Diffraction Model of First- and Second-Order Direct Nuclear Reactions*

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The Hankel-transform (inpact-parameter representation) method is used to simplify the derivation of the diffraction model of direct nuclear reactions. Both first- and second-order processes (represented by pole, triangle, and box diagrams) are discussed, and relatively simple formulas for the corresponding diffraction-model amplitudes are given. Inclusion of Coulomb effects and the extension to backward-peaked processes are also discussed.

I. INTRODUCTION

THE angular distributions of many direct nuclear reactions strongly resemble classical diffraction of light by an absorber. This resemblance is by no means accidental, and has been exploited by many authors in constructing a corpus of phenomenological reaction theories collectively known as the diffraction or strong-absorption model. In view of the excellent reviews of the subject by Austern[1] and by Frahn,[2] it would be superfluous to recapitulate the earlier work here. The intention of this paper is to derive diffraction-model formulas for several direct processes of current interest, using the Hankel-transform, or impact-parameter, method. This approach is useful first, because it simplifies the derivations, second, because it is easily generalized to second-order processes, and third, because the resulting expressions are relatively easy to evaluate.

The organization of the paper is as follows: The remainder of Sec. I defines the term "diffraction model" as it is used in this work. Section II applies the idea to the stripping process (pole diagram). In Secs. III and IV, the triangle and box diagrams (impulse approximation) are analyzed by this scheme, yielding both familiar and unfamiliar results applicable to the study of knockout, inelastic scattering, forbidden transitions, and even (with certain modifications) elastic scattering. Finally, Sec. V summarizes, draws conclusions about, and extends the ideas of this work.

What do we mean by a diffraction model? In physical terms, it is a model of a nuclear reaction in which the reaction itself occurs at the surface of the target nucleus. Consider a reaction with spinless particles: $a+A\to c+C$. Its amplitude $\mathcal{A}(E,Q)$ is a complex-valued function of the barycentric energy E, and momentum transfer Q, with certain well-defined analytic properties in each variable. These properties, besides enabling us to represent the amplitude by the Cauchy integral formula[3] (i.e., in terms of single- or double-dispersion relations), also permit us to expand it in an appropriate complete set of orthogonal functions such as Legendre polynomials or Bessel functions:

$$\mathcal{A}(E,Q)=\sum_l (2l+1)P_l(\cos\theta)\mathcal{A}_l(E) \tag{1}$$

and

$$\mathcal{A}(E,Q)=\int_0^\infty db\, bJ_0(bQ)\mathcal{A}(E,b). \tag{2}$$

At higher energies, Eq. (1) is a slowly converging series and so loses much of its usefulness. How do we know it converges slowly? In the forward direction, we have

$$|\mathcal{A}(E,Q)_{\theta=0}|=|\sum_l (2l+1)\mathcal{A}_l(E)|,$$

whereas at $\theta=\pi$, we have

$$|\mathcal{A}(E,Q)_{\theta=\pi}|=|\sum_l (2l+1)(-1)^l\mathcal{A}_l(E)|.$$

The strong forward peaking observed at medium-to-high energies requires the ratio

$$|\sum_l (2l+1)\mathcal{A}_l(E)|/|\sum_l (2l+1)(-1)^l\mathcal{A}_l(E)|$$

to be large, which is impossible if the terms $\mathcal{A}_l(E)$ decrease too rapidly with l. Thus, previous versions of the diffraction model have all had somehow to resum the higher-l terms of the partial-wave series [Eq. (1)] in order to obtain useful results. The advantages of the Hankel-transform representation [Eq. (2)] are first, that it seems to appear naturally in high-energy approximations,[4] and second, that it exhibits strong forward peaking with simple physical choices for the amplitude $\mathcal{A}(E,b)$. This latter feature is what makes the impact-parameter formalism useful in phenomenological analysis.

The diffraction-model amplitude is obtained from Eq. (2) by first, neglecting contributions to the integral from impact parameters b smaller than some radius R,

* Work supported in part by the U.S. Atomic Energy Commission.

[1] N. Austern, *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963).

[2] W. E. Frahn, *Fundamentals in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1967).

[3] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 563 ff.

[4] R. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Wiley-Interscience Publishers, Inc., New York, 1959), Vol. 1.

