Off-Shell T Matrix and the Jost Function

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An off-shell effective-rangelike theory is developed for the low-energy two-nucleon T matrix. It is shown that the parameters in the theory can be determined from on-shell scattering data. The parameters are determined from the $^3S_1$ and $^1S_0$ two-nucleon phase shifts, and the validity of the off-shell formula is tested by means of examples. For the cases considered, the formula is found to work very well. A new proof for the separability of the two-body T matrix near bound-state and resonance energies is given. This proof is based on the properties of the T matrix in configuration space. A theorem on the factorization of the Jost function is developed and used to solve the inversion problem for rank-two separable potentials. Results are given for phase shifts that become hard-core phase shifts at high energies and for tensor forces. These results allow one to construct a separable potential that has the same on-shell T matrix and bound-state wave function as a realistic local potential.

I. INTRODUCTION

In recent years it has become very clear that the T matrix is an important concept in nuclear physics. This operator has made its appearance in several different areas of nuclear physics. In particular, the T matrix plays an important role in the theory of three-body systems, the nucleon-nucleon bremsstrahlung amplitude can be expressed in terms of it, and the reaction matrix of Brueckner theory is related to it.

It was a mathematical difficulty which first brought the T matrix into play in the three-body problem; the difficulty being the presence of disconnected diagrams in the kernels of the Lippmann-Schwinger[1] equations for the three-body system. It was Faddeev[2,3] who showed that one could rearrange the equations so as to obtain integral equations with mathematically respectable kernels. In the Faddeev scheme the kernels depend explicitly on the off-shell two-body T matrix. Besides solving a mathematical problem, the appearance of the T matrix in the Faddeev equations led to a partial justification by Lovelace[4] of the separable approach to the three-body problem initiated by Mitra.[5]

Off-shell T matrix elements also appear in the formulation of the nucleon-nucleon bremsstrahlung problem given by Cromer and Sobel.[6] This formulation describes the half-off-shell T matrix in terms of the so-called quasiphase parameters. These parameters are not, in general, determinable from elastic nucleon-nucleon scattering data, and it is the sensitivity of the bremsstrahlung cross section to these parameters that is the central issue in the study of this phenomenon.

In systems with many particles, the T matrix has also played a role. The reaction or G matrix of Brueckner-Bethe-Goldstone (BBG) theory,[7,8] while not identical to the T matrix, is closely related to it. The G matrix differs from the T matrix mainly because of the Pauli principle.[9] Besides its application to nuclear matter, the BBG theory has also been used to study finite nuclei.[10]

It is thus very clear that the T matrix plays an important role in nuclear physics. Once one realizes that off-shell T matrix elements occur in nuclear physics calculations, two related questions arise. First, how much do we know about the off-shell two-nucleon T matrix, and, second, how sensitive are various calculations to the behavior of the off-shell T matrix?

We give a partial answer to the first question by developing a parametrization, of the low-energy two-nucleon T matrix. This parameterization is analogous to effective-range theory in that the parameters involved are the coefficients of a power series in the energy. Furthermore, it will be shown that the separability of the low-energy T matrix allows one to determine these parameters from on-shell data. In contrast to effective-range theory, these parameters depend on the phase shifts at high energies as well as those at low energies.

We attack the other question raised above by developing a systematic procedure for studying the sensitivity of various nuclear physics calculations to the off-shell T matrix. In particular, a method is given for constructing a separable potential that has a given Jost function.[11] The Jost function is determined by the phase shifts and the bound-state energy of the two-body system; hence this is equivalent to solving the inversion problem for separable potentials. The inversion problem for separable potentials has been studied by many authors.[12-17] The work that is most relevant to this paper is that of Fiedeldey,[17] who showed that a rank-two central separable potential is not

uniquely determined by the phase shift and bound-state energy. Rather, one has an arbitrary form factor, which allows one to vary the off-shell T matrix while keeping the on-shell behavior fixed. One can choose the form factor so that the potential reproduces a given two-body bound-state wave function. In this paper we will rederive Fiedeldey's results using a theorem for factoring the Jost function which does not appear to have been given before. Furthermore, we will generalize Fiedeldey's results in two ways. First, we will find a separable potential which reproduces exactly the on-shell behavior of a local potential with a hard core. Since it is possible to solve the three-body problem with hard-core potentials,[18-20] this generalization will allow one to compare the results for three-body quantities obtained from local hard-core potentials with those obtained from separable potentials with exactly the same on-shell T matrix. The second generalization that will be considered is tensor forces. The tensor force results include a procedure for finding a separable potential which, besides producing a given set of phase parameters, also reproduces a given bound-state wave function. These results will allow one to find a separable potential which reproduces the on-shell T matrix as well as the deuteron wave function of the realistic local potentials[21] (Hamada-Johnston, Yale, Reid).

In Sec. II we give the definition of the T matrix and the normalization we will use in subsequent sections. We also derive a power series expansion for the half-off-shell extension function.[22] This power series can be used to represent the half-off-shell T matrix at low energies. We give in Sec. III a new proof for the separability of the T matrix at bound-state and resonance energies. This proof, which is based on the properties of the Jost function, is much simpler than those given by other authors,[4,23] since it deals directly with the Schrödinger equation and does not rely on sophisticated mathematical concepts. The results of Sec. II and Sec. III are combined in Sec. IV to develop the off-shell effective-rangelike expansion mentioned above. The parameters in the expansion are determined from the experimental phase shifts. The validity of the expansion depends on two circumstances; being at low energies and being near a bound state or resonance. In Sec. V a theorem on the factorization of the Jost function is derived and used to generalize Fiedeldey's results to phase shifts which behave like hard-core phase shifts at high energies and to tensor faces. Section VI is a discussion of the results of the previous sections and gives suggestions for future work. Unless stated otherwise, we assume that $\hbar$ and the mass of the nucleon are unity.

II. T MATRIX

The T matrix can be defined by the operator equations

$$\begin{aligned} T(s) &= V + VG(s)V, \\ G(s) &= (s-H)^{-1}, \end{aligned} \tag{2.1}$$

where V is the two-body potential, s is a complex parameter, and H is the two-body Hamiltonian. The resolvent or Green's function $G(s)$ is easily shown to be the solution of the equations

$$\begin{aligned} G(s) &= G_0(s) + G_0(s)VG(s), \\ &= G_0(s) + G(s)VG_0(s), \end{aligned} \tag{2.2}$$

where

$$G_0(s) = (s - H_0)^{-1}. \tag{2.3}$$

H_0 is the kinetic-energy operator. The identity (2.2) allows one to show that the T matrix (2.1) is the solution of

$$\begin{aligned} T(s) &= V + VG_0(s)T(s), \\ &= V + T(s)G_0(s)V. \end{aligned} \tag{2.4}$$

For the central-force case we will normalize the eigenstates of H_0 and the orbital-angular-momentum operators L^2 and L_z according to

$$\langle \vec{r} \,|plm\rangle = (2\pi^2)^{-1/2} j_l(pr) Y_{lm}(\hat{r}). \tag{2.5}$$

j_l is the usual spherical Bessel function and Y_{lm} is a spherical harmonic. The normalization (2.5) implies the following result for the on-shell T matrix:

$$\langle klm \,|\, T(k^2 + i\epsilon) \,|\, klm\rangle = -(2\pi^2 k)^{-1} e^{i\delta_l} \sin\delta_l, \tag{2.6}$$

where δ_l is the phase shift for the lth partial wave. The normalization for tensor forces is given in Sec. V.

We now show that the half-off-shell T matrix can be expanded in a power series in the c.m. energy of the colliding particles. Consider the half-off-shell extension function given by

$$F_l(p,k) = T_l(p,k;\ k^2 + i\epsilon)/T_l(k,k;k^2+i\epsilon), \tag{2.7}$$

where the T_l's are the T matrix elements for the lth partial wave. Writing the matrix elements in the coordinate representation, (2.7) becomes

$$F_l(p,k) = \frac{\int_0^\infty j_l(pr) V(r) w_l(k,r) r^2 dr}{\int_0^\infty j_l(kr) V(r) w_l(k,r) r^2 dr}. \tag{2.8}$$

It has been assumed for simplicity that the potential V is local; it will become clear that the results will also obtain for nonlocal potentials.

$w_l(k,r)$ is the physical solution of the Schrödinger equation for the lth partial wave. If w_l is normalized so that

$$w_l(k,r) \underset{r\to\infty}{\sim} \frac{\sin(kr - \frac{1}{2}l\pi + \delta_l)}{kr\sin\delta_l}, \tag{2.9}$$

it is easy to show that

$$-k\int_0^\infty j_l(kr)V(r)w_l(k,r)r^2dr = 1. \tag{2.10}$$

Using (2.10), (2.8) can be written in the form

$$F_l(p,k) = (p/k)^l - k\int_0^\infty [j_l(pr) - (p/k)^l j_l(kr)] \times V(r)w_l(k,r)r^2dr. \tag{2.11}$$

Combining (2.11) and the series expansion for j_l, we arrive at the result

$$F_l(p,k) = (p/k)^l - kp^l\sum_{n=1}^\infty \frac{(-\frac{1}{2})^n(p^{2n}-k^{2n})}{n!(2l+1+2n)!!} \times \int_0^\infty r^{2n+l+2}V(r)w_l(k,r)dr. \tag{2.12}$$

To lowest order in the energies k^2 and p^2, the expansion becomes

$$F_l(p,k) = (p/k)^l[1 + \tfrac{1}{2}\lambda_l^2(k^2-p^2) + \cdots], \tag{2.13}$$

where

$$\lambda_l^2 = \frac{\int_0^\infty r^{l+4}V(r)w_l(0,r)dr}{(2l+3)\int_0^\infty r^{l+2}V(r)w_l(0,r)dr}. \tag{2.14}$$

We shall call the λ_l's the off-shell lengths (OSL). The normalization (2.10) has been made explicit in (2.14). We shall see in Sec. III that for the two-nucleon problem, λ_0^2 can actually be determined from the S-wave phase shifts. For the higher partial waves, one can estimate the OSL's by approximating the wave function w_l by the free wave j_l. This gives from (2.14)

$$\lambda_l^2 \approx \frac{\int_0^\infty r^{2l+4}V(r)dr}{(2l+3)\int_0^\infty r^{2l+2}V(r)dr}. \tag{2.15}$$

Clearly in this approximation, λ_l is sensitive only to the long-range part of the potential. Assuming $V(r)$ is a Yukawa potential of range μ^{-1}, (2.15) gives

$$\lambda_l^2 \approx (2l+2)/\mu^2. \tag{2.16}$$

III. SEPARABLE APPROXIMATION

In this section a new proof for the separability of the T matrix is given. In contrast to other proofs,[4,23] this proof is based on the behavior of the T matrix in configuration space.

