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Convenient Expansion for Local Potentials*

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We introduce a new separable expansion for local potentials. The expansion, called the unitary pole expansion, has real, energy-independent form factors, and satisfies the requirements of two-particle unitarity in all orders. The convergence of the expansion is investigated by comparing expanded and exact T matrices for negative energies, and by performing three-body bound-state calculations. In the latter case, a one-term approximation gives energies accurate to within 2% for potentials containing repulsion of the magnitude indicated by two-nucleon data.

I. INTRODUCTION

The solution of the Faddeev equations for systems interacting by means of local two-body potentials is a difficult numerical problem. To solve for the three-body bound state, we must solve a (in general coupled) set of two-dimensional integral equations. For the scattering states, these equations become singular.

Some progress[1] has been made towards the solution of the equations for the bound-state problem. For the scattering states no success has yet been reported.

To avoid the complications of two-dimensional

integral equations, a number of authors[2,3] have used separable expansions of the two-body interaction to reduce the problem to a coupled set of one-dimensional integral equations. These equations have been solved both for the three-body bound state and the three-body scattering states. In this paper we propose another expansion of the two-body interaction. The expansion, which we call the unitary pole expansion (UPE), is closely related to the Weinberg[4] series. However, it is considerably easier to use and shows better convergence properties. The form factors for the expansion are real and energy-independent, and the UPE satisfies the requirements of two-particle unitarity

in all orders.

In Sec. II we develop the UPE and investigate its convergence by comparing the expanded and exact T matrices for central potentials containing short-range repulsion; i.e., the potential V of Malfliet and Tjon[1] and the singlet S two-nucleon potential of Reid.[5] In Sec. III we investigate the convergence of the UPE by performing three-body bound-state calculations and examining the three-body bound-state energy as a function of the number of terms retained. A one-term approximation gives a three-body energy accurate to within 0.2 MeV for potentials with the amount of repulsion indicated by two-nucleon data. Results are also compared with the work of other authors. Section IV contains a discussion of our results.

II. THE UNITARY POLE EXPANSION

In this section, we develop our separable expansion of the two-body T matrix. The expansion uses the eigenfunctions of the homogeneous Lippmann-Schwinger equation to obtain a separable expansion of the two-body potential. For practical calculations this expansion is truncated and used in the Lippmann-Schwinger equation to obtain a separable T matrix which satisfies the requirements of two-particle unitarity, and which approximates the actual T matrix. We derive this expansion below and investigate its convergence by comparing the resulting T matrices with the actual T matrix for different local potentials.

For simplicity we consider expanding the S partial wave of the two-body T matrix for two identical spinless bosons. We use abstract vector notation to simplify the derivation. Actual calculations were performed in the momentum representation.

The two-body T matrix is the solution of the partial-wave Lippmann-Schwinger equation

$$T(s) = V + VG_0(s)T(s) , \qquad (2.1)$$

where V is the (S wave) two-body potential and $G_0(s)$ is the free two-particle Green's function. The homogeneous form of (2.1) is

$$|\psi_n(s)\rangle = \lambda_n(s)VG_0(s)|\psi_n(s)\rangle . \qquad (2.2)$$

In what follows we will fix s at some negative value, $-B$, and drop the s dependence of $|\psi_n\rangle$ and λ_n in (2.2).

It is also convenient to consider the equation adjoint to (2.2),

$$\langle \chi_n| = \mu_n\langle \chi_n|VG_0(-B) . \qquad (2.3)$$

It is then easy to show that the $|\psi_n\rangle$ and λ_n may be taken to be real and that the $|\chi_n\rangle$, μ_n and $|\psi_n\rangle$, λ_n are related by

$$|\chi_n\rangle = G_0(-B)|\psi_n\rangle , \qquad (2.4)$$

$$\lambda_n = \mu_n . \qquad (2.5)$$

Furthermore, the following orthonormality relation may be taken to hold:

$$\langle \psi_n|G_0(-B)|\psi_m\rangle = -\delta_{nm} . \qquad (2.6)$$

To prove the orthogonality relation we take the matrix element

$$\langle \psi_n|G_0(-B)VG_0(-B)|\psi_n\rangle = (1/\lambda_n)\langle \psi_n|G_0(-B)|\psi_m\rangle ,$$
$$(2.7)$$
$$= (1/\lambda_m)\langle \psi_n|G_0(-B)|\psi_m\rangle ,$$

where we have used (2.2)-(2.5). Then, barring any degeneracy, our freedom of normalization gives (2.6). If degeneracy exists, the functions may be made orthonormal by a generalization of the standard Schmidt procedure. The normalization to minus one is chosen because for n equal to m (2.6) takes the form

$$\int_0^\infty \frac{\psi_n(p)^2 p^2 dp}{-B-p^2} = -1 ,$$

and for real $\psi_n(p)$ the left-hand side is intrinsically negative.

Practically identical arguments hold for spin-dependent forces and for noncentral forces. In the latter case, we would consider the homogeneous form of a coupled set of integral equations.

Having proved the desired results, it is easy to show that the separable expansion

$$V = \sum_{m=1}^N -\frac{|\psi_m\rangle\langle\psi_m|}{\lambda_m} \qquad (2.8)$$

(with N possibly infinite) is at least formally consistent with (2.2) and (2.3); for using (2.8) and (2.2) gives

$$|\psi_n\rangle = \lambda_n \sum_{m=1}^N -\frac{|\psi_m\rangle\langle\psi_m|}{\lambda_m}G_0(-B)|\psi_n\rangle .$$

Using (2.6) reduces this to an identity. Truncating (2.8) to a finite number of terms gives a finite-rank separable potential which approximates the actual potential.

To obtain the unitary pole expansion (UPE) T matrix, we insert (2.8) (truncated at some finite N) into the Lippmann-Schwinger equation, (2.1). The resulting T matrix is then given by

$$T_{\text{UPE}}(s) = \sum_{n,m=1}^N |\psi_n\rangle\Delta_{n,m}(s)\langle\psi_m| \qquad (2.9)$$

with

$$-[\Delta(s)^{-1}]_{n,m} = \lambda_n\delta_{n,m} + \langle\psi_n|G_0(s)|\psi_m\rangle . \qquad (2.10)$$

Since the expansion (2.8) approximates the two-body potential, the UPE T matrix is then an approximation to the actual T matrix (found by solving the Lippmann-Schwinger equation with the full two-body potential).