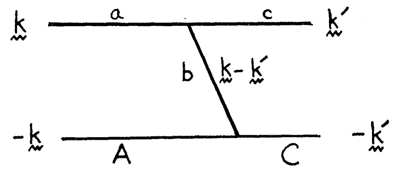

FIG. 1. Pole graph representing stripping or pickup.

and second, by replacing the exact impact-parameter amplitude $\mathcal{A}(E, b)$ by the Hankel transform of the corresponding lowest-order plane-wave Born approximation, $\mathcal{B}(E, b)$. In the case of elastic scattering, this procedure may still be followed, except that we must make explicit provision for elastic unitarity.[2,5] This amounts to assuming that $\mathcal{A}(E, b)$ is purely imaginary for $b \leq R$, and that it vanishes for $b > R$. A single phenomenological parameter is required in the sharp-cutoff approach described above. Additional parameters can be incorporated in the model in a variety of ways, most of them either well known, or obvious, or both. For the purpose of illustrating the Hankel-transform method as applied to various direct processes, this paper will be limited to the sharp-cutoff case. The only generalization will be to nonzero orbital angular momenta.

II. POLE DIAGRAMS

We begin with direct spinless particle transfer, illustrated diagrammatically in Fig. 1. The plane-wave Born approximation (PWBA) for this process is written (on the energy shell) as

$$\mathcal{B}(\mathbf{k}', \mathbf{k}) = -v_a(|\mathbf{k}' - c\mathbf{k}/a|^2) \times [\mathcal{E}_C + \hbar^2(\mathbf{k} - A\mathbf{k}'/C)^2/2m_{bA}]^{-1} v_C^*(|\mathbf{k} - A\mathbf{k}'/C|^2). \tag{3}$$

In Eq. (3), v_a is the vertex function (virtual-decay amplitude) for the process $a \to b + c$, whereas v_C^* plays a corresponding role with regard to $A + b \to C$. The binding energy of particle C is $\mathcal{E}_C$, and the reduced mass m_{bA} of particle b is just $m_b A/(A+b)$. (Here and subsequently, mass ratios such as m_A/m_C are abbreviated A/C.) Energy conservation implies the relation

$$(\hbar^2/2m_{bA})(\mathbf{k} - A\mathbf{k}'/C)^2 + \mathcal{E}_C = (\hbar^2/2m_{bc})(\mathbf{k}' - c\mathbf{k}/a)^2 + \mathcal{E}_a, \tag{4}$$

and so either $\mathbf{Q} \equiv \mathbf{k} - A\mathbf{k}'/C$ or $\mathbf{q} \equiv \mathbf{k}' - c\mathbf{k}/a$ may be used to represent the amplitude as a Hankel transform [Eq. (2)]. That is, the expression (3) is completely symmetric with respect to time reversal. Suppose we choose to expand as a function of Q. Then the diffraction-model amplitude for this process is

$$\mathcal{A}(E, Q) \simeq \int_R^\infty db\, bJ_0(bQ)\mathcal{B}(b), \tag{5}$$

where

$$\mathcal{B}(b) = -(2m_{bA}/\hbar^2) \int_0^\infty dQ\, QJ_0(bQ) \times v_a(\alpha Q^2 + \beta^2)(\kappa_C^2 + Q^2)^{-1} v_C^*(Q^2), \tag{6}$$

and we have written $\alpha = cC/aA$, $\beta^2 = 2m_{bc}(\mathcal{E}_C - \mathcal{E}_a)\hbar^2$, and $\kappa_C^2 = 2m_{bA}\mathcal{E}_C/\hbar^2$. Under the usual assumptions on the analyticity of the vertex functions,[6] we can write

$$\mathcal{B}(b) = -(2m_{bA}/\hbar^2\alpha) \int_{\mu_a}^\infty dx\, \sigma_a(x) \int_{\mu_C}^\infty dy\, \sigma_C(y) \times \{(x^2 - \kappa_C^2)^{-1}(y^2 - \kappa_C^2)^{-1} K_0(b\kappa_C) + (y^2 - x^2)^{-1} \times [K_0(bx)/(x^2 - \kappa_C^2) - K_0(by)/(y^2 - \kappa_C^2)]\}. \tag{7}$$

We have used here the identity[7]

$$(Q^2 + x^2)^{-1} \equiv \int_0^\infty db\, bJ_0(bQ)K_0(bx), \tag{7′}$$

where $K_0(z)$ is the modified Bessel function with exponentially decreasing behavior for large positive z.[8] The constants μ_a and μ_C represent the inverse ranges of the longest-range parts of the corresponding vertex functions. In any reasonable theory, $\mu_a > \kappa_C$ and $\mu_C > \kappa_C$. In fact, for many cases of practical interest, both these constants are much larger than κ_C. Thus, when we put Eq. (7) into Eq. (5), the only important term will be the one proportional to $K_0(\kappa_C b)$ in Eq. (7), i.e., from the longest-range part of the amplitude. Under these conditions we obtain

$$\mathcal{A}(E, Q) \simeq -(2m_{bA}/\hbar^2) v_a(\beta^2 - \alpha\kappa_C^2) v_C^*(-\kappa_C^2) \times [\kappa_C R K_1(\kappa_C R) J_0(QR) - K_0(\kappa_C R) QRJ_1(QR)] \times (Q^2 + \kappa_C^2)^{-1}. \tag{8}$$