The separability of the T matrix arises from the separability of the Green's function $G(s)$ which appears in (2.1) when s is in the vicinity of a resonance or bound-state energy. In configuration space, the Green's function for the lth partial wave is the solution of

$$\left(k^2 + \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} - V(r)\right) G_l^+(k;r,r') = \delta(r-r'), \tag{3.1}$$

with an outgoing-wave boundary condition. The Green's function is constructed from the solutions of the Schrödinger equation

$$\left(k^2 + \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r)\right)\psi = 0. \tag{3.2}$$

The construction is given in Newton's book,[11,24] so we will simply quote the relevant definitions and results. One introduces two classes of solutions of (3.2); the regular solutions defined by the boundary condition

$$\lim_{r\to 0} r^{-l-1}\phi_l(k,r) = 1, \tag{3.3}$$

and the irregular solutions defined by

$$\lim_{r\to\infty} e^{\mp ikr}f_{l\pm}(k,r) = 1. \tag{3.4}$$

The Wronskian[25] of these solutions gives the so-called Jost functions $F_{l\pm}$ according to

$$F_{l\pm}(k) = w(f_{l\pm}, \phi_l). \tag{3.5}$$

The regular solution ϕ_l is related to the irregular solutions $f_{l\pm}$ by

$$\phi_l(k,r) = (2ik)^{-1}[F_{l-}(k)f_{l+}(k,r) - F_{l+}(k)f_{l-}(k,r)]. \tag{3.6}$$

The Green's function is given by

$$G_l^+(k;r,r') = -\frac{\phi_l(k,r_<)f_l(k,r_>)}{F_l(k)}, \tag{3.7}$$

where

$$f_l(k,r) = f_{l+}(k,r), \tag{3.8}$$

$$F_l(k) = F_{l+}(k). \tag{3.9}$$

F_l and F_{l-} are related by

$$F_{l-}(k) = F_l(-k). \tag{3.10}$$

The separability of G_l^+ when k^2 is near a bound-state or resonance energy k_0^2 follows from the fact that the bound-state and resonance energies are determined by

$$F_l(k_0) = 0. \tag{3.11}$$

From (3.6)–(3.11) it then follows that

$$\phi_l(k_0,r)=F_l(-k_0)f_l(k_0,r)/2ik_0\,, \tag{3.12}$$

and

$$G_l^+(k;r,r')\approx -\frac{F_l(-k_0)f_l(k_0,r)f_l(k_0,r')}{2ik_0F_l(k)}, \quad k\approx k_0\,. \tag{3.13}$$

Putting (3.13) in (2.1) and writing the T matrix in momentum space, we have

$$T_l(p,q;k^2+i\epsilon)\approx -\frac{F_l(-k_0)g_l(p)g_l(q)}{2ik_0F_l(k)}, \quad k\approx k_0\,, \tag{3.14}$$

where

$$g_l(p)=(2\pi^2)^{-1/2}\int_0^\infty rj_l(pr)V(r)f_l(k_0,r)dr\,. \tag{3.15}$$

The T matrix (3.14) has the normalization given by (2.6).

We now examine the nature of the zero of (3.11); more precisely, we write

$$F_l(k)=\dot{F}_l(k_0)(k-k_0)+\cdots \tag{3.16}$$

(the dot means derivative) and try to determine whether or not $\dot{F}_l(k_0)$ vanishes. For the case of a bound state where $k_0=i|k_0|$, Newton[11,24] has shown that $\dot{F}_l(k_0)$ cannot vanish. His proof is based on demonstrating that

$$\frac{2ik_0\dot{F}_l(k_0)}{F_l(-k_0)}=-2k_0\int_0^\infty dr f_l^2(k_0,r)\,. \tag{3.17}$$

From (3.12) and the positive definite nature of the integrand in (3.17), the result follows immediately. For a resonance, k_0 is in the lower half of the k plane; clearly the integral in (3.17) does not exist in this case, since $f_l(k_0,r)$ will blow up exponentially at infinity. Newton's derivation of (3.17) can be easily modified, however, to show that

$$2ik_0\dot{F}_l(k_0)/F_l(-k_0)=-i+k_0\rho\,, \tag{3.18}$$

where

$$\rho=2\int_0^\infty[e^{2ik_0r}-f_l^2(k_0,r)]dr\,. \tag{3.19}$$

We see that $\dot{F}_l$ does not vanish if $k_0\rho$ is not identically i.[26] Assuming $k_0\rho\neq i$, (3.14) becomes, using (3.16) and (3.18),

$$T_l(p,q;k^2+i\epsilon)\approx\frac{g_l(p)g_l(q)}{(i-k_0\rho)(k-k_0)}, \quad k\approx k_0\,. \tag{3.20}$$

Everything in (3.20) is determined from the function $f_l(k_0,r)$ and, of course, k_0. We shall use the results of this section in the next in order to obtain a convenient parametrization of the two-nucleon T matrix at low energies.

IV. OFF-SHELL EFFECTIVE-RANGE THEORY

We begin by defining another Jost function[11,24]:

$$f_l(k)\equiv\frac{k^le^{-i\pi l/2}F_l(k)}{(2l+1)!!}\,. \tag{4.1}$$

This Jost function has the high-energy behavior

$$\lim_{|k|\to\infty}f_l(k)=1\,, \tag{4.2}$$

and is related to the phase shift δ_l by

$$f_l(k)=|f_l(k)|\,e^{-i\delta_l(k)}\,. \tag{4.3}$$

This Jost function is determined completely by the phase shift and the bound-state energies E_n by the formula

$$f_l(k)=\prod_n\left(1-\frac{E_n}{E}\right)\exp\left(\frac{1}{\pi}\int_0^\infty\frac{dE'\delta_l(E')}{E+i\epsilon-E'}\right); \tag{4.4}$$

hence, in principle, it is determinable from experiment.

Using (2.6) and (4.3), we can write the on-shell T matrix in the form

$$T_l(k,k;k^2+i\epsilon)=\frac{\mathrm{Im}f_l(k)}{2\pi^2kf_l(k)}\,. \tag{4.5}$$

At low energies the two-nucleon T matrix can be approximated by

$$T(p,q;k^2+i\epsilon)=-\frac{f(-k_0)g(p)g(q)}{2ik_0f(k)}, \tag{4.6}$$

where we have used (3.14) and (4.1) and have dropped the subscript $l=0$, since we are only interested in s waves. Comparing the on-shell limit of (4.6) with (4.5), one can easily show that

$$T(p,q;k^2+i\epsilon)\approx F(p,k)T(k,k;k^2+i\epsilon)F(q,k)\,, \tag{4.7}$$

where

$$F(p,k)=\left(\frac{k\,\mathrm{Im}f(p)}{p\,\mathrm{Im}f(k)}\right)^{1/2}. \tag{4.8}$$

Thus from (4.4), (4.7), and (4.8), we see that the two-nucleon T matrix at low energies is determined by on-shell data and the deuteron binding energy. Note, however, that it depends on the high-energy phase shift since the integral in (4.4) extends over all positive energies. The approximation (4.7) is the separable form suggested by Kowalski and Noyes.[22] What we have shown is that the half-off-shell extension function F at low momenta can be determined from the phase shift and deuteron binding energy.

We note that (4.8) is exact for a one-term separable potential. We also point out that the notation $\mathrm{Im}f(k)$ is, strictly speaking, valid only for k

real. As will become clear below, one must obtain $\mathrm{Im}f(k)$ for k real and then analytically continue the function so obtained off the real axis.

