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Analysis of the Distribution of the Spacings Between Nuclear Energy Levels*

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An empirical spacing distribution is always based on a finite, and usually small, number of observed levels. Thus, even if the spacing of levels were described exactly by the random-matrix model, the observed distribution would necessarily fluctuate about the theoretical mean – the Gaudin-Mehta distribution. A statistic $\Lambda(n)$ is defined to enable one to judge whether the magnitude of the observed fluctuations about the Gaudin-Mehta distribution is compatible with the random-matrix model. It is found that the correlations between the spacings implied by the model tend to reduce the expected fluctuations significantly. The statistical properties of $\Lambda(n)$ are studied by means of a Monte Carlo calculation with matrices of order 100 sampled from the Gaussian orthogonal ensemble. An illustrative analysis of the published neutron resonances observed in U^{239} by Garg *et al.*, reveals no obvious discrepancy between theory and experiment up to neutron kinetic energies of about 2 keV.

INTRODUCTION

Although the statistical theory of energy levels based on Wigner's Gaussian orthogonal ensemble of real symmetric matrices of high dimensionality[1-4] seems to be in good qualitative agreement with the experimental data obtained in slow-neutron resonance spectroscopy, the attempts at truly quantitative tests of the statistical model have been rather limited. We consider it important to continue the investigation of statistics aimed at the detection of deviations from the model, a study which was initiated by Dyson and Mehta.[5] Small deviations from the standard theory could arise for many reasons. One possibility that has recently been considered is the violation of time-reversal invariance of nuclear interactions.[6]

One of the statistical properties of an energy-level series most commonly studied experimentally is the probability distribution of the spacings between successive energy levels of the same spin and parity in a highly excited nucleus. The "observed" probability distribution, usually presented in the form of a histogram,[7] is inevitably based on a rather limited number of observed energy levels. Thus, even if the theoretical model were exact, the measured distribution would necessarily deviate from the theoretical Gaudin-Mehta[8] distribution. In this work we will investigate the question of whether or not the fluctuations of an empirical spacing distribution about the mean distribution have a magnitude in the range expected on the basis of the random-matrix model. The question is answered in terms of a statistic, denoted by $\Lambda(n)$, which represents a particular measure of the deviation between an observed distribution derived from n successive experimental spacings and the Gaudin-Mehta distribution. The following paragraphs are preliminary to the definition of $\Lambda(n)$ in Eq. (1.6).

Let $s_1, s_2, \ldots, s_n$ denote the values (usually expressed in eV) of successive spacings between $n+1$ adjacent energy levels of the same spin and parity (a "single" series). Let D denote the average value of the spacings. If n is sufficiently large, the arithmetic mean will provide a sufficiently precise estimate of D.[5] As is well known, the value of D depends very much on the structural details of the particular nucleus, on the excitation energy, on the value of the spin of the series, etc. The present discussion is concerned with the values of the normalized spacings defined by

$$t_i = s_i/D, \qquad i = 1, 2, \ldots, n. \tag{1.1}$$

It is a basic assumption that the division by D largely removes the dependence on energy and on the details of the structure, provided attention is focused on an energy interval within which the value of D remains virtually constant. In a heavy nucleus, such intervals may nevertheless contain hundreds and even thousands of energy levels. The fluctuations reflected by the dimensionless numbers t_i in Eq. (1.1) then follow statistical laws which are believed to be the same or nearly the same for many nuclei.

Let $F^*(x; t_1, t_2, \ldots, t_n)$, sometimes abbreviated as $F^*(x, n)$, denote the cumulative distribution inferred from n successive normalized spacings. It is defined by the relations

$$F^*(x; t_1, t_2, \ldots, t_n) = \frac{1}{n}\sum_{i=1}^{n} \theta(x - t_i), \tag{1.1a}$$

$$\begin{aligned}\theta(x-t) &= 0 \quad \text{if} \quad x - t < 0, \\ &= 1 \quad \text{if} \quad x - t \geq 0.\end{aligned} \tag{1.1b}$$

Wigner suggested that $F^*(x, n)$ should be regarded as a random variable in the Gaussian orthogonal ensemble E_G of real symmetric matrices H having (high) dimensionality N and being defined by the probability density

$$P(H) \propto \exp[-\mathrm{tr}(H^2/4\tau^2)], \tag{1.2a}$$

with respect to the volume element $d[H]$ defined by

$$d[H] \equiv \prod_{1 \le i \le j \le N} dH_{ij}. \tag{1.2b}$$

For sufficiently large values of the dimensionality N, the over-all density of eigenvalues follows Wigner's "semicircle" law.[9] The semicircle law is qualitatively quite different from the roughly exponential energy dependence of the nuclear level density. However, this fact does not affect the applicability of the random-matrix model to energy intervals over which the density of levels remains constant.[10] What is done is to identify a small region of the semicircle with the energy interval that is subjected to experimental investigation. It has been found that the flat central region of the semicircle is particularly convenient for this purpose.

Let $P(t_1, t_2, \ldots, t_n)$ denote the probability density of n successive spacings at the center of the semicircle. The expectation value of $F^*(x, n)$, denoted by $F(x)$, is defined by the relation

$$F(x) \equiv \int_0^\infty \cdots \int_0^\infty P(t_1, t_2, \ldots, t_n) F^*(x; n) \prod_{1 \le i \le n} dt_i. \tag{1.3}$$

It would be a natural requirement that $F(x)$ defined by (1.3) should be independent of n, and this "translational invariance" fortunately holds at the center of the semicircle. The form of $F(x)$ and its derivative $P(x)$ implied by E_G at the center of the semicircle was derived by Gaudin and Mehta.[8] A simple, but very good, approximation to the exact but rather complicated functions is given by the Wigner distribution

$$P(x) \approx P_W(x) = \tfrac{1}{2}\pi e^{-(1/4)\pi x^2},$$
$$F(x) \approx F_W(x) = 1 - e^{-(1/4)\pi x^2}. \tag{1.4}$$

Wigner surmised that for heavy nuclei at high excitation, and for a sufficiently large number n, the empirical spacing distribution would approach the mean, i.e., that

$$F^*(x, t_1, t_2, \ldots, t_n) \to F(x). \tag{1.5}$$

We now finally define the statistic which will serve as a measure of the inevitable deviation of F^* from F. The statistic $\Lambda(t_1, t_2, \ldots, t_n)$, often abbreviated $\Lambda(n)$, is defined by the relation[11]

$$\Lambda(t_1, t_2, \ldots, t_n) \equiv n \int_0^\infty [F^*(x, n) - F(x)]^2 dx. \tag{1.6}$$

Evidently the number $\Lambda(n)$ can be calculated for a given set of spacings provided the mean spacing D can also be estimated. From what has already been said, it is clear that $\Lambda(n)$ may also be regarded as a random variable whose distribution is, in principle, determined by the joint probability density $P(t_1, t_2, \ldots, t_n)$ at the center of the semicircle in the Gaussian orthogonal ensemble E_G.

