High-Energy Optical Potential and Nuclear Correlations*

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The high-energy elastic scattering of nucleons by nuclei is studied using the optical-potential operator provided by Watson's multiple-scattering theory. The main motivation is to investigate the possibility of detecting two-body correlations in terms of an optical potential. We have obtained an expression for the second-order potential in coordinate space which explicitly exhibits its nonlocality in terms of the two-body correlations arising from the Pauli principle and the repulsive core of the nucleon-nucleon interaction. We have performed a self-consistent analysis of the elastic scattering of 1-GeV protons on ^{4}He using a reasonable nucleon-nucleon scattering amplitude, the one-body density obtained from elastic electron scattering, and a parametrization for the two-body density consistent with the one-body density and dependent upon a parameter identifiable as a correlation length. We have also attempted to discriminate among several sets of nucleon-nucleon phase shifts at 630 MeV. The scattering of 1-GeV protons by ^{12}C and ^{16}O is studied in order to investigate the relative importance of Pauli and short-range correlations. We also critically compare our work with that of several others.

INTRODUCTION

For approximately the last twenty years the intermediate-energy (~100 MeV), small-angle scattering of nucleons by heavy nuclei has been successfully fitted using potentials of the form[1]

$$V(r) = (V_{CR} + iV_{CI})\rho(r) + \left(\frac{1}{m_\pi^2}\right)(V_{SR} + iV_{SI})\frac{1}{r}\frac{d\rho(r)}{dr}\vec{\sigma}\cdot\vec{l}, \qquad \text{(I.1)}$$

where $\rho(r)$ is simply related to the one-body density measured in elastic electron scattering. For light nuclei and large scattering angles the above potential is not valid, and Kerman, McManus, and Thaler[2] (KMT) attribute this discrepancy to nucleon-nucleon correlation effects. It then seems feasible to obtain information about nuclear correlations from the optical potential, provided we understand how it is related to the nucleon-nucleon interaction and the structure of the target nucleus.

The one-body density or momentum distribution is essentially determined by elastic electron scattering. There are several methods for directly measuring the two-body density, but all of these are fairly difficult.[3] In our work we study the possibility of using high-energy elastic nucleon-nucleus scattering to investigate two-body correlations. Because the nucleon is a strongly interacting particle and the nucleon-nucleon scattering amplitude is a fairly rapidly decreasing function of the momentum transfer, multiple scattering dominates at large angles. As a result of the nucleus being strongly virtually excited, we obtain information about nuclear correlations, such effects being most marked at large momentum transfers.

The experimental work along these lines was initiated by Palevsky *et al.*[4] who measured the elastic scattering of 1-GeV protons by ^{4}He, ^{12}C, and ^{16}O. The most interesting results were obtained for p-^{4}He scattering, the corresponding differential cross section exhibiting a sharp diffraction-like structure with a minimum at $q^2 \sim 5$ fm^{-2}. In order to fit the differential cross section, they needed a central potential with a sharp transition region at the nuclear surface. Assuming the first-order optical potential to be valid, this is in strong disagreement with the one-body density derived from electron scattering by Frosch *et al.*[5] For ^{12}C and ^{16}O they obtained reasonably good fits using central Saxon-Woods potentials consistent with the electron scattering data.

These experiments have been fairly successfully fitted by Czyz and Lesniak,[6] and Bassel and Wilkin[7] using the multiple-scattering theory of Glauber.[8] However, several of the approximations involved, such as the eikonal approximation and the additivity of the phase shifts, have recently been investigated by Feshbach[9] and Schiff,[10] and sizable corrections have been suggested. Another drawback of the above approach is that it requires writing down a wave function which is more than one wants to consider at present, and furthermore the calculation of many-body operators with any kind of realistic wave function is quite difficult. From his analysis of p-^{4}He elastic scattering at 1 GeV, Cromer[11] concludes that the Glauber model is not reliable enough beyond the first diffraction minimum to provide quantitative results. There is

then some doubt on the applicability of the Glauber multiple-scattering theory for extracting information on nuclear correlations.

The scattering of a nucleon by a nucleus considered as a many-body system may be described using Watson's multiple-scattering theory,[12] which expresses the nucleon-nucleus scattering amplitude as a power series in the nucleon-nucleon scattering matrix. In this series the first term involves the one-body density, the next term the two-particle correlations, and so on. For high-energy elastic scattering the above theory provides for the optical potential a series expansion which converges rapidly. Up to now, only the first-order potential has been extensively investigated while the second-order potential, which arises from corrections to the impulse and multiple-scattering approximations, has only been roughly estimated and is now being studied.

Following a series of approximations which involved replacing certain integrals over correlation functions by correlation lengths, Johnston and Watson,[13] and Johnston[14] obtained an equivalent local potential for the second-order potential in coordinate space. Using this local potential, McDonald and Hull[15] studied nucleon-nucleus scattering in the energy range 95 to 350 MeV for targets ranging from carbon to lead, and found the corrections due to the second-order potential to be significant. For large scattering angles and light nuclei, these approximations need to be refined, and we will see under which conditions one recovers their result.