We now look for a convenient way of parametrization F at low momenta. According to (2.12) and (2.13), it is possible to expand F as a power series in p^2 and k^2. In order to do this we consider the analytic structure of the Jost function at low momenta. It is shown in Ref. 11 that if the potential satisfies the conditions

$$\int_0^\infty dr\, r\,|V(r)|<\infty,$$

$$\int_0^\infty dr\, r^2\,|V(r)|<\infty,$$

$$\int_0^\infty dr\, r\,|V(r)|e^{2ar}<\infty,$$

then the Jost function $f(k)$ is analytic in

$$\mathrm{Im}k>-a.$$

We can therefore expand the Jost function in a power series about the origin with a radius of convergence a; i.e., we have

$$f(k)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}k^n, \quad |k|<a. \tag{4.9}$$

For a potential with a one-pion-exchange tail the radius of convergence corresponds to a c.m. energy of about 5 MeV. Using the fact that the Jost function satisfies[11,24]

$$f(-k)=f^*(k), \quad \mathrm{Im}k=0, \tag{4.10}$$

it is straightforward to show that

$$\frac{\mathrm{Im}f(k)}{k}=-i\sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}k^{2n}. \tag{4.11}$$

From the ratio test for the convergence of a power series, it follows immediately that (4.11) has the same radius of convergence as (4.9). Writing (4.11) in the form

$$\frac{\mathrm{Im}f(k)}{k}=\mathrm{const}\sum_{n=0}^{\infty}\Lambda_n{}^{2n}(-)^nk^{2n}, \tag{4.12}$$

with

$$\Lambda_0=1,$$

and then substituting into (4.8), we find

$$F(p,k)=1-\tfrac{1}{2}\Lambda_1{}^2(p^2-k^2)-\tfrac{1}{4}\Lambda_1{}^4k^2(p^2-k^2)+(\tfrac{1}{2}\Lambda_2{}^4-\tfrac{1}{8}\Lambda_1{}^4)(p^4-k^4)+\cdots. \tag{4.13}$$

By comparing (2.12), (4.12), and (4.13), we see that the expansion coefficients for the half-off-shell extension function are approximately given by the expansion coefficients for the imaginary part of the Jost function. This approximate equality depends, of course, on just how good the separable approximation (4.8) is. In order to get a feeling for the expansion coefficients Λ_n, it is convenient to consider the Jost function corresponding to a phase shift δ which is given exactly by effective-range theory; i.e.,

$$k\cot\delta=-1/a+\tfrac{1}{2}r_0k^2. \tag{4.14}$$

For this phase shift, one can very easily do the integral in (4.4) by contour integration and show that

$$f(k)=(k-i\beta_1)/(k+i\beta_2), \tag{4.15}$$

where

$$\beta_1=[1-(1-2r_0/a)^{1/2}]/r_0, \tag{4.16}$$

$$\beta_2=[1+(1-2r_0/a)^{1/2}]/r_0. \tag{4.17}$$

One then finds from (4.15) that all the expansion coefficients Λ_n have the value

$$\Lambda_n=\beta_2{}^{-1}, \quad n=1,2,3,\ldots. \tag{4.18}$$

If we take for the scattering lengths and the effective ranges the values

$$\begin{aligned} a_t&=5.396\ \mathrm{F},\\ r_{0t}&=1.726\ \mathrm{F},\\ a_s&=-23.678\ \mathrm{F},\\ r_{0s}&=2.7\ \mathrm{F}, \end{aligned} \tag{4.19}$$

we find

$$\begin{aligned} \beta_{2t}{}^{-1}&=1.079\ \mathrm{F},\\ \beta_{2s}{}^{-1}&=1.281\ \mathrm{F}. \end{aligned} \tag{4.20}$$

We thus expect that for the actual phase shift all the Λ_n's are of order of magnitude 1 F.

We now turn to finding the lowest Λ_n's for a realistic set of $^3S_1$ and $^1S_0$ phase shifts. Below 10 MeV (lab) we use effective-range theory with the parameters (4.10). Above 10 MeV we use the Yale phase parameters.[27] One could, in principle, obtain the Λ_n's by doing the integral in (4.4) numerically, however, it has been found to be more convenient to fit the phase shifts with simple potentials and then to extract the Λ_n's from the Jost functions the potentials give rise to. Two potentials have been considered: A hard-core square-well (HCSW) potential defined by

$$\begin{aligned} V(r)&=\infty, \quad 0\le r<c,\\ &=-V_0, \quad c<r<b+c,\\ &=0, \quad r>b+c; \end{aligned} \tag{4.21}$$

and the Morse potential[28]

$$V(r)=V_0(e^{-2(r-c)/b}-2e^{-(r-c)/b}). \tag{4.22}$$

Using (3.5) and (4.1), it is easy to show that for a

well-behaved potential the S-wave Jost function is simply given by [see (3.4) and (3.8)]

$$f(k)=f(k,0)\,. \tag{4.23}$$

For a hard-core potential such as (4.21), the S-wave Jost function[29] is given by

$$f(k)=f(k,c)\,. \tag{4.24}$$

Solving the S-wave Schrödinger equation with the boundary condition given by the upper signs in (3.4), we find the Morse-potential Jost function to be

$$\begin{aligned} f_M(k)&=e^{-\xi/2}M(A,B,\xi)\,,\\ \xi&=2V_0^{1/2}b\,e^{c/b}\,,\\ A&=\tfrac{1}{2}(1-2ikb)-V_0^{1/2}b\,,\\ B&=1-2ikb\,. \end{aligned} \tag{4.25}$$

We are using units in which $\hbar$ and the nucleon mass are unity. The function M in (4.25) is the confluent hypergeometric function.[30] The parameter ξ turns out to be quite large, so we can use the asymptotic form for M. This gives

$$f_M(k)\approx e^{\xi/2}\Gamma(B)\xi^{A-B}/\Gamma(A)\,. \tag{4.26}$$

From (4.3) we can determine the phase shift and hence the effective-range parameters. These turn out to be

$$\begin{aligned} a&=c+b[2\gamma+\ln(2V_0^{1/2}b)+\psi^{(0)}(x)]\,,\\ r_0&=\tfrac{2}{3}a-\tfrac{2}{3}(b^3/a^2)[8\zeta(3)+\tfrac{1}{2}\psi^{(2)}(x)]\,,\\ x&=\tfrac{1}{2}-V_0^{1/2}b\,, \end{aligned} \tag{4.27}$$

where γ is Euler's constant, the $\psi^{(n)}$'s are polygamma functions,[30] and ζ is the Riemann zeta function. Formulas for Λ_1 and Λ_2 were obtained from (4.12) and (4.26) and are given below:

$$\begin{aligned} \Lambda_1^2&=\tfrac{1}{2}a(a-r_0)+b^2[\tfrac{1}{3}\pi^2-\tfrac{1}{2}\psi^{(1)}(x)]\,,\\ \Lambda_2^4&=\tfrac{3}{8}a^4-\tfrac{3}{4}a^3r_0+\tfrac{1}{4}a^2r_0^2-aPr_0^3\\ &\quad+\tfrac{1}{2}a(a-r_0)b^2[\tfrac{1}{3}\pi^2-\tfrac{1}{2}\psi^{(1)}(x)]\\ &\quad+[\tfrac{1}{10}\pi^4-\tfrac{1}{6}\pi^2\psi^{(1)}(x)-\tfrac{1}{24}\psi^{(3)}(x)+\tfrac{1}{8}\psi^{(1)2}(x)]b^4\,, \end{aligned} \tag{4.28}$$

where P, the shape parameter, is given by

$$\begin{aligned} P&=\tfrac{1}{4}(a/r_0)-\tfrac{1}{2}(a/r_0)^2+\tfrac{1}{5}(a/r_0)^3\\ &\quad-(b^5/a^2r_0^3)[\tfrac{32}{5}\zeta(5)+\tfrac{1}{120}\psi^{(4)}(x)]\,. \end{aligned}$$

The three parameters in the potential (4.22) were determined by solving (4.27) for b and c and then doing a least-squares fit to the phase shifts above 10 MeV (lab) so as to find the best value for V_0. The results are given in Table I.

The Jost function for the hard-core potential (4.21) turns out to be

$$\begin{aligned} f_{\rm HC}(k)&=e^{ik(b+c)}(\mathcal{K}\cos\mathcal{K}b-ik\sin\mathcal{K}b)\,,\\ \mathcal{K}&=(V_0+k^2)^{1/2}\,. \end{aligned} \tag{4.29}$$

The parameters of interest are given below

$$\begin{aligned} a&=b+c-(\tan V_0^{1/2}b)/V_0^{1/2}\,,\\ r_0&=b+2c-\frac{(b+c)^2(b-2c)}{3a^2}-\frac{2c(b+c)}{a}-\frac{a-c}{V_0a^2}\,,\\ \Lambda_1^2&=\frac{a-c}{2V_0a}+\frac{(a-b-c)(b+c)c}{2a}+\frac{(b+c)^3}{6a}\,,\\ \Lambda_2^4&=\frac{3(a-c)}{8V_0^2a}+\frac{b^3}{8V_0a}+\frac{(a-c)(3b+2c)c}{8V_0a}\\ &\quad-\frac{(b+c)^3(a-b-c)(b-c)}{24a}+\frac{(b+c)^5}{120a}\,. \end{aligned} \tag{4.30}$$

The potential parameters were determined by following the same procedure as for the Morse potential. The results are presented in Table II. It is seen by comparing Tables I and II that the two sets of expansion parameters do not agree with each other; however, if one puts numbers into (4.13) it is found that the values of the half-off-shell extension function calculated with the two different sets agree with each other to better than 1% in the low-energy region (up to 20 MeV in the lab frame). Even though either set can be used we choose for the sake of definiteness the $^3S_1$ parameters given by the HCSW potential and the $^1S_0$ parameters given by the Morse potential, since the corresponding potentials give the best fits. For convenience, we give these values in Table III.

TABLE I. Parameters for the Morse potential.

State	V_0 (MeV)	b (F)	c (F)	Λ_1 (F)	Λ_2 (F)
$^3S_1$	109.61	0.3584	0.8531	0.9035	0.8132
$^1S_0$	43.05	0.4722	1.0112	1.1796	1.0723

TABLE II. Parameters for the hard-core square-well potential.

