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Binding Energy of a Λ Particle in Nuclear Matter

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The binding energy of a Λ particle in nuclear matter, $B_\Lambda(\infty)$, is calculated self-consistently with the help of the Brueckner theory. In the K -matrix equations for the Λ - N interaction, pure kinetic energies in the intermediate states are used. The K -matrix equations are solved numerically. The rearrangement energy is taken into account. The values of $B_\Lambda(\infty)$ calculated with several central Λ - N potentials $v_{\Lambda N}$ — though smaller than the values obtained by other authors — are, in general, larger than the empirical value of $B_\Lambda(\infty)$. An agreement with this value is obtained only if $v_{\Lambda N}$, adjusted to the binding energy of ${}^5_\Lambda\text{He}$, has a sufficiently large hard core and is sufficiently suppressed in odd states. Possible ways of reducing the calculated value of $B_\Lambda(\infty)$ are discussed. A critical discussion of the independent-pair approximation applied by other authors in calculating $B_\Lambda(\infty)$ is presented.

I. INTRODUCTION

The binding energy of a Λ particle in nuclear matter, $B_\Lambda(\infty)$, is a quantity of considerable interest in the phenomenological analysis of the Λ -nucleon interaction $v_{\Lambda N}$. Whereas the binding energies of light hypernuclei are determined primarily by the S -wave part of the Λ - N interaction, the binding energies of heavy hypernuclei depend on the Λ - N interaction in higher-angular-momentum states. A direct calculation of the binding energy of a Λ particle in a heavy but finite nucleus would be a most difficult task. It is much easier to calculate $B_\Lambda(\infty)$, the binding energy of a Λ particle in an infinite nuclear medium, i.e., in nuclear matter.

An empirical estimate of $B_\Lambda(\infty)$ is a nontrivial problem. One has to extrapolate the measured binding energies of a Λ particle in hypernuclei with finite values of A , $B_\Lambda(A)$ to the limiting case $A \rightarrow \infty$. The early estimates¹⁻⁴ of $B_\Lambda(\infty)$ were based on identified hypernuclei which, however, are all light (up to ${}_\Lambda\text{C}^{13}$), and consequently the extrapolation was of considerable uncertainty. Obviously,

much more important in determining $B_\Lambda(\infty)$ is the knowledge of binding energies of Λ in heavy hyperfragments, which play an essential role in all the newer estimates⁵⁻¹² of $B_\Lambda(\infty)$. Unfortunately, there are considerable ambiguities in determining $B_\Lambda(A)$ of mesonically decaying heavy hyperfragments. Namely, neither the mass of the hyperfragments nor the precise decay modes can be determined accurately. These uncertainties are reflected in the $B_\Lambda(\infty)$ values obtained by different authors. The notable estimates are those by Lagnaux *et al.*⁷ (27 ± 3 MeV), Lemonne *et al.*⁹ (27.2 ± 1.3 MeV), Goyal¹⁰ (32 ± 2 MeV), Bhowmik *et al.*¹¹ (30.6 ± 0.6 MeV), and Kang and Zaffarano¹² (27.7 ± 0.6 or 28.3 ± 0.3 MeV). All these results seem to indicate a tentative estimate¹³:

$$B_\Lambda(\infty) = 30 \pm 5 \text{ MeV.} \quad (1.1)$$

As mentioned before, the importance of $B_\Lambda(\infty)$ in reconstructing $v_{\Lambda N}$ is connected with the sensitivity of $B_\Lambda(\infty)$ to the Λ - N interaction in higher-angular-momentum states. To visualize it let us consider the range of energies of the Λ - N system rel-

evant in determining $B_\Lambda(\infty)$. In the ground state of the system, Λ particle + nuclear matter, the Λ particle occupies the state with zero momentum, and the relative Λ - N velocity v_{rel} is equal to the nucleon velocity $p_N/\hbar\mathfrak{M}_N$, where p_N is the nucleon momentum (in units of \hbar) and \mathfrak{M}_N is the mass of the nucleon¹⁶ divided by \hbar^2 . This corresponds to the Λ -particle kinetic energy in the nucleon rest system (the laboratory system for Λ - N scattering):

$$T_\Lambda = \frac{1}{2}\hbar^2\mathfrak{M}_\Lambda v_{rel}^2 = (\mathfrak{M}_\Lambda/\mathfrak{M}_N)(p_N^2/2\mathfrak{M}_N), \quad (1.2)$$

where \mathfrak{M}_Λ is the mass of the Λ particle. The range of p_N extends from 0 to the Fermi momentum k_F , and we have

$$0 \leq T_\Lambda \leq (\mathfrak{M}_\Lambda/\mathfrak{M}_N)E_F, \quad (1.3)$$

where E_F is the Fermi energy of nuclear matter. The average value of p_N^2 in nuclear matter is $0.6k_F^2$, and we get for the average value of T_Λ :

$$\bar{T}_\Lambda = 0.6(\mathfrak{M}_\Lambda/\mathfrak{M}_N)E_F. \quad (1.4)$$

For the value of the spacing parameter $r_0 = 1.1$ F we have

$$k_F = 1.35 \text{ F}^{-1}, \quad (1.5)$$

and $E_F = 37.8$ MeV. For this value of E_F we have $0 \leq T_\Lambda \leq 45$ MeV, and for the most heavily weighted Λ laboratory energy, $\bar{T}_\Lambda \cong 27$ MeV. Now, the Λ -proton scattering experiments¹⁷ show that for Λ momenta bigger than 248 MeV/ c , i.e., for $T_\Lambda \geq 30$ MeV, the Λ - p scattering departs from isotropy. Thus around $T_\Lambda = 30$ MeV the Λ - N interaction in the P state starts to be effective, and consequently we expect that the P -state interaction will effect the value of $B_\Lambda(\infty)$.

Another way of reaching this conclusion is to consider the relative Λ - N momentum in nuclear matter, $p = \mathfrak{M}_\Lambda p_N / (\mathfrak{M}_\Lambda + \mathfrak{M}_N)$. Similarly as with T_Λ , we have

$$0 \leq p \leq (\mathfrak{M}_\Lambda/\mathfrak{M}_\Lambda + \mathfrak{M}_N)k_F, \quad (1.6)$$

and for the average value:

$$\bar{p} = (0.6)^{1/2}(\mathfrak{M}_\Lambda/\mathfrak{M}_\Lambda + \mathfrak{M}_N)k_F. \quad (1.7)$$

If we assume the range of $v_{\Lambda N}$ to be equal to the intrinsic range $b = 1.5$ F, corresponding to the two-pion exchange, we get $0 \leq bp \leq 1.1$ and $b\bar{p} \cong 0.85$ for the value of k_F [Eq. (1.5)]. By applying the classical argument with the impact parameter, we see that both S and P waves in the Λ - N system are expected to be important in calculating $B_\Lambda(\infty)$. This conclusion has been confirmed by all existing calculations of $B_\Lambda(\infty)$, which also show that the D state already plays a minor role. Actually, the role of the P wave is enhanced compared to the S wave. Namely, the short-range repulsion in $v_{\Lambda N}$ acts predominantly in the S state and cancels a large part of the S -wave contribution of the attractive tail of

$v_{\Lambda N}$. Thus the net contributions of the S and P states to $B_\Lambda(\infty)$ are of comparable magnitudes.

Several calculations of $B_\Lambda(\infty)$ have been published.^{2,18-31} In the case of a regular, purely attractive Λ - N potential, one may simply apply the perturbation expansion.^{18,19} However, in the case of a realistic Λ - N potential with a hard core, a more sophisticated method of calculating $B_\Lambda(\infty)$ is necessary. Two such methods have been applied in the existing calculations of $B_\Lambda(\infty)$: The Brueckner method^{2,20-27} and the Jastrow method.²⁷⁻³¹ In the present paper we shall restrict ourselves to the Brueckner method. Among the Brueckner-type calculations of $B_\Lambda(\infty)$, only the paper by Taharzadeh, Moszkowski, and Sood²⁰ follows a systematic approximation scheme based on the separation method.³² However, the results obtained in Ref. 32 are outdated because of the assumed form of $v_{\Lambda N}$ and of the single-particle energies. In all the remaining Brueckner-type calculations the so-called independent-pair approximation (IPA) has been applied. No precise justification of this approximation has been presented so far. The only attempt²⁵ to estimate corrections to the IPA has shown that probably they are large. Furthermore, the single-nucleon energies applied in all the existing calculations of $B_\Lambda(\infty)$ cannot be justified from the point of view of the present state of the theory of nuclear matter.