We now relate our results to separable approximations of other authors. We assume for the moment that there is a two-body bound state and take B to be the two-body binding energy. The equation for $|\chi_n\rangle$ is

$$|\chi_n\rangle = \lambda_n G_0(-B) V |\chi_n\rangle ,$$

which is just the bound-state Schrödinger equation with the potential $\lambda_n V$. Thus the $|\chi_n\rangle$ may be interpreted as bound states of energy $-B$ of the potentials $\lambda_n V$. However, by our assumption V has a bound state, $|B\rangle$, and hence we may choose λ_1 equal to 1 and

$$|\chi_1\rangle \propto |B\rangle . \tag{2.11}$$

If we retain only one term in (2.9), we find

$$T_{\text{UPE}}(s) = |\psi_1\rangle \Delta(s) \langle \psi_1| , \tag{2.12}$$

$$\Delta(s) = -[1 + \langle \psi_1 | G_0(s) | \psi_1 \rangle]^{-1} . \tag{2.13}$$

This is just the unitary pole approximation (UPA) discussed by Lovelace[6] and Fuda.[7] This approximation was investigated for the square-well potential by Levinger, Lu, and Stagat[8]; for the Hulthén potential by Kok, Erens, and Van Wageningen[9]; and by Brady et al.[10] and Harms and Levinger[10] for a number of examples. Levinger, Lu, and Stagat compared the exact square-well T matrix with various separable approximations to it, and found the UPA to be the best of the one-term approximations investigated.

From (2.6), (2.12), and (2.13), we see that the UPA T matrix has a pole at the two-body bound-state energy, $s = -B$, corresponding to the bound-state pole in the actual T matrix. It may also be shown that the residues of the exact and UPA T matrices are the same, and hence in the neighborhood of the pole, the exact and UPA T matrices agree. It is this agreement at the pole which is one of the principle justifications[6] for the use of separable potentials. (It will be understood from now on that if there is one two-body bound state, B will be taken as the two-body bound-state energy.)

What about keeping additional terms in (2.8)? We see from (2.6) that retaining N terms in (2.8) does not change the bound-state energy or wave function for the separable potential. As a result, the N-term UPE T matrix has the same pole position as the UPA T matrix and the same as the actual T matrix. However, since retaining additional terms in (2.8) gives a better approximation to

the potential, the N-term T matrix gives a better approximation to the actual T matrix. For any number of separable terms, (2.8) gives a separable potential which is real and symmetric. The UPE T matrix will, therefore, always be unitary. The UPE thus gives a systematic method of improving upon the UPA, while still retaining the proper pole behavior and satisfying unitarity.

A similar expansion method was developed by Weinberg.[4] His method retains the energy dependence of $|\psi_n(s)\rangle$ and $\lambda_n(s)$ in (2.2) and sets s equal to the desired argument of $T(s)$. The potential still has the expansion (2.8), but the T matrix is now given by

$$T_W(s) = \sum_n \frac{|\psi_n(s)\rangle \langle \psi_n(s)|}{1 - \lambda_n(s)} .$$

This expression is simpler than (2.12), but is obtained at the expense of having to solve (2.2) for each s desired. Since in the three-body problem T enters in the form $T(s - \frac{3}{4}p^2)$ with $0 \leqslant p^2 \leqslant \infty$, use of the Weinberg series requires the solution of (2.2) many times, with the subsequent increased difficulty in computation. One of the greatest advantages of the UPE is that this problem does not occur. We solve for the form factors $|\psi_n\rangle$ once, and then the T matrix is obtained by evaluating the integrals in Δ, a much easier task.

The Weinberg series also appears to behave poorly for large s. For example, consider the case of the local Hulthén potential. The Weinberg eigenvalues are given by Sitenko et al.[2]

$$\frac{\lambda_n(s)}{\lambda_1(s)} = \frac{n(2\sqrt{-s} + n\mu)}{2\sqrt{-s} + \mu} , \tag{2.14}$$

and $\psi_1(p, s)$ is given by

$$\psi_1(p, s) \propto \frac{1}{p^2 + (\sqrt{-s} + \mu)^2} , \tag{2.15}$$

where μ is the inverse range of the potential in coordinate space. The UPE eigenvalues and form factor are obtained by setting $s = -B$. Assuming that $B \approx 0$, we see that for small s the two series are practically the same. The Weinberg and UPE eigenvalues each are approximately proportional to n^2, and the form factor has a reasonable range. For large negative s, however, the Weinberg eigenvalues increase initially only as n. The form factor has a very long range and is unsuited for expanding the T matrix which has a range comparable to that of the potential. Being independent of s, the UPE eigenvalues and form factor do not have this difficulty.

When there is no two-body bound state, we may choose B to have any convenient value. If there is an antibound state near $s = 0$ (as for example in the singlet S two-nucleon system), and we choose B

= 0 in (2.2), we may expect that keeping a number of terms in (2.8) will give a good approximation to the antibound-state pole in the T matrix.[11]

For partial waves without poles near the physical region, the expansion method is still applicable. However, the pole dominance aspect of the expansion is then absent, and more terms would probably have to be retained.

We now apply the theory that has been developed to two different potentials, the potential V of Malfliet and Tjon,[1] and the Reid[5] singlet S two-nucleon potential. The convergence of the UPE is investigated by retaining different terms in (2.8) and comparing the resulting separable T matrices with the exact T matrices.[12] In Sec. III we examine the convergence of the UPE by using it in the three-body problem.

Both the potentials considered contain short-range repulsive cores. As a result, the λ's may be both positive or negative. The positive (attractive) λ's are associated with two-body "bound states" [see the discussion preceding (2.11)] supported by the attractive regions of the two-body potential. A negative (repulsive) eigenvalue means that the two-body "bound state" associated with this eigenvalue is being supported by the repulsive part of the potential which in $\lambda_n V$ is now attractive.

The Malfliet-Tjon potential is the sum of two Yukawa potentials

$$V(r) = -\lambda_A \frac{e^{-\mu_A r}}{r} + \lambda_R \frac{e^{-\mu_R r}}{r} \qquad (2.16)$$

with $\mu_A = 1.55$ F^{-1}, $\mu_R = 3.11$ F^{-1}, and $\lambda_R/\lambda_A = 2.522$. An over-all multiplication constant was determined by fixing the two-body binding energy at 0.35 MeV. All numerical calculations for this section were done using a 32-point Gauss-Legendre quadrature mesh mapped to the infinite interval by

$$p = \beta \frac{(1+x)}{(1-x)}$$

with $-1 \leq x \leq 1$ and $0 \leq p < \infty$.

The first four attractive eigenvalues for the Malfliet-Tjon potential are given in Table I. The ei-

TABLE I. Attractive eigenvalues. The first (in order of increasing absolute value) four positive eigenvalues of Eq. (2.2).

Eigenvalue	Malfliet-Tjon potential V [a]	Reid $^1S_0$ potential[b]
λ_1^A	1.0	1.0819
λ_2^A	7.22	8.27
λ_3^A	19.	21.
λ_4^A	37.	38.

[a]Malfliet and Tjon, Ref. 1 [see our Eq. (2.16)].
[b]Reid, Ref. 5 [see our Eq. (2.17)].

TABLE II. Repulsive eigenvalues. The first (in order of increasing absolute value) three negative eigenvalues of Eq. (2.2).