In this work we shall obtain some information about the statistical properties of $\Lambda(n)$ in E_G by means of a Monte Carlo calculation with matrices of order 100. In Sec. II we study the n dependence of the mean and the standard deviation of $\Lambda(n)$; it will be seen that the correlation between spacings has the effect of substantially reducing the values of both.

In Sec. III, the values of $\Lambda(n)$ computed for the neutron-capture levels observed in U^{239} will be compared with the values of $\Lambda(n)$ expected from the statistical model. Concluding remarks are contained in Sec. IV.

II. STATISTICAL PROPERTIES OF Λ (n) IN THE GAUSSIAN ORTHOGONAL ENSEMBLE

A. Introduction

The precise functional form of the probability distribution of n successive spacings and consequently also of the probability distribution of the random variable $\Lambda(n)$ in the Gaussian orthogonal ensemble is not known. The information reported here concerning the distribution of $\Lambda(n)$ is based on our Monte Carlo calculation with a set of 180 real symmetric matrices of order 100, chosen at random from the ensemble E_G defined by Eqs. (1.2). All the calculations were carried out on the CDC-3600 digital computer located at this laboratory.

Since the machine calculations must be done with matrices of finite – in fact, rather low – dimensionality, it is necessary to take account of the variation in eigenvalue density if one wishes to utilize a substantial fraction of the eigenvalues available from the matrix diagonalizations. As in the past,[12] Wigner's semicircle law was used to make this correction. Let $\lambda_1^{(i)} \le \lambda_2^{(i)} \le \cdots \le \lambda_N^{(i)}$ denote the ordered eigenvalues belonging to the ith random matrix ($i = 1, 2, \ldots, T$) drawn from E_G. The density of the "reduced" eigenvalues, given by

$$\mu_j^{(i)} = \lambda_j^{(i)}/(2\tau N^{1/2}), \quad j = 1, 2, \ldots, N, \tag{2.1}$$

follows the semicircle law approximately. The mth *central* spacing, corrected by means of the semicircle law, is defined by the expression

$$s^{(l)}(m)=[\,|1-\tfrac{1}{4}(\mu_{\frac{1}{2}N-m+1}+\mu_{\frac{1}{2}N-m})^2|\,]^{1/2}\times[\mu_{\frac{1}{2}N-m+1}-\mu_{\frac{1}{2}N-m}]. \quad (2.2)$$

For example, the set of n central spacings [on the basis of which $\Lambda(n)$ will be computed] was obtained by letting $m=0, \pm1, \pm2, \ldots, \pm\frac{1}{2}(n-1)$, where n is an odd number. The mean value $D(n)$ of the spacings was estimated by computing the arithmetic mean for the entire set of T matrices, i.e.,

$$D(n)=\frac{1}{nT}\sum_{l=1}^{T}\sum_{m=0}^{\pm(n-1)}s^{(l)}(m)\,. \quad (2.3)$$

Next, the set of n *normalized* spacings

$$t^{(l)}(m)=s^{(l)}(m)/D, \quad m=0, \pm1, \pm2, \ldots, \pm\tfrac{1}{2}(n-1), \quad (2.4)$$

was calculated for each of the random matrices. Finally, one value of $\Lambda(n)$ was computed by numerical integration of Eq. (1.6) for each of the $T=180$ random matrices, and the statistical properties of $\Lambda(n)$ were estimated on the basis of the resulting sample of 180 values of $\Lambda(n)$.

B. Dependence of $\langle\Lambda(n)\rangle$ on n; The Effect of Correlations

Next we will discuss the value of the mean $\langle\Lambda(n)\rangle$ in the Gaussian-orthogonal ensemble. In Appendix A we derive an expression for $\langle\Lambda(n)\rangle$ by using only the translational invariance[13] of the energy level statistics. The result is

$$\langle\Lambda(n)\rangle=\langle\Lambda(1)\rangle+\frac{2}{n}\sum_{k=1}^{n-1}(n-k)I(k), \quad (2.5)$$

where

$$\langle\Lambda(1)\rangle=\int_0^\infty[F(x)-F^2(x)]\,dx, \quad (2.6)$$

and

$$I(k)\equiv\int_0^\infty dx\int_0^x\int_0^x[P_{1,1+k}(s,t)-P(s)P(t)]\,ds\,dt. \quad (2.7)$$

In the above, $P(s)$ and $F(x)$ denote, respectively, the probability density and cumulative distribution for a single spacing, and $P_{1,1+k}(t_1,t_{1+k})$ denotes the joint probability density of the two spacings t_1 and t_{1+k}. It will be recalled that the $t_i(i=1,2,\ldots,n)$ denote values of *successive* spacings. The functional form is known for only one of these joint probability densities, namely for $P_{1,2}(s,t)$, the density for two *adjacent* spacings.[4] The integrals $I(k)$ are particularly interesting quantities because they provide a measure of the correlations between the spacings.

The expression (2.5) consists of two parts. The first part, $\langle\Lambda(1)\rangle$, is independent of n and of correlations and may be evaluated (approximately) with Wigner's formula (1.5), the result being

$$\langle\Lambda(1)\rangle\approx\int_0^\infty[F_W(x)-F_W^2(x)]\,dx=1-\tfrac{1}{2}\sqrt{2}=0.2929. \quad (2.8)$$

The value 0.2929 – plotted as the line (b) in Fig. 1 – would be the value of $\langle\Lambda(n)\rangle$ if the spacings were statistically independent, i.e., if for all $k\geq1$,

$$P_{1,1+k}(s,t)=P(s)P(t). \quad (2.9)$$

However, it is known[14] that the relation of independence (2.9) certainly does not hold for nearby spacings, although it could conceivably become true for sufficiently large n, i.e., for distant spacings.