Equation (8) is effectively the Butler approximation.[9] That is, the inequalities $\mu_a \gg \kappa_C$, $\mu_C \gg \kappa_C$ give us the right to neglect the variation with q at vertex "a," and to replace the coordinate-space wave function $\Psi_C(r)$ by its asymptotic limit, $N_C e^{-\kappa_C r}/r$. Furthermore, it is clear that Q may be a more useful momentum transfer to use as a Hankel-transform variable than q. For phenomenological purposes, it is essential to formulate the problem in a manner which is as insensitive as possible to the (undetermined) details of the process. In many stripping and pickup reactions, the masses and physical sizes of the various particles are such that the angular distribution is primarily determined

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[5] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 776 (1962).

[6] I. S. Shapiro, *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963); see also H. J. Schnitzer, Rev. Mod. Phys. **37**, 666 (1965).

[7] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed., p. 425.

[8] Reference 7, p. 78.

[9] S. T. Butler, Proc. Roy. Soc. (London) **208**, 559 (1959).

by the variation of only one of the vertex functions. Thus, if we choose to transform with respect to the wrong momentum transfer, the corresponding impact-parameter amplitude will have to be a more complicated function in order to determine the same angular distribution. Moreover, all of the phenomenological analysis will become correspondingly more sensitive to the phenomenological parameters, a feature which is most undesirable. We note that at physical values of the momentum transfer, the J_0 term of Eq. (8) is usually sufficient to give the angular distribution. However, the J_1 term must be kept when extrapolating to the pole in Q^2, in order that the residue be independent of R.[10] Finally, the vertex functions appearing in Eq. (8) have been evaluated at the propagator pole. Under the conditions for which Eq. (8) is valid, we could just as well replace their arguments by the appropriate physical momentum transfers, if this were desired for reasons of convenience or esthetics.

Next, consider stripping with spin. The only major complication results from the orbital angular momenta, since intrinsic spins enter in an essentially straightforward manner. The Born approximation becomes

$$\mathcal{B}_{LM,lm}(Q)=-(2m_{bA}/\hbar^2)v_{a,l}(q)Y_{lm}(\hat{q})(Q^2+\kappa_C^2)^{-1}\times v_{C,L}^*(Q)Y_{LM}^*(\hat{Q}). \quad (9)$$

When taking the Hankel transform of Eq. (9) with respect to Q, for example, we may hold $\hat{Q}$ fixed. However, since $\mathbf{q}=u\mathbf{Q}+v\mathbf{k}$ (where u and v are constants), $\hat{q}$ is not constant as Q varies (in fact, it rotates from $\hat{k}$ to $\hat{Q}$ as Q increases). But since the amplitude [Eq. (9)] is expected to be strongly forward peaked, we make only a relatively small error by letting $\hat{q}=\hat{k}$ in the factor $Y_{lm}(\hat{q})$ (as compared to all the other approximations of the diffraction model).

The simplest way to convert Eq. (9) into a diffraction-model amplitude is to write the propagator in the form of Eq. (7′) and truncate the integral. That is, we assume that the propagator dominates the angular distribution, and this is reasonable in many cases. The result is

$$\mathcal{A}_{LM,lm}(E,Q)\simeq-(2m_{bA}/\hbar^2)Y_{lm}(\hat{k})v_{a,l}(q)\times Y_{LM}^*(\hat{Q})v_{C,L}^*(Q)\int_R^\infty db\, bJ_0(bQ)K_0(b\kappa_C). \quad (10)$$

The next simplest thing to do is to ignore the variation of (say) the $a\to b+c$ vertex, thereby obtaining (see Sec. III for details)

$$\mathcal{A}_{LM,lm}(E,Q)\simeq v_{a,l}(q)Y_{lm}(\hat{k})Y_{LM}^*(\hat{Q})\times\int_R^\infty db\, bJ_L(bQ)b^L\int_0^\infty dz(z^2+b^2)^{-L/2}\Psi_{C,L}^*(z^2+b^2)^{1/2}. \quad (11)$$

[10] R. D. Amado, Phys. Rev. Letters **2**, 399 (1959).