Eigenvalue	Malfliet-Tjon potential V [a]	Reid $^1S_0$ potential[b]
λ_1^R	-0.4342	-0.06286
λ_2^R	-1.78	-0.252
λ_3^R	-3.9	-0.55

[a]Malfliet and Tjon, Ref. 1 [see our Eq. (2.16)].
[b]Reid, Ref. 5 [see our Eq. (2.17)].

genvalues increase considerably more rapidly than is the case for purely attractive potentials [see (2.14) or Ref. 12] leading to a comparatively rapid convergence of (2.8). The first three repulsive eigenvalues are given in Table II. We see that the first repulsive eigenvalue is less than unity in absolute value, reflecting the large amount of repulsion present in the Malfliet-Tjon potential.

In Fig. 1 we plot the first two attractive form factors and the first two repulsive form factors as a function of momentum. The presence of a node in the first attractive form factor, $\psi_1^A(p)$, is quite different from the behavior of the standard Yamaguchi-type[13] form factor, (2.15). The repulsive form factors are greatest for relatively large momenta since they are associated with the repulsion in the two-body potential for large momenta.

In Tables III–V we give a comparison between the exact Malfliet-Tjon T matrix and the UPE T
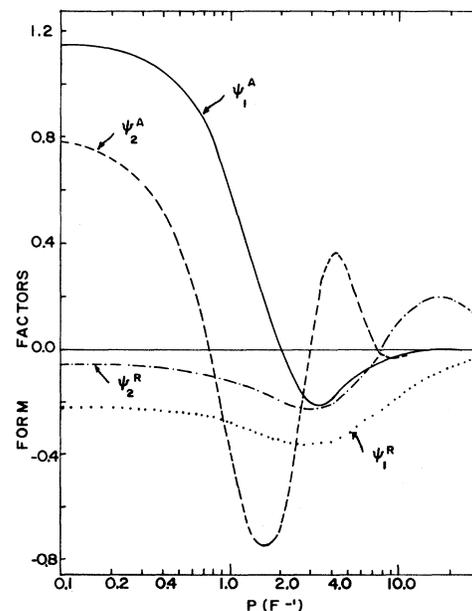
FIG. 1. The first two attractive, ψ_n^A, and the first two repulsive, ψ_n^R, form factors for the Malfliet-Tjon potential.

TABLE III. Separable and exact T matrices for the Malfliet-Tjon potential: $s = 0.0$. Columns 1 and 2 contain the momentum values (in F^{-1}). Columns 3-7 are UPE T matrices $T(p,q:s)$. $2A + 2R$ means, for example, that the first two attractive and the first two repulsive terms have been retained in the expansion (2.8). Exact T-matrix values are given in the last column. $s = 1$ corresponds to 41.46 MeV.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	7.7482	7.7482	7.7476	7.7477	7.7474	7.7468
0.170	0.004	7.6102	7.6103	7.6100	7.6101	7.6100	7.6099
0.170	0.170	7.4747	7.4748	7.4747	7.4747	7.4747	7.4746
0.630	0.004	6.0558	6.0554	6.0591	6.0590	6.0608	6.0645
0.630	0.170	5.9480	5.9479	5.9492	5.9491	5.9496	5.9500
0.630	0.630	4.7332	4.7362	4.7129	4.7134	4.7021	4.6787
1.560	0.004	1.4242	1.4224	1.4310	1.4307	1.4323	1.4321
1.560	0.170	1.3989	1.3982	1.4012	1.4011	1.4015	1.4015
1.560	0.630	1.1131	1.1259	1.0714	1.0735	1.0632	1.0647
1.560	1.560	0.2618	0.3151	0.1876	0.1966	0.1872	0.1876
3.426	0.004	−1.4349	−1.4376	−1.4414	−1.4418	−1.4436	−1.4431
3.426	0.170	−1.4094	−1.4106	−1.4118	−1.4118	−1.4123	−1.4121
3.426	0.630	−1.1215	−1.1029	−1.0791	−1.0760	−1.0647	−1.0680
3.426	1.560	−0.2638	−0.1859	−0.1301	−0.1169	−0.1067	−0.1043
3.426	3.426	0.2657	0.3795	0.3552	0.3745	0.3632	0.3655
7.639	0.004	−0.3254	−0.3271	−0.3272	−0.3272	−0.3276	−0.3274
7.639	0.170	−0.3196	−0.3202	−0.3202	−0.3202	−0.3203	−0.3203
7.639	0.630	−0.2543	−0.2428	−0.2422	−0.2423	−0.2397	−0.2405
7.639	1.560	−0.0598	−0.0116	−0.0102	−0.0105	−0.0082	−0.0125
7.639	3.426	0.0603	0.1307	0.1301	0.1298	0.1272	0.1200
7.639	7.639	0.0137	0.0573	0.0573	0.0573	0.0567	0.0583

matrices, keeping different numbers of attractive (designated by A) and repulsive (designated by R) terms in (2.8). We examine cases for $s = 0.0$, -0.5, and -3.0 ($s = 1$ corresponds to 41.46 MeV).

In Figs. 2-5 we compare the different T matrices. Figures 2-4 give the diagonal T matrices $T(p,p:s)$ as a function of the momentum p, for $s = 0.0$, -0.5, and -3.0. Figure 5 gives off-diagonal T matrices $T(p,k:s)$ as a function of p, for $s = -0.5$ and $k = 0.630$, 1.56, and 3.43 F^{-1}. The UPA results are given as a solid line. The dotted curves are a three-term UPE ($2A + 1R$ in the tables), keeping one attractive and one repulsive correction term. The points enclosed in circles give the exact T-matrix values.

FIG. 2. Diagonal Malfliet-Tjon T matrices $T(p,p:s)$ for $s = 0.0$. The solid line is the one-term UPE (UPA) T matrix. The dotted line is the UPE T matrix obtained keeping in addition one attractive and one repulsive correction term. The exact T-matrix values are given by the points enclosed in circles. One energy unit corresponds to 41.46 MeV.
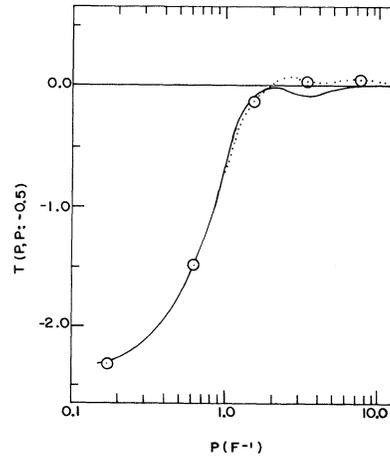
FIG. 3. Diagonal Malfliet-Tjon T matrices for $s = -0.5$. Curves have the same meaning as in Fig. 2.