The second part in expression (2.5), consisting of the sum over k, arises from the correlation between the spacings; and we will find that this gives a significant contribution to $\langle\Lambda(n)\rangle$. The solid circles along curve (a) of Fig. 1 represent the mean values of $\langle\Lambda(n)\rangle$, based on the 180 random matrices, as a function of n; and these val-

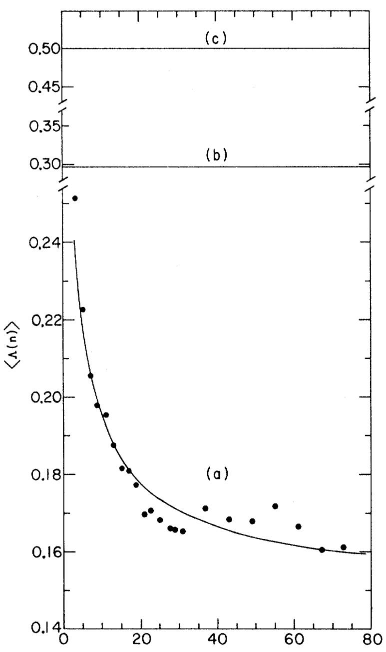

Fig. 1. Plots of $\langle\Lambda(n)\rangle$ versus n on the basis of three assumptions. (a) The statistics of n successive spacings are governed by the random-matrix model E_G; the solid circles represent values of $\langle\Lambda(n)\rangle$ estimated by means of a Monte Carlo calculation; the solid curve is a plot of the function (2.15). (b) The n spacings are statistically independent and follow the Wigner distribution (1.4). (c) The n spacings are statistically independent and follow the exponential distribution. Of the three assumptions, (a) is by far the most plausible for the spectrum of a heavy nucleus on both theoretical and empirical grounds, as discussed in Sec. III.

ues include, of course, the effects of the correlations between the spacings. The following points should be noted. (1) The correlations have the effect of making $\langle\Lambda(n)\rangle$ substantially less than the "uncorrelated" value 0.2929; for $n \gtrsim 20$, it is reduced to about half this value. (2) The values of $\langle\Lambda(n)\rangle$ decrease with increasing values of n. (3) For $n \gtrsim 25$, $\langle\Lambda(n)\rangle$ seems to approach an asymptotic value of about 0.16. We cannot say whether the variation of $\langle\Lambda(n)\rangle$ for values of n between 25 and 70 is real or is only a fluctuation resulting from the small size of the sample (180 matrices). In this connection it should be noted that successive points are not statistically independent. For example, the same 43 spacings are used in the estimates of both $\langle\Lambda(43)\rangle$ and $\langle\Lambda(49)\rangle$.

The system of linear equations (2.5) may be inverted so as to express the interesting integrals $I(n)$ as linear combinations of the mean values $\langle\Lambda(n)\rangle$. The results are

$$I(1) = -\langle\Lambda(1)\rangle + \langle\Lambda(2)\rangle \tag{2.10}$$

and for $n \geq 2$,

$$I(n) = \tfrac{1}{2}(n-1)\langle\Lambda(n-1)\rangle - n\langle\Lambda(n)\rangle + \tfrac{1}{2}(n+1)\langle\Lambda(n+1)\rangle. \tag{2.11}$$

Formula (2.10) is actually a special case of (2.11) provided we adopt the interpretation that for $n=1$,

$$\tfrac{1}{2}(n-1)\langle\Lambda(n-1)\rangle = 0. \tag{2.12}$$

Our Monte Carlo calculation yielded values of $\langle\Lambda(n)\rangle$ as a function of n, and we are therefore in a position to estimate the values of $I(n)$ by means of the relation (2.11). The accuracy of these estimates is very much limited by the fact that (2.11) expresses $I(n)$ as a (small) difference between two large numbers, and meaningful results can be obtained only for small values of n. Our estimated values of $I(n)$ for $n=1$, 2, and 3, are listed in Table I along with the values of $\langle\Lambda(n)\rangle$ which were used in the computation. The value of $\langle\Lambda(1)\rangle$ is known virtually exactly from the random-matrix model and is given by formula (2.8). The values of $\langle\Lambda(3)\rangle$ and $\langle\Lambda(5)\rangle$ were obtained from the Monte Carlo data. Unfortunately, we did not compute, at the time when it would have been convenient to do so, the values of $\langle\Lambda(n)\rangle$ for even values of n. However, we obtained an accurate value of $\langle\Lambda(2)\rangle$ with the help of Mehta's[4] work on the joint probability density $P_{1,2}(s,t)$. A numerical integration on the basis of Mehta's tabulated function yielded the value $I(1) \approx -0.025$ which is entered in our Table I. The value of $\langle\Lambda(2)\rangle$ computed from formula (2.10) was found to lie on a smooth curve connecting the values of $\langle\Lambda(1)\rangle$, $\langle\Lambda(3)\rangle$, and $\langle\Lambda(5)\rangle$. Finally, the value of $\langle\Lambda(4)\rangle$ was obtained by a rough interpolation.

The limited results entered in Table I suggest, but certainly do not prove, that $I(n)$ may be negative for all n and decreases in absolute value with increasing n.

Inspection of Fig. 1 suggests that $\langle\Lambda(n)\rangle$ may be a relatively smooth function of n. If we treat n as a continuous variable and approximate differences by derivatives, the relation (2.11) may be written in the form

$$I(n) \approx \frac{1}{2}\frac{d^2}{dn^2}[n\langle\Lambda(n)\rangle]. \tag{2.13}$$

From the form of (2.13) we can draw the conclusion that if $\langle\Lambda(n)\rangle$ is proportional to $n^{-\alpha+1}$, then $I(n)$ follows the power law $n^{-\alpha}$ and has the same sign as $(1-\alpha)(2-\alpha)$ for all values of n.