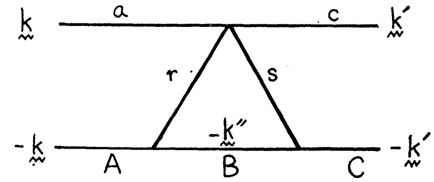

FIG. 2. Triangle diagram applicable to second-order particle transfer (knockout), inelastic and lowest-order elastic scattering.

[Here $\Psi_{C,L}(r)$ is the radial bound-state wave function of the b-A system.] The forms of Eqs. (11) and (10) clearly agree in the forward direction, as well as in the limit where we ignore variations of the vertex v_C. [In view of the great difficulty of calculating the Hankel transform of the function

$$v_{a,l}(q)v_{C,L}^*(Q)(Q^2+\kappa_C^2)^{-1}$$

in the most general case, it is indeed fortunate that in almost all physical applications this is not necessary.]

III. TRIANGLE DIAGRAMS

The simplest second-order process can be represented in PWBA by the triangle graph (Fig. 2). This is actually a special case of the impulse approximation in which the off-shell t matrix for the process $a+r\to c+s$ is replaced by a (complex) constant g^2. This amplitude has been studied in considerable detail by Shapiro[6] and by Blokhintsev *et al.*[11] The triangle mechanism has an elegant representation in the diffraction model. We shall exhibit it for the case where particles a, c, r, s, and A have spin 0 and positive parity, and C has $J^P=L^{(-)^L}$. The PWBA (Fig. 2) is

$$\mathcal{B}_{LM}(Q)=g^2\int d^3r\,\exp(i\mathbf{Q}\cdot\mathbf{r})\Psi_A(r)\Psi_{C,L}^*(r)Y_{LM}^*(\hat{r}), \quad (12)$$

where we have defined the momentum transfer as

$$Q^2=(B\mathbf{k}/A-B\mathbf{k}'/C)^2. \quad (13)$$

The angular integration in Eq. (12) is easily performed, yielding

$$\mathcal{B}_{LM}(Q)=4\pi g^2 i^L Y_{LM}^*(\hat{Q})\times\int_0^\infty dr\, r^2\Psi_A(r)\Psi_{C,L}^*(r)j_L(Qr). \quad (14)$$

Using the identity[12]

$$\int_0^\infty dQ\, QJ_L(bQ)j_L(rQ)=b^L r^{-L-1}(r^2-b^2)^{-1/2}\theta(r-b), \quad (15)$$

[11] L. D. Blokhintsev, E. I. Dolinsky, and V. S. Popov, Nucl. Phys. **40**, 117 (1963).
[12] Bateman Manuscript Project, in *Tables of Integral Transforms* (McGraw-Hill Book Co., New York, 1954), Vol. II, p. 48.

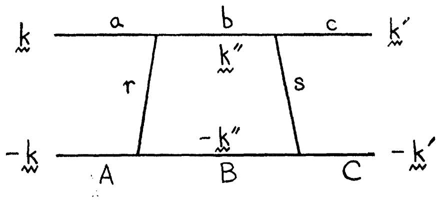

FIG. 3. Box graph—same as the triangle graph, but with a different approximation for the $a+r\to c+s$ off-shell t matrix.

we obtain, as in Sec. II, the diffraction-model result

$$\mathcal{A}_{LM}(Q)=4\pi g^2 i^L Y_{LM}^*(\hat{Q})\int_R^\infty db\, b^{L+1}J_L(bQ)$$
$$\times\int_0^\infty dz\,(z^2+b^2)^{-L/2}\Psi_A[(z^2+b^2)^{1/2}]\Psi_{C,L}^*[(z^2+b^2)^{1/2}]. \quad (16)$$

For large b, we may write

$$\Psi_A(r)\Psi_{C,L}^*(r)\simeq N_A N_C \exp[-(\kappa_A+\kappa_C)r]/r^2. \quad (16')$$

That is, we approximate the wave functions by their asymptotic forms, which is justified when the cutoff radius R is as large as, or larger than, the size of the nuclear interactions which give rise to the bound states Ψ_A and Ψ_C. The approximation [Eq. (16′)] leads to the amplitude

$$\mathcal{A}_{LM}(Q)=4\pi g^2 i^L N_A N_C Y_{LM}^*(\hat{Q})\int_R^\infty db$$
$$\times bJ_L(bQ)\int_{\kappa_A+\kappa_C}^\infty dx\, Ki_L(bx), \quad (17)$$

where $Ki_L(z)$ is defined[13]