In the present paper³³ we solve exactly the equations of the Brueckner theory for the Λ - N interaction. In accordance with the present state of the theory of nuclear matter,³⁴ we use pure kinetic energies in the intermediate states. The pure nuclear-matter problem is considered to be solved with the resulting saturation density and energy in agreement with experiment. The precise shape of single-nucleon energies of the occupied states turns out to be not important provided its average value is consistent with the known binding energy per nucleon in nuclear matter.

First calculations^{35,36} of $B_\Lambda(\infty)$ along these lines have been performed with the help of the improved Moszkowski-Scott separation method³⁷ of solving the K -matrix equation [Eq. (2.5)]. Recently, an essentially identical approach has been reported by Bodmer and Rote.³⁸ The main difference between our approach and that of Bodmer and Rote is that we use the integral form of the wave function equation [Eq. (2.13)], whereas Bodmer and Rote work with the integrodifferential form of this equation.

The whole scheme of calculating $B_\Lambda(\infty)$ for the case of a central Λ - N potential considered in the present paper is presented in Sec. 2. In Sec. 3 we discuss the choice of $v_{\Lambda N}$ used in the present calculations. The computational procedure is de-

scribed in Sec. 4. The results obtained are presented and discussed in Sec. 5. Finally, in the Appendix we present a critical discussion of the IPA.

2. K -MATRIX THEORY OF $B_\Lambda(\infty)$

We assume that the Λ - N interaction $v_{\Lambda N}$ is central and charge symmetric. It is equal to ${}^3v(r)$ in the spin triplet state and ${}^1v(r)$ in the spin singlet state, where r is the distance between Λ and N . Nuclear matter is assumed to have in each occupied state four nucleons: Two protons and two neutrons with spin up and down. In such a system with no neutron excess, the charge-symmetry-breaking (CSB) component of $v_{\Lambda N}$ is expected to have a minor effect only. For instance, the CSB potential of the form considered by Herndon and Tang³⁹ gives a zero contribution to $B_\Lambda(\infty)$ in the first-order approximation in the attractive part of the CSB potential. In the presence of a neutron excess [and the heavy hyperfragments used in the estimates of $B_\Lambda(\infty)$ have a neutron excess] the CSB potential could play a role which, however, is not considered in the present paper.

Let us introduce the following notation. By \vec{m}_N , \vec{m}_Λ we shall denote the nucleon and Λ momenta (in units of \hbar) of the occupied states; by \vec{k}_N , \vec{k}_Λ the momenta of the excited states; and by \vec{p}_N , \vec{p}_Λ general momenta without any restrictions. As mentioned before,

$$\vec{m}_\Lambda = 0. \quad (2.1)$$

Furthermore, we shall denote by μ the reduced mass of the Λ - N system:

$$\mu = \mathfrak{M}_N \mathfrak{M}_\Lambda / (\mathfrak{M}_N + \mathfrak{M}_\Lambda). \quad (2.1')$$

The binding energy of the Λ particle in nuclear matter, $B_\Lambda(\infty)$, is defined by

$$-B_\Lambda(\infty) = E(A+1_\Lambda) - E(A), \quad (2.2)$$

where $E(A)$ and $E(A+1_\Lambda)$ are the ground-state energies of nuclear matter and of the system: nuclear matter + Λ particle. By applying the Brueckner-theory expressions for $E(A)$ and $E(A+1_\Lambda)$, one obtains⁴⁰

$$-B_\Lambda(\infty) = V_\Lambda + V_R, \quad (2.3)$$

where V_Λ is the Λ single-particle potential (usually denoted in the literature by $-D$) and V_R is the rearrangement potential. First, we shall explain the method of calculating V_Λ , and afterwards, at the end of this section, we shall discuss the rearrangement potential V_R .

A. Calculation of V_Λ

For V_Λ we have⁴⁰

$$V_\Lambda = V_\Lambda(m_\Lambda) = \sum_{\vec{m}_N} [(\vec{m}_N \vec{m}_\Lambda | {}^1K | \vec{m}_N \vec{m}_\Lambda) + 3(\vec{m}_N \vec{m}_\Lambda | {}^3K | \vec{m}_N \vec{m}_\Lambda)], \quad (2.4)$$

where 1K and 3K are the Λ - N singlet- and triplet-state K matrices, i.e., the effective Λ - N interactions in nuclear matter. They are determined by the following equation:

$$(\vec{p}_N \vec{p}_\Lambda | {}^sK | \vec{m}_N \vec{m}_\Lambda) = (\vec{p}_N \vec{p}_\Lambda | {}^s v | \vec{m}_N \vec{m}_\Lambda) + \sum_{\vec{k}_N \vec{k}_\Lambda} \frac{(\vec{p}_N \vec{p}_\Lambda | {}^s v | \vec{k}_N \vec{k}_\Lambda)(\vec{k}_N \vec{k}_\Lambda | {}^sK | \vec{m}_N \vec{m}_\Lambda)}{e_N(m_N) + e_\Lambda(m_\Lambda) - \epsilon_N(k_N) - \epsilon_\Lambda(k_\Lambda)}, \quad (2.5)$$

where $s=1, 3$; and ϵ_N , ϵ_Λ are the N and Λ kinetic energies; and e_N , e_Λ are the N and Λ single-particle energies of the occupied states:

$$e_N(m_N) = \epsilon_N(m_N) + V_N(m_N), \quad (2.6)$$

$$e_\Lambda(m_\Lambda) = V_\Lambda(m_\Lambda), \quad (2.7)$$

where the last equation is a consequence of Eq. (2.1). The single-nucleon potential V_N will be discussed later.

Equation (2.5) differs from the traditional form⁴¹ of the K -matrix equation by the appearance of the kinetic energies ϵ_N and ϵ_Λ in the intermediate states. The point is that a systematic approach in terms of the number of hole lines requires that particle self-energy diagrams should be considered together with other three-body diagrams. The total contribution to the energy per nucleon of the entire class of the three-body diagrams is probably not larger than about 1 MeV in the case of nuclear matter, and one should expect a similar situation in our case. It seems then that the most reasonable procedure is to disregard the self-energy insertions into the particle lines, i.e., to use kinetic energies in the intermediate states. The probably small error thus committed may be left to a perturbative estimate.^{34, 42, 43} Needless to say, by using $\epsilon_N(k_N)$, $\epsilon_\Lambda(k_\Lambda)$ in Eq. (2.5) we essentially simplify the problem of solving this equation.