A somewhat different treatment of the second-order potential has been presented by Chalmers and Saperstein.[16] By working in momentum space, they were able to treat the problem without having to make the approximations involved in obtaining an equivalent local potential. However, in our work we are interested in explicitly exhibiting the optical potential in coordinate space.

Foldy and Walecka[17] have recently investigated high-energy elastic scattering by nuclei using separable potentials for the interaction. This permits them to solve exactly for the many-body scattering amplitude and to systematically investigate the optical potential to any degree of accuracy. This interesting work is mostly oriented towards investigating the multiple-scattering problem, and our work, although different in approach, leads to several of their conclusions.

The second-order potential is worth investigating because it contains much information about the structure of the target nucleus, namely the two-body correlations. We have obtained an exact expression for the second-order potential which explicitly exhibits its dependence on two-body correlations arising from either the Pauli principle or dynamics. In coordinate space this constitutes an energy-dependent, nonlocal potential directly related to the correlation function. We then proceed to obtain an effective equivalent local potential, and the problem is thereby reduced to solving a simple differential Schrödinger equation properly modified to take into account relativistic kinematics.

Using the approach briefly described above we have tried to understand the high-energy elastic scattering experiments of Palevsky *et al.*[4] We have performed a thorough and self-consistent analysis of the elastic scattering of 1-GeV protons on ^{4}He using a reasonable nucleon-nucleon scattering amplitude, the one-body density obtained from elastic electron scattering, and a parametrization for the two-body density consistent with the one-body density and dependent upon a parameter identifiable as a correlation length. The scattering of 1-GeV protons by ^{12}C and ^{16}O was studied in order to investigate the relative importance of Pauli and short-range correlations. Finally, we attempted to discriminate among several sets of nucleon-nucleon phase shifts at 630 MeV.

We now briefly outline the contents of the other sections. In Sec. II we present a derivation of the optical potential following the formulation of KMT. In Sec. III we study the first- and second-order potentials. In Sec. IV we present an expansion which effectively replaces the central and spin-dependent nonlocal potentials by equivalent local potentials. In Sec. V we discuss both the elementary-particle and nuclear-structure aspects of the problem. In Sec. VI we explicitly exhibit the potentials and also compare our work with the work of other authors. In Sec. VII we compare our calculated results with the experimental data, and finally, in Sec. VIII we present our conclusions.

II. OPTICAL-POTENTIAL OPERATOR

Even though our formulation is essentially that of KMT, we briefly review it for the sake of completeness, as well as to critically examine the approximations involved.

The scattering of a particle by a nucleus is described by the following Schrödinger equation:

$$(H_0 + V)\Psi_\epsilon(\vec{r}_0; \vec{r}_1, \ldots, \vec{r}_N) = \epsilon\Psi_\epsilon(\vec{r}_0; \vec{r}_1, \ldots, \vec{r}_N), \tag{II.1}$$

where

$$V = \sum_{i=1}^{N} v(\vec{r}_0, \vec{r}_i), \quad H_0 = H_T + K_0,$$

H_T being the target Hamiltonian, and K_0 the kinetic energy of the projectile;

$$\Psi_\epsilon = \sum_\alpha \chi_\alpha(\vec{r}_0)\Phi_\alpha(\vec{r}_1, \ldots, \vec{r}_N),$$

χ_α represents the scattered wave of the projectile, and Φ_α the eigenstates of H_T.

Equation (II.1) is equivalent to an integral equation for T:

$$T = V[1 + (1/d)T], \tag{II.2a}$$

where

$$d = \epsilon + i\delta - K_0 - H_T.$$

For fully antisymmetrized wave functions we may set $V = Nv$ provided we write

$$T = Nv[1 + (1/\alpha)T] = [N/(1 - Nv/\alpha)]v, \tag{II.2b}$$

where $1/\alpha = \mathcal{A}/d$ and $\mathcal{A}$ is the projection operator for completely antisymmetrized states.

It is convenient to introduce the scattering matrices

$$\tau = v[1 + (1/\alpha)\tau] \tag{II.3a}$$

and

$$t = v[1 + (\epsilon^0 - K_0 - K_j + i\delta)^{-1}t], \tag{II.3b}$$

where t is the free two-nucleon scattering operator. Making use of Eq. (II.3a), we express v in terms of τ, and Eq. (II.2b) becomes

$$T = N[1 - (N-1)\tau/\alpha]^{-1}\tau. \tag{II.4}$$

Defining $T' = [(N-1)/N]T$, we obtain an integral equation for T':

$$T' = U^{(0)}[1 + (1/\alpha)T'], \tag{II.5a}$$

where

$$U^{(0)} = (N-1)\tau. \tag{II.5a}$$

Despite its appearance Eq. (II.5) contains the full complexity of the $(N+1)$ many-body problem. For elastic scattering, with the target nucleus in the state $|0\rangle$, we wish to obtain the following type of equation for T':

$$\langle 0\vec{k}' | T' | 0\vec{k}\rangle \equiv T'_{00} = U_{00}(1 + \langle 1/\alpha\rangle_{00} T'_{00}). \tag{II.6}$$