We tried to reproduce the variation of $\langle\Lambda(n)\rangle$ with n, indicated by the solid circles of Fig. 1(a), by postulating that

$$I(k) = -\beta k^{-\alpha}, \quad k = 1, 2, \ldots. \tag{2.14}$$

The sum over k which occurs in (2.5) was approximately evaluated on the basis of assumption (2.14) by retaining the first three terms in the Euler-MacLaurin summation formula, and this led to the expression

$$\langle\Lambda(n)\rangle \approx 0.2929 - 2\beta\left[\frac{(n-1)^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} + \frac{1}{2} + \frac{1}{2}(n-1)^{-\alpha} - \frac{1}{n}\frac{(n-1)^{2-\alpha}}{2-\alpha} + \left(\frac{1}{n}\right)\frac{1}{2-\alpha} - \frac{1}{2n} - \frac{(n-1)^{1-\alpha}}{2n}\right]. \tag{2.15}$$

The values of the parameters $\alpha = 1.74$ and $\beta = 0.0396$ were determined by a least-squares fit of (2.15) to the values of $\langle\Lambda(n)\rangle$ represented by the solid circles in Fig. 1(a). The solid curve is a plot of the function (2.15) evaluated on the basis of the minimizing parameter values.

Assumption (2.14) would yield the value $I(1) = -0.0396$, whereas the correct value, obtained from Mehta's work, is close to -0.025. The disagreement is not surprising, since there is no reason to expect relations (2.14) and (2.15) to be good approximations for small values of n.

TABLE I. Values of $I(n)$ deduced by means of relation (2.10).

n	$\langle\Lambda(n)\rangle$	$I(n)$
1	0.2929	-0.025
2	0.2679	-0.013
3	0.2514	-0.009
4	0.238	...

Occasionally, the simple form $P(x)=e^{-x}$ has been considered. The probability distribution of spacings approaches an exponential when a large number of level systems having different spins are randomly superposed.[15] In this case successive spacings are not correlated and $\langle\Lambda(n)\rangle=\frac{1}{2}$. For comparison, this value is also plotted as line (c) in Fig. 1.

C. Dependence of the Standard Deviation of $\Lambda(n)$ on n; Effects of Correlations

In this part we will discuss the standard deviation of the random variable $\Lambda(n)$ much as its mean value was treated in the preceding part. The standard deviation is defined as

$$\sigma[\Lambda(n)] \equiv [\langle\Lambda^2(n)\rangle - \langle\Lambda(n)\rangle^2]^{1/2}. \qquad (2.16)$$

The dependence of σ on n was estimated from the random sample of 180 matrices. The solid circles in Fig. 2(a) represent the values of σ obtained in this way, and they include, of course, the correlations implied by the matrix ensemble. It will be noted that the values of σ decrease rapidly with increasing values of n, and σ seemingly approaches an asymptotic value of about 0.1 for $n \gtrsim 15$. As in the case of the mean value $\langle\Lambda(n)\rangle$, we observe a variation in the values of σ between $n\approx 25$ and $n=70$; and again we cannot decide whether this variation is real or results from fluctuations due to the finite size of the sample. Again we must emphasize that the values of σ were not obtained from statistically independent samples.

In Appendix B it is shown that if the spacings $t_i (i=1, 2, \ldots, n)$ *were* statistically independent, then the standard deviation $\sigma[\Lambda(n)]$ would be given by the expression

$$\sigma^2[\Lambda(n)] = 2A + B/n, \qquad (2.17)$$

where

$$A \equiv \int_0^\infty\int_0^\infty [F(x)F(y) - F(z)]^2 dx dy, \qquad (2.18)$$

$$B \equiv \int_0^\infty\int_0^\infty [F(x)F(y) - F(z)][4F(x) + 2F(z) - 6F(x)F(y) - 1]dx dy, \qquad (2.19)$$

and

$$z = x \text{ if } x < y, \quad = y \text{ if } x \geqslant y. \qquad (2.20)$$

As before, $F(x)$ denotes the cumulative distribution of a single spacing.

If we adopt the approximate form (1.5) for $F(x)$ which, aside from very small corrections, follows from E_G, the independence assumption enables one to obtain

$$A = \tfrac{1}{2} - \sqrt{2} + (4\sqrt{2}/\pi)\tan^{-1}\sqrt{2} + 2/\pi = 0.028367, \qquad (2.21)$$

$$B = 10\sqrt{2} - 6 - 4/\pi - (16\sqrt{2}/\pi)\tan^{-1}\sqrt{2} = -0.011801, \qquad (2.22)$$

and

$$\sigma[\Lambda(n)] \approx 0.2382[1 - 0.1040/n]. \qquad (2.23)$$

The integrations leading to the above results were carried out with the help of the formula[16]

$$\int_0^\infty \phi(cx)e^{-\alpha x^2}dx = (\pi\alpha)^{-1/2}\tan^{-1}(c\alpha^{-1/2}), \quad \phi(z) \equiv (2/\pi^{1/2})\int_0^z e^{-t^2}dt. \qquad (2.24)$$

The values of $\sigma[\Lambda(n)]$ from Eq. (2.23) are plotted as a function of n as curve (b) in Fig. 2. Comparison of (a) and (b) of Fig. 2 lead to the conclusion that the correlation between spacings cuts the standard deviation $\sigma[\Lambda(n)]$ about in half.

For comparison we have also calculated $\sigma[\Lambda(n)]$ on the basis of the assumptions (1) that $P(x)=e^{-x}$, $F(x)=1-e^{-x}$ and (2) that the spacings are statistically independent. This yields

$$A = \tfrac{1}{12}, \quad B = \tfrac{1}{6}, \quad \sigma[\Lambda(n)] = 0.4082(1 + 0.5/n). \qquad (2.25)$$

The rather large standard deviation is plotted as curve (c) in Fig. 2.

