, with $\mathbf{k}'$, $\mathbf{k}$, and k real and $|\,\mathbf{k}'\,|=|\,\mathbf{k}\,|=k$, is given by

$$\langle\mathbf{k}'\,|\,t(q)\,|\,\mathbf{k}\rangle=\lim_{\epsilon\to0^+}\langle\mathbf{k}'\,|\,t(k+i\epsilon)\,|\,\mathbf{k}\rangle. \tag{3.10}$$

If the expansions (3.1)

$$\langle\mathbf{r}\,|\,\mathbf{k}\rangle=(2\pi)^{-3/2}\sum_{l=0}^{\infty}(2l+1)\,i^lP_l(\hat{k}\cdot\hat{r})\,j_l(kr) \tag{3.11}$$

and

$$\langle\mathbf{r}\,|\,\Psi_{q,\mathbf{k}}\rangle=(2\pi)^{-3/2}\sum_{l=0}^{\infty}(2l+1)\,i^lP_l(\hat{k}\cdot\hat{r})\,\frac{u_l(k\,|\,q\,|\,r)}{r} \tag{3.12}$$

are inserted into the coordinate representation of (3.9), one obtains

$$u_l(k\,|\,q\,|\,r)=rj_l(kr)-\int_0^{\infty}dr'G_{l,q}(r\,|\,r')\,U_l(r')\,u_l(k\,|\,q\,|\,r'), \tag{3.13}$$

where $U_l(r')=(2\mu/\hbar^2)\,V_l(r')$, and

$$G_{l,q}(r\,|\,r')=\frac{2}{\pi}\int_0^{\infty}dp\,\frac{(prj_l(pr))(pr'j_l(pr'))}{p^2-q^2}. \tag{3.14}$$

For $q^2\to q^2$ (real and positive)$+i\epsilon$ and $\epsilon\to0^+$, we find

$$G_{l,q}(r\,|\,r')=-[qr_>h_l^{(+)}(qr_>)\,qr_<j_l(qr_<)/iq], \tag{3.15}$$

where $h_l^{(+)}(x)$ is the lth-order spherical Hankel function of the first kind whose asymptotic behavior is $-(i/x)\exp[i(x-l\pi/2)]$ and $r_>$, $r_<$ are, respectively, the greater and lesser of r and r'. The radial Green's function $G_{l,q}(r\,|\,r')$ satisfies the equation

$$\left(\frac{d^2}{dr^2}+q^2-\frac{l(l+1)}{r^2}\right)G_{l,q}(r\,|\,r')=-\delta(r-r'). \tag{3.16}$$

If the expansions (3.1), (3.11), (3.12), and

$$\langle\mathbf{k}'\,|\,t(q)\,|\,\mathbf{k}\rangle=-\frac{h^2}{2\mu}\frac{1}{2\pi^2}\times\sum_{l=0}^{\infty}(2l+1)\,t_l(k'\,|\,q\,|\,k)\,P_l(\hat{k}'\cdot\hat{k}) \tag{3.17}$$

are substituted in (3.8), we find

$$t_l(k'\,|\,q\,|\,k)=-\int_0^{\infty}dr\,rj_l(k'r)\,U_l(r)\,u_l(k\,|\,q\,|\,r). \tag{3.18}$$

For the on-shell case, the asymptotic behavior of u_l is

$$u_l(k\mid k\mid r)\sim rj_l(kr)+t_l(k\mid k\mid k)\exp[i(kr-l\pi/2)], \quad (3.19)$$

with

$$t_l(k\mid k\mid k)=\exp[i\delta_l(k)]\sin\delta_l(k)/k, \quad (3.20)$$

where $\delta_l(k)$ is the real partial-wave phase shift.

We now generalize the basic equations, (3.13) and (3.17), to the case of the BCM.

First consider $V_l(r)$ to be $V(r)$ in Fig. 1 with V_0 and V_1 finite. It is then evident from (3.13) that for $0\leq r\leq r_0^+$,

$$\{(d^2/dr^2)+q^2-[l(l+1)/r^2]\}u_l(k\mid q\mid r) =(q^2-k^2)rj_l(kr)+[U_0\Theta(r_0-r)-U_1r_0\delta(r-r_0)]\times u_l(k\mid q\mid r), \quad (3.21)$$

with

$$u_l(k\mid q\mid 0)=0. \quad (3.22)$$

$\Theta(x)$ is defined as

$$\Theta(x>0)=1,\qquad \Theta(x<0)=0. \quad (3.23)$$

Thus, in the interval $0\leq r<r_0$,

$$u_l(k\mid q\mid r)=\frac{-k^2+q^2}{-k^2+q^2-U_0}rj_l(kr)+\lambda_l\frac{r\,j_l(i(U_0-q^2)^{1/2}r)}{r_0\,j_l(i(U_0-q^2)^{1/2}r_0)}, \quad (3.24)$$