In solving Eq. (2.5) we follow closely Ref. 41. We introduce the relative and center-of-mass momenta:

$$\vec{m} = \frac{\mathfrak{M}_\Lambda \vec{m}_N - \mathfrak{M}_N \vec{m}_\Lambda}{(\mathfrak{M}_N + \mathfrak{M}_\Lambda)} = \frac{\mu \vec{m}_N}{\mathfrak{M}_N}, \quad (2.8)$$

$$\vec{M} = \vec{m}_N + \vec{m}_\Lambda = \vec{m}_N. \quad (2.9)$$

The last parts of these equations follow from Eq. (2.1). Notice that

$$\vec{M} = \mathfrak{M}_N \vec{m} / \mu, \quad (2.10)$$

i.e., the center-of-mass momentum is proportional to the relative momentum. Because of momen-

tum conservation, we have

$$(\vec{p}_N \vec{p}_\Lambda | {}^s K | \vec{m}_N \vec{m}_\Lambda) = \delta_{\vec{p} \vec{M}} (\vec{p} | {}^s K | \vec{m}), \quad (2.11)$$

where \vec{p} and \vec{P} are expressed by \vec{p}_N , \vec{p}_Λ , by equations analogous to Eqs. (2.8) through (2.9). Because \vec{M} is determined by \vec{m} , we shall drop the subscript \vec{M} at the ${}^s K$ matrix and also at the wave function ${}^s \Psi_{\vec{m}}$ defined by the equation

$${}^s K | \vec{m} \rangle = {}^s v | {}^s \Psi_{\vec{m}} \rangle. \quad (2.12)$$

Equation (2.5) may be written as the following integral equation for the wave function ${}^s \Psi_{\vec{m}}$ in the configuration space:

$${}^s \Psi_{\vec{m}}(\vec{r}) = e^{i \vec{m} \cdot \vec{r}} + \int d\vec{r}' G_{\vec{m}}(\vec{r}, \vec{r}') {}^s v(r') {}^s \Psi_{\vec{m}}(\vec{r}'), \quad (2.13)$$

where the Green function

$$G_{\vec{m}}(\vec{r}, \vec{r}') = \left(\frac{1}{2\pi} \right)^3 \int d\vec{p} Q(\vec{M}, \vec{p}) \frac{e^{i \vec{p} \cdot (\vec{r} - \vec{r}')}}{z(m) - p^2/2\mu}, \quad (2.14)$$

where the exclusion-principle operator

$$\begin{aligned} Q(\vec{M}, \vec{p}) &= 1 \text{ for } |(\mu \vec{M} / \mathfrak{M}_\Lambda) + \vec{p}| > k_F, \\ &= 0 \text{ otherwise,} \end{aligned} \quad (2.15)$$

and where

$$z(m) = e_N(m_N) + e_\Lambda(m_\Lambda) - \frac{M^2}{2(\mathfrak{M}_N + \mathfrak{M}_\Lambda)}. \quad (2.16)$$

Notice that all the momenta on the right-hand side of Eq. (2.16) are determined by m [Eqs. (2.1), (2.8), and (2.10)].

With the help of the wave function ${}^s \Psi_{\vec{m}}(\vec{r})$, which describes the relative Λ - N motion in nuclear matter (with relative momentum \vec{m} and in the spin state s), we may write Eq. (2.4) in the form:

$$\begin{aligned} V_\Lambda = \left(\frac{1}{2\pi} \right)^3 \left(\frac{\mathfrak{M}_N}{\mu} \right)^3 \int d\vec{m} [\langle \vec{m} | {}^s K | \vec{m} \rangle + 3 \langle \vec{m} | {}^s K | \vec{m} \rangle] \\ m < \mu k_F / \mathfrak{M}_N, \end{aligned} \quad (2.17)$$

where

$$\langle \vec{m} | {}^s K | \vec{m} \rangle = \int d\vec{r} e^{-i \vec{m} \cdot \vec{r}} {}^s v(r) {}^s \Psi_{\vec{m}}(\vec{r}). \quad (2.18)$$

To be able to analyze Eq. (2.13) into uncoupled partial wave equations we approximate the exclusion-principle operator Q by its angle average:

$$Q(\vec{M}, \vec{p}) \cong \bar{Q}(m, p) = (1/4\pi) \int d\hat{M} \hat{Q}(\vec{M}, \vec{p}), \quad (2.19)$$

which may be calculated easily with the result:

$$\begin{aligned} \bar{Q}(m, p) &= 0 \text{ for } p < k_F - \mathfrak{M}_N m / \mathfrak{M}_\Lambda, \\ &= 1 \text{ for } p > k_F + \mathfrak{M}_N m / \mathfrak{M}_\Lambda, \\ &= \frac{(p + \mathfrak{M}_N m / \mathfrak{M}_\Lambda)^2 - k_F^2}{4(\mathfrak{M}_N / \mathfrak{M}_\Lambda) m p} \text{ otherwise.} \end{aligned} \quad (2.20)$$

Notice that in writing Eq. (2.20) we have replaced M by m [Eq. (2.10)]. The approximation (2.19)

which restores the rotational invariance of Eq. (2.13) has been checked to be fairly accurate in the nuclear-matter case⁴⁴ and in the Λ + nuclear-matter case.³⁸

With the approximation (2.19) we may expand $G_{\vec{m}}$ into spherical harmonics:

$$G_{\vec{m}}(\vec{r}, \vec{r}') = \sum_l \left(\frac{4\pi}{2l+1} \right)^{1/2} G_m^l(r, r') Y_{l0}(\hat{r} \hat{r}'), \quad (2.21)$$

where

$$G_m^l(r, r') = \frac{1}{2\pi^2} \int dp p^2 \bar{Q}(m, p) j_l(pr) j_l(pr'). \quad (2.22)$$

Similarly, we expand

$${}^s \Psi_{\vec{m}}(\vec{r}) = \sum_l [4\pi(2l+1)]^{1/2} i^l {}^s R_l(m, r) Y_{l0}(\hat{r}), \quad (2.23)$$

and from Eq. (2.13) obtain the following equations for the radial functions ${}^s R_l$:

$$\begin{aligned} {}^s R_l(m, r) \\ = j_l(mr) + 4\pi \int_0^\infty dr' r'^2 G_m^l(r, r') {}^s v(r') {}^s R_l(m, r'). \end{aligned} \quad (2.24)$$

For a potential with a hard core of radius r_C , we encounter the difficulty of having a product vR of the indeterminate form $\infty \times 0$ inside of the hard core in Eq. (2.24). We bypass this difficulty by approximating the hard core by the hard shell of the same radius r_C .^{41, 45} This approximation has been shown to be quite satisfactory.⁴⁶ By applying this approximation, we get in place of Eq. (2.24)⁴⁷

$$\begin{aligned} {}^s R_l(m, r) = s_l(m, r) \\ + 4\pi \int_{r_C}^\infty dr' r'^2 F_m^l(r, r') {}^s v(r') {}^s R_l(m, r'), \end{aligned} \quad (2.25)$$

where

$$s_l(m, r) = j_l(mr) - \frac{j_l(mr_C) G_m^l(r, r_C)}{G_m^l(r_C, r_C)} \quad (2.26)$$

is the solution of Eq. (2.24) for the case of a pure hard-shell interaction, and where the new Green functions

$$F_m^l(r, r') = G_m^l(r, r') - \frac{G_m^l(r, r_C) G_m^l(r_C, r')}{G_m^l(r_C, r_C)}. \quad (2.27)$$

Equation (2.18) may be written in the form

$$\langle \vec{m} | {}^s K | \vec{m} \rangle = \langle \vec{m} | K | \vec{m} \rangle_C + \langle \vec{m} | {}^s K | \vec{m} \rangle_A, \quad (2.28)$$

where the pure hard-core contribution⁴⁸

$$\begin{aligned} \langle \vec{m} | K | \vec{m} \rangle_C &= \int d\vec{r} e^{-i \vec{m} \cdot \vec{r}} v_C(r) \Psi_{\vec{m}}^C(\vec{r}), \\ &= - \sum_l \frac{(2l+1) j_l(mr_C)^2}{G_m^l(r_C, r_C)} \end{aligned} \quad (2.29)$$

(where v_C is the hard-core potential) and the contribution of ${}^s v_A$, the attractive tail of ${}^s v$, is

$$\begin{aligned} \langle \hat{m} | {}^s K | \hat{m} \rangle_A &= \int d\vec{r} \Psi_{\hat{m}}^C(\vec{r}) {}^s v_A(r) {}^s \Psi_{\hat{m}}(\vec{r}), \\ &= 4\pi \sum_l (2l+1) \int_{r_C}^{\infty} dr r^2 s_l(m, r) \\ &\quad \times {}^s v_A(r) {}^s R_l(m, r). \end{aligned} \quad (2.30)$$

In both equations (2.29) and (2.30) we use the notation

$$\Psi_{\hat{m}}^C(\vec{r}) = \sum_l [4\pi(2l+1)]^{1/2} i^l s_l(m, r) Y_{l0}(\hat{m}\hat{r}) \quad (2.31)$$

for the solution of Eq. (2.13) in the case of a pure hard-core potential $v = v_C$ (with v_C approximated by a hard-shell potential).⁴⁸ Notice that $\langle K \rangle_C$ and s_l , Ψ^C do not depend on the Λ - N spin because in all the Λ - N potentials to be considered, the hard-core radius r_C is the same in the triplet and singlet states.