TABLE IV. Separable and exact T matrices for the Malfliet-Tjon potential: $s = -0.5$.
Columns have the same meaning as in Table III.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	−2.3824	−2.3801	−2.4052	−2.4048	−2.4146	−2.4293
0.170	0.004	−2.3400	−2.3380	−2.3602	−2.3599	−2.3683	−2.3802
0.170	0.170	−2.3983	−2.2966	−2.3163	−2.3160	−2.3233	−2.3329
0.630	0.004	−1.8620	−1.8632	−1.8559	−1.8561	−1.8521	−1.8430
0.630	0.170	−1.8289	−1.8299	−1.8235	−1.8236	−1.8202	−1.8128
0.630	0.630	−1.4553	−1.4547	−1.4568	−1.4568	−1.4584	−1.4642
1.560	0.004	−0.4379	−0.4488	−0.3964	−0.3981	−0.3904	−0.3936
1.560	0.170	−0.4301	−0.4396	−0.3932	−0.3948	−0.3881	−0.3906
1.560	0.630	−0.3423	−0.3366	−0.3519	−0.3510	−0.3542	−0.3517
1.560	1.560	−0.0805	−0.0291	−0.1383	−0.1298	−0.1359	−0.1361
3.426	0.004	0.4412	0.4245	0.4048	0.4020	0.3932	0.3985
3.426	0.170	0.4333	0.4187	0.4012	0.3988	0.3912	0.3957
3.426	0.630	0.3448	0.3535	0.3593	0.3606	0.3642	0.3614
3.426	1.560	0.0811	0.1603	0.2015	0.2150	0.2220	0.2255
3.426	3.426	−0.0817	0.0403	0.0248	0.0463	0.0384	0.0398
7.639	0.004	0.1001	0.0898	0.0906	0.0906	0.0886	0.0899
7.639	0.170	0.0983	0.0893	0.0900	0.0900	0.0883	0.0894
7.639	0.630	0.0782	0.0835	0.0833	0.0833	0.0841	0.0835
7.639	1.560	0.0184	0.0668	0.0652	0.0652	0.0668	0.0627
7.639	3.426	−0.0185	0.0560	0.0566	0.0566	0.0549	0.0472
7.639	7.639	−0.0042	0.0413	0.0413	0.0413	0.0409	0.0422

In Fig. 2 the pole contribution to the T matrix is positive, accounting for the large positive values of the T matrix. Since we are very near the pole, the UPA is very close to the exact T matrix. The corrections from additional terms occur at momenta where the UPA form factor is small. It is interesting to note the bump in the actual T matrix which occurs around $p = k = 3$ F^{-1}. Its origin in ψ_1^A is clear from Fig. 1. In Fig. 3 we are on the other side of the pole, and its contribution to the T matrix is now negative. The relative difference between the UPA and the exact T matrix appears larger than in Fig. 2. This is because the pole con-

tribution to the T matrix is less dominant here. (We are about 20 MeV away from the pole.) The three-term UPE picks up most of the difference, however, and forms an excellent approximation to the T matrix. Even in Fig. 4, where we are about 125 MeV away from the pole the UPA provides a reasonable approximation to the T matrix, and the three-term UPE is practically indistinguishable from the true values. The UPA and UPE fits are

FIG. 4. Diagonal Malfliet-Tjon T matrices for $s = -3.0$. Curves have the same meaning as in Fig. 2.
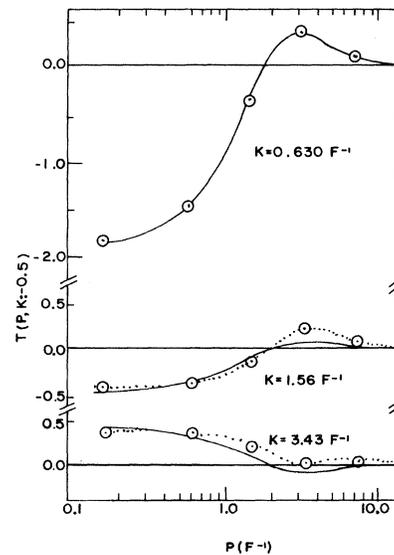
FIG. 5. Off-diagonal Malfliet-Tjon T matrices $T(p, k:s)$ for $s = -0.5$ and $k = 0.630$, 1.56, and 3.43 F^{-1}. Curves have the same meaning as in Fig. 2.

TABLE V. Separable and exact T matrices for the Malfliet-Tjon potential: $s = -3.0$.
Columns have the same meaning as in Table III.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	−1.6799	−1.6784	−1.7285	−1.7283	−1.7460	−1.7668
0.170	0.004	−1.6499	−1.6487	−1.6946	−1.6944	−1.7103	−1.7277
0.170	0.170	−1.6206	−1.6195	−1.6615	−1.6614	−1.6756	−1.6902
0.630	0.004	−1.3129	−1.3143	−1.3160	−1.3162	−1.3146	−1.3066
0.630	0.170	−1.2896	−1.2907	−1.2923	−1.2924	−1.2910	−1.2844
0.630	0.630	−1.0262	−1.0249	−1.0250	−1.0248	−1.0249	−1.0285
1.560	0.004	−0.3088	−0.3180	−0.2473	−0.2487	−0.2390	−0.2437
1.560	0.170	−0.3033	−0.3110	−0.2463	−0.2476	−0.2394	−0.2427
1.560	0.630	−0.2413	−0.2328	−0.2305	−0.2291	−0.2299	−0.2273
1.560	1.560	−0.0568	0.0008	−0.0988	−0.0890	−0.0937	−0.0935
3.426	0.004	0.3111	0.2972	0.2746	0.2722	0.2612	0.2689
3.426	0.170	0.3056	0.2938	0.2731	0.2711	0.2613	0.2680
3.426	0.630	0.2432	0.2561	0.2553	0.2574	0.2584	0.2570
3.426	1.560	0.0572	0.1446	0.1764	0.1923	0.1979	0.2027
3.426	3.426	−0.0576	0.0749	0.0647	0.0904	0.0835	0.0858
7.639	0.004	0.0706	0.0620	0.0648	0.0647	0.0621	0.0634
7.639	0.170	0.0693	0.0621	0.0646	0.0645	0.0622	0.0634
7.639	0.630	0.0551	0.0631	0.0631	0.0632	0.0635	0.0628
7.639	1.560	0.0130	0.0665	0.0626	0.0634	0.0647	0.0606
7.639	3.426	−0.0131	0.0682	0.0694	0.0706	0.0690	0.0611
7.639	7.639	−0.0030	0.0469	0.0467	0.0468	0.0464	0.0472

also good for the off-diagonal T matrices shown in Fig. 5.

For our other example we consider the Reid soft-core singlet S potential. In coordinate space this potential is given by ($x = \mu r$, $\mu = 0.7$ F^{-1})

$$V(r) = -10.463\, e^{-x}/x - 1650.6\, e^{-4x}/x$$

$$+ 6484.2\, e^{-7x}/x \text{ MeV .} \qquad (2.17)$$

In this case there is no two-body bound state, and

we must decide upon a value for B in (2.2). We have chosen to use $B = 0.0$. As mentioned, the anti-bound state is very near to $s = 0.0$, so this choice should be good enough.