where λ_l is a constant. The continuity of $u_l(k\mid q\mid r)$, and the discontinuity of $-U_1r_0u_l(k\mid q\mid r_0)$ in the derivative of $u_l(k\mid q\mid r)$ at $r=r_0$, implies that

$$[(-k^2+q^2)/(-k^2+q^2-U_0)]r_0j_l(kr_0)+\lambda_l=u_l(k\mid q\mid r_0^+), \quad (3.25)$$

$$-\frac{-k^2+q^2}{-k^2+q^2-U_0}\frac{d}{dr}(rj_l(kr))\Big|_{r=r_0}-\lambda_l\frac{d}{dr}\left(\frac{r\,j_l(i(U_0-q^2)^{1/2}r)}{r_0\,j_l(i(U_0-q^2)^{1/2}r_0)}\right)\Bigg|_{r=r_0}+\frac{d}{dr}u_l(k\mid q\mid r)\,\Big|_{r=r_0^+}=-U_1r_0u_l(k\mid q\mid r_0). \quad (3.26)$$

In the BCM limit (2.7) and (2.8), Eqs. (3.25) and (3.26) give

$$\lambda_l=u_l(k\mid q\mid r_0^+), \quad (3.27)$$

$$r_0\frac{(d/dr)u_l(k\mid q\mid r)}{u_l(k\mid q\mid r)}\Bigg|_{r=r^+}=f_l, \quad (3.28)$$

so that the BCM condition (2.9) also holds for the off-shell radial wave function. In deriving (3.27) and (3.28), use has been made of the following asymptotic behavior:

$$\frac{r\,j_l(i(U_0-q^2)^{1/2}r)}{r_0\,j_l(i(U_0-q^2)^{1/2}r_0)}\sim\exp[-(U_0)^{1/2}(r_0-r)] \quad (3.29)$$

as $(U_0)^{1/2}r$, $(U_0)^{1/2}r_0\gg l$, $U_0\gg q^2$.

Multiplication of $u_l(k\mid q\mid r)$ by $U_{l,\,\mathrm{BCM}}(r)$, and the use of (3.24), (3.27), and (3.29), yields

$$U_{l,\,\mathrm{BCM}}(r)u_l(k\mid q\mid r)=\lim_{\substack{U_0,U_1\to\infty\\ f_l=r_0(\sqrt{U_0}-r_0U_1)}}\{(k^2-q^2)rj_l(kr)\Theta(r_0-r)+u_l(k\mid q\mid r_0^+)U_0\exp[-(U_0)^{1/2}\Theta(r_0-r)]-U_1r_0\delta(r-r_0)u_l(k\mid q\mid r)\}. \quad (3.30)$$

In order to evaluate the limit in (3.30), we use the result

$$U_0\exp[-(U_0)^{1/2}(r_0-r)]\approx(U_0)^{1/2}\delta(r-r_0)+(d/dr)\delta(r-r_0), \quad (3.31)$$

which holds for large U_0. (3.31) may be easily verified by considering the integration by parts of

$$\int_0^{r_0}f(r)\,U_0\exp[-(U_0)^{1/2}(r_0-r)]dr, \quad (3.32)$$

with $f(r)$ an arbitrary function with a continuous derivative over the range of integration. The $(U_0)^{1/2}$ term in (3.31) may now be combined with the U_1 term in (3.30) to give

$$U_{l,\,\mathrm{BCM}}(r)u_l(k\mid q\mid r)=(k^2-q^2)rj_l(kr)\Theta(r_0-r)+u_l(k\mid q\mid r_0^+)[(f_l/r_0)\delta(r-r_0)+(d/dr)\delta(r-r_0^-)]. \quad (3.33)$$

We now assume that $U_{l,\,\mathrm{BCM}}(r)$ in (3.13) is given by

$$U_l(r)=U_{l,\,\mathrm{BCM}}(r)+\tilde{U}_l(r), \quad (3.34)$$

where $\tilde{U}_l(r)=0$ for $r<r_0$ and goes to zero faster than $1/r$ as $r\to\infty$. Thus,

$$u_l(k\mid q\mid r)=rj_l(kr)-u_l(k\mid q\mid r_0^+)\left(\frac{f_l}{r_0}G_{l,q}(r_0\mid r_0)-\frac{d}{dr'}G_{l,q}(r\mid r')\,\Big|_{r'=r_0}\right)-(k^2-q^2)\int_0^{r_0}dr'G_{l,q}(r\mid r')r'j_l(kr')-\int_{r_0}^{\infty}dr'G_{l,q}(r\mid r')\tilde{U}_l(r')u_l(k\mid q\mid r'). \quad (3.35)$$

Setting $r=r_0^+$ in (3.35), we find

$$u(k\mid q\mid r_0^+)=\left(r_0 j_l(kr_0)-\int_{r_0}^{\infty}dr\,G_{l,q}(r_0\mid r)\tilde{U}_l(r)u_l(k\mid q\mid r)\right.$$

$$\left.+(q^2-k^2)\int_0^{r_0}dr\,G_{l,q}(r_0\mid r)rj_l(kr)\right)\Big/\left[1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)\right],\quad(3.36)$$

where

$$G_{l,q}'(r\mid r_0)=(d/dr')G_{l,q}(r\mid r')\,|_{r'=r_0}.\quad(3.37)$$

It may be easily checked that $u_l(k\mid q\mid r)$, as given by (3.35) and (3.36), vanishes for $r<r_0$.