Finally, in Table II we have listed the asymptotic values ($n\to\infty$) of the mean $\langle\Lambda\rangle$, the standard deviation $\sigma[\Lambda]$, and the "figure of merit" $\phi[\Lambda] \equiv \sigma^2[\Lambda]/\langle\Lambda\rangle^2$. Each has been calculated under three alternative assumptions: (a) a Gaussian orthogonal ensemble, (b) statistically independent spacings fol-

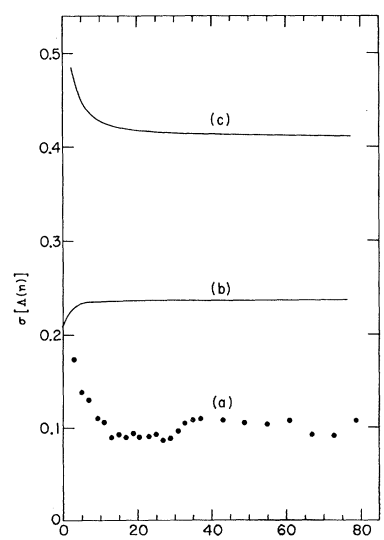

Fig. 2. Plots of the standard deviations $\sigma[\Lambda(n)]$ vs n on the basis of the three assumptions that led to curves (a), (b), and (c) of Fig. 2.

lowing the Wigner distribution, and (c) statistically independent spacings following the exponential distribution.

D. Probability Distribution of Λ (n)

In Secs. IIB and IIC we described estimates of the mean and the standard deviations of $\Lambda(n)$ as a function of n on the basis of the random sample of 180 matrices. It is of course possible to estimate the values of still higher moments, and one may even obtain a rough estimate of $P[\Lambda(n)]$, the probability distribution of $\Lambda(n)$. As an example, $P[\Lambda(55)]$ is represented as a histogram in Fig. 3. Inspection of Fig. 3 suggests that $P[\Lambda]$ may be approximated by a χ^2 distribution, namely

$$P[\Lambda]=\left(\frac{k}{2^{(1/2)k}\Gamma(\frac{1}{2}k)\langle\Lambda\rangle}\right)\left(\frac{k\Lambda}{\langle\Lambda\rangle}\right)^{(1/2)k-1} e^{-(1/2)k\Lambda/\langle\Lambda\rangle}. \tag{2.26}$$

The values of the two independent parameters k and $\langle\Lambda\rangle$, which are the number of degrees of freedom and the mean value, respectively, were varied until we obtained a least-squares fit of the cumulative distribution determined by the 180 values of $\Lambda(55)$. In this way we found that the probability distribution of $\Lambda(55)$ is closely approximated by a χ^2 distribution with $k=8.70$ and $\langle\Lambda\rangle=0.163$. That distribution is plotted as the smooth curve in Fig. 3. Two other (derived) parameters, the most probable value Λ_0 and the standard deviation σ, have the values

$$\Lambda_0(55)=\langle\Lambda\rangle(1-2/k)=0.125, \tag{2.27}$$

$$\sigma[\Lambda(55)]=\langle\Lambda\rangle(2/k)^{1/2}=0.078. \tag{2.28}$$

The values of the mean and of the standard deviation can, of course, also be estimated more directly by computing the arithmetic mean of the first and second powers of $\Lambda(55)$. (This direct procedure was, in fact, used to obtain the values discussed in the preceding Secs. IIB and IIC.) The values of $\langle\Lambda(55)\rangle$ and $\sigma[\Lambda(55)]$ obtained by the least-squares fit are respectively 5 and 20% lower than the values of these quantities obtained by the direct method.

TABLE II. Expectation value, standard deviation, and figure of merit of statistic Λ for $n \gtrsim 20$ for three statistical models.

$P(t_1,t_2,\ldots,t_n)$	$\langle\Lambda\rangle$	$\sigma(\Lambda)$	$\phi(\Lambda)$
(a) Implied by E_G	~0.16	~0.08	~0.25
(b) $\prod_{i=1}^{n} P_W(t_i)$	0.29	0.24	0.67
(c) $\prod_{i=1}^{n} e^{-t_i}$	0.50	0.41	0.67

Although we have not made a detailed analysis of $P[\Lambda(n)]$ as a function of n, we believe that $P[\Lambda(n)]$ will roughly follow a χ^2 distribution corresponding to the values of the mean and standard deviation which are given in Figs. 1 and 2.

III. COMPARISON WITH OBSERVED SERIES OF NEUTRON RESONANCE LEVELS IN U^{239}

This section illustrates how the statistic $\Lambda(n)$ might be useful in the analysis of data. We shall analyze the series of nearly 200 resonances observed by Garg *et al.*[7] in the reaction $U^{238}+n$ in the range of neutron kinetic energies from zero to approximately 4 keV. These data probably represent the most thoroughly investigated *long* series of neutron resonances. Nevertheless, the authors label 37 levels as "doubtful." In addition, it is to be expected that the probability of missing a weak level increases with excitation energy, because of the deterioration in the resolution of the time-of-flight spectrometer.

Since the spin of U^{238}, the target nucleus, is zero and the resonances are formed predominantly by s-wave interactions, the majority of observed levels are assumed to have spin and parity $\frac{1}{2}^+$. Such levels form a "single" series, the statistics of which should be governed by the Gaussian orthogonal ensemble. Indeed, the observed spacing distribution is in good qualitative agreement with the Gaudin-Mehta distribution, as may be seen by inspection of Fig. 12 of Ref. 7. Our aim is to see what quantitative statements regarding the agreement can be made with the help of the statistic $\Lambda(n)$.

The statistic $\Lambda(n)$, defined by Eq. (1.6), was computed by numerical integration on the basis of the lowest $n+1$ resonance levels (n spacings)

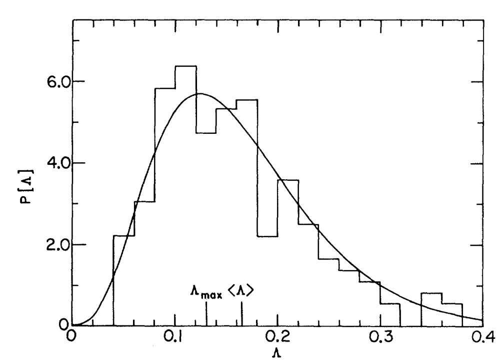

Fig. 3. Histogram representing the probability distribution $P[\Lambda(n)]$ for $n=55$. The histogram is based on 180 matrices of order 100 chosen at random from the Gaussian orthogonal ensemble. The solid curve is a plot of a χ^2 distribution with the parameter values determined by a least-squares fit.

observed in the reaction. The value of n was varied from 3 to the maximum value of about 200.