The attractive eigenvalues are given in Table I, and the repulsive eigenvalues in Table II. The attractive eigenvalues are very similar to the Malfliet-Tjon results. The repulsive eigenvalues are, however, considerably smaller in absolute value than the Malfliet-Tjon eigenvalues, reflecting the great amount of repulsion in the Reid potential. (In momentum space the Reid potential is everywhere positive.)

FIG. 6. Diagonal Reid T matrices $T(p,p:s)$ for $s = 0.0$. Curves have the same meaning as in Fig. 2.
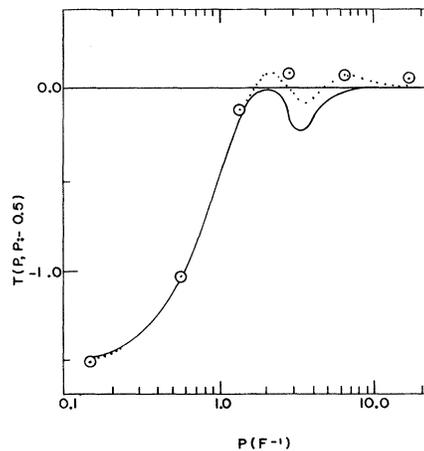
FIG. 7. Diagonal Reid T matrices for $s = -0.5$. Curves have the same meaning as in Fig. 2.

In Table VI we give the first two attractive and the first repulsive form factors for the Reid potential. They are quite similar to the Malfliet-Tjon potential form factors. [The Reid UPA form factor, $\psi_1^A(p)$, has an additional node at about 7 F^{-1}.] In Tables VII–IX, we give the UPE and exact T matrices for various momenta with $s = 0.0$, -0.5, and -3.0. Figures 6 and 7 give diagonal T matrices for $s = 0.0$ and -0.5. Figure 8 gives off-diagonal T matrices $T(p,k{:}s)$ as a function of p, for $s = -0.5$ and for $k = 0.567$, 1.41, and 3.08 F^{-1}.

Looking at the figures we see that except for a somewhat slower convergence of the repulsive terms, the Reid results are very similar to the Malfliet-Tjon case. The three-term approximation provides a very good representation of the true T matrix.

III. THREE-BODY CALCULATIONS

In this section we investigate the convergence of the unitary pole expansion by using it in three-body-state calculations. For simplicity we consider a model problem in which three identical spinless bosons interact by means of the S-wave part of a local potential. Our procedure is to retain different numbers of terms in the expansion (2.8), and investigate the convergence of the three-body bound-state energy as a function of the number of terms retained.

Retaining N terms in (2.8), the three-body bound-state equations become
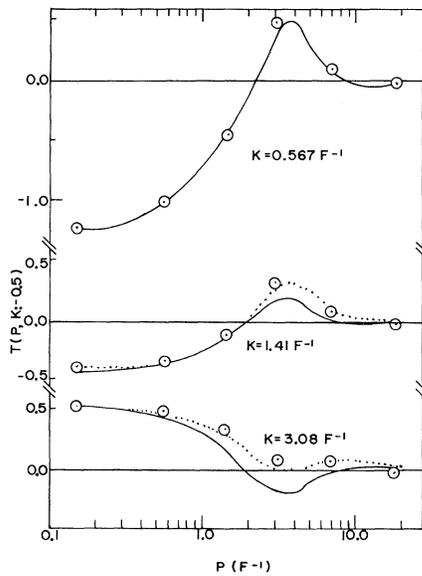
FIG. 8. Off-diagonal Reid T matrices $T(p,k{:}s)$ for $s = -0.5$ and $k = 0.567$, 1.41, and 3.08 F^{-1}. Curves have the same meaning as in Fig. 2.

$$\chi_i(p) = \sum_{j,k=1}^{N} \int_0^\infty Z_{i,j}(p,q{:}E_T)\Delta_{j,k}(E_T - \tfrac{3}{4}q^2)\chi_k(q)q^2dq , \quad (3.1)$$

with

$$Z_{i,j}(p,q{:}E) = \int_{-1}^1 \frac{\psi_i(|\vec{q}+\vec{p}/2|)\psi_j(|\vec{p}+\vec{q}/2|)}{E -p^2 -q^2 -\vec{p}\cdot\vec{q}} d\cos\theta_{pq} , \quad (3.2)$$

and

$$-[\Delta(\xi)^{-1}]_{j,k} = \lambda_j\delta_{j,k} + \int_0^\infty \frac{\psi_j(q)\psi_k(q)q^2dq}{\xi -q^2} . \quad (3.3)$$

All integrations indicated in the above equations were done numerically. Angular integrals were performed using the Gauss-Legendre formulas. For the most part, momentum integrations were performed using the Gauss-Gegenbauer formulas.[10,14] These formulas take advantage of the asymptotic form of the three-body kernels. We found that a ten-point Gauss-Gegenbauer mesh was extremely accurate, giving energies reliable

TABLE VI. UPE form factors for the Reid potential. Form factors are obtained as solution of (2.2) in the momentum representation with $s = 0.0$. $\psi_n A$ (ψ_n^R) is the form factor associated with the nth attractive (repulsive) eigenvalue. Momentum p is in F^{-1}.

p	ψ_1^A	ψ_1^R	ψ_2^A
3.84 − 3	9.472 − 1	3.211 − 1	−7.463 − 1
2.03 − 2	9.469 − 1	3.211 − 1	−7.454 − 1
5.02 − 2	9.455 − 1	3.212 − 1	−7.408 − 1
9.42 − 2	9.414 − 1	3.212 − 1	−7.271 − 1
1.53 − 1	9.322 − 1	3.214 − 1	−6.966 − 1
2.28 − 1	9.145 − 1	3.217 − 1	−6.395 − 1
3.21 − 1	8.847 − 1	3.223 − 1	−5.468 − 1
4.33 − 1	8.388 − 1	3.231 − 1	−4.121 − 1
5.67 − 1	7.733 − 1	3.242 − 1	−2.370 − 1
7.27 − 1	6.853 − 1	3.257 − 1	−3.133 − 2
9.16 − 1	5.729 − 1	3.276 − 1	1.866 − 1
1.14 − 0	4.356 − 1	3.297 − 1	3.912 − 1
1.40 − 0	2.757 − 1	3.320 − 1	5.499 − 1
1.72 − 0	1.007 − 1	3.340 − 1	6.224 − 1
2.09 − 0	−7.421 − 2	3.350 − 1	5.675 − 1
2.54 − 0	−2.249 − 1	3.342 − 1	3.624 − 1
3.08 − 0	−3.225 − 1	3.300 − 1	3.995 − 2
3.75 − 0	−3.434 − 1	3.207 − 1	−2.806 − 1
4.56 − 0	−2.814 − 1	3.048 − 1	−4.202 − 1
5.58 − 0	−1.725 − 1	2.810 − 1	−2.759 − 1
6.88 − 0	−6.027 − 2	2.492 − 1	1.198 − 2
8.56 − 0	6.549 − 3	2.106 − 1	1.367 − 1
1.08 − 1	2.039 − 2	1.679 − 1	3.828 − 2
1.38 − 1	9.991 − 3	1.248 − 1	−2.245 − 2
1.81 − 1	2.030 − 3	8.537 − 2	−4.497 − 4
2.44 − 1	1.857 − 4	5.288 − 2	4.694 − 4
3.44 − 1	1.784 − 5	2.913 − 2	−3.193 − 5
5.12 − 1	1.537 − 6	1.389 − 2	8.781 − 7
8.32 − 1	−6.573 − 7	5.454 − 3	−6.598 − 9
1.56 − 2	−3.285 − 7	1.583 − 3	−1.392 − 10
3.86 − 2	−5.993 − 8	2.608 − 4	4.718 − 12
2.04 − 3	−3.325 − 9	9.284 − 6	3.998 − 13

