These forms indicate a relative strong MS influence on C_{nnn} and $C_{n\times k\,Kn}$ and a weak influence on the remaining two coefficients.

IV. SUMMARY AND CONCLUSIONS

To summarize, a formalism was developed to include the MS interaction in the neutron-proton scattering problem. This procedure is of general interest ecause the MS force causes channel-spin nonconservation. The scattering amplitude matrix was derived, including this singlet-triplet mixing, and its elements were related to the Wolfenstein coefficients. The nine most common scattering observables were also related to these coefficients. The effect of the MS interaction on phase shifts was evaluated through the use of a perturbation calculation for low l waves, and a Born-approximation calculation for high l waves. Calculations for energies from 25–210 MeV were performed to find the influence of the MS force on the scattering observables where it was assumed that the phase shifts determined by the lowest χ^2 fits to the scattering data for the nucleon-nucleon problem were due to the nuclear interaction only.

From this study it can be concluded that all nine scattering observables are markedly influenced by the MS interaction, but for all but the lowest energies this influence is confined to small scattering angles ($<5°$). At low energy, the influence on polarization and the correlation coefficient C_{KP}, and possibly the correlation coefficients C_{nnn} and $C_{n\times k\,Kn}$, extends to beyond 10°. Thus, phase shifts as determined from scattering data and reported in the literature will not be altered by inclusion of the MS force in the formalism unless small angle measurements are included in the data.

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Self-Consistent Calculation of p-Shell Hypernuclear Binding Energies*

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Calculations of the binding energies of p-shell hypernuclei have been carried out treating all the nucleons and the Λ self-consistently. The Tabakin potential is used for the NN interaction and a simple central force which fits ΛN scattering is used for the Λ-N system. The binding energies obtained exceed those observed. The dynamical aspects of the hypernuclear system are also discussed, as well as various correction terms. Finally, a comparison is made with other hypernuclear calculations.

I. INTRODUCTION

BINDING energies of Λ hypernuclei in the p shell may provide valuable information about certain properties of the Λ-N interaction, which have not been or cannot be measured in free Λ-N scattering. Indeed Λ-N scattering data is quite sparse and is subject to a wide variety of interpretations. If it is assumed that the Λ-N potential is central then the binding energies of s-shell hypernuclei exceed the experimental values, independent of the particular model employed.[1] This has led to the suggestion that the Λ-N interaction is severely suppressed in a hypernucleus relative to the free Λ-N force. Alternatively, this difficulty might be overcome by including a tensor component or an exchange part in a phenomenological Λ-N interaction. Such attempts have so far been unsuccessful. The binding energy of the Λ in ${}_{\Lambda}He^5$, for example, was found[2] to be insensitive to an additional short-range tensor force and an exchange interaction would have little, if any, effect on a calculation for this system. It is likely that such additional components of the interaction will have a more pronounced effect in p-shell hypernuclear calculations but so far the problem of overbinding remains for ${}_{\Lambda}He^5$ and also, probably, for the $A=4$ hypernuclei.

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[1] R. C. Herndon and Y. C. Tang, Phys. Rev. **159**, 853 (1967); S. Ali, M. E. Grypeos, and L. P. Kok, Phys. Letters **24B**, 543 (1967); R. K. Bhaduri, Y. Nogami, and W. VanDijk, Phys. Rev. **155**, 1671 (1967).

[2] J. Law, M. R. Gunye, and R. K. Bhaduri, in *Proceedings of the International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969* (to be published).

It has been suggested[3,4] that this difficulty might be overcome if three-body forces are included in the analysis. Specifically, there are irregularities in the Λ binding energies of p-shell hypernuclei, notably the relatively low binding of ${}_{\Lambda}Be^{9}$ and to some extent ${}_{\Lambda}C^{13}$ (see Fig. 1). These might be related to a repulsive part in the ΛNN interaction rather than implying a huge spin dependence of the Λ-N force.[5] Scattering experiments, of course, give no information about the existence or nature of ΛNN forces and one must invoke a model, like two-pion exchange (TPE), in order to estimate three-body effects. The result of such estimates[4] in ${}_{\Lambda}He^{5}$ is a reduction in the binding energy so that reasonable agreement with experiment is obtained. Such a treatment has, however, other difficulties. First of all, the dominance of the TPE diagram (even at small interparticle distances) is surely unjustifiable, and other exchanges would have to be considered. Second, use of the three-body interaction presupposes a knowledge of the nuclear correlations which are, in fact, unknown. For p-shell hypernuclei the uncertainties of such calculations are even larger, since noncentral aspects of the ΛNN force would have to be considered.