State	V_0 (MeV)	b (F)	c (F)	Λ_1 (F)	Λ_2 (F)
$^3S_1$	44.57	1.776	0.1402	0.8701	0.7118
$^1S_0$	17.84	2.306	0.1372	1.1185	0.9118

TABLE III. Best values for the off-shell expansion parameters.

State	Λ_1 (F)	Λ_2 (F)
$^3S_1$	0.87	0.71
$^1S_0$	1.18	1.07

TABLE IV. Comparison of $^3S_1$ half-off-shell extension functions $F(p,k)$ for the hard-core square-well potential.

$\hbar^2p^2/M$ (MeV)	Exact $F(p,k)$ [a]	Jost-function approximation[b]	Series expansion[c]
0	0.98	0.98	0.98
5	0.94	0.94	0.94
10	0.89	0.89	0.89
15	0.85	0.85	0.85
20	0.81	0.81	0.81
25	0.77	0.78	0.78
30	0.73	0.74	0.74
35	0.70	0.70	0.71
40	0.66	0.67	0.67

[a] $F(p,k)$ is calculated from (2.7) using $\hbar^2k^2/M = -2.225$ MeV.
[b] This column is calculated from (4.8).
[c] This column is calculated from (4.13) using Λ_1 and Λ_2 from Table III.

In Table IV, a comparison is made for the case of the $^3S_1$ HCSW potential of the exact half-off-shell extension function F [see (2.7) and (2.8)], the Jost function approximation for F given by (4.8), and the series for F given by (4.13). The comparison is made for $\hbar^2k^2/M$ equal to the binding energy of the deuteron. It is seen that the three sets of values for F agree extremely well. In Table V, a comparison is made for the case of the $^1S_0$ HCSW potential of the same three functions for $\hbar^2k^2/M$ equal to 10 MeV. It is seen that the values of the Jost function approximation to F agree very well with the exact values over the energy range considered, and the series expansion is in good agreement for values of $\hbar^2p^2/M$ up to about 20 MeV (c.m.). Comparisons of this type have also been made at other energies in the range from −40 MeV up to +40 MeV, and it has been found that the accuracy of the approximations indicated by Tables IV and V is typical. The Jost-function approximation is very good in both spin states; the series expansion of F gives very good values in the $^3S_1$ state and reasonably good values for the $^1S_0$ state. The fact that the series expansion does somewhat better in the $^3S_1$ state can be traced to the smaller values for the $^3S_1$ expansion parameters (see Table III).

TABLE V. Comparison of $^1S_0$ half-off-shell extension functions $F(p,k)$ for the hard-core square-well potential.

$\hbar^2p^2/M$ (MeV)	Exact $F(p,k)$ [a]	Jost function approximation[b]	Series expansion[c]
0	1.17	1.17	1.17
5	1.08	1.08	1.08
10	1.00	1.00	1.00
15	0.92	0.92	0.93
20	0.85	0.85	0.88
25	0.78	0.78	0.83
30	0.71	0.72	0.80
35	0.65	0.66	0.78
40	0.59	0.61	0.77

[a] $F(p,k)$ is calculated from (2.7) using $\hbar^2k^2/M = 10$ MeV.
[b] This column is calculated from (4.8).
[c] This column is calculated from (4.13) using Λ_1 and Λ_2 from Table III.

V. LOCAL AND NONLOCAL POTENTIALS WITH THE SAME JOST FUNCTION

A. Central Forces

In this section we will consider the problem of finding a central separable potential that has the same Jost function as a given central local potential. From (4.4) it follows that the Jost function is determined from the phase shifts and bound-state energies; therefore, this amounts to solving the inversion problem for separable potentials. The inversion problem for one-term separable potentials has been studied by many authors for central as well as tensor forces.[12-17] Fiedeldey[17] has considered the inversion problem for two-term separable central potentials. Thus, the necessary relations for finding a separable potential which has a given Jost function can easily be obtained in the literature. However, we prefer to derive the results from scratch, since we feel our derivation is concise, and also it is the basis for our generalization to tensor forces and potentials with hard cores.

Unless otherwise stated, we will assume the Jost function $f_l(k)$ [see (4.1) and (3.5)] is analytic in the upper half of the k plane and satisfies the relations[11,24]

$$\lim_{|k|\to\infty} f_l(k) = 1, \quad \mathrm{Im}k \geq 0, \tag{5.1}$$

$$f_l(-k) = f_l^*(k), \quad \mathrm{Im}k = 0. \tag{5.2}$$

These assumptions allow us to write the dispersion relation

$$\mathrm{Re}f_l(k) = 1 + \frac{2}{\pi}\mathrm{P}\int_0^\infty dk' \frac{k'\,\mathrm{Im}f_l(k')}{k'^2 - k^2}, \tag{5.3}$$

where P means principal value. Our normalization (see Sec. II) is such that the partial-wave T matrix equation takes the form

$$T_l(p,q;s) = V_l(p,q) + \int_0^\infty V_l(p,x)\frac{4\pi x^2 dx}{s - x^2} T_l(x,q;s). \tag{5.4}$$

We begin by considering the simplest case; namely, a one-term separable potential

$$V_l(p,q) = g_l(p)\lambda_l g_l(q). \tag{5.5}$$

From (5.4) it follows that this potential gives rise

to the T matrix

$$T_l(p,q;k^2+i\epsilon)=V_l(p,q)/D_l(k)\,, \tag{5.6}$$

where

$$D_l(k)=1+\int_0^\infty \frac{\lambda_l g_l^{\,2}(x)4\pi x^2dx}{x^2-k^2-i\epsilon}\,,$$

$$=1+\mathrm{P}\int_0^\infty \frac{\lambda_l g_l^{\,2}(x)4\pi x^2dx}{x^2-k^2}+2\pi^2ik\lambda_l g_l^{\,2}(k)\,. \tag{5.7}$$

The second form of D_l follows from the identity

$$\frac{1}{x^2-k^2-i\epsilon}=\mathrm{P}\frac{1}{x^2-k^2}+\pi i\delta(x^2-k^2)\,. \tag{5.8}$$

The function $D_l(k)$ is the Jost function for the potential (5.5). If we want $D_l(k)$ to be the Jost function $f_l(k)$ for some local potential we simply choose

$$\lambda_l g_l^{\,2}(k)=(2\pi^2k)^{-1}\mathrm{Im}f_l(k)\,, \tag{5.9}$$

or using (4.3)

$$\lambda_l g_l^{\,2}(k)=-(2\pi^2k)^{-1}|f_l(k)|\sin\delta_l(k)\,. \tag{5.10}$$

If we use (5.3) it then follows that $D_l(k)=f_l(k)$. From (3.5) and (4.1) it follows that if one can solve the radial Schrödinger equation for a local potential, one can easily obtain the Jost function and hence from (5.9) the factors in the separable potential which produces a phase shift and bound-state energy which are identical to those produced by the local potential. Of course [see (5.10)] this procedure only works if $\sin\delta_l(k)$ has one sign for all positive energies, otherwise $g_l(k)$ will be real for some momenta and imaginary for others. One can get around this by allowing the coupling constant λ_l to change sign as $\sin\delta_l$ changes, however, this makes the potential energy dependent.

In order to avoid using energy-dependent potentials, we consider separable potentials of higher rank, in particular, two-term separable potentials. The basic idea we will use is to factor the Jost function into pieces, each of which has a phase [see (4.3)] whose sine is of one sign; i.e., we write

$$f_l(k)=\prod_{i=1}^{n} f_l^{(n)}(k)\,, \tag{5.11}$$

where

$$f_l^{(n)}(k)=|f_l^{(n)}(k)|e^{-i\delta_l^{(n)}(k)}$$

and $\sin\delta_l^{(n)}(k)$ is of one sign. This procedure can be carried out by means of an identity which we now derive. In Newton[11,24] it is shown that the Jost function $f_l(k)$ is identical to the Fredholm determinant of the operator that appears in the Lippmann-Schwinger equation (5.4). In a highly symbolic operator notation we have

$$f_l(k)=\det[1-G_0(s)V]\,, \tag{5.13}$$

where

$$G_0(s)=(s-H_0)^{-1}\,.$$

H_0 is the kinetic-energy operator. It is understood that in working out the determinant in (5.13), we work only in the subspace of the lth partial wave. Let us divide the potential V into two parts:

$$V=V_1+V_2\,. \tag{5.14}$$

Corresponding to the first part V_1, we have a Green's function or resolvent $G_1(s)$ which satisfies the equations

$$G_1(s)=G_0(s)+G_0(s)V_1G_1(s)\,,$$
$$=G_0(s)+G_1(s)V_1G_0(s)\,. \tag{5.15}$$

Using (5.15) it is easy to show that

$$1-G_0(s)V=[1-G_0(s)V_1][1-G_1(s)\,V_2]\,. \tag{5.16}$$

Combining (5.16) with (5.13), we have

$$f_l(k)=f_l^{(1)}(k)f_l^{(2)}(k)\,,$$

where

$$f_l^{(1)}(k)=\det[1-G_0(s)V_1]\,,$$
$$f_l^{(2)}(k)=\det[1-G_1(s)V_2]\,.$$

This result can easily be generalized. In general, we let

$$V=\sum_{i=1}^{n} V_i \tag{5.17}$$

and define $G_i(s)$ as the solution of

$$G_i(s)=G_{i-1}(s)+G_{i-1}(s)V_iG_i(s)$$
$$=G_{i-1}(s)+G_i(s)V_iG_{i-1}(s)\,. \tag{5.18}$$

We then have

$$1-G_0(s)V=\prod_{i=1}^{n}[1-G_{i-1}(s)V_i] \tag{5.19}$$

and

$$f_l(k)=\prod_{i=1}^{n} f_l^{(i)}(k) \tag{5.20}$$

with

$$f_l^{(i)}(k)=\det[1-G_{i-1}(s)V_i]\,. \tag{5.21}$$

If we assume that each of the V_i are separable, i.e.,

$$V_i=|i\rangle\lambda_i\langle i|\,, \tag{5.22}$$

then it is easy to invert the operator $1-G_{i-1}(s)V_i$ and find its determinant. The result is

$$f_l^{(i)}=1-\lambda_i\langle i|G_{i-1}(s)|i\rangle\,. \tag{5.23}$$

Also it is a straightforward matter to show from (5.18) and (5.23) that

$$G_i(s) = G_{i-1}(s) + G_{i-1}(s)\frac{|i\rangle\lambda_i\langle i|}{f_l^{(i)}(k)}G_{i-1}(s). \quad (5.24)$$

Thus, each G_i can be obtained from the one below it.