TABLE VII. Separable and exact T matrices for the Reid potential: $s = 0.0$.
Columns have the same meaning as in Table III.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	−10.9533	−10.8563	−10.9329	−10.9085	−10.9348	−10.9529
0.153	0.004	−10.7799	−10.6827	−10.7542	−10.7297	−10.7531	−10.7642
0.153	0.153	−10.6092	−10.5119	−10.5787	−10.5541	−10.5748	−10.5812
0.567	0.004	−8.9424	−8.8444	−8.8687	−8.8433	−8.8425	−8.8170
0.567	0.153	−8.8008	−8.7027	−8.7254	−8.6999	−8.6992	−8.6774
0.567	0.567	−7.3006	−7.2017	−7.2094	−7.1830	−7.1830	−7.1808
1.405	0.004	−3.1890	−3.0887	−3.0322	−3.0043	−2.9890	−2.9760
1.405	0.153	−3.1385	−3.0381	−2.9854	−2.9574	−2.9438	−2.9299
1.405	0.567	−2.6036	−2.5023	−2.4843	−2.4553	−2.4557	−2.4363
1.405	1.405	−0.9285	−0.8248	−0.8664	−0.8344	−0.8434	−0.8258
3.084	0.004	3.7302	3.8300	3.8341	3.8619	3.8443	3.8626
3.084	0.153	3.6712	3.7710	3.7748	3.8027	3.7872	3.8050
3.084	0.567	3.0454	3.1461	3.1474	3.1763	3.1769	3.1927
3.084	1.405	1.0861	1.1891	1.1861	1.2179	1.2282	1.2548
3.084	3.084	−1.2704	−1.1679	−1.1681	−1.1364	−1.1482	−1.1224
6.878	0.004	0.6971	0.7724	0.7736	0.7785	0.7843	0.7706
6.878	0.153	0.6860	0.7614	0.7626	0.7675	0.7726	0.7593
6.878	0.567	0.5691	0.6451	0.6455	0.6507	0.6505	0.6398
6.878	1.405	0.2029	0.2808	0.2799	0.2855	0.2822	0.2664
6.878	3.084	−0.2374	−0.1600	−0.1601	−0.1545	−0.1506	−0.1750
6.878	6.878	−0.0444	0.0141	0.0141	0.0151	0.0138	0.0294

to within 0.01 MeV for four coupled equations.

For our first examples, we present the results of a number of calculations using the local Hulthén potential. The form factors for this potential may be obtained in analytic form. A number of other authors have done three-body calculations using this potential. Sitenko, Karchenko, and Petrov,[2] and Lu and Levinger[2] have used the Weinberg series to calculate the energy of the model three-body problem. In Table X we compare the convergence of the UPE with that of the Weinberg series, as reported by these authors. (The Sitenko, Karchenko, and Petrov potential has $B = 2.225$ MeV and $\beta = 1.3797$ F^{-1}. Lu and Levinger's potential has B

TABLE VIII. Separable and exact T matrices for the Reid potential: $s = -0.5$.
Columns have the same meaning as in Table III.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	−1.5391	−1.5388	−1.5580	−1.5579	−1.5700	−1.5908
0.153	0.004	−1.5147	−1.5143	−1.5313	−1.5312	−1.5414	−1.5561
0.153	0.153	−1.4908	−1.4902	−1.5052	−1.5051	−1.5137	−1.5244
0.567	0.004	−1.2566	−1.2551	−1.2518	−1.2515	−1.2464	−1.2311
0.567	0.153	−1.2367	−1.2348	−1.2319	−1.2314	−1.2272	−1.2161
0.567	0.567	−1.0259	−1.0199	−1.0205	−1.0187	−1.0208	−1.0324
1.405	0.004	−0.4481	−0.4435	−0.4098	−0.4088	−0.3971	−0.3985
1.405	0.153	−0.4410	−0.4352	−0.4054	−0.4040	−0.3941	−0.3945
1.405	0.567	−0.3658	−0.3471	−0.3529	−0.3470	−0.3519	−0.3457
1.405	1.405	−0.1305	−0.0716	−0.1306	−0.1108	−0.1222	−0.1112
3.084	0.004	0.5242	0.5324	0.5285	0.5302	0.5165	0.5179
3.084	0.153	0.5159	0.5262	0.5228	0.5252	0.5137	0.5152
3.084	0.567	0.4279	0.4612	0.4619	0.4720	0.4776	0.4825
3.084	1.405	0.1526	0.2571	0.2639	0.2980	0.3112	0.3400
3.084	3.084	−0.1785	0.0068	0.0060	0.0645	0.0491	0.0921
6.878	0.004	0.0979	0.1030	0.1027	0.1031	0.1066	0.1038
6.878	0.153	0.0964	0.1028	0.1026	0.1030	0.1060	0.1035
6.878	0.567	0.0800	0.1006	0.1006	0.1027	0.1012	0.0998
6.878	1.405	0.0285	0.0933	0.0938	0.1006	0.0971	0.0854
6.878	3.084	−0.0334	0.0815	0.0815	0.0931	0.0972	0.0699
6.878	6.878	−0.0062	0.0650	0.0650	0.0673	0.0663	0.0794

TABLE IX. Separable and exact T matrices for the Reid potential: $s = -3.0$.
Columns have the same meaning as in Table III.