It is by no means obvious that the irregularities in the B_{Λ} values for p-shell hypernuclei do not originate from something other than three-body forces. It is, in fact, known that p-shell nuclei are deformed, but the effect of the Λ particle on the deformation and of the deformation on the binding of the Λ has not been carefully considered. Since these are much simpler than considering three-body forces it is natural to determine to what extent the irregularities can be explained by taking the dynamical aspects of deformation into account.

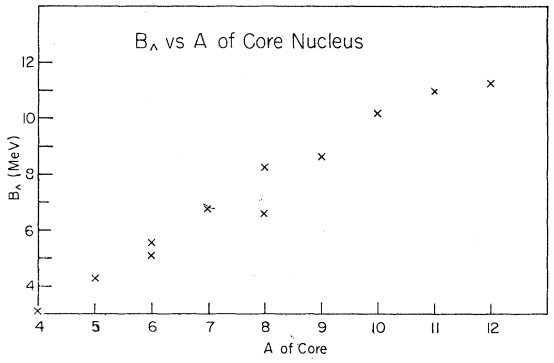

FIG. 1. Observed B_{Λ} for p-shell hypernuclei [D. Davis and J. Sacton, invited talk at the *International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969* (to be published)]. There are two hypernuclei whose core nuclei have $A=6$ and two with $A=8$. The more tightly bound are ${}_{\Lambda}Li^{7}$ and ${}_{\Lambda}Li^{9}$. The other hypernuclei shown for $A=6$ and $A=8$ are ${}_{\Lambda}Be^{7}$ and ${}_{\Lambda}Be^{9}$. The statistical error associated with all shown points never exceeds 0.2 MeV.

[3] A. Gal, Phys. Rev. **152**, 975 (1966).
[4] R. K. Bhaduri, B. Loiseau, and Y. Nogami, Ann. Phys, (N. Y.) **44**, 55 (1967).
[5] A. Gal, Phys. Rev. Letters **18**, 568 (1967).

In this paper, the Hartree-Fock (HF) method is used to calculate the binding energies of hypernuclei assuming a simple, central ΛN force that fits the scattering data. Unlike other methods, HF allows one to consider, in a consistent way, the mutual polarization effects of the Λ-nucleus system. Thus, one maintains a dynamical description of the Λ and the nucleons present in a hypernucleus. In particular, the compression and deformation affected by adding a Λ to a nucleus can be calculated and whether or not these effects lead to the observed irregularities can be determined.

II. METHOD OF CALCULATION

The Hartree-Fock method has been applied to many nuclei with various degrees of generality.[6] The method is fully described elsewhere[7] and essentially entails the diagonalization of the matrix

$$\langle i \mid h \mid j\rangle \equiv \langle i \mid t \mid j\rangle + \sum_{k=1}^{N} \langle ik \mid V_A \mid jk\rangle, \qquad (1)$$

where i and j indicate an arbitrary basis, t is the kinetic energy operator, V_A indicates that the matrix element is antisymmetrized, and the summation is restricted to occupied orbitals. An iteration process is necessary since the occupied orbitals k are just the lowest eigenfunctions of the h matrix. In practice, the i, j representation is usually taken to be the harmonic-oscillator functions comprising the first few shells of the shell model, and k is expanded in these same functions. The coefficients of the expansion are then determined self-consistently by iteration. The Hartree-Fock energy is then given by

$$E_{\mathrm{HF}} = \tfrac{1}{2} \sum_{k=1}^{N} (\epsilon_k + \langle k \mid t \mid k\rangle), \qquad (2)$$

where the ϵ's are the eigenvalues of h.

The method can be generalized in a straight-forward manner to include Λ particles. The Hamiltonian for a system of nucleons and Λ's is

$$\begin{aligned} H = &\sum_{ij} \langle i \mid t \mid j\rangle a_i^{\dagger} a_j + \sum_{\alpha\beta} \langle \alpha \mid \tilde{t} \mid \beta\rangle b_{\alpha}^{\dagger} b_{\beta} \\ &+ \tfrac{1}{2} \sum_{ijkl} \langle ij \mid V \mid \bar{k}l\rangle a_i^{\dagger} a_j^{\dagger} a_l a_k \\ &+ \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta \mid U \mid \gamma\delta\rangle b_{\alpha}^{\dagger} b_{\beta}^{\dagger} b_{\delta} b_{\gamma} \\ &+ \sum_{\alpha\beta mn} \langle \alpha m \mid W \mid \beta n\rangle b_{\beta}^{\dagger} a_m^{\dagger} b_{\beta} a_n, \qquad (3) \end{aligned}$$

where a, $a^{\dagger}$ create and destroy nucleons, b, $b^{\dagger}$ create and destroy Λ particles. V is the N-N interaction, U the Λ-Λ interaction, W the Λ-N interaction, and t and $\tilde{t}$ are the N and Λ kinetic energy operators. If we consider