$$Ki_L(z)=\int_0^\infty dt\,\exp(-z\cosh t)/(\cosh t)^L. \quad (18)$$

It is interesting to compare Eq. (17) with the result of making a similar approximation in Eq. (11). The stripping angular distribution is determined by the function

$$F(Q)=\int_R^\infty db\, bJ_L(Qb)Ki_L(B\kappa_C b/A). \quad (19)$$

When Eqs. (17) and (19) pertain to the same reaction (so that the orbital angular momentum transfer, momentum transfer, and cutoff radii would be the same for both expressions), and under the conditions which usually obtain in direct reactions, the angular distributions from pole and triangle diagrams are nearly identical, being approximately proportional to $[J_L(QR)]^2$. The angular distribution predicted by Eq. (17) is indistinguishable from the Blair[14] result for inelastic scattering.

Finally, we observe that whereas Eq. (19) has simple poles at $Q=\pm iB\kappa_C/A$, Eq. (17) has logarithmic branch points at the points $Q=\pm i(\kappa_A+\kappa_C)$. (Here we have redefined the momentum transfer for the pole diagram to agree with that of the triangle diagram.)

IV. BOX DIAGRAMS

The box graph can be considered another special case of the impulse approximation, in which the t matrix for $a+r\to c+s$ has a pole corresponding to a bound state or resonance "b" in the $a+r$ (and also in the $c+s$) subsystem. The t matrix is replaced by its pole contribution, and may or may not be unitary.[15] The example we consider has, for simplicity, constant vertices and a simple pole as the off-shell t matrix. The on-shell box amplitude is proportional to (see Fig. 3)

$$\mathcal{B}(E,Q)=-\int d^3K''\,[\alpha^2+(\mathbf{K}''-\mathbf{p})^2]^{-1}(K''^2+\Delta^2)^{-1}$$
$$\times[\beta^2+(\mathbf{K}''-\mathbf{p}')^2]^{-1}. \quad (20)$$

In Eq. (20), $\mathbf{p}=B\mathbf{k}/A$, $\mathbf{p}'=B\mathbf{k}'/C$, and α^2, β^2, and Δ^2 are given by

$$\alpha^2=2m_r(B/A\hbar^2)(m_B+m_r-m_A)c^2, \quad (21a)$$
$$\beta^2=2m_s(B/C\hbar^2)(m_B+m_s-m_C)c^2, \quad (21b)$$
$$-\Delta^2=k^2(bB/aA)+2m_b[B/\hbar^2(b+B)]$$
$$\times(m_a+m_A-m_b-m_B)c^2. \quad (21c)$$

Using Feynman parameterization,[16] Eq. (20) can be integrated in closed form, yielding

$$\mathcal{B}(E,Q)=-(\pi^2/\alpha\beta\Delta)[1-\tfrac{1}{4}(\rho^2+\sigma^2+\tau^2-\rho\sigma\tau)]^{-1/2}$$
$$\times\tan^{-1}\{[1-\tfrac{1}{4}(\rho^2+\sigma^2+\tau^2-\rho\sigma\tau)]/[1+\tfrac{1}{2}(\rho+\sigma+\tau)]\}, \quad (22)$$

where

$$\rho=(\alpha^2+p^2+\Delta^2)/\alpha\Delta, \quad (23a)$$
$$\sigma=(\beta^2+p'^2+\Delta^2)/\beta\Delta, \quad (23b)$$
$$\tau=(\alpha^2+\beta^2+Q^2)/\alpha\beta, \quad (23c)$$

and $Q^2=(\mathbf{p}-\mathbf{p}')^2=(B\mathbf{k}/A-B\mathbf{k}'/C)^2$ is just the momentum-transfer variable which appeared naturally in the triangle amplitude. Equation (22) possesses a more transparent representation

$$B(E,Q)=\int_0^\infty db\, bJ_0(bQ)h(E,b), \quad (24)$$

[13] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (National Bureau of Standards, Washington, D.C., 1964), p. 483.

[14] J. S. Blair, Phys. Rev. **115**, 928 (1959).

[15] Reference 3, p. 754 ff.

[16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), p. 155. See also Ref. 11, as well as the Appendix, where this result is briefly rederived.

where

$$h(E,b)=-\pi^2[(\alpha\beta)^{1/2}/\Delta]\int_0^\infty du\,u^{-1/2}\int_0^\infty dv\,v^{-1/2}\times\exp[-b\mu(u,v)]/\mu(u,v) \quad (25)$$

and

$$\mu^2=(\alpha+\beta)^2+\alpha\beta[\rho v+\sigma u+uv+(u-v)^2/uv]. \quad (26)$$

The derivation of Eq. (25) is given in detail in the Appendix. We see from Eqs. (25), (26), and (23) that the minimum value of the modulus of $\mu(u,v)$ is $\alpha+\beta$, independent of whether Δ^2 is positive or negative. That is, the box amplitude exhibits (independent of energy) logarithmic branch points in the complex Q plane, at $Q=\pm i(\alpha+\beta)$, exactly as did the corresponding triangle diagram.