Let us summarize the procedure of calculating V_Λ . We simply solve Eqs. (2.25) for the radial functions ${}^s R_l(m, r)$. Next, we calculate the K matrix elements [Eqs. (2.28) through (2.30)] and insert them into expression (2.17) for V_Λ . Obviously, we have to start the whole procedure with the calculation of the Green functions $G_m^l(r, r')$ and $F_m^l(r, r')$ [see Eqs. (2.22) and (2.27)]. To do it we have to apply a certain form of the single-nucleon energy $e_N(m_N)$. We shall discuss this point in a moment. Right now, let us assume that we know $e_N(m_N)$. The only unknown element in the Green function G_m^l is then V_Λ , and obviously we are faced with a self-consistency problem. It is, however, a very simple self-consistency problem because it concerns only one number, the value of V_Λ . We have to assume a certain input value of V_Λ to compute the Green functions. After applying the whole procedure we get a certain output value of V_Λ . Self-consistency is achieved when the output and input values of V_Λ are equal. Practically, one has to perform the whole calculation for a few input values of V_Λ . In this way one obtains the output value of V_Λ as a function of the input value of V_Λ . From this function one then determines the point of self-consistency.

B. Single-Nucleon Energy e_N

There are a few ways of fixing the spectrum of $e_N(m_N)$:

(i) Let us assume that the Brueckner theory is correct, and with the proper form of the N - N interaction it leads to the observed value of the binding energy per nucleon in nuclear matter, ϵ_{vol} , at the observed value of the equilibrium density of nuclear matter determined by k_F . Then

$$\bar{\epsilon}_N + \frac{1}{2} \bar{V}_N = \frac{1}{2} (\bar{\epsilon}_N + \bar{\epsilon}_N) = -\epsilon_{\text{vol}}, \quad (2.32)$$

where the bars denote average values in the Fermi sea.⁴⁹ If we apply the effective-mass approximation for e_N ,

$$e_N(m_N) = m_N^2 / 2\mathfrak{M}_N^* + A, \quad (2.33)$$

we need, besides the condition (2.32), one condition more to fix the two constants \mathfrak{M}_N^* and A . Let us consider the condition

$$e_N(k_F) = -\epsilon_{\text{vol}}. \quad (2.34)$$

At the equilibrium density, the energy per nucleon in nuclear matter, $-\epsilon_{\text{vol}}$, is equal to the separation energy. Hence it may be stated, based on Eq. (2.34), that the single-nucleon energy at the Fermi surface is equal to the separation energy. Actually, in the Brueckner theory, the rearrangement energy should be added to $e_N(k_F)$ in Eq. (2.34).⁵⁰ Nevertheless, we shall use both conditions (2.32) and (2.34) to fix $e_N(m_N)$, assumed in the form (2.33).

(ii) In the crudest approach we may approximate $e_N(m_N)$ in the energy denominator of G_m^l [Eqs. (2.22) and (2.16)] by its average value:

$$e_N(m_N) \cong \bar{e}_N = -\left[\frac{3}{5} \epsilon_N(k_F) + 2\epsilon_{\text{vol}}\right], \quad (2.35)$$

calculated from Eq. (2.32).

(iii) We may use the values of $e_N(m_N)$ obtained in one of the successful nuclear-matter calculations. Here, we shall use $e_N(m_N)$ obtained by Brueckner and Gammel.⁴¹ In order that Eq. (2.32) be satisfied for the values of k_F and ϵ_{vol} applied in the present work, we shall shift the whole Brueckner-Gammel spectrum a little bit (by less than 6 MeV).

Actually, the calculated value of V_Λ turns out to be not sensitive to the particular method, (i)-(iii), of fixing $e_N(m_N)$, provided Eq. (2.32) is satisfied. After this was determined, the e_N spectrum (i) was applied throughout the present work.

C. Rearrangement Potential V_R

A detailed derivation of the expression for the rearrangement potential V_R has been presented in Ref. 40. The source for V_R is the difference in the nucleon-nucleon K matrix in the case of nuclear matter plus Λ particle and in the case of pure nuclear matter: namely, the presence of a Λ particle changes the single-nucleon energies slightly. In the present approach, we tacitly assume that the nucleon-nucleon K matrix equation also contains pure kinetic energies in the intermediate states, and thus only the single-nucleon energies $e_N(m_N)$ of the occupied states are affected by the presence of the Λ particle. Consequently, the total rearrangement potential V_R is equal to the hole rearrangement potential V_{R_h} (the particle rear-

rangement potential V_{Rp} vanishes). By applying the simple and accurate approximate expression for V_{Rth} derived in Ref. 40, we may write

$$V_R = V_{Rth} \cong \kappa V_\Lambda, \quad (2.36)$$

where the dimensionless parameter

$$\kappa = \rho \left\{ \int d\vec{r} \left[\frac{3}{8} |\chi_{\vec{m}^+}(\vec{r})|^2 + \frac{3}{8} |\chi_{\vec{m}^-}(\vec{r})|^2 + \frac{1}{8} |\chi_{\vec{m}^+}(\vec{r})|^2 + \frac{9}{8} |\chi_{\vec{m}^-}(\vec{r})|^2 \right] \right\}_{Av}, \quad (2.37)$$

where ρ is the density of nuclear matter. The nucleon-nucleon difference function is

$${}^s\chi_{\vec{m}^\pm}(\vec{r}) = {}^s\varphi_{\vec{m}^\pm}(\vec{r}) - (e^{i\vec{m} \cdot \vec{r}})^\pm, \quad (2.38)$$

where ${}^s\varphi_{\vec{m}^\pm}$ is the wave function of the relative motion of two nucleons in nuclear matter in a state with the relative momentum $\vec{m} = (\vec{m}_1 - \vec{m}_2)/2$, with the parity \pm , and with the spin defined by s ($s=1, 3$ for the singlet, triplet state), and where $(e^{i\vec{m} \cdot \vec{r}})^\pm$ is the part of the unperturbed wave function $e^{i\vec{m} \cdot \vec{r}}$ with the parity \pm (\vec{r} denotes the relative position vector of the two nucleons). The subscript Av at the nucleon-nucleon correlation volume $\{ \}$ indicates the average value in the Fermi sea, i.e., one has to average over the momenta \vec{m}_1 and \vec{m}_2 of the two nucleons. Actually, Eq. (2.37) is an obvious generalization of the corresponding expression derived in Ref. 40 in the case of a spin-independent Serber N - N potential.

3. CHOICE OF $v_{\Lambda N}$

The central, spin-dependent, charge-symmetric Λ - N interactions used in the present calculation of $B_\Lambda(\infty)$ are listed in Table I. The parameters characterizing the interactions are: The hard-core radius r_C ; the intrinsic range b ; the singlet and triplet well-depth parameters s_s, s_t ; the singlet and triplet scattering lengths a_s, a_t ; and the corresponding effective ranges r_s, r_t . In all the potentials considered, r_C and b have the same values in the singlet and triplet states. Some of the poten-

tials are weaker in odd-angular-momentum states, i.e., their strength in these states is equal to y times the strength in even states. The shape of the attractive part of $v_{\Lambda N}$ is also shown in Table I.

The HTS potential is an old potential fitted by Herndon, Tang, and Schmid²³ to the binding energies of the S -shell hypernuclei. Its intrinsic range is equal to the intrinsic range of a purely attractive two-pion-exchange Yukawa potential. Typically, HTS is much stronger in the singlet than in the triplet state.

The AGK potential has been fitted by Ali, Grypos, and Kok⁵¹ to Λ - p scattering under the assumption of a common intrinsic range b for both the singlet and triplet interactions. Charge symmetry has been assumed. No fit to hyperfragment energies has been attempted. It should be mentioned that the recent analysis¹⁷ shows that, by fitting the Λ - p scattering without the assumption concerning b , one gets a wide spectrum of acceptable values of a_s, a_t, r_s, r_t .