[6] I. Kelson and C. A. Levinson, Phys. Rev. **134**, B269 (1964); S. J. Krieger, M. Baranger, and K. T. R. Davies, Nucl. Phys. **84**, 545 (1966).
[7] M. Baranger, in *Cargèse Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1963); A. K. Kerman, in *Lectures Given at the Summer School of Cargèse, Corse, 1968* (to be published).

hypernuclei with only one Λ particle the trial wave function is

$$\psi=b_\mu^\dagger \prod_{k=1}^{N} a_k^\dagger \mid 0\rangle \tag{4}$$

and the U term will not contribute. Variations with respect to the k and μ orbitals lead to the set of simultaneous equations

$$\langle j \mid h \mid r\rangle \equiv \langle j \mid t \mid r\rangle + \sum_{k=1}^{N} \langle jk \mid V_A \mid rk\rangle + \langle \mu j \mid W \mid \mu r\rangle = 0, \tag{5}$$

$$\langle \eta \mid g \mid \mu\rangle \equiv \langle \eta \mid \tilde{t} \mid \mu\rangle + \sum_{k=1}^{N} \langle \eta k \mid W \mid \mu k\rangle = 0, \tag{6}$$

where μ is the occupied Λ orbital, η an unoccupied Λ orbital, k and r are occupied orbitals, and j is an unoccupied nucleon orbital. As in the usual HF method, the above condition that h and g not connect occupied to unoccupied orbitals is imposed by requiring h and g to be diagonal. Thus, the generalized HF equations are

$$\langle i \mid h \mid j\rangle \equiv \langle i \mid t \mid j\rangle + \sum_{k=1}^{N} \langle ik \mid V_A \mid jk\rangle + \langle \mu i \mid W \mid \mu j\rangle = \epsilon_i \delta_{ij}, \tag{7}$$

$$\langle \nu \mid g \mid \mu\rangle \equiv \langle \nu \mid \tilde{t} \mid \mu\rangle + \sum_{k=1}^{N} \langle \nu k \mid W \mid \mu k\rangle = \omega_\mu \delta_{\mu\nu}, \tag{8}$$

and the corresponding energy is given by

$$E=\tfrac{1}{2}\sum_{k=1}^{N} (\epsilon_k + \langle k \mid t \mid k\rangle) + \tfrac{1}{2}(\omega_\mu + \langle \mu \mid \tilde{t} \mid \mu\rangle). \tag{9}$$

The iterative procedure for solving these equations consists of choosing initial nuclear orbits k, diagonalizing g to obtain the Λ orbital μ, and substituting this μ and the k's into h. The procedure is then repeated with the resulting eigenfunctions of h until consistency is obtained.

In practice, both the Λ and nucleon orbitals are expanded in harmonic-oscillator bases

$$\mid k\rangle = \sum_i c_i^k \mid (nl_j m\tau_t)_i\rangle, \tag{10}$$

$$\mid \mu\rangle = \sum_\alpha d_\alpha^\mu \mid (nl_j m)_\alpha\rangle, \tag{11}$$

where the set i contains $1s$, $1p$, $2s$-$1d$, and $2p$-$1f$ shells and α contains the $1s$ and the $2s$-$1d$ shells. This truncation introduces a dependence on the oscillator constants which must both be determined so as to minimize E and, in general, will have different optimum values. It should be noted, however, that this space is sufficiently large that many deformation modes can be simulated for Be^8, C^{12} and somewhat heavier nuclei. This view is supported by the small change in all relevant nuclear quantities, including the quadrupole moment, when a larger space is employed.[8]

[8] W. H. Bassichis, B. A. Pohl, and A. K. Kerman, Nucl. Phys. **A112**, 360 (1968).

The evaluation of the matrix elements of V in Eq. (7) is implemented by a transformation from the coordinates of the two nucleons to their relative and c.m. coordinates. If the potentials generating the wave functions are different, as is the case for the W matrix elements, the usual transformation must be generalized. The existence of such generalized transformation brackets[9] enables one to evaluate the W matrix elements in the standard way.

III. CHOICE OF INTERACTIONS

The nucleon-nucleon interaction V was chosen to be that of Tabakin.[10] The force has the advantage of fitting nucleon-nucleon scattering reasonably well, while being smooth enough so that its matrix elements are finite. This force has been used extensively in nuclear HF calculations and is known to have certain shortcomings, which will be discussed later. The Coulomb force has been neglected in all of the calculations.