p	q	$1A$	$1A + 1R$	$2A + 1R$	$2A + 2R$	$3A + 2R$	Exact
0.004	0.004	−1.1588	−1.1559	−1.1885	−1.1877	−1.2079	−1.2341
0.153	0.004	−1.1404	−1.1373	−1.1669	−1.1660	−1.1837	−1.2031
0.153	0.153	−1.1224	−1.1190	−1.1458	−1.1448	−1.1603	−1.1752
0.567	0.004	−0.9460	−0.9401	−0.9422	−0.9404	−0.9378	−0.9224
0.567	0.153	−0.9311	−0.9246	−0.9265	−0.9245	−0.9223	−0.9105
0.567	0.567	−0.7724	−0.7602	−0.7604	−0.7564	−0.7567	−0.7643
1.405	0.004	−0.3374	−0.3229	−0.2811	−0.2764	−0.2629	−0.2626
1.405	0.153	−0.3320	−0.3162	−0.2783	−0.2731	−0.2613	−0.2597
1.405	0.567	−0.2754	−0.2457	−0.2431	−0.2327	−0.2344	−0.2243
1.405	1.405	−0.0982	−0.0252	−0.0787	−0.0517	−0.0607	−0.0433
3.084	0.004	0.3946	0.4188	0.4183	0.4261	0.4101	0.4166
3.084	0.153	0.3884	0.4148	0.4143	0.4229	0.4089	0.4154
3.084	0.567	0.3222	0.3718	0.3718	0.3891	0.3911	0.4008
3.084	1.405	0.1149	0.2368	0.2374	0.2825	0.2932	0.3318
3.084	3.084	−0.1344	0.0690	0.0690	0.1443	0.1315	0.1903
6.878	0.004	0.0737	0.0890	0.0902	0.0922	0.0970	0.0921
6.878	0.153	0.0726	0.0892	0.0903	0.0925	0.0967	0.0921
6.878	0.567	0.0602	0.0915	0.0916	0.0959	0.0953	0.0921
6.878	1.405	0.0215	0.0984	0.0968	0.1081	0.1049	0.0926
6.878	3.084	−0.0251	0.1032	0.1032	0.1220	0.1258	0.0988
6.878	6.878	−0.0047	0.0763	0.0762	0.0809	0.0797	0.0902

$= 0.43$ MeV and $\beta = 1.149$ F^{-1}.) The entry for an infinite number of terms was obtained by assuming that the three-body energy E_T depends upon the number of terms n retained according to the expression

$$E_T(n) = E_\infty + An^{-p} . \qquad (3.4)$$

The parameter p is determined by fitting for finite n.

In Table X the UPE energy for a finite number of terms is always more negative than the Weinberg value and hence closer to the actual (infinite number of terms) value. This is especially noticeable when comparing the first few terms. In both cases the one-term UPE is almost as good as the two-term Weinberg series. Using an appropriate extrapolation, it would appear that both methods are capable of giving the three-body energy to within at least 0.1 MeV.

Kok, Erens, and Van Wageningen[9] have used the equivalent two-body method to calculate the binding energy of three identical spinless bosons interacting by means of a local Hulthén potential. In Table XI we compare their results with UPE calculations for the same potentials. The results, shown in the last two rows, are in very good agreement .

Note added in proof: L. P. Kok [Ph.D. thesis, State University of Groningen, 1969 (unpublished)] has also performed three-body bound-state calculations using the UPE, obtaining results in accord with those given here. We would like to thank Dr. Kok for sending us his results.

We now move on to the consideration of more realistic local potentials containing some type of repulsion at small interparticle distances. Our first example is the potential V of Malfliet and Tjon (see Sec. II). Using this potential in the Faddeev

TABLE X. Comparison of three-body energies obtained with the UPE and Weinberg series for the Hulthén potential. The Weinberg series results are taken from Sitenko, Karchenko, and Petrov[2] and Lu and Levinger.[2] The UPE results were obtained as part of this work. All energies are in MeV.

Number of terms	Three-body energies			
	Potential of Sitenko *et al.*		Potential of Lu	
	Weinberg series	UPE	Weinberg series	UPE
1	−18.37	−24.35	−6.66	−9.36
2	−25.74	−26.36	−9.88	−10.20
3	−27.13	−27.24	−10.46	−10.62
4	−27.41	−27.43	−10.64	−10.74
∞	−27.61	−27.61	−10.77	−10.83

TABLE XI. Comparison of UPE results with the calculations of Kok, Erens, and Van Wageningen. The first five rows contain the calculated three-body bound-state energies retaining different numbers of terms in the UPE and an estimate of the local-potential result. The results of Kok *et al.*, Ref. 9, are given in the last row. Hulthén potential parameters are given in Kok *et al.*

Number of terms	Potential 2 E_T (MeV)	Potential 3 E_T (MeV)
1	−12.48	−25.41
2	−13.62	−27.52
3	−14.22	−28.46
4	−14.41	−28.70
∞	−14.55	−28.87
Kok *et al.*	−14.59 ± 0.1	−28.91 ± 0.1

equations, these authors found a three-body bound state at −7.3 MeV and estimated their answer to be accurate to within 0.1 MeV ($1\frac{1}{2}$%).

In Table XII we present the results of our UPE calculations with the Malfliet-Tjon potential. The first group of three columns gives the three-body energy when we retain one attractive and zero, one, two, or three repulsive terms in (2.8). The column ΔE_T gives the change in E_T brought about by adding the last term. The second three columns give the results keeping up to four attractive terms, while the last three columns show the effect of adding repulsive terms when two attractive terms are retained.

It is clear that the repulsive terms converge very rapidly. The $1A + \infty R$ entry is obtained from (3.4) using $p = 2$. The attractive terms seem to converge somewhat more slowly. The extrapolated ∞A entry is an average of the results obtained using $p = 1$ and 2 in (3.4).

Comparing the last column with column three, we see that the changes in energy brought about by the addition of repulsive terms is the same whether one or two attractive terms are retained. This means we can assess the effect of the attractive

and repulsive correction terms independently. The correction to the one-term energy due to other attractive terms is then found by subtracting the one-term results from the extrapolated infinite-attractive-terms result giving −0.204 MeV. Similarly, the correction for repulsive terms is +0.092 MeV. The actual three-body energy for the local S-wave Malfliet-Tjon potential should then be the sum of the one-term result and these corrections, or −7.55 MeV (entered as E_{UPE} at the bottom of Table XII). This result is in reasonable agreement with the Malfliet-Tjon value of −7.3 ± 0.1 MeV, but is outside their quoted errors.

One source of error in our calculations is the uncertainty involved in the extrapolation procedure. It would appear, however, that this is much less than 0.1 MeV, because of the smallness of the computed corrections and the consistency of the extrapolated results.

A second source of error arises from the numerical solution of (3.1). The presence of a large number of nodes in the two-body form factors could produce significant numerical errors. To test this, calculations were performed using a 16-point mesh. The largest discrepancy between the 10- and 16-point results was less than 0.005 MeV.

Another source of error arises in solving (2.2) for the two-body form factors. It would appear that the UPA form factor is obtained quite accurately with the numerical methods used, as is the first repulsive form factor. The determination of other form factors becomes difficult. However, since their effect is small, a rough determination should be sufficient.

Because of the cancellation between attractive and repulsive corrections, the −7.44-MeV result of the one-term UPE (the UPA) gives an excellent approximation[1] to the three-body energy differing by only 0.11 MeV or about $1\frac{1}{2}$% of E_{UPE}. A three-term approximation containing one attractive and one repulsive correction ($2A + 1R$ in Table XII) is even better, giving an energy of −7.49 MeV. The

TABLE XII. Unitary pole expansion calculations with the Malfliet-Tjon potential. E_T is the three-body bound-state energy obtained retaining different numbers of attractive (A) and repulsive (R) terms in the UPE for the Malfliet-Tjon potential. ΔE_T is the change in E_T brought about by the addition of a correction term. E_{UPE} is an estimate of the actual three-body energy using the full potential.