Potentials E, E', H, F' have been determined by Herndon and Tang³⁹ in the following way. For a given b and r_C , the spin- and charge-dependent $v_{\Lambda N}$ has been adjusted to $B_\Lambda(\Lambda H^3), B_\Lambda(\Lambda H^4), B_\Lambda(\Lambda He^4)$. Next, the parameter y has been adjusted to Λ - p scattering data. Among the potentials thus obtained we have chosen the best potential H and also the potential E . An alternative set of potentials, which we denote by primes, has been obtained by applying the same procedure, except that their charge-independent components have been adjusted to $B_\Lambda(\Lambda H^3)$ and $B_\Lambda(\Lambda He^5)$. The best of the alternative set are the potentials E' and F' , although they lead to total Λ - p cross sections about 20% too small compared to the experiment. In the present work we shall also consider potentials identical with those of Ref. 39 except that they are not suppressed in odd states ($y=1$). They are denoted by the letters NX (for no exchange) added to their original symbols. All the potentials of Ref. 39 are

TABLE I. Parameters of the charge-symmetric Λ - N potentials. All lengths are in fermis. The figures in parentheses are the corresponding values of the parameters of the Λ - p potentials (with the CSB components included).

Ref.	Shape	r_C	b	s_s	s_t	a_s	a_t	r_s	r_t	y	Symbol
23	Exp	0.4	1.5	0.865	0.675	-2.89	-0.71	1.94	3.75	1.0	HTS
51	Yuk	0.4	2.07	0.805	0.669	-3.35	-1.47	2.84	4.07	1.0	AGK
39	Exp	0.45	2.0	0.790 (0.761)	0.703 (0.712)	-2.63 (-2.16)	-1.52 (-1.60)	2.92 (3.15)	3.71 (3.61)	1.0 (0.6)	ENX E
39	Exp	0.45	2.0	0.803 (0.774)	0.662 (0.672)	-2.91 (-2.31)	-1.19 (-1.26)	2.82 (3.03)	4.27 (4.14)	1.0 (0.6)	E' NX E'
39	Exp	0.6	2.1	0.834 (0.810)	0.792 (0.800)	-2.78 (-2.25)	-1.96 (-2.08)	3.04 (3.29)	3.49 (3.40)	1.0 (0.6)	HNX H
39	Exp	0.6	2.0	0.852 (0.830)	0.766 (0.773)	-2.79 (-2.29)	-1.36 (-1.44)	2.84 (3.05)	3.91 (3.79)	1.0 (0.6)	F' NX F'
21	Exp	0.4	1.9	0.578	0.578	-0.75	-0.75	5.49	5.49	1.0	DW

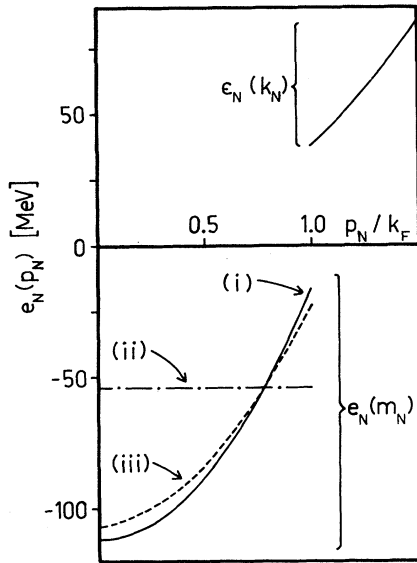


FIG. 1. The three single-nucleon spectra, adjusted to $k_F = 1.35 \text{ F}^{-1}$ and $\epsilon_{\text{vol}} = 15.8 \text{ MeV}$. The parameters of the (i) spectrum are $A = -112.0 \text{ MeV}$, $\mathfrak{M}_N^* = 0.393 \mathfrak{M}_N$. The constant value of the (ii) spectrum is $e_N(m_N) = -54.3 \text{ MeV}$.

charge dependent. The parameters⁵² in Table I characterize the charge-independent components of these potentials used in the present work (compare with the remarks at the beginning of Sec. 2). These parameters differ from those characterizing the full Λ - p potentials of Ref. 39 (which contain the CSB component) shown in parentheses in Table I.

The DW potential is one of the old spin-independent potentials considered by Downs and Ware²¹ in their calculation of $B_\Lambda(\infty)$. We include this potential into the present work for the sake of a comparison with the independent-pair approximation discussed in the Appendix.

4. NUMERICAL PROCEDURE

The present calculations have been performed for $k_F = 1.35 \text{ F}^{-1}$, which corresponds to the spacing parameter $r_0 = 1.12 \text{ F}$. In fixing the single-nucleon spectrum (Sec. 2B) we have used the value $\epsilon_{\text{vol}} = 15.8 \text{ MeV}$ for the coefficient of the volume term in the semiempirical mass formula. The three energy spectra considered are shown in Fig. 1. As mentioned in Sec. 2B, the spectrum (i) has been used in the present calculations.

To compute Green functions we transform Eq. (2.22) into the form⁴⁵

$$G_m^l(r, r') = \frac{1}{2\pi^2} \int \frac{dp p^2 [\bar{Q}(m, p) - 1] j_l(pr) j_l(pr')}{z(m) - p^2/2\mu} + (\mu/2\pi) a h_2^{(1)}(iar_>) j_l(iar_<), \quad (4.1)$$

where $r_> = \max(r, r')$, $r_< = \min(r, r')$, and $a = [-2\mu z(m)]^{1/2}$. Notice that $z(m) < 0$ [see Eq. (2.16)]. The form (4.1) of G_m^l has the advantage that, because of the $[\bar{Q} - 1]$ factor, the integral in Eq. (4.1) extends over a finite range of p values, namely, $p \leq k_F + \mu k_F / \mathfrak{M}_\Lambda < 1.5 k_F$ [see Eq. (2.20)].

All integrations have been performed by means of Simpson's rule. The following meshes have been used:

(a) *Green functions* [Eq. (4.1)]:

$$p = 0.0(0.015)1.5k_F, \quad (4.2)$$

$$r, r' = r_C(0.05)r_C + 0.3(0.1)r_C + 1.1(0.2)r_C + 1.9(0.5)r_C + 3.9 \text{ F}. \quad (4.3)$$

(b) *Wave functions*: Eqs. (2.25) have been solved by iteration with the zero-order ansatz:

$${}^S R_l(m, r) = s_l(m, r). \quad (4.4)$$

The r mesh has been the same as in Eq. (4.3).

The sixth-order iteration has been found to be sufficiently accurate.

(c) $\langle {}^S K \rangle_A$ *matrices* [Eq. (2.30)]:

$$r = r_C(0.05)r_C + 0.3(0.1)r_C + 1.1(0.2)r_C + 1.9(0.5)r_C + 7.9 \text{ F}, \quad (4.5)$$

with

$${}^S R_l(m, r) = j_l(mr) \text{ for } r > r_C + 3.9 \text{ F}. \quad (4.6)$$

(d) V_Λ [Eq. (2.17)]:

$$m = 0.0(0.25)1.0\mu k_F / \mathfrak{M}_N. \quad (4.7)$$

It was found to be sufficient to calculate V_Λ twice: first with an input value of V_Λ of about -40 MeV , and next with an input value of V_Λ equal to the output value of the first run. The self-consistent value of V_Λ was then determined by a linear interpolation.

All the numerical calculations have been performed on the IBM7044 computer of the Centro di Calcolo dell' Università di Trieste.

5. RESULTS AND DISCUSSION

The results of our calculations are shown in Table II. We have restricted ourselves to calculating the contributions of the first three partial waves S , P , D to V_Λ , and our results show that there is no need to go beyond the D wave. The remarkable size of the P -state contributions is the result of the hard core, which acts predominantly in the S state and partly cancels the S -state contribution of the attractive part of $v_{\Lambda N}$. An example of typical behavior of the wave functions is presented in Fig. 2, showing the j_l , s_l , and the singlet 1R_l functions in the case of the potential H for $m_N = 0.25k_F$.