The Λ-N force used was the sum of an attractive and a repulsive Gaussian,

$$W(r) = -V_0 \exp(-r^2/a^2) + V_1 \exp(-r^2/b^2), \tag{12}$$

where V_0 and V_1 are positive. The range parameters, a and b, were chosen so that they give the intrinsic range parameters corresponding to two-pion exchange and K-meson exchange, respectively, i.e.,

$$a^2=1.063 \text{ fm}^2, \qquad b^2=0.349 \text{ fm}^2. \tag{13}$$

The soft repulsive part of $W(r)$ was supposed to simulate the effects of a strong short-range repulsion. It was found *a posteriori* that, unless such a repulsion is included, the Λ particle wave function tended to be concentrated farther inside the nuclear interior than is reasonable. The values of the strengths V_0 and V_1, were fixed by solving the S-wave scattering problem and comparing the results with the low-energy scattering data of Alexander *et al.*[11] Examples of such comparisons are given in Table I. The fits shown are quite reasonable except for the first potential. The effect of a small repulsive part is seen by comparing the first two rows. If the strength of the first potential were reduced by about 4%, reasonable agreement would be obtained also in that case.

IV. DETERMINATION OF OSCILLATOR PARAMETERS

In principle, the oscillator parameters for the nucleon wave functions γ_N and the oscillator parameter for the Λ wave functions γ_Λ should both be treated as variables to be determined by the minimization of the energy. [The correspondence between this notation and that of Ref. 9 is given by $\nu\gamma=1$, where the Gaussian factor in

[9] A. Gal, Ann. Phys. (N.Y.) **49**, 341 (1968).
[10] F. Tabakin, Ann. Phys. (N. Y.) **30**, 51 (1964).
[11] G. Alexander *et al.*, Phys. Rev. **173**, 1452 (1968).

TABLE I. (a) Calculated s-wave Λ-N total cross sections (in mb) for different Λ-N central potentials. The strengths V_0 and V_1 are defined by Eq. (12). (b) Experimental Λ-p total cross section (Ref. 11).

(a)

	Potential strength (MeV)		Calculated σ_Λ for S waves (mb) Incident Λ momentum (MeV/c)										
	V_0	V_1	120	140	160	180	200	220	240	260	280	300	320
(a)	55.2	0	219	195	172	151	133	116	102	90	78	68	60
(b)	55.2	9.2	188	168	150	133	117	103	91	80	70	62	54
(c)	59.8	27.6	190	170	150	132	116	102	90	78	68	60	53
(d)	69.0	64.4	196	174	154	134	118	102	90	78	68	59	51
(e)	73.6	82.8	202	178	156	136	119	104	90	78	67	58	56

(b)

Incident Λ momentum (MeV/c)	Experimental $\sigma_{\Lambda P}$ (mb)
120–170	180±22
170–200	130±17
200–220	118±16
220–240	101±12
240–260	83±9
260–320	57±9

the wave functions is of the form $\exp(-r^2/2\gamma)$.] Since, however, the optimum γ_N for light nuclei was found to be 2.6 fm^2, this value was used throughout the calculations.[12] The Λ oscillator parameter was, however, varied. Calculations were performed for ${}_\Lambda$He^5 and ${}_\Lambda$O^{17} for a number of forces that fit the scattering data and for different values of γ_Λ in order to determine the sensitivity of the results to these parameters. The results are summarized in Table II. For ${}_\Lambda$He^5 the minimum occurs, for each of the forces used, at $\gamma_\Lambda=2.0$ fm^2, though a curve of B_Λ as a function of γ_Λ is rather flat between $\gamma_\Lambda=1.7$ fm^2 and $\gamma_\Lambda=3.0$ fm^2. The binding energy of the Λ, B_Λ is defined by

$$-B_\Lambda=[\tfrac{1}{2}\sum_k(\epsilon_k+\langle k\mid t\mid k\rangle)+\tfrac{1}{2}(\omega_\mu+\langle\mu\mid\tilde{t}\mid\mu\rangle)-\tfrac{1}{2}\sum_{k_0}(\epsilon_{k_0}+\langle k_0\mid t\mid k_0\rangle)]. \quad (14)$$

Here the subscript zero indicates that the eigenvalues and eigenfunctions are those obtained without a Λ present. For ${}_\Lambda$O^{17} the minimum occurs at $\gamma_\Lambda=1.7$ fm^2, though again the B_Λ curve is quite flat. It should be noted that, though the B_Λ curves may be relatively flat, other quantities, like $\langle Q_\Lambda\rangle$ and $\langle r_\Lambda{}^2\rangle$, may not be so insensitive to changes in γ_Λ. Thus, it is important when working in any finite space to determine the optimum γ_Λ. The sensitivity of various quantities to variations in γ_Λ is shown in Table III.