We are interested in the energy dependence of the box diagram near the threshold for real production of the intermediate state $b+B$, i.e., near $\Delta^2=0$. Because $\mu(u,v)$ is such a complicated function of the integration variables, we can only give here a qualitative discussion of the behavior of $B(E,Q)$. We see that the asymptotic behavior of $h(E,b)$ in Eq. (25), as a function of the impact parameter b, is exponential with a range determined by the average value of $\mu(u,v)$. When $\Delta^2>0$, (i.e., below threshold), the term $\alpha\beta[\rho v+\sigma u]$ is real and positive, and therefore tends to increase μ_{av}. To analytically continue to $\Delta^2<0$, we set $\Delta=-i\kappa$, and then the $\alpha\beta[\rho v+\sigma u]$ term becomes purely imaginary. This reduces μ_{av}; thus, the amplitude $B(E,Q)$ receives its main contribution from larger values of b. That is, although the nearest momentum-transfer singularity is always at $\pm i(\alpha+\beta)$ {and hence the longest-range part of $h(E,b)$ is the tail $\propto\exp[-b(\alpha+\beta)]$}, the "effective range" of the amplitude is always somewhat less than $(\alpha+\beta)^{-1}$. Furthermore, we expect this effective range to increase rapidly as the energy surpasses the threshold of the $b+B$ intermediate state. In the presence of strong absorption, this effect will naturally be accentuated as the interaction region does or does not extend beyond the strong-absorption radius. Moreover, the phase of the box amplitude varies rapidly as the energy exceeds threshold, making possible interference with a more slowly varying background amplitude, if such is present. Thus the sudden increase of the diffraction-modified box amplitude above threshold could produce either a bump or a dip in the cross section. We also expect, by analogy with certain three-particle, K-matrix and dispersion-theoretic calculations,[17] that the coupling between the $b+B$ channel and other open channels (e.g., elastic $a+A$, $c+C$, and inelastic $b+B$, as well as other rearrangements) will, through unitarity, absorb the flux from the $b+B$ channel and thereby quickly damp the $a+A\to c+C$ cross section after its rise at threshold. That is, if the box amplitude dominates the cross section, the net effect of an opening channel in the intermediate state should be a sharp bump.

The diffraction-model amplitude derived from Eq. (24) is

$$A(E,Q)=\int_R^\infty db\,bJ_0(bQ)h(E,b). \quad (27)$$

This suggests how the diffraction-modified box amplitude can be generalized to the case where the $B+s\to C$ vertex has orbital angular momentum L, without the necessity for deriving first the corresponding PWBA expression: It should be adequate to replace the impact-parameter amplitude

$$\int_{\kappa_A+\kappa_C}^\infty dx\,Ki_L(bx)=Ki_{L+1}(b(\kappa_A+\kappa_C))/b,$$

appearing in Eq. (17) by the corresponding function

$$-\pi^2[(\alpha\beta)^{1/2}/\Delta]\int_0^\infty du\,u^{-1/2}\int_0^\infty dv\,v^{-1/2}\exp[-b\mu(u,v)]\times[(\alpha+\beta)/\mu(u,v)]^L/\mu(u,v).$$

V. SUMMARY AND CONCLUSIONS

The previous sections have described how to use the Hankel-transform method to represent various types of direct-reaction amplitudes in the sharp-cutoff diffraction model. Most cases of interest can be built up from an appropriate superposition of the simpler forms considered in this paper. It is also clear how to incorporate additional parameters to represent such effects as nuclear "skin-thickness." Probably it is best not to introduce too many extra parameters, since we do not wish to create a phenomenological theory which is merely an analog computer for curve fitting. There are a host of effects, all of about the same magnitude, which can be neglected in a rough treatment, but it is not clear which of them we are justified in including while ignoring others. In such a situation, perhaps the best thing to do is to include none.