Using (3.18), (3.35), and (3.36), we may decompose $t_l(k'\mid q\mid k)$ as

$$t_l(k'\mid q\mid k)=t_{l,\,\mathrm{BCM}}(k'\mid q\mid k)+\tilde{t}_{l,\,\mathrm{BCM}}(k'\mid q\mid k),\quad(3.38)$$

where

$$\tilde{t}_{l,\,\mathrm{BCM}}(k'\mid q\mid k)=\left(\int_{r_0}^{\infty}dr\,G_{l,q}(r_0\mid r)\tilde{U}_l(r)u_l(k\mid q\mid r)\Big/1+\frac{f_l}{r_0}G_{l,q}(r_0\mid r)-G_{l,q}'(r_0^+\mid r_0)\right)$$

$$\times\left(\frac{f_l}{r_0}r_0 j_l(k'r_0)-\frac{d}{dr'}(r'j_l(k'r'))\,|_{r'=r_0}\right)-\int_{r_0}^{\infty}dr\,rj_l(k'r)\tilde{U}_l(r)u_l(k\mid q\mid r),\quad(3.39)$$

and $t_{l,\,\mathrm{BCM}}(k'\mid q\mid k)$, the t matrix for the "pure" BCM ($\tilde{U}=0$), is given by

$$t_{l,\,\mathrm{BCM}}(k'\mid q\mid k)=(q^2-k^2)\int_0^{r_0}dr\left[rj_l(k'r)-\left(\frac{f_l}{r_0}r_0j_l(k'r_0)-\frac{d}{dr'}(r'j_l(k'r'))\,|_{r'=r_0}\right)\right.$$

$$\left.\times\frac{G_{l,q}(r_0\mid r)}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}\right]rj_l(kr)-\left(\frac{f_l}{r_0}r_0j_l(k'r_0)-\frac{d}{dr}(rj_l(k'r))\,|_{r=r_0}\right)$$

$$\times\frac{r_0j_l(kr_0)}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}.\quad(3.40)$$

$t_{l,\,\mathrm{BCM}}(k'\mid q\mid k)$ and $\tilde{t}_{l,\,\mathrm{BCM}}(k'\mid q\mid k)$ are symmetric under the interchange of k and k', as is required by space-reflection and time-reversal invariance. In the half-off-shell case ($q=k'$), (3.40) is equivalent to the reaction-matrix result of Hoenig and Lomon.[16]

We may derive an integral equation for $\tilde{t}_{l,\,\mathrm{BCM}}(k'\mid q\mid k)$, the contribution to $t_l(k'\mid q\mid k)$ coming from $\tilde{U}_l(r)$, by substituting (3.35) into (3.39), using (3.14), and identifying a portion of one of the integrands as $\tilde{t}_l(p\mid q\mid k)$, p being an integration variable. After some straightforward algebra, the resulting integral equation is found to be

$$\tilde{t}_{l,\,\mathrm{BCM}}(k'\mid q\mid k)=V_{l,1}(k'\mid q\mid k)-\frac{2}{\pi}\int_0^{\infty}\frac{p^2dp}{q^2-p^2}V_{l,2}(k'\mid q\mid p)\tilde{t}_{l,\,\mathrm{BCM}}(p\mid q\mid k),\quad(3.41)$$

where

$$V_{l,1}(k'\mid q\mid k)=-\int_{r_0}^{\infty}dr\left[rj_l(k'r)-\left(\frac{f_l}{r_0}r_0j_l(k'r_0)-\frac{d}{dr'}(r'j_l(k'r'))\,|_{r'=r_0}\right)\frac{G_{l,q}(r_0\mid r)}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}\right]$$

$$\times\tilde{U}_l(r)\left[rj_l(kr)-r_0j_l(kr_0)\left(\frac{(f_l/r_0)G_{l,q}(r_0\mid r)-G_{l,q}'(r\mid r_0)}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}\right)\right.$$

$$\left.+(q^2-k^2)\int_0^{r_0}dr'\,\frac{G_{l,q}(r\mid r')r'j_l(kr')}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}\right]\quad(3.42)$$

and

$$V_{l,2}(k'\mid q\mid p)=-\int_{r_0}^{\infty}dr\left[rj_l(k'r)-\left(\frac{f_l}{r_0}r_0j_l(k'r_0)-\frac{d}{dr'}(r'j_l(k'r'))\,|_{r'=r_0}\right)\right.$$

$$\left.\times\frac{G_{l,q}(r_0\mid r)}{1+(f_l/r_0)G_{l,q}(r_0\mid r_0)-G_{l,q}'(r_0^+\mid r_0)}\right]\tilde{U}_l(r)rj_l(pr).\quad(3.43)$$