In the case of the potential H we have calculated

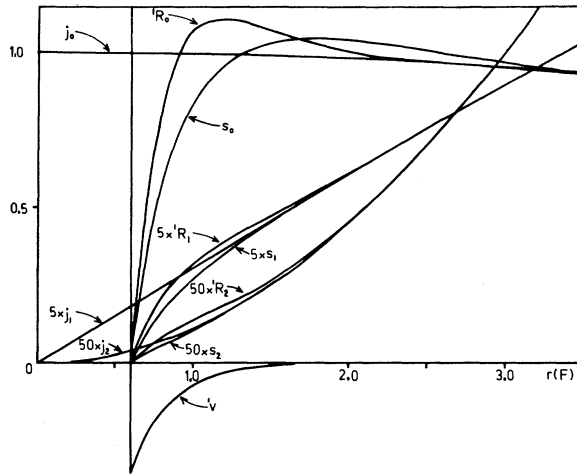


FIG. 2. The j_l , s_l and the singlet 1R_l wave functions in the case of the H potential for $m_N = 0.25k_F$ ($k_F = 1.35 \text{ F}^{-1}$). The H potential is also shown (the v curve, in an arbitrary scale).

V_Λ with the help of the two other single-nucleon spectra, (ii) and (iii), discussed in Sec. 2B (see Fig. 1). We have found a negligible difference of less than 0.1 MeV in the resulting values of V_Λ compared with the case of the spectrum (i).

Our method of calculating V_Λ is expected to contain all terms of order ρ ,^{34,42,43} and obviously the value of V_Λ is sensitive to the value of k_F . Let us consider a change of our value of $k_F = 1.35 \text{ F}^{-1}$ into the value of $k_F = 1.366 \text{ F}^{-1}$ applied in most of the existing calculations of V_Λ . This change in k_F by $\sim 1.2\%$ corresponds to an increase in ρ by $\sim 3.6\%$, and we would expect a corresponding increase in the depth of V_Λ . We have confirmed this conclusion by calculating V_Λ in the case of the DW potential with $k_F = 1.366 \text{ F}^{-1}$, with the result $V_\Lambda = -36.3 \text{ MeV}$ (see Table III).

Within the present scheme, the calculation of V_Λ requires no detailed information concerning nuclear matter. We only need the value of the density ρ and of the binding energy per nucleon, ϵ_{vol} [to fix $e_N(m_N)$]. However, to calculate the rearrangement potential V_R [Eq. (2.36)] we must know κ , determined by the nucleon-nucleon correlation volume in nuclear matter, $\int d\vec{r} |\chi|^2$, a quantity which depends on the nucleon-nucleon interaction. The nucleon-nucleon correlation volume has been calculated by several authors with results for κ between about 0.07 and 0.21 (see Ref. 48, where references to some of the earlier calculations may be found). The largest value, $\kappa \cong 0.21$, has been obtained in Ref. 48 with the hard-core Reid N - N potential. The main source for this large value is the large 3S contribution to κ , connected primarily with the large size of the hard core ($r_C = 0.51 \text{ F}$) of the Reid potential in the 3S state. A reduction of r_C to the usually accepted

value of about 0.4 F would reduce the 3S contribution to κ to the magnitude of the 1S contribution, and this would reduce the value of κ about twice. Also the value $\kappa = 0.15$ of Ref. 40 should probably be considered as an overestimate because the Bethe-Goldstone⁵⁴ function φ applied in Ref. 40 in the calculation of the nucleon-nucleon correlation volume heals slower than a more realistic function obtained with the type of single-nucleon energies shown in Fig. 1. On the other hand, the low estimates of $\kappa \cong 0.07$ (e.g. those of Ref. 37 and Köhler⁵⁵) neglect the contributions to κ of partial waves higher than S and probably are too low. We feel that at the moment the most reasonable estimate is

$$\kappa \cong 0.10. \quad (5.1)$$

This is a tentative estimate only, but our present knowledge of the N - N interaction does not allow us to make a more precise estimate. However, the error of this estimate does not seem to be larger than about ± 0.05 . The corresponding uncertainty in $B_\Lambda(\infty)$ would then be about $\pm 2 \text{ MeV}$. In the present work we have assumed for κ the value of 0.1. Equations (2.36) and (2.3) then lead to the values of $B_\Lambda(\infty)$ shown in Table II.

Results of other calculations are shown also in Table II. In principle, the calculation of Bodmer and Rote³⁸ is equivalent to our calculation. The essential difference is in the procedure of determining the K matrix, namely, an integrodifferential form of Eq. (2.13) is applied in Ref. 38. It is then encouraging to see that the agreement between the results of the two calculations is very good indeed. Actually, the differences in the results obtained in the two calculations for $-V_\Lambda$ are surprisingly small, especially when we consider the following three differences in the two calculations. (1) Bodmer and Rote calculate $-V_\Lambda$ for a few single-nucleon spectra, all of them being of the form (2.33). The results of Bodmer and Rote given in Table II have been obtained with that single-nucleon spectrum $e_N^{\text{BR}}(m_N)$, which is closest to our spectrum $e_N(m_N)$, in the sense that its average value, $\bar{e}_N^{\text{BR}} = -45.0 \text{ MeV}$, is closest to the average value of our spectrum, $\bar{e}_N = -54.3 \text{ MeV}$, determined from Eq. (2.35). [Let us note that inserting \bar{e}_N^{BR} into Eq. (2.35) leads to the binding energy per nucleon in nuclear matter, $\epsilon_{\text{vol}}^{\text{BR}} = 10.9 \text{ MeV}$, which is about 5 MeV smaller than the empirical value.] This shallower spectrum e_N^{BR} should produce a deeper single- Λ -particle potential than our spectrum e_N (e.g., in the case of the potential H we would expect an increase in $-V_\Lambda$ of about 1–2 MeV).

(2) There is the difference in the treatment of the hard core which in our calculation has been replaced by a hard shell. The exact treatment of the hard core by Bodmer and Rote is expected to di-

minish slightly their value of $-V_\Lambda$ compared with ours. Thus the two differences, (1) and (2), act in the opposite directions. (3) The value of $k_F = 1.366 \text{ F}^{-1}$ of Ref. 38 differs slightly from our value, $k_F = 1.35 \text{ F}^{-1}$. As we have noted, $-V_\Lambda$ is approximately proportional to $\rho \sim k_F^3$. We have assumed this proportionality in correcting the results of Ref. 38 for this difference in k_F , and consequently the Bodmer and Rote results of Table II are approximate.

The IPA results, similarly to the results of Ref. 38, do not contain the rearrangement energy and thus should be compared with our results for V_Λ . On the other hand, the Jastrow-method results should be compared directly with our results for $B_\Lambda(\infty)$. As seen from Table II, the Jastrow method values of $B_\Lambda(\infty)$ are systematically and quite appreciably larger than our results. So far, we do not know why the Jastrow method results for $B_\Lambda(\infty)$ are so large. Certainly, it would be very interesting to find out the reason. The independent-pair-approximation results for V_Λ are, in general, larger in absolute value than our results, although in the case of the HTS potential the situation is reversed. It seems to us that the independent-pair approximation is not reliable for reasons explained in the Appendix.

Before discussing our results, let us try to advocate the accuracy of our method of calculating $B_\Lambda(\infty)$. The general scheme of the contemporary Brueckner-type theory of an infinite system consists of grouping diagrams according to the num-

ber of hole lines.^{34,42,43} In the case of nuclear matter plus one Λ particle we always have one Λ hole line [at least in the calculation of $B_\Lambda(\infty)$], and in the successive steps of the scheme we increase the number of nucleon hole lines. Thus the small parameter of the whole approximation scheme is the same as in the case of pure nuclear matter, namely a quantity of order κ [Eq. (2.37)]. In the present calculation we have included all two-hole-line diagrams in V_Λ and the hole-self-energy class of the three-hole-line diagrams in V_R . The remaining three-hole-line diagrams form the class of the so-called three-body (ΛNN) diagrams. We assume that similarly as in the case of pure nuclear matter, the contribution of the three-body diagrams is small, of the order of 1 MeV.⁵⁶ This would mean that in our procedure all the important three-hole-line diagrams are included, and we would expect that by going one step further and considering four-hole-line diagrams we would get a correction to our values of $B_\Lambda(\infty)$ of the order of κ^2 , i.e., of the order of one percent.