The binding energy of ${}_\Lambda$He^5, as shown in Table II, exceeds the observed binding energy by an amount similar to that obtained in other calculations. No remedy for this is attempted in the present calculations. The incremental binding energy of roughly 3 MeV per nucleon between ${}_\Lambda$He^5 and ${}_\Lambda$O^{17} is also too large. This is directly a result of the diagonal matrix element of the Λ-N interaction between a $1s$ Λ and a $1p$ nucleon being approximately 2.5 MeV for $\gamma_\Lambda=2.0$ fm^2. An attempt to reduce this will be discussed in a later section.

TABLE II. (a) The B_Λ for ${}_\Lambda$He^5 as a function of the oscillator parameter γ_Λ and the potential strengths V_0 and V_1 (see Table I). (b) The B_Λ for ${}_\Lambda$O^{17} as a function of γ_Λ and the strengths V_0 and V_1.

(a)

γ_Λ (fm^2)	B_Λ for ${}_\Lambda$He^5 (MeV) Force b	Force d	Force e
1.0	4.15	4.39	4.47
1.7	5.81	6.14	6.25
2.0	5.95	6.29	6.41
3.0	5.85	6.23	6.35
4.0	5.46	5.86	5.99

(b)

	B_Λ for ${}_\Lambda$O^{17} (MeV)		
1.0	39.1	41.1	41.8
1.7	39.3	41.3	42.0
2.0	39.1	41.1	41.7
3.0	37.5	39.3	40.0
4.0	35.8	37.5	38.1

TABLE III. Sensitivity of Q_Λ and R_Λ to variations in the oscillator parameter.

Core nucleus	Q_Λ (mb) $\gamma_\Lambda=1.7$	$\gamma_\Lambda=2.0$	R_Λ (fm) $\gamma_\Lambda=1.7$	$\gamma_\Lambda=2.0$
Li^6	3.75	4.41	1.75	1.80
Li^7	6.99	8.24	1.75	1.80
Be^8	10.70	12.00	1.75	1.80
B^{10}	6.24	6.40	1.63	1.66
C^{12}	−4.85	−4.44	1.57	1.58

[12] A. K. Kerman, J. P. Svenne, and F. M. H. Villars, Phys. Rev. **147**, 710 (1966).

TABLE IV. Results of HF calculations using the optimum γ_Λ. From ${}_\Lambda He^5$ to ${}_\Lambda He^8$ the optimum γ_Λ is 2.0 fm^2 and for the heavier hypernuclei shown it is 1.7 fm^2. The Q_{nucleus} and R_{nucleon} values in parentheses are those obtained by HF for the core nucleus alone. The * indicates that a c.m. correction has been applied. The relevant definitions for the quantities listed are found in Eqs. (14) and (15).

Core nucleus	B_Λ (MeV)	Q_{nucleus} (b)		Q_Λ (mb)	R_{nucleon} (fm)		R_Λ (fm)
He^4	6.41	0	(0)	0	1.58*	(1.70)*	1.62*
He^6	9.72	0.149	(0.190)	4.44	2.16	(2.36)	1.81
Li^6	10.00	0.144	(0.186)	4.41	2.14	(2.35)	1.80
Li^7	11.88	0.241	(0.320)	8.24	2.22	(2.45)	1.80
He^7	11.94	0.097	(0.131)	2.73	2.24	(2.44)	1.77
Be^8	14.32	0.332	(0.479)	12.00	2.24	(2.52)	1.80
Li^8	14.53	0.197	(0.267)	6.34	2.24	(2.46)	1.75
He^8	14.12	0.045	(0.083)	1.14	2.28	(2.48)	1.74
B^9	17.28	0.285	(0.412)	8.61	2.24	(2.51)	1.70
B^{10}	21.24	0.230	(0.331)	6.24	2.21	(2.45)	1.63
B^{11}	24.39	−0.222	(−0.296)	−4.51	2.21	(2.43)	1.60
C^{12}	28.32	−0.244	(−0.325)	−4.85	2.20	(2.41)	1.57
C^{13}	31.53	−0.162	(−0.232)	−3.06	2.19	(2.39)	1.53
N^{14}	36.23	0.019	(0.035)	0.38	2.17	(2.33)	1.49
N^{15}	39.27	0.00	(0.00)	0.00	2.18	(2.34)	1.47
O^{16}	42.00	0	(0)	0	2.19	(2.35)	1.50
Ne^{20}	48.77	0.497	(0.508)	2.96	2.40	(2.48)	1.47
Ca^{40}	76.54	0	(0)	0	2.76	(2.79)	1.48
Zr^{80}	120.8	0	(0)	0	3.12	(3.12)	1.46

In Sec. V, the results will be given for the hypernuclei between ${}_\Lambda He^5$ and ${}_\Lambda O^{17}$ for two values of γ_Λ, namely, 1.7 and 2.0 fm^2. Since the results are rather insensitive to V_0 and V_1, the force designated by (e) in Table I will be used throughout.