We now apply these identities to the inversion problem. Assume we have a separable potential V_1 [see (5.22)] which produces a phase shift δ_1. There are, of course, several types of V_1's to consider: V_1 can be attractive or repulsive; if attractive, it may or may not produce a bound state. For now we will assume V_1 is attractive and produces a bound state. In particular we choose

$$V_1 = \frac{U|B\rangle\langle B|U}{\langle B|U|B\rangle}, \quad (5.25)$$

where U is the potential whose on-shell T matrix is to be reproduced by the separable potential we will find, and $|B\rangle$ is the bound state of the potential U, i.e.,

$$(H_0 + U)|B\rangle = |B\rangle B. \quad (5.26)$$

The potential V_1 is the so-called unitary-pole approximation for the potential U.[31] From (5.23) the Jost function for V_1 is

$$f_l^{(1)}(k) = 1 - \frac{\langle B|UG_0(s)U|B\rangle}{\langle B|U|B\rangle}. \quad (5.27)$$

One easily sees from (5.26) that $f_l^{(1)}(k)$ vanishes at k_0 where

$$k_0^2 = B. \quad (5.28)$$

Let $f_l(k)$ be the Jost function corresponding to U. We want our two-term separable potential to have the same Jost function as U. Let

$$f_l^{(2)}(k) = f_l(k)/f_l^{(1)}(k). \quad (5.29)$$

Since $f_l(k)$ and $f_l^{(1)}(k)$ both vanish at k_0, $f_l^{(2)}(k)$ is analytic in the upper half of the k plane. It also satisfies the conditions (5.1) and (5.2); hence it obeys the dispersion relation (5.3), i.e.,

$$\mathrm{Re}f_l^{(2)}(k) = 1 + \frac{2}{\pi}\mathrm{P}\int_0^\infty dk' \frac{k'\,\mathrm{Im}f_l^{(2)}(k')}{k'^2 - k^2}. \quad (5.30)$$

We now introduce the continuum eigenstates of the potential V_1. These are given by

$$|klm]^{(\pm)} \equiv [1 + G_0(k^2 + i\epsilon)T_1(k^2 + i\epsilon)]\,|klm\rangle, \quad (5.31)$$

where T_1 is the T matrix of V_1 which is

$$T_1(s) = V_1/f_l^{(1)}(k), \quad s = k^2 + i\epsilon. \quad (5.32)$$

It is well known that the incoming and outgoing eigenstates are related by the S matrix.[11,32] The relation is

$$|klm]^{(+)} = |klm]^{(-)}e^{2i\delta_l^{(1)}(k)}, \quad (5.33)$$

where $\delta_l^{(1)}$ is the phase shift produced by V_1. It is convenient to use the states defined by

$$|klm] = |klm]^{(\pm)}e^{\mp i\delta_l^{(1)}(k)}. \quad (5.34)$$

These states obey the relation

$$[plm|klm] = \delta(p - k)/4\pi pk. \quad (5.35)$$

We assume the second separable potential V_2 [see (5.22)] has its factors given by

$$|2\rangle = \int_0^\infty |klm]4\pi k^2 dk\, v_l(k). \quad (5.36)$$

Using the fact that

$$G_1(s) = (s - H_0 - V_1)^{-1}, \quad (5.37)$$

it follows from (5.23) and (5.36) that

$$f_l^{(2)}(k) = 1 - \int_0^\infty \frac{\lambda_2 v_l^2(x)4\pi x^2 dx}{k^2 + i\epsilon - x^2}. \quad (5.38)$$

It is clear from (5.30) that we must have

$$\lambda_l^2 v_l^2(k) = (2\pi^2 k)^{-1}\mathrm{Im}f_l^{(2)}(k). \quad (5.39)$$

The two-term separable potential $(V_1 + V_2)$ will now reproduce the Jost function $f_l(k)$. From (4.3) and the similar relation for $f_l^{(1)}(k)$, we have from (5.29) that

$$f_l^{(2)}(k) = \frac{|f_l(k)|e^{-i(\delta_L - \delta_l^{(1)})}}{|f_l^{(1)}(k)|}, \quad (5.40)$$

and therefore from (5.39)

$$\lambda_2 v_l^2(k) = -\frac{|f_l^{(2)}(k)|\sin[\delta_l(k) - \delta_l^{(1)}(k)]}{2\pi^2 k}.$$

Thus in order for (5.39) to make sense, $\delta_l - \delta_l^{(1)}$ must always be of one sign.

Using (5.8), (5.31), and (5.34), we can write (5.36) in the form

$$\langle plm|2\rangle = v_l(p)\cos\delta_l^{(1)}(p) + \mathrm{P}\int_0^\infty \frac{4\pi k^2 dk\, T_1(p,k;k^2+i\epsilon)}{k^2 - p^2}\, e^{-i\delta_l^{(1)}(k)}v_l(k). \quad (5.41)$$

This is the final result for the factor in the second separable potential. From (5.36) it is clear that

$$V_2|B\rangle = 0;$$

hence the two-term potential has $|B\rangle$ as its bound state.

Although the formula (5.41) was derived assuming V_1 was attractive and produces a bound state, it is clearly also valid if V_1 does not produce a bound state or even if V_1 is repulsive. It is only necessary to write, instead of (5.25),

$$V_1 = |1\rangle\lambda_1\langle 1|.$$

The factor $|1\rangle$ and strength λ_1 can be almost anything, as long as $\delta_l - \delta_l^{(1)}$ is of one sign.

B. Hard-Core Potentials and Tensor Forces

The case of hard-core potentials requires special consideration, since the Jost function for such potentials does not satisfy (5.1). From (4.3) it is clear that this is related to the fact that the phase shift for a hard-core potential does not vanish at high energies. This difficulty can be circumvented. We break up the separable potential that is going to reproduce the hard-core potential's phase shift into two parts: The first part is taken to be a separable potential which produces a pure hard-core phase shift; the second part is chosen then to give a total phase shift which is identical to the phase shift of the original hard-core potential. The difference between the total phase shift and the pure hard-core phase shift is, of course, well behaved at high energies. The separable potential which gives a pure hard-core phase shift is the so-called hard-shell potential given by

$$V(\vec{r},\vec{r}') = \lambda \sum_{l,m} \delta(r-c) Y_{lm}(\hat{r}) Y_{lm}^*(\hat{r}') \delta(r'-c) \tag{5.42}$$

with the understanding that the strength λ becomes infinite. If we insert the momentum representation of (5.42) into (5.4), it is easy to show that the hard-shell T matrix is given by[33]

$$T_l^{\mathrm{HS}}(p,q;k^2+i\epsilon) = \frac{j_l(pc)j_l(qc)}{2\pi^2 k j_l(kc) h_l^{(+)}(kc)}. \tag{5.43}$$

In order to obtain a sensible Jost function[29] for the hard-shell potential and for hard-core potentials, one must simply change the definition of the regular solution of the radial Schrödinger equation from (3.3) to

$$\begin{aligned} &\phi_l(k,c) = 0, \\ &\phi_l'(k,c) = 1. \end{aligned} \tag{5.44}$$

One can still use (3.4), (3.5), and (4.1). The hard-shell and pure hard-core Jost functions are the same and are given by

$$f_l^{(1)}(k) = k^{l+1} c h_l^{(+)}(kc)/(2l+1)!!. \tag{5.45}$$

One easily checks that the negative of the phase of $f_l^{(1)}$ is the hard-core phase shift. The Jost function for a potential with forces outside the core is given by[29]

$$f_l(k) = \frac{k^l e^{-i\pi l/2} f_l(k,c)}{(2l+1)!!}, \tag{5.46}$$

where $f_l(k,c)$ is obtained from the irregular solution defined by the upper signs in (3.4). The separable potential that must be added to the hard-shell potential is obtained from (5.45), (5.46), (5.29), (5.39), and (5.41). The steps leading to (5.41) are legitimate, since $f_l^{(2)}(k)$ [see (5.29)] behaves properly for infinite momentum. In (5.41) we identify $\delta_l^{(1)}$ as the pure hard-core phase shift and T_1 as the hard-shell T matrix (5.43).