For general complex q^2 with Im $q\neq0$, the integral equation (3.41) is well defined in the Fredholm sense,

TABLE I. The on- and off-energy-shell t matrix for the s-wave Herzfeld potential with $r_0=0.4$ F, $r_1=1.737$ F, and $V_1=73$ MeV computed from (4.1), (4.2), and (4.3). These results are to be compared with the exact results (in brackets) computed from (4.11). The variables q^2, k^2, and k'^2 are in units of F^{-2} with $\hbar^2/m=41.47$ MeV F^2. The adjustable constant α is set equal to 10.

q^2 (F^{-2})	$k^2=0.1208$ $k'^2=0.1208$	$k^2=0.1208$ $k'^2=1.5542$	$k^2=0.1208$ $k'^2=33.304$	$k^2=1.5542$ $k'^2=1.5542$	$k^2=1.5542$ $k'^2=33.304$	$k^2=33.304$ $k'^2=33.304$
−0.1	19.3595 (19.3691)	9.4616 (9.4671)	−1.8196 (−1.8207)	4.3264 (4.3291)	−0.9695 (−0.9701)	0.0277 (0.0278)
−1.5	3.1086 (3.1082)	1.4993 (1.4992)	−0.2942 (−0.2942)	0.5960 (0.5960)	−0.2793 (−0.2794)	−0.1358 (−0.1357)
−33.0	0.9253 (0.9249)	−0.1581 (−0.1582)	−0.7804 (−0.7804)	−0.7711 (−0.7711)	−0.7988 (−0.7988)	−0.4582 (−0.4582)

since the kernel is bounded, i.e.,

$$\int_0^\infty dk' \int_0^\infty dp \left| \frac{p^2 V_{l,2}(k' \mid q \mid p)}{q^2-p^2} \right|^2 < \infty, \quad (3.44)$$

as can be seen from the expression (3.43) for $V_{l,2}(k' \mid q \mid p)$. For a bounded kernel, it is well known that the solution of (3.41) exists for all q^2 except for an at most denumerable set of points q_i^2, and furthermore that the solutions of (3.41) are unique.[17]

For negative real q^2, as is appropriate for the bound-state problem, the kernel of (3.41) does not develop a singularity, and hence (3.41) is an ordinary Fredholm integral equation for which several standard methods of solution are available. For $q^2 \to q^2$ (real and positive) $+i\epsilon$, $\epsilon \to 0^+$, as is appropriate for the scattering problem, (3.41) can be reduced to a Fredholm integral equation by a well-known reduction method.[12]

IV. HARD-CORE LIMIT

In this section, we consider local potentials with hard cores as a limiting case of the BCM. As a special case, we consider a local potential with a hard core and an attractive square well outside (the Herzfeld potential or hard-core square-well potential) and derive the exact expression for the complete off-shell t matrix.

For local potentials with hard cores, we take limits $U_0 \to \infty$ and $U_1 \to 0$ or equivalently $f_l \to \infty$ in (3.40), (3.42), and (3.43). We then have

$$t_l(k' \mid q \mid k) = t_{l,\,\mathrm{HC}}(k' \mid q \mid k) + \tilde{t}_{l,\,\mathrm{HC}}(k' \mid q \mid k), \quad (4.1)$$

where the first term on the right-hand side is the pure hard-core contribution given by

$$t_{l,\,\mathrm{HC}}(k' \mid q \mid k) = (q^2-k^2) \int_0^{r_0} dr \times \left(r j_l(k'r) - r_0 j_l(k'r_0) \frac{G_{l,q}(r_0 \mid r)}{G_{l,q}(r_0 \mid r_0)} \right) r j_l(kr) - \frac{r_0 j_l(k'r_0) r_0 j_l(kr_0)}{G_{l,q}(r_0 \mid r_0)}. \quad (4.2\text{a})$$

For the special case of $l=0$ (s wave), (4.2a) reduces to

$$t_{0,\,\mathrm{HC}}(k' \mid q \mid k) = (q^2-k^2) \int_0^{r_0} \frac{\sin k'r}{k'} \frac{\sin kr}{k} dr + iq \left(\frac{\sin kr_0}{k} - \frac{\cos kr_0}{iq} \right) \frac{\sin k'r_0}{k'}. \quad (4.2\text{b})$$