V. RESULTS

The results of the hypernuclear HF calculations for the Λ-N force (e) of Table I are given in Table IV. The results are given for two values of γ_Λ which are in the neighborhood of the optimum γ_Λ. In He^4 the rms radii have been corrected by subtracting the spurious contribution of the c.m. coordinate. This correction has not been made for other nuclei, but the effect should begin to be rather small for $A \geq 8$.

The precise definitions of the quantities appearing in Tables III and IV are given below:

$$\begin{aligned} Q_\Lambda &= \langle 3z_\Lambda^2 - r_\Lambda^2 \rangle, \\ R_\Lambda^2 &= \langle r_\Lambda^2 \rangle, \\ R_N^2 &= A^{-1} \sum_{i=1}^{A} \langle r_i^2 \rangle, \\ Q_N &= \sum_{i=1}^{A} \langle 3z_i^2 - r_i^2 \rangle. \end{aligned} \tag{15}$$

The binding energies in the region ${}_\Lambda He^5$ to ${}_\Lambda C^{13}$, where experimental binding energies are known, exceed the observed values by an appreciable amount. It is likely that such overbinding will occur even with more elaborate central Λ-N interactions if they fit the low-energy scattering data. The overbinding would be only slightly reduced if the force employed were such that the optimum γ_Λ occurred at a larger value. The matrix element of the Λ-N force that plays the dominant role in the calculation of the incremental binding energy of p-shell hypernuclei is, obviously, that between a $1s$ Λ and a $1p$ nucleon. In Table V, this matrix element is given for various values of γ_Λ, and one sees that a very large γ_Λ would have to be used for this matrix element, and with it the incremental binding energy in the p-shell, to be less than 2 MeV.

Another attempt to reduce the overbinding consisted in suppressing the Λ-N force by a factor of 2 in the odd relative angular momentum states. The rather small effect of this suppression is shown in Table VI.

As seen from Table IV, the orbit of the Λ particle is deformed (a nonzero Q_Λ) in a way that follows the nuclear deformation, i.e., Q_Λ and Q_N have the same sign. Furthermore, the sign of Q_N was always the same in the hypernucleus as it was in the nucleus alone. The main effects of the Λ particle on the nuclear properties was a radial compression of about 10% and a reduction

TABLE V. The variation of an important ΛN matrix element with γ_Λ. The incremental binding energy in the p shell is dominated by the $\langle 1s1p \mid W \mid 1s1p \rangle$ matrix element. The optimum γ_Λ for hypernuclei considered here is either 1.7 or 2.0 fm^2.

γ_Λ (fm^2)	1.0	1.7	2.0	2.6	3.0	4.0
$\lvert \langle 1s\,1p \mid W \mid 1s\,1p \rangle \rvert$ (MeV)	2.80	2.58	2.50	2.31	2.20	1.95

in angular deformation (Q_N/R_N^2) by about the same amount.

The large values of Q_N as compared to Q_Λ are a result of the fact that nuclear deformation in the p shell is a result of mixing states from the same major shell ($p_{1/2}$ and $p_{3/2}$), while the Λ deformation can only result from major shell mixing (e.g., $1s$ with $1d$).

One sees from Table IV that for a fixed γ_Λ the rms radius of the Λ decreases as the atomic number increases. This is due to two effects. First, as A increases the number of Λ-N bonds increases. Second, the N-N force employed is known to yield radii which do not rise as fast as $A^{1/3}$.[8] Even if the force were such that the calculated nuclear radii went as $A^{1/3}$, this might not compensate for the first effect. This deficiency of the Tabakin force is also evident from the results of the calculations of the heavier hypernuclei which have not been observed, as seen in Table IV. The binding energy of the Λ seems to increase without limit as A increases. This lack of saturation can also be traced to high nuclear densities characteristic of the Tabakin force.[13] Thus, the nuclear potential well felt by the bound Λ has a depth of about $A\rho(0)$, where $\rho(0)$ is the density at the origin. For actual nuclei $A\rho(0)$ is roughly independent of A for heavy nuclei but with the Tabakin interaction $A\rho(0)$ continues to increase. (This has been checked quantitatively by using a square well to represent the effect of the nucleus on the Λ particle.)