We now turn our attention to tensor forces. Of course we need only consider those two-nucleon states of total angular momentum J whose spin S is 1 and whose parity is $(-)^{J\pm1}$. For such states there is a coupling between states of orbital angular momentum $L = J \pm 1$. For uncoupled states the previous analyses can be used. As our basis functions we choose

$$|pLSJM\rangle = |pL\rangle\,|LSJM\rangle, \tag{5.47}$$

where

$$\langle r|pL\rangle = (2\pi^2)^{-1/2} j_L(pr)$$

and $|LSJM\rangle$ is a vector spherical harmonic.[34] In order to simplify notation, we introduce a two-component row matrix

$$|\underline{\Omega}(k)\rangle = [\,|kJ-1\,1\,JM\rangle\,|kJ+1\,1\,JM\rangle] \tag{5.48}$$

and its adjoint

$$\langle\underline{\Omega}(k)| = \begin{bmatrix} \langle k & J-1 & 1 & JM| \\ \langle k & J+1 & 1 & JM| \end{bmatrix}. \tag{5.49}$$

Any operator whose matrix elements are taken with respect to the two-component entities (5.48) and (5.49) will be underlined; e.g., we write

$$\underline{T}(p,q;s) = \langle\underline{\Omega}(p)|T(s)|\underline{\Omega}(q)\rangle. \tag{5.50}$$

It is well known[35] that on the energy shell the T matrix (5.50) can be written in the form

$$\underline{T}(k,k;k^2+i\epsilon) = -(2\pi^2 k)^{-1}\underline{U}(k) \times \begin{bmatrix} e^{i\delta_\alpha}\sin\delta_\alpha & 0 \\ 0 & e^{i\delta_\beta}\sin\delta_\beta \end{bmatrix} \tilde{\underline{U}}(k), \tag{5.51}$$

where the real orthogonal matrix $\underline{U}$ is given by

$$\underline{U}(k) = \begin{bmatrix} \cos\epsilon(k) & -\sin\epsilon(k) \\ \sin\epsilon(k) & \cos\epsilon(k) \end{bmatrix}. \tag{5.52}$$

Here the tilde means transpose; $\delta_{\alpha,\beta}$ and ϵ are the Blatt-Biedenharn eigenphases and coupling parameter.

The T matrix (5.50) is the solution of the equation

$$\underline{T}(p,q;s) = \underline{V}(p,q) + \int_0^\infty \underline{V}(p,x)\frac{4\pi x^2 dx}{s-x^2}\underline{T}(x,q;s). \tag{5.53}$$

It is very easy to give a solution of the tensor-force inversion problem if one only requires the

separable potential to have a certain on-shell behavior and to produce a bound state at a certain energy. It is more difficult to solve the inversion problem if one demands that the separable potential also give a particular bound-state wave function. We consider the simpler problem first. The on-shell T matrix [see (5.51)] is obviously diagonalized by the orthogonal matrix U; i.e., we have

$$\underline{\tilde{U}}(k)\underline{T}(k,k;k^2+i\epsilon)\underline{U}(k)=\underline{T}'(k,k;k^2+i\epsilon), \quad (5.54)$$

where T' is the diagonal matrix that appears explicitly in (5.51). We assume that the off-shell T matrix and the potential are also diagonalized by $\underline{U}$; i.e., we assume

$$\underline{\tilde{U}}(p)\underline{T}(p,q;s)\underline{U}(q)=\underline{T}'(p,q;s), \quad (5.55)$$

$$\underline{\tilde{U}}(p)\underline{V}(p,q)\underline{U}(q)=\underline{V}'(p,q), \quad (5.56)$$

where the primes on the right-hand side of (5.55) and (5.56) indicate that the matrices are diagonal. With this assumption, the equations in (5.53) can be uncoupled by applying the transformation $\underline{U}$. The uncoupled equations are

$$T'_\gamma(p,q;s)=V'_\gamma(p,q)+\int_0^\infty V'_\gamma(p,x)\frac{4\pi x^2dx}{s-x^2}T'_\gamma(x,q;s), \quad \gamma=\alpha,\beta. \quad (5.57)$$

T'_γ and V'_γ are the diagonal matrix elements of $\underline{T}'$ and $\underline{V}'$. We define one Jost function for the α channel and one for the β channel by [see (4.4)]

$$f_\gamma(k)=\prod_n\left(1-\frac{E_n^{(\gamma)}}{E}\right)\exp\left[\frac{1}{\pi}\int_0^\infty\frac{dE'\delta_\gamma(E')}{E-E'}\right], \quad \gamma=\alpha,\beta. \quad (5.58)$$

We can apply the central-force analysis to each channel separately in order to find the separable $V_\gamma(p,q)$'s that give the Jost functions (5.58).

We now turn our attention to the more difficult inversion problem. We assume the separable potential is the sum of two parts, i.e.,

$$V=V_1+V_2. \quad (5.59)$$

Corresponding to this separation of the potential, the T matrix can be separated into two parts according to the relations[36]

$$T(s)=T_1(s)+[1+T_1(s)G_0(s)]T_2(s)[1+G_0(s)T_1(s)], \quad (5.60)$$

$$T_1(s)=V_1+V_1G_0(s)T_1(s), \quad =V_1+T_1(s)G_0(s)V_1, \quad (5.61)$$

$$T_2(s)=V_2+V_2G_1(s)T_2(s), \quad =V_2+T_2(s)G_1(s)V_2. \quad (5.62)$$

$G_1(s)$ is given by the solution of (5.15). We assume that V_1 is given by

$$V_1=\frac{U|B\rangle\langle B|U}{\langle B|U|B\rangle}. \quad (5.63)$$

Here U is the local potential which produces the bound state $|B\rangle$. From (5.61) it follows that

$$T_1(s)=V_1/f_\alpha^{(1)}(k), \quad (5.64)$$

where

$$f_\alpha^{(1)}(k)=1-\frac{\langle B|UG_0(s)U|B\rangle}{\langle B|U|B\rangle}. \quad (5.65)$$

The subscript α in (5.65) indicates that we are assuming the bound state is in the α channel. We now become more explicit and assume that $|B\rangle$ stands for the deuteron. We define [see (5.47)]

$$C(p)=\langle p\ 0\ 11\ M|(H_0+B)|B\rangle, \quad T(p)=\langle p\ 2\ 11\ M|(H_0+B)|B\rangle. \quad (5.66)$$

It is easy to show using the results of a previous paper[37] that the T matrix (5.64) in the representation of (5.47)–(5.49) is given by the relations:

$$\underline{T}_1(p,q;s)=\underline{U}_1(p)\underline{T}'_1(p,q;s)\underline{\tilde{U}}_1(q),$$

$$\underline{U}_1(k)=\begin{bmatrix}\cos\epsilon_1(k) & -\sin\epsilon_1(k)\\ \sin\epsilon_1(k) & \cos\epsilon_1(k)\end{bmatrix},$$

$$\tan\epsilon_1(k)=T(k)/C(k), \quad (5.67)$$

$$\underline{T}'_1(p,q;s)=-\frac{1}{f_\alpha^{(1)}(k)}\begin{bmatrix}g(p)g(q) & 0\\ 0 & 0\end{bmatrix},$$

$$g(p)=(-\langle B|U|B\rangle)^{-1/2}[C^2(p)+T^2(p)].$$

The eigenphases for T_1 are given according to (5.51) by

$$e^{i\delta_\alpha^{(1)}}\sin\delta_\alpha^{(1)}=2\pi^2kg^2(k)/f_\alpha^{(1)}(k), \quad \delta_\beta^{(1)}=0. \quad (5.68)$$

The continuum eigenstates of the potential (5.63) are given by

$$|\psi^{(\pm)}(k)\rangle=[1+G_0(k^2+i\epsilon)T_1(k^2+i\epsilon)]|\Omega(k)\rangle. \quad (5.69)$$

Here, because of the definition of $|\Omega(k)\rangle$, $|\psi^{(\pm)}(k)\rangle$ is a two-dimensional row matrix. It can be shown[11,32] that the incoming and outgoing states are related by

$$|\psi^{(+)}(k)\rangle=|\psi^{(-)}(k)\rangle\underline{S}_1(k), \quad (5.70)$$

where $\underline{S}_1(k)$ is the S matrix corresponding to the potential V_1. This can be written in the form

$$\underline{S}_1(k)=\underline{U}_1(k)\underline{S}'_1(k)\underline{\tilde{U}}_1(k), \quad (5.71)$$

where $\underline{S}'_1$ is the diagonal matrix

$$\underline{S}'_1(k)=\begin{bmatrix}e^{2i\delta_\alpha^{(1)}(k)} & 0\\ 0 & 1\end{bmatrix}. \quad (5.72)$$

We introduce the square root and inverse square root of the S matrix by the relations:

$$\underline{S}_1(k)^{\pm 1/2} = \underline{U}_1(k)\underline{S}_1'(k)^{\pm 1/2}\tilde{\underline{U}}_1(k),$$
$$\underline{S}_1'(k)^{\pm 1/2} = \begin{bmatrix} e^{\pm i\delta_\alpha^{(1)}(k)} & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.73}$$

Using (5.73), we introduce another set of continuum eigenstates by the relation

$$|\psi(k)\rangle = |\psi^{(\pm)}(k)\rangle\underline{S}_1'(k)^{\mp 1/2}. \tag{5.74}$$

These states are normalized so that

$$\langle\psi(p)|\psi(k)\rangle = \delta(p-k)\underline{1}/4\pi pk. \tag{5.75}$$

We now construct the on-energy-shell matrix elements of (5.60) in the representation given by (5.47)–(5.49). Using (5.69) and (5.74), we obtain the relation

$$\underline{T}(k) = \underline{T}_1(k) + \underline{S}_1(k)^{1/2}\underline{T}_2(k)\underline{S}_1(k)^{1/2}, \tag{5.76}$$

where

$$\underline{T}_2(k) = \langle\psi(k)|T_2(k^2+i\epsilon)|\psi(k)\rangle, \tag{5.77}$$

and $T(k)$ and $T_1(k)$ are the on-shell versions of (5.50) and (5.67). The S matrix corresponding to T and the S matrix corresponding to T_1 are given by the relations

$$\underline{S}(k) = \underline{1} - 4\pi^2 ik\underline{T}(k),$$
$$\underline{S}_1(k) = \underline{1} - 4\pi^2 ik\underline{T}_1(k). \tag{5.78}$$