The second term of (4.1) satisfies the integral equation

$$\tilde{t}_{l,\,\mathrm{HC}}(k' \mid q \mid k) = V_{l,1}(k' \mid q \mid k) - \frac{2}{\pi} \int_0^\infty \frac{p^2 dp}{q^2-p^2} V_{l,2}(k' \mid q \mid p) \tilde{t}_{l,\,\mathrm{HC}}(p \mid q \mid k), \quad (4.3)$$

where

$$V_{l,1}(k' \mid q \mid k) = -\int_{r_0}^\infty dr \times \left(r j_l(k'r) - r_0 j_l(k'r_0) \frac{G_{l,q}(r_0 \mid r)}{G_{l,q}(r_0 \mid r_0)} \right) \times \tilde{U}_l(r) \left(r j_l(kr) - r_0 j_l(kr_0) \frac{G_{l,q}(r_0 \mid r)}{G_{l,q}(r_0 \mid r_0)} \right) \quad (4.4)$$

and

$$V_{l,2}(k' \mid q \mid p) = -\int_{r_0}^\infty dr \times \left(r j_l(k'r) - r_0 j_l(k'r_0) \frac{G_{l,q}(r_0 \mid r)}{G_{l,q}(r_0 \mid r_0)} \right) \times \tilde{U}_l(r) r j_l(kr). \quad (4.5)$$

Note that Eq. (4.3) reduces to the Lippmann-Schwinger equation for a single potential as $r_0 \to 0$.

We now consider as a special case the Herzfeld potential for which Eqs. (4.1)–(4.5) are valid. For each partial wave l, we define the parameters of the Herzfeld potential as follows:

$$\begin{aligned} V_l(r) &= (\hbar^2/2\mu) U_0 = \infty && \text{for } r<r_0, \\ V_l(r) &= (\hbar^2/2\mu) \tilde{U}(r) = (\hbar^2/2\mu) U_1 < 0 && \text{for } r_1>r>r_0, \\ V_l(r) &= (\hbar^2/2\mu) U_2 = 0 && \text{for } r>r_1. \end{aligned} \quad (4.6)$$

[17] F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).

r_0 is the core radius previously defined, r_1 is the radius of the outer edge of the square-well potential, and U_i's are constants proportional to the interaction strengths.

For the Herzfeld potential, it is well known that the exact expression for the t matrix can be derived by a method which does not involve the solution of an integral equation. Van Leeuwen and Reiner have given a detailed description of the method.[9] We will briefly outline the derivation here and give the final explicit expression for the t matrix.

We take U_0 to be finite initially and then take the limit $U_0\to\infty$ at the end. With the $V_l(r)$ defined in (4.6), the differential equation for $u_l(k\mid q\mid r)$ defined by (3.13) is

$$\left(\frac{d^2}{dr^2}+q^2-\frac{l(l+1)}{r^2}\right)u_l(k\mid q\mid r)=(q^2-k^2)rj_l(kr)+V_l(r)u_l(k\mid q\mid r). \quad (4.7)$$

For $q^2\to q^2$ (real and positive)$+i\epsilon$, $\epsilon\to 0^+$, the solution of (4.7), which has an outgoing scattered wave for $r>r_1$ [as is implied by (3.13) and (3.15)], is

$$u_l(k\mid q\mid r)=A_0rj_l(kr)+B_0^+rh_l^{(+)}(\alpha_0r)+B_0^-rh_l^{(-)}(\alpha_0r),\qquad 0\le r\le r_0$$

$$u_l(k\mid q\mid r)=A_1rj_l(kr)+B_1^+rh_l^{(+)}(\alpha_1r)+B_1^-rh_l^{(-)}(\alpha_1r),\qquad r_0<r<r_1 \quad (4.8)$$

$$u_l(k\mid q\mid r)=rj_l(kr)+B_2^+rh_l^{(+)}(qr),\qquad r>r_1.$$

The $A_i(i=0,1)$ are given by

$$A_i=(q^2-k^2)/(q^2-k^2-U_i), \quad (4.9)$$

and the $\alpha_i(i=0,1,2)$ by

$$\alpha_i=(q^2-U_i^2)^{1/2}. \quad (4.10)$$

The $h_l^{(\pm)}(x)$ are the lth-order spherical Hankel functions of the first (second) kind. The $B_i^\pm$ $(i=0,1,2)$ are complex coefficients, independent of r, which are determined from the vanishing of $u_l(k\mid q\mid r)$ at $r=0$ and the continuity of derivative and value of $u_l(k\mid q\mid r)$ at $r=r_0$ and $r=r_1$. In the limit $U_0\to\infty$, the t matrix defined by (3.18) is thus given by

$$\begin{aligned}t_l^H(k'\mid q\mid k)=&(q^2-k^2)\int_0^{r_0}dr\,rj_l(k'r)rj_l(kr)\\&-A_1\left(r_0j_l(k'r_0)\frac{d}{dr'}[r'j_l(kr')]\big|_{r'=r_0}+U_1\int_{r_0}^{r_1}dr\,rj_l(k'r)rj_l(kr)\right)\\&-B_1^+\left(r_1j_l(k'r_1)\frac{d}{dr'}[r'h_l^{(+)}(\alpha_1r')]\big|_{r'=r_1}+U_1\int_{r_0}^{r_1}dr\,rj_l(k'r)rh_l^{(+)}(\alpha_1r)\right)\\&-B_1^-\left(r_1j_l(k'r_1)\frac{d}{dr'}[r'h_l^{(-)}(\alpha_1r')]\big|_{r'=r_1}+U_1\int_{r_0}^{r_1}dr\,rj_l(k'r)rh_l^{(-)}(\alpha_1r)\right),\end{aligned} \quad (4.11)$$