It is interesting to note the difference in B_Λ between ${}_\Lambda\mathrm{Be}^7$ and ${}_\Lambda\mathrm{Li}^7$ in Table IV. Such differences are often attributed to a term $\Delta\mathbf{S}_N\cdot\mathbf{S}_\Lambda$ in a parameterization of the B_Λ curve.[14] Thus, since the nuclear spin $\mathbf{S}_N$ is zero for Be^6 and is one for Li^6, the experimentally observed difference in B_Λ leads to the conclusion that $\Delta\approx 0.5$ MeV. In these HF calculations, with a purely central spin-independent force, there should be no difference in the B_Λ's for ${}_\Lambda\mathrm{Be}^7$ and ${}_\Lambda\mathrm{Li}^7$, if it is indeed only a spin effect. Thus, the calculated 0.3-MeV difference, which must be dynamical in origin (reflecting a slight variation in densities and compressibilities) and of the order of the observed difference, casts some doubt on the procedure of determining such parameters as Δ from the data.[15]

Also to be noted in Table IV is that the relatively low binding energies of the Λ observed in ${}_\Lambda\mathrm{Be}^9$ and ${}_\Lambda\mathrm{C}^{13}$ are not reproduced in this calculation. Instead the calculated B_Λ for the $A=9$ systems show the same dynamical differences as were found in the $A=7$ hypernuclei.

TABLE VI. The increase in R_Λ and the decrease in B_Λ when the ΛN interaction is suppressed by factor of 2 in odd partial waves. The decrease in the magnitude of the $\langle 1s\,1p \mid W \mid 1s\,1p\rangle$ matrix element due to this suppresion is about 10%.

Core nucleus	ΔR_Λ (fm) $\gamma_\Lambda=2$	$\gamma_\Lambda=2.6$	$-\Delta B_\Lambda$ (MeV) $\gamma_\Lambda=2$	$\gamma_\Lambda=2.6$
Li^6	0.01	0.02	0.47	0.36
Li^7	0.03	0.04	0.68	0.54
Be^8	0.05	0.06	0.93	0.75
B^{10}	0.07	0.06	1.85	1.40
O^{16}	0.08	0.05	4.98	4.03

VI. CORRECTIONS

There are a number of corrections which should be made in these calculations, just as in the ordinary nuclear HF calculations. For example, an error was introduced by not subtracting the kinetic energy operator for c.m. from the Hamiltonian.[16] The binding energy of the Λ, as given by Eq. (14), should therefore be corrected by the difference between the c.m. energy in the nucleus and the hypernucleus. In nuclear calculations, it has been found that subtraction of the c.m. kinetic energy results in an increase of the calculated binding energy by about 12 MeV, independent of A.[8] For hypernuclear calculations the expectation value of the c.m. energy is easily found to be

$$\langle E^{\mathrm{c.m.}}\rangle = m/M\langle E_{\mathrm{nuc}}{}^{\mathrm{c.m.}}\rangle + \tilde{m}/M\langle \tilde{l}\rangle, \qquad (16)$$

where m, $\tilde{m}$, and M are the masses of the nucleon, Λ, and the hypernucleus, respectively. Because the Λ compresses the nucleus (Table IV) it is expected that $\langle E_{\mathrm{nuc}}{}^{\mathrm{c.m.}}\rangle$ will be about 20% larger in a hypernucleus than for the bare nucleus. Since $\langle\tilde{l}\rangle$ rises from about 12 MeV in ${}_\Lambda\mathrm{He}^5$ to about 17 MeV in ${}_\Lambda\mathrm{N}^{15}$ and since $m+\tilde{m}=M$, $\langle E^{\mathrm{c.m.}}\rangle$ should also be approximately 20% larger in a hypernucleus than in the corresponding nucleus. Thus, the binding energy of the Λ as given in Table IV should be corrected by about an additional 2.5-MeV binding.

Another correction which should be made is connected with the fact that the determinantal wave functions obtained via HF are not eigenstates of total angular momentum. Thus, a true binding energy should be given by

$$-B_\Lambda = \langle\psi_H \mid HP^J \mid \psi_H\rangle - \langle\psi_N \mid HP^J \mid \psi_N\rangle, \qquad (17)$$

where P^J projects out the angular momentum of the ground state and ψ_H and ψ_N are the HF determinants for the hypernucleus and the corresponding nucleus.[17] The fact that the presence of the Λ in general reduces the deformation of the nucleus might indicate that this rotational correction is smaller in the hypernucleus than in the nucleus and this would tend to decrease the overbinding. However, this correction is not present for the spherical nuclei. A similar projection procedure is necessary for isospin.

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TABLE VII. The comparison of the results of different methods for calculating B_Λ. The column labelled B_Λ is the result of the self-consistent calculation [e.g., (14)]. ω is the eigenvalue of the HF matrix for the Λ [e.g., (8)] and the difference between $-\omega$ and B_Λ represents the energy lost by nucleus due to rearrangement. $-\omega^0$ is the binding energy of the Λ obtained by a static calculation. R_Λ and $R_\Lambda{}^0$ are radii as obtained self-consistently and with a static calculation, respectively.