If we make use of (5.76), (5.78), and the fact that S and S_1 are symmetric unitary matrices, it is easy to show that the matrix

$$\underline{S}_2(k) = \underline{1} - 4\pi^2 ik\underline{T}_2(k) \tag{5.79}$$

is a symmetric unitary matrix and obeys the relation

$$\underline{S}(k) = \underline{S}_1^{1/2}(k)\underline{S}_2(k)\underline{S}_1^{1/2}(k). \tag{5.80}$$

Clearly $\underline{S}_2$ can be determined from (5.80) and (5.73). Since $\underline{S}_2$ is a symmetric unitary matrix, it can be written in the form

$$\underline{S}_2(k) = \underline{U}_2(k)\underline{S}_2'(k)\tilde{\underline{U}}_2(k), \tag{5.81}$$

where $\underline{U}_2$ is a real orthogonal matrix of the same type as $\underline{U}$ and $\underline{U}_1$, and $\underline{S}_2'$ is the diagonal matrix

$$\underline{S}_2'(k) \equiv \begin{bmatrix} e^{2i\delta_\alpha^{(2)}(k)} & 0 \\ 0 & e^{2i\delta_\beta^{(2)}(k)} \end{bmatrix}. \tag{5.82}$$

We assume that $\underline{T}_2(p,q;s)$ and $\underline{V}_2(p,q)$ are diagonalized by the transformation $\underline{U}_2$; i.e., we assume

$$\begin{aligned} \underline{T}_2(p,q;s) &\equiv \langle\psi(p)|T_2(s)|\psi(q)\rangle \\ &= \underline{U}_2(p)\underline{T}_2'(p,q;s)\tilde{\underline{U}}_2(q), \\ \underline{V}_2(p,q) &\equiv \langle\psi(p)|V_2|\psi(q)\rangle \\ &= \underline{U}_2(p)\underline{V}_2'(p,q)\tilde{\underline{U}}_2(q), \end{aligned} \tag{5.83}$$

where $\underline{T}_2'$ and $\underline{V}_2'$ are diagonal matrices. If one assumes that

$$V_2|B\rangle = 0, \tag{5.84}$$

it is easy to derive from (5.62) and (5.83) the equations satisfied by the diagonal matrix elements of $\underline{T}_2'$. These are

$$T_{2\gamma}'(p,q;s) = V_{2\gamma}'(p,q) + \int_0^\infty V_{2\gamma}'(p,x)\frac{4\pi x^2 dx}{s-x^2}T_{2\gamma}'(x,q;s), \quad \gamma = \alpha,\beta. \tag{5.85}$$

If we assume

$$V_{2\gamma}'(p,q) = \lambda_\gamma u_\gamma(p)u_\gamma(q), \quad \gamma = \alpha,\beta, \tag{5.86}$$

and define Jost functions by

$$f_\gamma^{(2)}(k) = \exp\left[\frac{1}{\pi}\int_0^\infty \frac{dE'\delta_\gamma^{(2)}(E')}{E-E'}\right], \quad \gamma = \alpha,\beta, \tag{5.87}$$

it is clear from the central-force analysis that we must have

$$\lambda_\gamma u_\gamma^2(k) = (2\pi^2 k)^{-1}\mathrm{Im}f_\gamma^{(2)}(k), \quad \gamma = \alpha,\beta. \tag{5.88}$$

The separable potential $\underline{V}_2$ will be given by

$$\underline{V}_2(p,q) = \underline{U}_2(p)\begin{bmatrix} \lambda_\alpha u_\alpha(p)u_\alpha(q) & 0 \\ 0 & \lambda_\beta u_\beta(p)u_\beta(q) \end{bmatrix}\tilde{\underline{U}}_2(q). \tag{5.89}$$

The eigenphases $\delta_{\alpha,\beta}^{(2)}$, which are needed to construct $u_{\alpha,\beta}$ from (5.88) and (5.87), and the real orthogonal matrix $\underline{U}_2$, which is needed to find $\underline{V}_2$, are obtained by solving (5.8) for $\underline{S}_2(k)$ and by writing $\underline{S}_2$ in the form (5.81). Of course the potential (5.89) is in the representation supplied by the potential V_1 [see (5.83)], and in applications we, in general, want V_1 in the representation of (5.47)–(5.49). This is easily done by using the completeness relation for the eigenstates produced by V_1 and the assumption (5.84). We have

$$V_2 = \int_0^\infty\int |\psi(x)\rangle 4\pi x^2 dx\, \underline{V}_2(x,y) 4\pi y^2 dy \langle\psi(y)|, \tag{5.90}$$

where $|\psi\rangle$ is given by (5.74), (5.73), (5.69), and (5.67); and V_2 is given by (5.89). It is easy to show that

$$\langle\Omega(p)|\psi(k)\rangle = \underline{U}_1(p)\begin{bmatrix} \phi_\alpha(p,k) & 0 \\ 0 & \delta(p-k)/4\pi pk \end{bmatrix}\tilde{\underline{U}}_1(k), \tag{5.91}$$

where

$$\phi_\alpha(p,k) = (4\pi pk)^{-1}\delta(p-k)\cos\delta_\alpha^{(1)}(k) - \mathrm{P}\frac{1}{k^2-p^2}\frac{g(p)g(q)}{|f_\alpha^{(1)}(k)|}. \tag{5.92}$$

Assuming that $\delta_\alpha^{(2)}$ and $\delta_\beta^{(2)}$ are of one sign, it follows from (5.88)–(5.92) that

$$\langle \Omega(p) | V_2 | \Omega(q) \rangle$$

is a real matrix. It is clear that we now have a separable potential which will reproduce a given S-matrix and bound-state wave function. It will be of great interest to see if the T matrices of the two-term separable potential and the local potential from which the separable potential is generated have significantly different off-shell elements.

VI. DISCUSSION

We have developed an effective-rangelike theory for the low-energy off-shell two-nucleon T matrix. This theory is based on a power-series expansion of the half-off-shell extension function $F(p,k)$ (the ratio of the half-off-shell T matrix to the on-shell T matrix), and on the approximate separability of the low-energy T matrix. We have given a new proof of this approximate separability. This proof is based on the behavior of the T matrix in the coordinate representation. The simplicity of the proof suggests the usefulness of not always dealing with the T matrix in the momentum representation. The approximate separability of the T matrix allowed us to relate $F(p,k)$ to the imaginary part of the Jost function. Since the Jost function can be obtained from the phase shift and the two-body binding energy, we were able to determine the parameters in the expansion of $F(p,k)$ from experimental data on the two-nucleon system. This theory suggests that potentials which produce almost the same S-wave phase shift and binding energy will have the same low-energy T matrix, on and off the energy shell. This suggestion is actually supported by our method for determining the expansion parameters. They were obtained from the Jost function of two different potentials (Morse and HCSW) that were fitted to the $^3S_1$ and $^1S_0$ phase shifts.[27] Whereas the two parameter sets (see Tables I and II) were somewhat different, the values of $F(p,k)$ calculated with one set agreed in the low-energy range (up to about 20 MeV in the lab frame) with the values obtained from the other set.

The separable approximation upon which the off-shell theory is based is of the form suggested by Kowalski and Noyes.[22] This form has been tested[38,39] as a separable approximation to the square-well T matrix and has been found to give good results (roughly good to 1%) in the low-energy range. We have also checked the approximation for the HCSW-potential T matrix (see Tables IV and V) and found it to be a good one. It is clear, however, that more extensive tests should be made. In particular, the theory should be checked by seeing how well the formulas reproduce the low-energy behavior of the T matrix arising from realistic potentials.[21,40] Such calculations are being carried out.

Using the off-shell expansion formula, one should be able to find a one-term separable potential that gives better results than the conventional Yamaguchi shape.[41] This could be done by finding a separable potential that reproduces the off-shell lengths. Such a potential might improve Amado's model[42] of the three-nucleon system by making it unnecessary to employ a renormalization factor. The off-shell expansion formula can also be used to calculate the low-energy n-p bremsstrahlung cross section by means of Sobel and Cromer's[6] T-matrix formalism. If the formula can be modified to include Coulomb effects, it could also be used for low-energy p-p bremsstrahlung.

In an effort to develop methods for studying the off-shell T matrix, we have obtained a factorization theorem for the Jost function. In this theorem the total potential is written as a sum of partial potentials, and each partial potential is shown to correspond to a factor in the Jost function. Since the phase of the Jost function is the negative of the phase shift, this theorem associates a *piece* of the phase shift with a *piece* of the potential.

Using the factorization theorem, we were able to solve the inversion problem for two-term central separable potentials in a way that differs from Fiedeldey's[17] approach. Furthermore, we have solved the inversion problem for two cases not considered by Fiedeldey: namely, phase shifts which become pure hard-core phase shifts at high energies, and tensor forces. These extensions allow one to find a separable potential which has the same on-shell T matrix as a realistic local potential.[21] The interesting feature of the two-term separable potential is that it is *not* uniquely determined by the phase shifts and two-body binding energy; hence one can produce an infinite number of separable potentials with the same on-shell behavior. A comparison of central separable potentials with the same on-shell T matrix has been made by Fiedeldey,[17] and he has found large difference in the off-shell scattering matrices. One can exploit the nonuniqueness of the two-term separable potential by constructing it so that it reproduces a given two-body bound-state wave function. Fiedeldey has given the procedure for doing this with central forces, and we have shown that this can be done for tensor forces.