with

$$B_1^+=(-\alpha_1^2/D)[-A_1r_0j_l(kr_0)X_l(\alpha_1^-\alpha_2^+\mid r_1)+(A_1-1)r_0h_l^{(-)}(\alpha_1r_0)X_l(k\alpha_2^+\mid r_1)] \quad (4.12)$$

and

$$B_1^-=(\alpha_1^2/D)[-A_1r_0j_l(kr_0)X_l(\alpha_1^+\alpha_2^+\mid r_1)+(A_1-1)r_0h_l^{(+)}(\alpha_1r_0)X_l(k\alpha_2^+\mid r_1)]. \quad (4.13)$$

The X_l's are Wronskians defined by

$$X_l(k\alpha_2^+\mid r_1)=\begin{vmatrix} r_1j_l(kr_1) & r_1h_l^{(+)}(\alpha_2r_1)\\ (d/dr)[rj_l(kr)]\big|_{r=r_1} & (d/dr)[rh_l^{(+)}(\alpha_2r)]\big|_{r=r_1}\end{vmatrix} \quad (4.14)$$

and similar expressions for $X_l(\alpha_1^+,\alpha_2^+\mid r_1)$ and $X_l(\alpha_1^-,\alpha_2^+\mid r_1)$. The symbol D stands for the function $D_l(r_0,r_1;\alpha_1,\alpha_2)$ defined as

$$D_l(r_0,r_1;\alpha_1,\alpha_2)=-\alpha_1^2[r_0h_l^{(+)}(\alpha_1r_0)X_l(\alpha_1^-\alpha_2^+\mid r_1)-r_0h_l^{(-)}(\alpha_1r_0)X_l(\alpha_1^+\alpha_2^+\mid r_1)]. \quad (4.15)$$

As in (3.20), the normalization is such that on the energy shell we have $t_l^H(k\mid k\mid k)=e^{i\delta_l(k)}[\sin\delta_l(k)/k]$. As expected, $t_l^H(k'\mid q\mid k)$ is symmetric in k' and k. For a finite value of q^2, as k (or k') approaches infinity, $t_l^H(k'\mid q\mid k)$ approaches zero as $1/k$ (or $1/k'$). For a very large value of q^2, the behavior of $t_l^H(k'\mid q\mid k)$ is mainly determined by the first term on the right-hand side of Eq. (4.11). As $q^2\to\infty$, $t_l^H(k'\mid q\mid k)$ approaches infinity as q^2.

Although $t_l^H(k'\mid q\mid k)$ is, in general, complex, it is real for the negative (real) q^2, as is expected from general properties of the t matrix. This is not obvious from Eq. (4.11), especially for the case of $0>q^2>U_1$ with $U_1<0$. However, we can show for this case that the function $D_l(r_0,r_1;\alpha_1,\alpha_2)$ is purely imaginary and, furthermore, that $B_1^+=(B_1^-)^*$, so that, when all terms on the right-hand side of Eq. (4.11) are summed, the imaginary parts cancel, and hence $t_l^H(k' \mid q \mid k)$ turns out to be real.

Apparent singularities due to $(q^2-k^2-U_1)$ in the denominator of A_1 in Eq. (4.11) are fictitious and cancel out when all the contributions to t_l^H are summed. However, for negative real q^2, genuine singularities develop for values of U_1 which are large enough to give bound states. That is, zeros of $D_l(r_0, r_1; \alpha_1, \alpha_2)$ as a function of $q^2<0$ give rise to poles in $t_l^H(k' \mid q \mid k)$. The binding energies can be obtained as the values of negative q^2 at which $D_l(r_0, r_1; \alpha_1, \alpha_2)$ vanishes. As an example we give a simple expression for the binding energy in the case of an s-wave interaction. Setting $D_0(r_0, r_1; \alpha_1, \alpha_2)=0$, we obtain for $U_1<q^2<0$

$$(|q^2|)^{1/2}\sin[(q^2-U_1)^{1/2}(r_1-r_0)] + (q^2-U_1)^{1/2}\cos[(q^2-U_1)^{1/2}(r_1-r_0)]=0. \quad (4.16)$$

It is easy to show that $D_0(r_0r_1; \alpha_1, \alpha_2)$ does not develop zeros for the values of q^2 such that $q^2<U_1<0$. It is of some interest to find parameters r_0, r_1, and U_1 of the s-wave Herzfeld potential which gives zero binding energy. For this case, we must have

$$(-U_1)^{1/2}\cos[(-U_1)^{1/2}(r_0-r_1)]=0,$$

which yields the well-known formula[18]

$$V=(\tfrac{1}{2}\pi)^2(\hbar^2/mb^2), \quad (4.17)$$

where $b=r_1-r_0$ and $V=-U_1(\hbar^2/m)$ with $m=2\mu$.