There is no need to discuss our value of $B_\Lambda(\infty)$ obtained with the very old DW potential, included in the present work only for the sake of the discussion of the IPA (see the Appendix). The value of $B_\Lambda(\infty) = 60.1 \text{ MeV}$ obtained with the AGK potential seems to disfavor this potential as a realistic representation of the Λ - N interaction (compare also the remarks in Sec. 3). The value $B_\Lambda(\infty) = 36.9 \text{ MeV}$ calculated with the help of the HTS potential is reasonably close to the range of the empirically

TABLE II. The calculated values of $-V_\Lambda$ and $B_\Lambda(\infty)$. The results of other authors are also shown (IPA stands for the independent-pair approximation). All energies are in MeV.

$v_{\Lambda N}$	Partial-wave contributions to $-V_\Lambda$						$-V_\Lambda$	$B_\Lambda(\infty)$	Results of other authors ^a		
	¹ S	³ S	¹ P	³ P	¹ D	³ D			Bodmer and Rote ^b	IPA	Jastrow method
								$-V_\Lambda$	$-V_\Lambda$	$B_\Lambda(\infty)$	
HTS	14.4	17.4	2.9	5.9	0.1	0.2	41.0	36.9	36.3 ^c	45.9 ^d	
AGK	15.4	28.2	6.1	14.8	0.6	1.6	66.8	60.1			
ENX	12.6	25.5	5.4	14.0	0.4	1.1	59.0	53.1	59.6		
E	12.8	26.2	2.8	7.2	0.4	1.1	50.6	45.5	50.1	60.3–60.6 ^e	
E'NX	13.4	20.4	5.5	13.0	0.4	1.1	53.7	48.3	54.1		
E'	13.6	21.1	2.9	6.6	0.4	1.1	45.6	41.0	45.5	54.9–55.1 ^e	
F'NX	10.4	14.8	5.6	13.9	0.3	0.9	45.9	41.3			
F'	11.0	16.3	2.2	5.4	0.3	0.9	36.1	32.5			
HNX	9.5	20.5	6.1	16.9	0.4	1.2	54.7	49.2	54.8	69.1 ^f	
H	10.1	22.3	2.6	7.1	0.4	1.2	43.8	39.4	44.0	61.7–62.9 ^{e,f}	
DW	5.2	15.7	3.3	9.9	0.3	0.8	35.2	31.7	37.8 ^g		

^aCorrected for the difference in their (1.366 F^{-1}) and our (1.35 F^{-1}) value of k_F with the help of the factor $(1.35/1.366)^3$.

^bThe Bodmer and Rote (see Ref. 38) results for their first set of nuclear-matter parameters [in our notation, Eq. (2.33), $A = -81.4 \text{ MeV}$, $\mathfrak{N}_N^* = 0.638 \mathfrak{N}_N$].

^cSee Ref. 23.

^dSee Ref. 53.

^eSee Ref. 31.

^fSee Ref. 27.

^gSee Ref. 21.

determined values of $B_{\Lambda}(\infty)$, especially if we recall the uncertainty in the value of κ . However, the HTS potential, which is not fitted to the Λ - p scattering data, has been outdated by the newer potentials of Herndon and Tang.³⁹ As is seen from Table II, a large part of $B_{\Lambda}(\infty)$ results from the P -state Λ - N interaction. Consequently, the suppression of the interaction in the odd- l states, introduced in Ref. 39 to fit the Λ - p scattering, essentially reduces the values of $B_{\Lambda}(\infty)$. Among the potentials of Ref. 39 applied to the present work, one has to distinguish between the unprimed potentials E, H , determined by fitting the three- and four-body hypernuclear data, and the primed potentials E', F' , determined by fitting the three- and five-body hypernuclear data. Herndon and Tang³⁹ argue that the unprimed potentials form a more reliable representation of an effective central interaction in an isolated Λ - N system. Namely, the inclusion of the hypernucleus ${}_{\Lambda}\text{He}^5$ in the determination of the primed potentials may lead to an underestimate of the triplet interaction due to tensor and isospin suppression effects.^{57,58} However, these effects are expected to operate in the Λ + nuclear-matter system in much the same way as in ${}_{\Lambda}\text{He}^5$. Consequently, in a calculation of $B_{\Lambda}(\infty)$, the use of the primed potentials seems to be more meaningful. It is then encouraging to see that the F' potential gives $B_{\Lambda}(\infty) = 32.5$ MeV, which is in agreement with the empirical estimate [Eq. (1.1)]. We conclude then that a Λ - N central interaction with a sufficiently large hard core ($r_C = 0.6$ F), which reproduces the experimental binding energy of ${}_{\Lambda}\text{He}^5$, also reproduces the empirical value of $B_{\Lambda}(\infty)$, provided the interaction in odd-angular-momentum states is sufficiently suppressed ($y = 0.6$). Notice that the large size of the hard core is essential for our conclusion. The E' potential with $r_C = 0.45$ F, but otherwise similar to the F' potential, leads to too large a value; $B_{\Lambda}(\infty) = 41.0$ MeV. This is connected with the higher kinetic energies of the Λ - N system in nuclear matter compared with the case of ${}_{\Lambda}\text{He}^5$ (see Sec. 1). At these higher energies the hard core is more effective in reducing $B_{\Lambda}(\infty)$.

All the unprimed potentials of Ref. 39 lead to overbinding of a Λ particle in nuclear matter. The

H potential, considered by Herndon and Tang to be the best representation of the interaction in an isolated Λ - N system, gives $B_{\Lambda}(\infty) = 39.4$ MeV. And the problem obviously remains of how to reconcile the Λ - N interaction in an isolated system (in particular the Λ - p scattering) with properties of systems such as ${}_{\Lambda}\text{He}^5$ and Λ + nuclear matter. A reduction in the calculated value of $B_{\Lambda}(\infty)$ by a few MeV could be achieved with a larger value of κ and a smaller value of y , which determines the odd-state suppression (neither κ nor y is known precisely). Let us mention some other possibilities of reducing the value of $B_{\Lambda}(\infty)$:

Tensor forces. In the case of pure nuclear forces it is well known that tensor forces effective in an isolated N - N system are much less effective, i.e., suppressed in nuclear matter. In principle, a similar situation might be expected in the Λ - N case (see, however, Ref. 38).⁵⁹

Three-body ΛNN interaction. No doubt, there are theoretical reasons to expect the existence of an appreciable ΛNN interaction, and several authors have investigated its possible effects in hypernuclei.^{19,60} Obviously, with a repulsive ΛNN force, one should be able to reconcile the Λ - p scattering data with the hypernuclear data. The unpleasant thing about introducing ΛNN forces $v_{\Lambda NN}$ is that with the present amount of hypernuclear data, with the present possibilities of solving the hypernuclear few- and many-body problem, and with the present possibilities of deriving theoretically $v_{\Lambda NN}$, the task of determining $v_{\Lambda NN}$ seems to be extremely difficult.

Isospin suppression effect. This effect, pointed out by Bodmer,⁵⁸ would lead to the conclusion that the whole idea of treating hypernuclei as systems of nucleons and a Λ particle with the same $v_{\Lambda N}$ as in an isolated ΛN system is wrong. Instead, one should use an effective Λ - N interaction, which would depend on the hypernucleus considered. If this turns out to be the case, as it very well might be, any attempt to correlate hypernuclear data would become most difficult.