A generalization which is clearly necessary is the inclusion of Coulomb effects. Any method that accounts for most of the effect and introduces no additional undetermined parameters is probably satisfactory. A way that suggests itself is the impact-parameter version of the Durand-Chiu-Sopkovitch approximation[18] in which the amplitude is written

$$\mathcal{A}(E,b)\simeq[S_f(E,b)]^{1/2}\mathcal{B}(E,b)[S_i(E,b)]^{1/2}. \quad (28)$$

In Eq. (28), $S_f(E,b)$ and $S_i(E,b)$ are the impact-

[17] R. Aaron, D. Teplitz, R. Amado, and J. Young, Phys. Rev. **187**, 2047 (1969); R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, London, 1962), p. 122 ff; J. Ball and W. Frazer, Phys. Rev. Letters **7**, 204 (1961); R. F. Peierls, *ibid.* **6**, 641 (1961).

[18] N. J. Sopkovitch, Nuovo Cimento **26**, 186 (1962); L. Durand and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

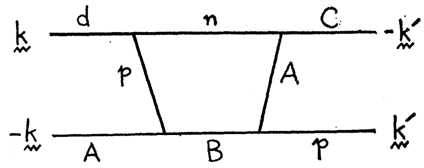

FIG. 4. Backward-peaked second-order contribution to (d, p) reactions.

parameter "S matrices" for elastic scattering in final and initial channels of the reaction being considered, and $\mathcal{B}(E, b)$ is the Hankel transform of its PWBA amplitude. The S matrices define the elastic differential cross sections in their respective channels through the relation

$$\frac{d\sigma}{d\Omega}=\left|ik\int_0^\infty db\, bJ_0(2k\sin\tfrac{1}{2}\theta)[1-S(E,b)]\right|^2. \quad (29)$$

In the case of interest, for $b>R$ we may approximate $S(E, b)$ by

$$S(E,b)\simeq\exp[2i\eta\ln(kb)-2i\phi], \quad (30)$$

where

$$\eta=(1/137)z_1z_2M_rc/\hbar k \quad (31)$$

and ϕ is a constant (real) phase angle resulting from the long-range cutoff we apply to the Coulomb potential.[19] The approximation [Eq. (30)] follows from the eikonal approximation[20] and agrees with the analytic continuation of the partial-wave Coulomb phase shift

$$\sigma_l(E)=\arg\Gamma(l+1+i\eta) \quad (32)$$

to $l=kb-\frac{1}{2}$. This approximation of the Coulomb effects is similar both to the method of Bethe[21] in proton-proton scattering (based on the WKB approximation) and to the Blair model.[22] Furthermore, there is evidence[23] that the Durand-Chiu-Sopkovitch approximation gives a good account of direct reaction amplitudes in the partial-wave representation, and there is no reason to believe that the impact-parameter representation will invalidate it. Application of expression (28) to analysis of several reactions is currently underway, and the results will be reported elsewhere.

Finally, the Hankel-transform approach can easily be used to represent *backward*-peaked reaction mechanisms. We would expect, by analogy with the partial-wave representation of a backward-peaked process, that the Hankel transform, with respect to the variable Q, will turn out to be a rapidly oscillatory function of b. Therefore it is prudent to transform with respect to an appropriate exchange, or "crossed-channel" momentum transfer, rather than the direct momentum transfer Q. That is, in a process such as that illustrated by Fig. 4, we should write the amplitude as

$$\mathcal{A}(E,\tilde{Q})=\int_0^\infty db\, bJ_0(b\tilde{Q})\mathcal{A}(E,b), \quad (33)$$

where

$$\tilde{Q}=|n\mathbf{k}/d+n\mathbf{k}'/C|. \quad (34)$$

With this slight modification, all the results of this paper can be applied directly to the phenomenological analysis of backward-peaked graphs.

Many of the methods of this paper have appeared earlier in the literature, particularly in high-energy applications.[24-28] To my knowledge, no analogous discussion of second-order processes has been given previously. In particular, I believe the comparison of first- and second-order angular distributions and the discussion of threshold effects in the box graph (as a range phenomenon) are new. In view of the impossibility of searching the entire, vast literature on the diffraction model, I apologize in advance to anyone whose work I may have inadvertently duplicated and to which I have not referred.