In order to compute the observed values of $\Lambda(n)$ according to (1.6), it is necessary to estimate the mean value of the spacing D for U^{239}. That was done by two methods. The first method consisted in computing the arithmetic mean of the n spacings under consideration. In the second method, the value of D was determined by the condition that $\Lambda(n)$ shall be a minimum as a function of D. The two methods yielded values of $\Lambda(n)$ which usually differed only by a few per cent. From this study we concluded that the inevitable uncertainty in our knowledge of the value of the mean spacing D will not have any significant effect on the conclusions we shall draw on the basis of the statistic $\Lambda(n)$.

The results of calculations that are pertinent to the comparison between experiment and theory are summarized in Fig. 4. The open circles represent the "observed" values of $\Lambda(n)$ as a function of n for U^{239}, as inferred from the set of levels from which the "doubtful" levels (possibly p-wave) were deleted. The solid circles represent the "observed" values of $\Lambda(n)$ for the level series that includes the levels labeled as "doubtful" by Garg *et al.*[7]

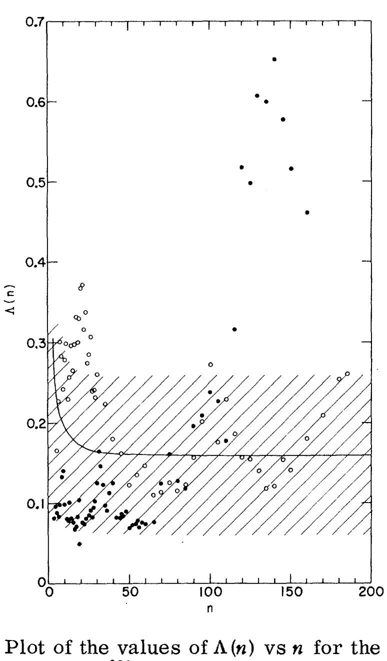

Fig. 4. Plot of the values of $\Lambda(n)$ vs n for the resonance levels observed in U^{239} by Garg *et al.* (Ref. 7). The open circles represent the "observed" values of $\Lambda(n)$ for the set of levels from which the "doubtful" resonances were omitted. The solid circles represent the values of $\Lambda(n)$ for the series of resonances that includes the doubtful levels. The continuous curve is a plot of $\langle\Lambda(n)\rangle$ in the Gaussian orthogonal ensemble, and the shaded region corresponds to the values of $\Lambda(n)$ that lie between the limits $\langle\Lambda(n)\rangle \pm \sigma[\Lambda(n)]$ according to the same random-matrix model. The Monte Carlo calculations from which these results were obtained are described in Sec. II.

For the purpose of comparing the observed values of $\Lambda(n)$ with the results based on the Gaussian orthogonal ensemble E_G, we have plotted the solid curve which represents the expectation value $\langle\Lambda(n)\rangle$ in the ensemble E_G. The shaded area in Fig. 4 encompasses the values of $\Lambda(n)$ that lie between the limits $\langle\Lambda(n)\rangle \pm \sigma[\Lambda(n)]$ in the plot of $\Lambda(n)$ versus n. The theoretical results leading to this plot were obtained by means of the Monte Carlo calculations described in Sec. II.

Inspection of Fig. 4 does not reveal any obvious discrepancy between theory and experiment up to $n \approx 100$. It would seem that the series with the doubtful levels included is in somewhat better agreement with the theory than the series with the doubtful levels omitted. Thus, if we make the assumption that the theory is exact, then we would be tempted to infer that many of the doubtful levels up to $n \approx 100$ belong to the s-wave series.

The observed values of $\Lambda(n)$ up to $n \approx 100$ are good evidence that the correlations between the spacings implied by E_G are present in the experimental spectrum. According to line (b) of Table II, if the correlations between spacings were eliminated (without affecting the marginal distribution of a single spacing) we would expect to find $\Lambda(n) = 0.29 \pm 0.24$. That would place virtually all of the observed values of $\Lambda(n)$ for $n \lesssim 100$ below the theoretical mean value 0.29, an unacceptably improbable occurrence.

As the neutron kinetic energy increases, we would expect to find considerable deviations from the model because the missing of s-wave levels and the erroneous inclusion of p-wave resonances would tend to destroy the correlations and would even modify the form of $P(t)$ in the direction of the exponential distribution. The steep rise in the values represented by the solid circles (with "doubtful" levels) is qualitatively understandable on this basis. However, the good agreement between theory and the observed values of $\Lambda(n)$ represented by the open circles (without "doubtful" levels) is puzzling. The naive conclusion on the basis of our study would be that the doubtful levels beyond $n \approx 100$ do not, for the most part, belong to the s-wave series, and that once the doubtful levels have been deleted from the observed series, the remainder gives a rather complete and accurate representation of the spectrum of U^{239} in the range from 0 to 4 keV.

IV. CONCLUDING REMARKS

We have estimated both the mean and the standard deviation of the statistic $\Lambda(n)$ as a function of

n in the Gaussian orthogonal ensemble, and have found that some of the best data (namely the first 100 neutron-capture levels observed in U^{239}) follow an empirical spacing distribution which fluctuates about the Gaudin-Mehta distribution by an amount [measured by $\Lambda(n)$] which is acceptable.

The correlation between the spacings reduces both the mean and the standard deviation of $\Lambda(n)$ to about half of what these values would be if the spacings were statistically independent but followed the Gaudin-Mehta distribution. The data clearly disagree with a model in which the correlations are neglected. No one has proposed such a model, and this point is made merely as an indication of the sensitivity of the statistic $\Lambda(n)$ for the purpose of testing the theory based on the Gaussian orthogonal ensemble.

There exists a large body of experimental data[17] consisting of short series of successive energy levels ($n<10$). Our Monte Carlo calculation yielded an estimate of the values of both the mean and standard deviation of $\Lambda(n)$ in this range, and short experimental level series may therefore by analyzed in terms of the results presented explicitly in this work.