Terms	E_T (MeV)	ΔE_T (MeV)	Terms	E_T (MeV)	ΔE_T (MeV)	Terms	E_T (MeV)	ΔE_T (MeV)
$1A$	−7.437		$1A$	−7.437		$2A$	−7.562	
$1A + 1R$	−7.361	0.076	$2A$	−7.562	−0.125	$2A + 1R$	−7.486	0.076
$1A + 2R$	−7.349	0.012	$3A$	−7.594	−0.032	$2A + 2R$	−7.474	0.012
$1A + 3R$	−7.347	0.002	$4A$	−7.609	−0.015			
$1A + \infty R$	−7.345		∞A	−7.64				

$$E_{UPE} = -7.55 \text{ MeV}$$

difference here of 0.06 MeV is less than 1% of E_{UPE}. This agreement is not too surprising in view of the excellent agreement obtained between the expanded and exact T matrices in Sec. II.

Our next series of calculations uses the Reid singlet S potential, (2.17). Here the two-body system does not support a bound state, and $B = 0.0$ is used in (2.2). The results of these calculations are given in Table XIII. The repulsive terms are extrapolated nicely with $p = 1$ in (3.4). The attractive terms appear to converge slightly more slowly than with $p = 1$. The $p = 1$ value is given in Table XIII. The correction for repulsive terms is 0.149 MeV, that for attractive terms is -0.074 MeV, and the resulting energy, -1.02 MeV, is given at the bottom of the table. The UPA result, -1.094 MeV, differs by 0.07 MeV or 7% of the final value. However, the three-term result of -1.026 MeV is in excellent agreement with E_{UPE}.

We have also done a calculation with a two-term separable potential of Mongan.[15] We use his case II fit to the singlet S nucleon-nucleon phase shifts. The resulting three-body energy is -2.064 MeV. This result differs considerably from that of the Reid potential and indicates a strong dependence of the three-body energy upon the off-shell extrapolation of the two-body T matrix.

IV. DISCUSSION

The results presented here are interesting for a number of reasons. First of all, the accuracy of a pole approximation for central potentials containing repulsion is apparent. Not only is the three-body energy reproduced well, but the two-body T matrix is approximated fairly accurately also. This would seem to indicate that separable-potential calculations are more closely related to local-potential calculations than might otherwise have been thought. It should be noted, however, that the UPA form factors found here differ from the standard Yamaguchi-type form factor and most other form factors used in the literature.[10]

In this paper we have concerned ourselves primarily with central potentials. The accuracy of a pole approximation for a potential containing non-central components still needs investigation. We also need information on the accuracy of the approximation for scattering states.

The unitary pole approximation appears to be a good starting point for three-body calculations. The UPE gives a method of improving upon this ap-

TABLE XIII. Unitary-pole-expansion calculations with the Reid potential. E_T is the three-body bound-state energy obtained retaining different numbers of attractive (A) and repulsive (R) terms in the UPE for the Reid singlet S potential. E_{UPE} is an estimate of the actual three-body energy using the full potential.

Terms	E_T (MeV)	Terms	E_T (MeV)
$1A$	-1.094	$1A$	-1.094
$1A + 1R$	-0.998	$2A$	-1.122
$1A + 2R$	-0.972	$3A$	-1.136
$1A + 3R$	-0.963	$4A$	-1.144
$1A + \infty R$	-0.945	∞A	-1.168
$E_{\text{UPE}} = -1.02$ MeV			

proximation. The excellent agreement seen in Sec. II between the three-term UPE T matrices and the exact T matrices is quite encouraging. One other point that should be stressed is the comparative simplicity of the UPE approach to the three-body problem. Using a separable expansion we need only deal with coupled one-dimensional integral equations. Furthermore, once we have determined the UPE form factors, the T matrix is obtained by performing the integrals in Δ. Since the two-body T matrix appears in the Faddeev equations in the form $T(E - \frac{3}{4}p^2)$ where $0 \le p^2 < \infty$, this saves considerable effort as compared to the direct inversion of the Lippmann-Schwinger equations. It is also possible to obtain good analytic approximations to the UPE form factors so that deformation of contour methods may be used in the solution of three-body scattering problems.

Intimately related to the question of how to attack the three-body problem is the problem of relativistic corrections and multiparticle forces. Estimates place their contribution to the bound-state energy at as much as 1 MeV or more. With such uncertainties it may be unnecessary, if not fruitless, to try to solve the three-body problem for a specified nonrelativistic two-body potential to an accuracy much better than a few percent. Such accuracies should be obtainable from an approach based upon separable potentials.

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Hydrodynamical Model of the Nucleus with Surface Effects*

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An attempt is made to improve the predictions of the A dependence of giant dipole energies of the two-fluid hydrodynamical models by the introduction of a more realistic surface behavior of the ground-state densities. The Euler-Lagrange equations of motion for the dipole oscillation are obtained by the expansion of the energy functional $\epsilon[\rho, (\vec{\nabla}\rho)^2, \alpha(\vec{r})]$, where minimization yields acceptable neutron and proton densities, in quadratic powers of the densities and their derivatives. The equations are solved under certain limiting conditions, but within these limitations a much better A dependence is obtained.

I. INTRODUCTION

In recent years the statistical model of the nucleus has had some success[1] in relating the binding energies, proton and neutron densities, and surface properties of finite nuclei to infinite nuclear-matter calculations. An energy functional $\epsilon[\rho, \alpha]$ is obtained in the latter calculations for variable nuclear density ρ, and neutron excess $\alpha = (N - Z)/A$. Because of the finite range of nuclear forces a density variation correction proportional to $(\vec{\nabla}\rho)^2$ must be added when finite nuclei are considered. The quantities $\rho(\vec{r})$ and $\alpha(\vec{r})$ are, in principle, obtainable from the Euler-Lagrange equations arising from requiring

$$\delta \int d^3r \, \epsilon[(\vec{\nabla}\rho)^2, \alpha] = 0. \tag{1}$$

In many cases one can estimate the nuclear binding energy and charge and matter distribution from using trial functions $\rho(\vec{r})$ and $\alpha(\vec{r})$ in the variational principle expressed in Eq. (1).

In principle, one can obtain a Lagrangian for collective oscillations of the nucleus by expanding the energy functional through quadratic terms in the density fluctuation and adding a kinetic-energy term. The early work of Steinwedel and Jensen[2] on a hydrodynamical model of the giant dipole oscillation represented an attempt in this direction. They took the ground state of the nucleus to be a spherical matter distribution that is spatially a constant inside a rigid surface. Using the symmetry energy term in the semiempirical mass formula as the origin of the restoring force, they