V. NUMERICAL SOLUTION

We present in this section a numerical example which demonstrates the practicality of solving the integral equation (4.3) for the Herzfeld potential. The use of other local potentials $\tilde{U}_l(r)$, which approach zero faster than $1/r$ as $r\to\infty$, does not present any numerical difficulties. For simplicity, we consider the case of negative real q^2. Extension to the positive q^2 case is slightly more complicated but straightforward.[12] Also for simplicity, we consider only the case of the two-nucleon system interacting through an s-wave Herzfeld potential.

The parameters of the potential as defined in (4.6) are chosen to be $r_0=0.4$F, $r_1=1.737$F, and $V_1=-(\hbar^2/m)U_1=73$ MeV with $(\hbar^2/m)=41.47\text{F}^2$ MeV, where $m=2\mu$ is the nucleon rest mass. For these values of parameters, the potential gives the experimental triplet s-wave scattering length and effective range, and the deuteron binding energy[19] (tensor coupling being ignored).

For negative real q^2, the integral equation (4.3) is of the Fredholm type. We adopt a standard numerical method of matrix inversion. We approximate the integral part of (4.3) by a finite sum using Gaussian

TABLE II. Successive approximations for the determinants at different values of two-body energy parameter q^2 (in units of $\hbar^2/m=41.47$ MeV F^2) as a function of the total number N of Gaussian quadrature points. The adjustable constant α is set equal to 10.

$q^2(\text{F}^{-2})$	$N=16$	$N=20$	$N=24$	$N=40$
−0.1	0.0577	0.0594	0.0598	0.0598
−0.8	0.3462	0.3431	0.3441	0.3449
−1.5	0.4477	0.4433	0.4445	0.4455
−3.0	0.5557	0.5503	0.5516	0.5529
−33.0	0.8253	0.8217	0.8228	0.8241

quadrature. It is convenient to make a change of variable so as to make the upper integration limit finite. The change of variable with $p=\alpha[t/(1-t)]$ shifts the integration limits of $(0, \infty)$ to $(0, 1)$, and with $p=\alpha\tan(\pi/4)(1+t)$ it is shifted to $(-1, +1)$. The parameter α is an adjustable constant. The first choice of $p=\alpha[t/(1-t)]$ has been used, and is known to work well for the Lippmann-Schwinger equation with the s-wave part of a local Yukawa potential[20] and for the Faddeev equation with s-wave separable potentials of the Yamaguchi type.[21] The second choice of $p=\alpha\tan(\pi/4)(1+t)$ appears to be equally suitable for our purpose.[22]

We choose the change of variable $p=\alpha[t/(1-t)]$ and rewrite Eq. (4.3) as

$$\tilde{t}_{0,\text{HC}}(t' \mid q \mid t'')=V_{0,1}(t' \mid q \mid t'') - \frac{2}{\pi}\int_0^1 \frac{\alpha^3 t^2 dt}{[q^2-\alpha^2t^2/(1-t)^2](1-t)^4} V_{0,2}(t' \mid q \mid t) \times \tilde{t}_{0,\text{HC}}(t \mid q \mid t''). \quad (5.1)$$

We approximate the integral part of (5.1) as

$$I=\int_0^1 dt\, K_q(t', t)\tilde{t}_{0,\text{HC}}(t \mid q \mid t'') \cong \sum_{i=1}^N W_i(t')K_q(t', t_i)\tilde{t}_{0,\text{HC}}(t_i \mid q \mid t''), \quad (5.2)$$

with

$$K_q(t', t)=-\frac{2}{\pi}\frac{\alpha^3t^2}{[q^2-\alpha^2t^2/(1-t)^2](1-t)^4} V_{0,2}(t' \mid q \mid t), \quad (5.3)$$

where t_i and W_i are the abscissas and weight factors for Gaussian quadrature. If we discretize the variables t' and t'' in the same way as t_i's with N points distributed over the interval $(0, 1)$, then Eq. (5.1) becomes a matrix equation for a given value of negative

[18] A. Bohr and B. R. Mottelson, in *Nuclear Structure* (W. A. Benjamin, Inc., New York, 1969), Vol. I, p. 245.
[19] H. Enge, *Introduction to Nuclear Physics* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1966).
[20] T. A. Osborn, SLAC Report No. 79, 1967 (unpublished).
[21] See paper of J. H. Hetherington and L. H. Schick in Ref. 7.
[22] G. E. Brown, A. D. Jackson, and T. T. S. Kuo, Nucl. Phys. **A133**, 481 (1969); M. Baranger, B. Girand, S. K. Mukhopdhyay, and P. U. Sauer (to be published).

TABLE III. Successive approximations for the deuteron binding energy q_0^2 computed from the s-wave hard-core square-well potential (the Herzfeld potential) as a function of the total number N of Gaussian quadrature points. These results are to be compared with the exact result of -5.3822×10^{-2} F^{-2} which corresponds to -2.2320 MeV with $\hbar^2/m=41.47$ MeV F^2. The adjustable constant parameter α is set to be 10.