Core nucleus	$-\omega$ (MeV)	B_Λ (MeV)	$-\omega^0$ (MeV)	$R_\Lambda{}^0$ (fm)	R_Λ (fm)
Li^6	11.38	10.00	8.43	1.93	1.80
Li^7	13.78	11.88	10.04	1.96	1.80
Be^8	16.74	14.32	11.63	2.03	1.80
B^{10}	23.83	21.17	18.32	1.83	1.66
O^{16}	44.88	41.70	38.04	1.57	1.50

By far, the largest correction to these calculations stems from the fact that the result of an HF calculation should only be considered as the first term in a perturbation expansion. This perturbation expansion has been carried out, up to second order, for various nuclei and, though the series appears to converge, the second-order terms are large.[18] In O^{16}, for example, the potential energy in first order is about 350 MeV and the second-order contribution is about 70 MeV. Thus, the second-order correction to B_Λ, given by

$$-\Delta B_\Lambda = \Delta E^{(2)}\ (\text{hypernucleus}) - \Delta E^{(2)}\ (\text{nucleus}), \quad (18)$$

is the difference between two large numbers and can be quite large itself. Since the expression for $\Delta E^{(2)}$ contains an infinite sum over an infinite number of unoccupied states, the calculation can only be done approximately and the results are difficult to estimate. Such approximate calculations are presently being undertaken.

VII. COMPARISON WITH OTHER METHODS

A basic assumption of all p-shell hypernuclear shell-model calculations[14] is that the variation with A of the Λ wave function is negligible. The HF method provides a means of determining the validity of this assumption. The results in Table IV indicate that the wave function becomes more concentrated at small distances as the atomic number increases. This variation is also reflected in the increase of the expectation value of the kinetic energy of the Λ, from 12 to 16 MeV, between ${}_\Lambda He^5$ and ${}_\Lambda C^{13}$. It should be pointed out, however, that part of this variation of the Λ wave function is due to the poor saturation properties of the Tabakin potential.[13] It is possible that a HF calculation with an N-N force that reproduces the $A^{1/3}$ dependence of the nuclear radius will validate the assumption.

Another assumption which may be checked is that of the rigid-core model, where the hypernucleus is treated as a two-body system composed of the core nucleus and a Λ. In the rigid-core model, the relevant coordinate is the distance between the Λ and the nuclear c.m. Because of the use of single-particle coordinates in the HF calculation, the comparison with rigid-core model calculations[15] is meaningful only for heavy hypernuclei where c.m. effects are negligible. In the HF framework, however, the basic idea of the rigid-core model can be assessed by using the results of the nuclear HF calculation for the occupied nuclear states in Eq. (8) and simply solving that equation for ω without iterating. This ω will be designated by ω^0. The difference in the results of such calculations and complete HF calculations is shown in Table VII. It is seen that the polarization of the nucleus by the Λ leads to an increase in B_Λ of about 2 MeV. The effect of the Λ on purely nuclear properties has been discussed in Table IV. In Table VII, the eigenvalue ω, obtained by solving both Eqs. (7) and (8) self-consistently, is also listed for each nucleus. The difference between these eigenvalues and the B_Λ as calculated from Eq. (14) represents the energy lost by rearrangement of the nucleus in moving from its optimum size and shape. This loss is seen to be about 2 MeV in all the hypernuclei considered.

VIII. CONCLUSIONS

The results of the HF calculations presented here are not meant to be compared directly with experiment. As has been pointed out, there are corrections concerned with second-order effects, spurious c.m. energy, and angular momentum and isospin projection. Furthermore, a N-N force must be found which leads to correct nuclear densities and more information must be obtained about the Λ-N interaction before realistic calculations can be carried out.

The aim of the calculations presented here was, therefore, to determine the type of effects, and their order of magnitude, that result from a dynamical treatment of hypernuclei. The main differences between these HF results and those of a static calculation were found to be an additional 2–3 MeV in the binding energy, and a 10% compression and a reduction in the deformation of the core nucleus. No exceptional dynamical effects were found for ${}_\Lambda Be^9$ or ${}_\Lambda C^{13}$ that would explain their relatively low binding energies.

The calculated binding energies in all cases exceed those observed and this is presumed to result from using a central ΛN interaction that fits low-energy scattering data. The use of a long-range tensor force or a three-body ΛNN interaction seems to be necessary in order to reduce the calculated B_Λ. In this respect, the conclusion of this calculation is similar to those of recent investigations of s-shell hypernuclei and of the Λ binding energy in nuclear matter.[19]

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