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The authors express their gratitude to Professor

TABLE III. Values of V_{Λ} (in MeV) calculated with the two types of single-particle spectra.

l	e_N, e_N [Eq. (A.3)]				e_N, e_{Λ} [Sec. 2]		
	Hard core	IPA	First order in v_A	Exact	Hard core	First order in v_A	Exact
0	52.5	-22.7	- 6.0	-17.1	64.1	- 12.8	- 21.2
1	2.6	-15.5	-14.5	-17.0	1.9	-12.7	-14.0
2	0.0	- 1.7	- 1.7	- 1.8	0.0	- 1.1	- 1.1
Total	55.2	-39.9	-22.1	-35.9	66.0	-26.6	-36.3

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APPENDIX

In an early paper by Gomes, Walecka, and Weisskopf,⁶¹ an approximation of the Brueckner theory has been applied. The essence of this approximation is the division of v into the hard-core part v_C and the attractive⁶² part v_A , an exact treatment of v_C , and a first-order approximation in v_A . In our case of $B_\Lambda(\infty)$, the approximation amounts to the replacement of the exact Eq. (2.30) by the approximate equation

$$\begin{aligned} \langle \vec{m} | K | \vec{m} \rangle_A &\cong \int d\vec{r} \Psi_m^C(\vec{r}) v_A(r) \Psi_m^C(\vec{r}), \\ &= 4\pi \sum_l (2l+1) \int_{r_C}^{\infty} dr r^2 s_l(m, r) v_A(r) s_l(m, r). \end{aligned} \quad (\text{A.1})$$

In almost all published Brueckner-type calculations of $B_\Lambda(\infty)$, a different approximation of Eq. (2.30) has been applied, namely

$$\begin{aligned} \langle \vec{m} | K | \vec{m} \rangle_A &\cong \int d\vec{r} e^{-i\vec{m} \cdot \vec{r}} v_A(r) \Psi_m^C(\vec{r}), \\ &= 4\pi \sum_l (2l+1) \int_{r_C}^{\infty} dr r^2 j_l(mr) v_A(r) s_l(m, r). \end{aligned} \quad (\text{A.2})$$

In other words, Ψ_m has been approximated in Eq. (2.18) by Ψ_m^C . This approximation is usually called the independent-pair approximation (IPA). In contradistinction to Eq. (A.1) the IPA is not a full first-order approximation⁶³ in v_A . This in itself is not a disadvantage of the IPA because a first-order approximation in v_A is not expected to be a good one. The trouble is, however, that the precise meaning of the IPA is not clear.

The most detailed presentation of the IPA has been given by Downs and Ware.²¹ To discuss the IPA more fully, let us investigate one of the Λ - N potentials considered by Downs and Ware, namely, the potential DW of Table I. To compare our results for the DW potential with those of Ref. 21, we shall use here the value of $k_F = 1.366 \text{ F}^{-1}$ and the single-particle energies appearing in the K -matrix equation (2.5) in the form

$$\begin{aligned} e_\Lambda(p_\Lambda) &= p_\Lambda^2 / 2\mathfrak{M}_\Lambda + C_\Lambda, \\ e_N(p_N) &= p_N^2 / 2\mathfrak{M}_N^* + C_N, \end{aligned} \quad (\text{A.3})$$

where the effective nucleon mass $\mathfrak{M}_N^* = 0.735\mathfrak{M}_N$,

and C_Λ, C_N are constants [they are canceled in the expression for the energy denominator of Eq. (2.5)]. Needless to say, this form for the single-particle energies, valid for all momenta, is difficult to justify from the viewpoint of the present state of the nuclear-matter theory.

We shall not discuss here how the wave function Ψ_m^C has been approximated in Ref. 21, because the difference between our function Ψ_m^C , calculated with the single-particle spectra (A.3), and those of Ref. 21 turns out to be of no practical importance.⁶⁴

Our results for V_Λ are shown in Table III. Besides the results obtained with the single-particle energy spectra of Eq. (A.3), Table III also contains the self-consistent results obtained with our original single-particle spectra of Sec. 2, adjusted to the value of $k_F = 1.366 \text{ F}^{-1}$. The "exact" results have been obtained by solving the wave-function equation (2.24), whereas the IPA and the first-order results in v_A have been obtained by applying the approximation (A.2) and (A.1), respectively. The small differences between our IPA results and those of Ref. 21 may be traced back to some additional approximations applied by Downs and Ware [$Q(\vec{M}, \vec{p}) \cong Q(0, \vec{p})$ and approximations for the Bethe-Goldstone function, in particular in the P state].

The difference between the exact results obtained for the total V_Λ with the two types of single-particle energies (-35.9 vs -36.3 MeV) is surprisingly small. However, the contributions of separate partial waves, as well as the pure hard-core contributions and the first-order results in v_A , do differ in the two cases.⁶⁵ The full first-order results in v_A are systematically much smaller than the exact results.

Our main purpose here is to compare the IPA results with the exact results, obtained with the help of the single-particle spectrum (A.3) (from now on we restrict ourselves to this spectrum). We have $V_\Lambda(\text{exact}) = -35.9$ MeV and $V_\Lambda(\text{IPA}) = -39.9$ MeV. The sizable differences in the contributions of separate partial waves are partially canceled, and we are left with a difference of 4 MeV. Thus the magnitude of this difference is accidental and depends on the form of $v_{\Lambda N}$ (see Table II). Still, we may ask why the difference is so small. To answer this question let us notice that the IPA amounts to replacing the R_j functions under the r integral of Eq. (2.30) by the j_l functions. Now, j_l is larger than R_l close to the hard core and smaller than R_l at larger distances but still within the range of v (compare with Fig. 2). Thus the net effect of the replacement is partly canceled in the course of the r integration. On the other hand, in the first-order approximation in v_A we replace R_l by s_l , which is smaller than R_l within the range of v , and thus we get $|V_\Lambda(\text{first order})| \ll |V_\Lambda(\text{exact})|$.

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Perturbation Effects and Morse Function Based on the Velocity-Dependent Potential*

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A velocity-dependent nucleon-nuclear potential approximated to the well-known Morse function has been employed to generate the single-particle wave functions and eigenvalues for all nuclei. A general method of treating the perturbation is prescribed which gives a very good agreement with the data on separation energies and with other theoretical work.

1. INTRODUCTION

The velocity-dependent potentials have appeared in the literature pertaining to the nuclear and particle physics. Most of the workers while dealing with velocity-dependent potentials have employed numerical techniques in solving the Schrödinger equation. Recently, in a previous paper¹ (to be designated by I from now on), a method of treating a velocity-dependent potential analytically was presented. It provided a simple technique for producing single-particle energy levels based on the velocity-dependent nucleon-nuclear potential. The nucleon-nuclear potential studied in I is based on a relatively realistic nucleon-nucleon interaction which is replaced by an effective nuclear potential. A simple prescription for approximating this so-called effective potential by an analytically solvable potential was given. The analytically solvable potential used for that purpose was of the well-known Morse-function type. The Morse parameters were expressed as functions of A and l explicitly in semiempirical formulas. As an application of this method, neutron energy states for nuclides with $2 \leq A^{1/3} \leq 7$ were obtained and are in general agreement with other works. At the outset of I, the primary concern was with the application of the method developed there and to discuss the results qualitatively. Accordingly, many

important terms, such as spin-orbit coupling, etc., which must be considered in predicting the nuclear properties, were ignored. The purpose of the present work is to inquire whether the model discussed in I would still be consistent when extended calculations are performed. In particular, the spin-orbit effect has been treated as a perturbation to the potential used in I, where the results were very encouraging. The splitting of levels thus obtained was reported in another paper² (hereafter referred to as II), which, in general, is consistent with experiments. In this work it is desired to develop a generalized technique for modifying the four parameters of the Morse-function-type potential in response to an arbitrary perturbation. The technique developed here works quite well if the perturbation is not outsized compared to the original Morse potential. The general formulation of the scheme is undertaken in the following section. Some particular forms of perturbation, such as the spin-orbit effect, are discussed in Sec. 3 and applied to several finite nuclei.

II. EFFECTIVE VELOCITY-DEPENDENT POTENTIAL

In I an effective potential equivalent to a realistic velocity-dependent nucleon-nuclear potential was used in the independent-particle model (IPM)