N	$q_0^2(10^{-2}\text{F}^{-2})$	Difference between calculated and exact results (%)
16	-5.5789	3.378
20	-5.4001	0.331
24	-5.3662	0.297
40	-5.3798	0.046

real q^2,

$$(\mathbf{1}-\mathbf{K})\mathbf{T}=\mathbf{V}. \tag{5.4}$$

$\mathbf{1}$, $\mathbf{K}$, and $\mathbf{V}$ are $N\times N$ matrices whose (i,j) elements are given by δ_{ij}, $W_j(t_i')K_q(t_i',t_j)$, and $V_{0,1}(t_i'\mid q\mid t_j'')$, respectively. The solution for $\tilde{t}_{0,\text{HC}}(t'\mid q\mid t'')$ is obtained as an $N\times N$ matrix $\mathbf{T}$ by inverting the matrix equation (5.4). Solutions for $\tilde{t}_{0,\text{HC}}(k'\mid q\mid k)$ thus obtained are then added to the pure hard-core t matrix $t_{0,\text{HC}}(k'\mid q\mid k)$ to obtain the complete off-shell t matrix. This is to be compared with the exact $t_0^H(k'\mid q\mid k)$ calculated from (4.11).

To test the stability of our solution, the total number N of Gaussian quadrature points is varied from 16 to 40 and the adjustable constant α from 2 to 10. A matrix inversion with $N=40$ and $\alpha=10$ yields results with an accuracy of about 0.05% as compared with the exact results calculated from Eq. (4.11). Comparisons of these results at a few selected momenta and energies are presented in Table I.

Another test of the accuracy of our solutions is to look at convergence of the determinant $|\mathbf{1}-\mathbf{K}|$ as a function of N. This test is expected to be a sensitive one because evaluation of the determinant involves a summation of many comparable terms with opposite signs. Furthermore, this test becomes very useful when we are dealing with the usual situation in which the exact solutions for the t matrix are not available. Table II shows successive approximations for the determinant at different values of q^2.

A similar test to the above one is to calculate zeros of the determinant as a function of N when there are bound states present in the two-particle system. We calculate the values of q^2 at which the determinant vanishes and compare the results with the exact binding energy obtained by solving Eq. (4.16). These results are presented in Table III. For $N=40$, our solutions for the binding energies differ only about 0.05% from the exact ones.

VI. SUMMARY

We have derived an integral equation for the complete off-energy-shell t matrix in the case of a singular core interaction described by the boundary-condition model, and an outside local potential. This work supplements the previous work of Lomon and his collaborators[16] who studied the half-off-shell t matrix for the same interaction. The total t matrix consists of the pure BCM part $t_{\text{BCM}}(k'\mid q\mid k)$ and the contribution $\tilde{t}_{\text{BCM}}(k'\mid q\mid k)$ which comes from the outside local potential. We have derived a Fredholm integral equation for $\tilde{t}_{\text{BCM}}(k'\mid q\mid k)$. The t matrix for a hard-core interaction and a local outside potential is obtained from our formalism as a special limiting case.

The integral equation for $\tilde{t}_{\text{BCM}}(k'\mid q\mid k)$ has been solved numerically for negative q^2 using the s-wave Herzfeld potential (hard core and outside square well) which gives the experimental nucleon-nucleon bound state and (low-energy) scattering parameters for the $^3S_1$ state (with tensor coupling to $^3D_1$ being ignored). For the Herzfeld potential, an exact analytic expression for the t matrix can be derived.[9] The numerical solution of the integral equation was compared to the exact result. With a reasonable number N of Gaussian quadrature points, we found satisfactory agreement ($\sim$0.5% accuracy with $N=24$ and $\sim$0.05% accuracy with $N=40$).

Extensions to the cases of real positive q^2 are slightly more complicated but straightforward. Inclusion of the Coulomb interaction in the two-body potential does not introduce any difficulties except complications arising from complexities of the Coulomb wave functions, which in this case replace the Bessel functions in our formalism.

It should be pointed out that there are several other works[13,14] in which the complete off-shell t matrix is derived and studied numerically for local hard-core potentials. One of the practical advantages of our method is that, for a given energy parameter q^2, the solution of the integral equation for $\tilde{t}(k'\mid q\mid k)$ determines simultaneously the complete k' and k dependences of $\tilde{t}(k'\mid q\mid k)$, whereas the method proposed by Laughlin and Scott[13] involves solving a differential equation for each value of k' and q, and the method by Fuda[14] involves an additional double integral in k' and k variables after solving an integral equation similar to ours.

The main objective of this paper was to develop a practical method of calculating the complete off-shell t matrix for local potentials with hard cores such as the Hamada-Johnston and Yale potentials[3] and also for potentials with the more general core interactions described by the BCM.[4] The methods presented in this paper are now being used in calculations of the properties of trinucleon scattering and bound states.