It should be noted that for large values of n (say, $n>50$) our statistic $\Lambda(n)$ does not provide as sensitive a test of the random-matrix model as does the best statistic, Δ_3, of Dyson and Mehta.[5] For very small values of n we are not able to compare $\Lambda(n)$ with Δ_3, since the precise values of the errors in the estimates of Dyson and Mehta are not known.

APPENDIX A. DERIVATION OF EQ. (2.5)

Our aim is to derive Eqs. (2.5) and (2.6) of the text. The joint probability distribution of n successive spacings, and therefore all the various marginal distributions derived from it, are assumed to have the property of translational invariance,[13] i.e.,

$$P_{i,j,k,\ldots}(t_i,t_j,t_k,\ldots) = P_{i+\delta,j+\delta,k+\delta,\ldots}(t_i,t_j,t_k,\ldots), \tag{A-1}$$

where δ represents an arbitrary integer. Particular cases of (A-1) which are relevant to our discussion are

$$P_1(s)=P_i(s), \quad i=1,2,\ldots,n, \tag{A-2}$$

and

$$P_{i,j}(s,t)=P_{i+\delta,j+\delta}(s,t), \quad 1\le i<j\le n. \tag{A-3}$$

The subscript may be safely omitted in the case of $P_i(s)$ but not in the case of $P_{i,j}(s,t)$. Next, we note the trivial relations

$$\theta(x-t)=\theta^2(x-t), \tag{A-4}$$

$$\int_0^\infty \theta(x-t)P(t)dt=\int_0^x P(t)dt\equiv F(x), \tag{A-5}$$

$$\int_0^\infty\int_0^\infty \theta(x-s)\theta(x-t)P_{i,j}(s,t) = \int_0^x\int_0^x P_{i,j}(s,t)dsdt. \tag{A-6}$$

With the introduction of the abbreviation

$$u(x,s)\equiv\theta(x-s)-F(x), \tag{A-7}$$

the expectation value of $\Lambda(n)$, defined by Eq. (2.5), may be expressed as

$$n\langle\Lambda(n)\rangle=\sum_{i=1}^{n}\int_0^\infty\int_0^\infty P(s)u^2(x,s)dsdx + 2\sum_{1\le i<j\le n}\int_0^\infty\int_0^\infty\int_0^\infty P_{i,j}(s,t)u(x,s) \times u(x,t)dsdtdx. \tag{A-8}$$

On the basis of translational invariance and Eqs. (A-4)–(A-6), it is readily seen that

$$\int_0^\infty P_i(s)u^2(x,s)ds=F(x)-F^2(x) \tag{A-9}$$

and

$$\int_0^\infty\int_0^\infty P_{i,j}(s,t)u(x,s)u(x,t)dsdt = \int_0^\infty\int_0^\infty [P_{1,j-i+1}(s,t)-P(s)P(t)]dsdt. \tag{A-10}$$

If expressions (A-9) and (A-10) are substituted into (A-8), and (A-3) is applied to the terms in the double summation, then the desired expression (2.5) is obtained.

APPENDIX B. DERIVATION OF EQS. (2.17) - (2.20)

Our aim is to derive Eqs. (2.17)–(2.20) of the text. We will assume that the joint probability density of n successive spacings is given by the simple product

$$P(t_1,t_2,\ldots,t_n)=\prod_{i=1}^{n}P(t_i), \tag{B-1}$$

and in what follows $F(x)$ will again denote the cumulative distribution

$$F(x)=\int_0^x P(t)dt. \tag{B-2}$$

With the assumption (B-1) it is relatively easy to obtain connections between the moments of $\Lambda(n)$ and the moments of $\Lambda(n+1)$. It will be convenient to introduce the symbol $S(x,n)$ defined by the expression

$$S(x,n) \equiv \sum_{i=1}^{n} u(x, t_i). \tag{B-3}$$

The quantity $u(x, t)$ was defined in Appendix A. The first moment of $\Lambda(n)$ is given by the expression

$$\langle\Lambda(n)\rangle = (1/n)\int_0^\infty \langle S^2(x,n)\rangle dx. \tag{B-4}$$

In the above, the angular parentheses $\langle\,\rangle$ denote expectation value with respect to the probability distribution (B-1). With the help of (B-4) and the fact that

$$\langle S(x,n)u(x, t_{n+1})\rangle = 0, \tag{B-5}$$

the recursion relation

$$(n+1)\langle\Lambda(n+1)\rangle - n\langle\Lambda(n)\rangle = \langle\Lambda(1)\rangle = \int_0^\infty [F(x) - F^2(x)]\,dx \tag{B-6}$$

is easily established. Equation (B-6) can easily be solved to yield

$$\langle\Lambda(n)\rangle = \langle\Lambda(1)\rangle = \int_0^\infty [F(x) - F^2(x)]\,dx. \tag{B-7}$$

The expression (B-7) for $\langle\Lambda(n)\rangle$ agrees with the expression (2.5) of the text if the terms contributed to (2.5) by the *correlations* between spacings are neglected.

In a similar fashion, a recursion formula for the second moment of $\Lambda(n)$ may be derived. The second moment of $\Lambda(n)$ is given by the expression

$$\langle\Lambda^2(n)\rangle = (1/n^2)\int_0^\infty\int_0^\infty \langle S^2(x,n)S^2(y,n)\rangle dxdy. \tag{B-8}$$

With the help of the relations such as

$$\langle u^2(x-s)\rangle = F(x) - F^2(x), \tag{B-9}$$

$$\langle S(x,n)S(y,n)\rangle = nF(z), \quad z = \min(x,y), \tag{B-10}$$

$$\langle S(x,n)\rangle = 0, \tag{B-11}$$

a relatively straightforward derivation leads to the recursion formula

$$(n+1)^2[\langle\Lambda^2(n+1)\rangle - \langle\Lambda(n+1)\rangle^2] - n^2[\langle\Lambda^2(n)\rangle - \langle\Lambda(n)\rangle^2] = 4nA + [\langle\Lambda^2(1)\rangle - \langle\Lambda(1)\rangle^2]. \tag{B-12}$$