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APPENDIX: EVALUATION OF THE BOX GRAPH

A form of Feynman parameterization[16] can be used to evaluate the integral in Eq. (20). The identity

$$a^{-1}=\int_0^\infty du\,\exp(-au) \quad (A1)$$

is used to elevate the three energy denominators into exponentials, transforming the integral into the form

$$J=\int_0^\infty du\int_0^\infty dv\int_0^\infty dw\int d^3K'' \times\exp[-\alpha^2u-\Delta^2v-\beta^2w-K''^2(u+v+w) +2\mathbf{K}''\cdot(\mathbf{p}u+\mathbf{p}'w)-p^2u-p'^2w]. \quad (A2)$$

By completing the square in the variable $\mathbf{K}''$ and transforming the origin in this variable, the K'' integral

[19] Reference 3, p. 265.
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may be performed, giving us

$$J=\pi^{3/2}\int_0^\infty du\int_0^\infty dv\int_0^\infty dw\ (u+v+w)^{-3/2}$$
$$\times\exp[(\mathbf{p}u+\mathbf{p}'w)^2/(u+v+w)-(\alpha^2+p^2)u-(\beta^2+p'^2)w-\Delta^2 v]. \quad (A3)$$

We now scale u and w with respect to v: $u\to vu$, $w\to vw$. Thus, after integrating with respect to v, we have

$$J=\pi^2\int_0^\infty du\int_0^\infty dw\ [\alpha^2u^2+\beta^2w^2+(\alpha^2+\beta^2+Q^2)wu+(\alpha^2+p^2+\Delta^2)u+(\beta^2+p'^2+\Delta^2)w+\Delta^2]^{-3/2}. \quad (A4)$$

Scaling once more with respect to $\alpha\Delta$ and $\beta\Delta$, and defining ρ, σ, and τ as in Eqs. (23), we find $J=\frac{1}{2}\pi^2 I/\alpha\beta\Delta$, where

$$I=\int_0^\infty du\int_0^\infty dw\ (u^2+w^2+\rho u+\sigma w+\tau uw+1)^{-3/2}. \quad (A5)$$

The integral with respect to (say) w can immediately be performed, and the transformation

$$u=-\tfrac{1}{2}\rho+\tfrac{1}{2}[x/(1+\tfrac{1}{2}\tau)-(1+\tfrac{1}{2}\tau)(1-\tfrac{1}{4}\rho^2)/x]$$

immediately reduces I to the form

$$I=2\int_{1+\frac{1}{2}(\rho+\sigma+\tau)}^\infty dx\,[x^2+1-\tfrac{1}{4}(\rho^2+\sigma^2+\tau^2-\rho\sigma\tau)]^{-1}, \quad (A6)$$

which is easily integrated to give Eq. (22).

Moreover, Eq. (A4) is easily put into the form of Eq. (25) by first, changing variables

$$u\to\Delta/\alpha u, \qquad w\to\Delta/\beta v,$$

which gives

$$J=\pi^2[(\alpha\beta)^{1/2}/\Delta]\int_0^\infty du\ u^{-1/2}\int_0^\infty dv\ \times v^{-1/2}\{\mu^2(u,v)+Q^2\}^{-3/2}, \quad (A7)$$

where $\mu^2(u, v)$ is as given in Eq. (26); and then substituting the expression[29]

$$\int_0^\infty db\ bJ_0(bQ)\ \exp(-b\mu)/\mu=(\mu^2+Q^2)^{-3/2} \quad (A8)$$

into (A7).

[29] Reference 12, p. 29.

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Threshold Electrodisintegration of the Deuteron*

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The electrodisintegration of the deuteron near threshold has been investigated by inelastic electron scattering at momentum transfers $q=0.35$ F^{-1} and $q=0.66$ F^{-1} and scattering angles $\theta=125°$, 140°, and 155°. The longitudinal and transverse matrix elements to the final $^1S_0$ and $^3S_1$ states have been compared with the Jankus-Durand theory using repulsive-core wave functions. The repulsive-core parameter, corresponding to a radius of about 0.2 F, has been adjusted to fit the elastic deuteron form factors up to $q=6$ F^{-1}. The computed inelastic cross sections give a statisfactory description of the existing data in the whole momentum-transfer range experimentally investigated: (0.35 F$^{-1}\leq q\leq 3$ F^{-1}).

INTRODUCTION

NEAR the electrodisintegration threshold, the inelastic electron scattering cross section from the deuteron $d^2\sigma/d\Omega dE_x$ is enhanced by a strong interaction between the neutron and the proton in the final $^1S_0$ and $^3S_1$ partial waves. This final-state interaction has been investigated theoretically by Jankus[1] and Durand[2] using a nonrelativistic potential model, and by Bosco[3] using the dispersive approach. Both of these methods neglect two-body exchange currents and in the Jankus-Durand method our knowledge of the nucleon-nucleon potential is still inadequate, especially in the repulsive-core region. The nucleon-nucleon scattering data from energies 0.5 to 450 MeV, fitted phenomenologically with energy-dependent phase shifts using separate $^1S_0$-wave phase shifts for the n-p and p-p systems, suggest some degree of charge dependence of the two-nucleon force.[4] Yet the attempts to fit the data with an energy-independent potential

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