Fiedeldey has found that separable central potentials with the same phase shifts and bound-state wave function have very similar off-shell scattering matrices. What will be of great interest will

be to compare the off-shell T matrices of a local potential with that of a separable potential that has been constructed so as to have the same on-shell T matrix and bound-state wave function as the local potential. Kok, Erens, and Van Wageningen[43] have found that a one-term separable potential (Yamaguchi) with the same phase shift as a local potential (the Bargmann potential that has the same Jost function as the Yamaguchi potential) gives about 15% *more* binding energy in a three-body system than the local potential. On the other hand, they have found that a one-term separable potential (Yamaguchi) with the same bound-state wave function as a local potential (Hulthén) gives about 15% *less* binding energy in a three-body system than the local potential. Since these differences are in opposite directions, it seems reasonable to anticipate that a separable potential with the same phase shift and bound-state wave function as a local potential will give about the same three-body binding energy as the local potential. There is other evidence to support this conjecture. A fair amount of work has been done in testing the so-called unitary-pole approximation (UPA) to the two-body T matrix.[31, 39, 44] In this approximation one uses a one-term separable potential that has the same bound-state wave function as a local potential. The general result of these tests seems to be that whereas purely attractive potentials with the same bound-state wave function can have significantly different off-shell T matrix elements, a local potential with a strong short-range repulsion and a one-term separable potential with the same bound-state wave function have very similar off-shell T matrices. In particular, Harms and Levinger[44] have found that the UPA for the soft-core Malfliet-Tjon[45] potential gives a three-body binding energy which agrees with Malfliet and Tjon's result to within 2%. A separable potential which also has the same phase shift as the local potential would probably remove even this 2% discrepancy.

Using our tensor-force results, it will be possible to find a separable potential of the Tabakin and Mongan type,[40] which reproduces the phase parameters and bound-state wave function of the soft-core Reid[21] potential. One can construct the potential so that it also gives the same residue function for the virtual bound state. Brady[46] has been able to solve the coupled equations that arise when one uses the tensor potentials of Tabakin and Mongan in the three-body problem (he includes the coupled $^3S_1$-$^3D_1$ and the $^1S_0$ states). Thus, it may be possible to do a three-body calculation with a potential whose T matrix elements are very similar to those of a local soft-core potential. We are studying this possibility.

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[1]B. A. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).

[2]L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [transl.: Soviet Phys. – JETP 12, 1014 (1961)].

[3]L. D. Faddeev, Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory, translated from Russian, the Israel Program for Scientific Translations, Jerusalem (Daniel Davey and Company, New York, 1965).

[4]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[5]A. N. Mitra, Nucl. Phys. 32, 529 (1962).

[6]A. H. Cromer and M. I. Sobel, Phys. Rev. 152, 1351 (1966); 162, 1174(E) (1966).

[7]K. A. Brueckner, Phys. Rev. 100, 36 (1955).

[8]H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 267 (1957).

[9]M. Baranger, in Proceedings of the International School of Physics "Enrico Fermi," Varenna, 1967, Course XXXX (Academic Press Inc., New York, 1969).

[10]R. L. Becker, A. D. MacKellan, and B. M. Morris, Phys. Rev. 174, 1264 (1968).

[11]R. G. Newton, Scattering Theory of Waves and Particles (McGraw-Hill Book Company, Inc., New York, 1966).

[12]M. Gourdin and A. Martin, Nuovo Cimento 6, 757 (1957); 8, 699 (1958); A. Martin, Nuovo Cimento 7, 607 (1958).

[13]K. Chadan, Nuovo Cimento 10, 892; 47A, 510 (1967).

[14]M. Bolsterli and J. MacKenzie, Physics 2, 141 (1965).

[15]G. C. Ghirardi and A. Rimini, J. Math. Phys. 5, 722 (1964).

[16]F. Tabakin, Phys. Rev. 177, 1443 (1969).

[17]H. Fiedeldey, Nucl. Phys. A135, 353 (1969).

[18]M. G. Fuda, Phys. Rev. 174, 1134 (1968); 178, 1682 (1969); 186, 1078 (1969).

[19]L. M. Delves, in Three-Particle Scattering in Quantum Mechanics, edited by J. Gillespie and J. Nuttall (W. A. Benjamin, Inc., New York, 1968).

[20]T. Ohmura, M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) 15, 222 (1956); 17, 326 (1957); 22, 34 (1959); R. Van Wageningen and L. P. Kok, Nucl. Phys. A98, 365 (1967).

[21]T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962); K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. 126, 882 (1962); R. V. Reid, Jr., Ann. Phys. (N.Y.) 50, 411 (1968).

[22]H. P. Noyes, Phys. Rev. Letters 15, 538 (1965); K. L. Kowalski, *ibid.* 15, 798 (1965).

[23]J. Nuttall, J. Math. Phys. 8, 873 (1967).

[24]See Chap. 12 of Ref. 11 for a treatment of Jost functions.

[25]The definition of the Wronskian is $w(f,g)=fg'-f'g$.

[26]For the case of the $^1S_0$ virtual state of the two-nucleon system it is obvious that ρ is approximately the singlet effective range. One easily checks that the right-hand side of (3.18) is not zero.

[27]R. E. Seamon, K. A. Friedman, G. Breit, R. D. Haracz, J. M. Holt, and A. Prakash, Phys. Rev. 165, 1579 (1968).
[28]P. M. Morse, Phys. Rev. 34, 57 (1929).
[29]See Sec. 14.2 of Ref. 11 for a discussion of the Jost function for hard-core potentials.
[30]Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover Publications, Inc., New York, 1965).
[31]T. Brady, M. Fuda, E. Harms, J. S. Levinger, and R. Stagat, Phys. Rev. 186, 1069 (1969).
[32]See Sec. 7.2 of Ref. 11.
[33]The spherical Bessel and Hankel functions are normalized, see A. Messiah, Quantum Mechanics, translated from French (Wiley-Interscience, New York, 1961–2).
[34]See, for example, M. A. Preston, Physics of the Nucleus (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).
[35]J. M. Blatt and L. C. Biedenharn, Phys. Rev. 86, 399 (1952).
[36]See Ref. 11, Sec. 7.2, in particular, Subsection 7.2.5.
[37]M. G. Fuda, Nucl. Phys. A116, 83 (1968).
[38]A. S. Reiner, Nuovo Cimento 51A, 1 (1967).
[39]J. S. Levinger, A. H. Lu, and R. Stagat, Phys. Rev. 179, 926 (1969).
[40]F. Tabakin, Ann. Phys. (N.Y.) 30, 51 (1964); T. R. Mongan, Phys. Rev. 178, 1597 (1969).
[41]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).
[42]R. D. Amado, in Three-Particle Scattering in Quantum Mechanics, edited by J. Gillespie and J. Nuttall (W. A. Benjamin, Inc., New York, 1968).
[43]L. P. Kok, G. Erens, and R. Van Wageningen, Nucl. Phys. A122, 684 (1968).
[44]E. Harms and J. S. Levinger, to be published.
[45]R. A. Malfliet and J. A. Tjon, Nucl. Phys. A127, 161 (1969).
[46]T. J. Brady, Bull. Am. Phys. Soc. 14, 512 (1969); private communication.

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Observation of Nonlocal Effects in Nuclear Scattering*

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The question of how nonlocal effects might be observed in elastic scattering is investigated under the assumption that the interaction is describable by a Hamiltonian. The energy dependence of the real part of the forward-scattering amplitude is related to a dynamic quantity with dimensions of length, which we refer to as the nonlocal distance. Possible off-energy-shell effects can be discussed in terms of this nonlocal distance.

This note addresses itself to the question of how a nonlocality in the nuclear interaction potential might manifest itself in measurements of elastic scattering. We shall assume for the sake of argument that the interaction potential may be expressed in coordinate space as

$$V_I = V(\vec{r},\vec{r}') \tag{1}$$

independent of the parametric energy. Such a potential, if it satisfies the requirements of rotational invariance, may be considered to be a function of three variables: $r=|\vec{r}|$, $r'=|\vec{r}'|$, and $\vec{r}\cdot\vec{r}'$ (or equivalently r, r', and $\rho=|\vec{\rho}|\equiv|\vec{r}'-\vec{r}|$). We write this energy-independent, rotationally invariant potential as

$$V_I = V(r, r'; \rho). \tag{2}$$

If V_I is local, then this becomes

$$V_I = V(r, r')\delta(\rho). \tag{3}$$

In the Born approximation, we may write the scattering amplitude as

$$T(\vec{k},\vec{k}')=V(\vec{k},\vec{k}')\equiv\int e^{-i\vec{k}\cdot\vec{r}}V(\vec{r},\vec{r}')e^{+i\vec{k}\cdot\vec{r}'}d\vec{r}\,d\vec{r}'. \tag{4}$$

For a local interaction this becomes

$$V(\vec{k},\vec{k}')=\int e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}}V(r,r)d\vec{r}=V(|\vec{q}|), \tag{5}$$

where $\vec{k}-\vec{k}'=\vec{q}$ is the momentum transfer. This is the usual argument that in the Born approximation the scattering amplitude for a local potential is a function of the momentum transfer alone. For a nonlocal potential of the form given in Eq. (2), this remark does not hold.

For a potential of the form given in Eq. (2), we may write