In the above, A is an abbreviation for a double integral, i.e.,

$$A = \int_0^\infty\int_0^\infty [F(z) - F(x)F(y)]^2 dxdy. \tag{B-13}$$

The recursion formula (B-12) has the solution

$$\langle\Lambda^2(n)\rangle - \langle\Lambda(n)\rangle^2 = \frac{2(n-1)}{n}A + \frac{1}{n}[\langle\Lambda^2(1)\rangle - \langle\Lambda(1)\rangle^2]. \tag{B-14}$$

The most tedious part of the entire analysis consists in the evaluation of the variance of $\Lambda(1)$. We find the expression

$$\langle\Lambda^2(1)\rangle - \langle\Lambda(1)\rangle^2 = \int_0^\infty\int_0^\infty F(z)[1 - 4F(x) + 4F(x)F(y)] \times dxdy + \int_0^\infty\int_0^\infty F(x)F(y) \times [4F(y) - 4F(x)F(y) - 1]\,dxdy. \tag{B-15}$$

When the expression (B-15) is substituted into (B-14), the Eqs. (2.17)–(2.20) are obtained.

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[6]See, for example, N. Rosenzweig, J. E. Monahan, and M. L. Mehta, Nucl. Phys. A109, 437 (1968). It should be noted, however, that the statistic $\Lambda(n)$, which is the subject of this work, was not designed specifically for revealing deviations resulting from the presence of a small time-reversal-odd term in the nuclear Hamiltonian.

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[9]See Ref. 1. See also, M. L. Mehta and M. Gaudin, Nucl. Phys. 18, 420 (1960) and Ref. 4, Appendix A. 29.

[10]It is possible to define matrix ensembles that will yield a realistic (in fact, prescribed) level-density law while at the same time the "local" energy-level statistics presumably remain unaffected. See, for example, R. Balian, Nuovo Cimento 57, B183 (1968).

[11]For preliminary work on $\Lambda(n)$, see J. E. Monahan and N. Rosenzweig, Bull. Am. Phys. Soc. 11, 305 (1966); J. Phys. Soc. Japan Suppl. 24, 715 (1968).

[12]See, for example, Ref. 6.

[13]Translational invariance, defined more precisely in Appendix A, is a requirement that the statistical model should satisfy. It is one of the aspects that led Dyson to introduce his "circular" ensembles [J. Math. Phys. 3, 140 (1963); reprinted in Ref. 3]. However, translational invariance holds also at the center of Wigner's semicircle. See the remarks following Eqs. (5.84) and (6.25)

and Appendix A. 31 of Ref. 4.

[14]For example, the correlation coefficient for two adjacent spacings has a value of about -0.25. See C. E. Porter, Nucl. Phys. 40, 167 (1963); P. B. Kahn, Nucl. Phys. 41, 159 (1963). Both of these papers are reprinted in Ref. 3.

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[16]P. B. Kahn and C. E. Porter, Nucl. Phys. 48, 385 (1963); reprinted in Ref. 3.

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Intermediate-Coupling Calculations in Odd-*A* Bismuth Isotopes*

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The intermediate-coupling version of the unified model has been used to calculate the properties of the low-lying levels of $Bi^{203,205,207,211}$. In each case, the quadrupole oscillations of the doubly even core have been coupled to the $h_{9/2}$ and $f_{7/2}$ single-particle states. Energy levels, transition rates, and magnetic dipole and electric quadrupole moments have been calculated for these four nuclei. Agreement between calculated nuclear properties and available experimental data is good for $Bi^{205,207,211}$, but Bi^{203} does not appear to fit within the framework of the intermediate-coupling model.

I. INTRODUCTION

In the intermediate-coupling approach to the unified nuclear model developed by Choudhury,[1] the nucleus is treated as an oscillating core coupled to the single-particle states of an extra odd nucleon. Intermediate-coupling calculations depend upon three parameters. The phonon energy of the core $\hbar\omega$ is a fixed parameter which is determined from the first excited state of the neighboring doubly even nucleus. The two adjustable parameters are ξ, the coupling strength between the quadrupole vibration of the core and the single-particle motion, and ϵ, the effective energy spacing between the single-particle states. In the present calculations, core excitations up to three phonons are coupled to two single-particle states. To obtain the wave function and energies of the states, the total Hamiltonian for the system is diagonalized. A detailed description of these calculations has been presented in several earlier papers.[2-4]

The intermediate-coupling model has been successful in predicting level structures, transition rates, and nuclear moments for heavy nuclei with several nucleons outside closed shells. Bismuth isotopes are of particular interest because they lie near the doubly-closed-shell nucleus Pb^{208}. Recent experimental investigations of Bi^{203}, Bi^{205}, and Bi^{207} reported by Hopke, Nauman, and Spejewski (HNS)[5] indicate that the experimentally observed properties of these isotopes could be accounted for by the intermediate-coupling theory. Recent studies of the decay of Bi^{211} by Gorodetzky *et al.*,[6] Davies and Hamilton[7] as well as shell-model calculations by Gabrakov[8] present evidence for mixing of single-particle states in the excited levels of Bi^{211}. Because experimental data have previously been unavailable, there has been a lack of theoretical calculations for these isotopes. The purpose of the present investigation is to extend the application of the intermediate-coupling model to odd-mass bismuth isotopes and to compare the results with the experimentally observed properties.

Each bismuth isotope is described as a coupled system consisting of the neighboring doubly even Pb core plus an odd proton. The level structures of all the doubly even Pb isotopes exhibit quadrupole vibrations except Pb^{208}. Since our model assumes the existence of cores capable of quadrupole vibrations, we shall not consider Bi^{209}. From shell-model considerations, the last odd proton is assumed to have available both the $h_{9/2}$ and the $f_{7/2}$ states.

II. ENERGY LEVELS

The total Hamiltonian was diagonalized for different values of the effective energy spacing ϵ while the coupling strength was varied from 0 to 4. The dimension of the matrices to be diagonalized ranged from 10×10 for $I=\frac{3}{2}$ to 17×17 for $I=\frac{9}{2}$. For the calculation, our basis vectors are the normalized states $|lj;NR\rangle$, where l is the