We remark that Eq. (II.5) is not of the above form because τ is not diagonal among nuclear states. However, it may be reduced to the form of Eq. (II.6) by making use of the projection operators

$$P_0 = |0\rangle\langle 0|, \quad Q_0 = \sum_{n \neq 0} |n\rangle\langle n|, \quad P_0 + Q_0 = 1. \tag{II.7}$$

In terms of the above projection operators it is easily seen that

$$U^{(0)}[1 + (1/\alpha)T'] = U[1 + (P_0/\alpha)T'], \tag{II.8}$$

where the optical-potential operator U satisfies the integral equation

$$U = U^{(0)}[1 + (1/\alpha)Q_0 U]. \tag{II.9}$$

The perturbation solution for the optical potential in the states $|0\rangle$ then reads

$$U_{00} = U^{(0)}_{00} + \sum_{n \neq 0} U^{(0)}_{0n}\langle 1/\alpha\rangle_{nn} U^{(0)}_{n0} + \cdots, \tag{II.10}$$

and we study it up to second order in t.

The first-order optical potential involves two approximations: (i) the multiple-scattering approximation which neglects the virtual excitation in intermediate states and depends strongly on two-body and higher correlations; (ii) the impulse approximation which sets $\tau = t$, thereby reducing the many-body dynamics to that of the two-body problem. The first-order potential for elastic scattering in the ground state is then given by

$$\langle \vec{k}' | V^{(1)} | \vec{k}\rangle = (N-1)\langle 0\vec{k}' | t | \vec{k}0\rangle. \tag{II.11a}$$

The above potential includes many of the Glauber multiple-scattering terms.[8] Neglecting the Fermi motion of the struck nucleon and terms of order $(1/N)$ in t allows for the factorization of the above matrix element:

$$\langle \vec{k}' | V^{(1)} | \vec{k}\rangle \simeq (N-1)\langle 0\vec{k}' | t | \vec{k}0\rangle\langle 0 | e^{i(\vec{k}' - \vec{k})\cdot\vec{r}_j} | 0\rangle, \tag{II.11b}$$

where $\langle 0 | e^{i\vec{q}\cdot\vec{r}_j} | 0\rangle$ is the nuclear form factor $F(q)$ and t now operates only on the spin and isospin of the target nucleons.

The second-order potential includes the corrections to the impulse and multiple-scattering approximations binary in t. In order to evaluate the sum over the intermediate states we make use of the closure approximation replacing H_T by some average target excitation $\bar{W}$. To get a physical feeling for $\bar{W}$ we examine Eq. (II.10). The multiple-scattering term consists of an excitation of the target by collision with particle i and a subsequent rescattering by particle j which returns the target to its ground state. We then expect the main contribution to come from intermediate states resulting from quasielastic scattering so that $\bar{W} \simeq (\vec{k} - \vec{k}'')^2/2M_t$, where $\vec{k}''$ is the momentum of the projectile in the intermediate state. The closure may then be approximated by considering the incident particle as having an effective reduced mass μ. The second-order potential in momentum space is then given by[2]

$$\begin{aligned}\langle \vec{k}' | V^{(2)} | \vec{k}\rangle = (N-1)^2 \int d\vec{k}'' [\epsilon - \epsilon(k'') + i\delta]^{-1} \\ \times [\langle 0 | t_I(\vec{k}', \vec{k}'') t_j(\vec{k}'', \vec{k}) e^{i(\vec{k}' - \vec{k}'')\cdot\vec{r}_I} e^{i(\vec{k}'' - \vec{k})\cdot\vec{r}_j} | 0\rangle \\ - \langle 0 | t_j(\vec{k}', \vec{k}'') e^{i(\vec{k}' - \vec{k}'')\cdot\vec{r}_I} | 0\rangle \\ \times \langle 0 | t_j(\vec{k}'', \vec{k}) e^{i(\vec{k}'' - \vec{k})\cdot\vec{r}_j} | 0\rangle].\end{aligned} \tag{II.12}$$

$\epsilon(k'')$ is the energy of the scattered particle with

mass μ and momentum k'' in the intermediate states. $t_i t_j$ is a matrix in spin and isospin-space but is not longer a coordinate-space operator.

The potential derived in this section gives rise to

$$T' = [(N-1)/N]T,$$

while the one derived by Watson gives rise to T. As shown by KMT this permits one to easily take into account the identity of the N target nucleons. The two potentials are related by

$$V^{(1)} = [(N-1)/N]V_W^{(1)}, \quad V^{(2)} = [(N-1)/N]^2 V_W^{(2)},$$

where $V_W^{(1)}$ and $V_W^{(2)}$ are the first- and second-order potentials obtained using Watson's scheme. This difference, although easily accounted for, is not trivial. In the Born approximation, the two results are equivalent only for the first-order potentials while the second-order potentials differ by terms of order $1/N$. However, the Born approximation is not expected to be valid for the cases of interest, and at large momentum transfers this difference may be sizable. In our work we have limited ourselves to the potential as given above and have concentrated on evaluating the second-order potential.

KMT provide a slightly different expansion for the optical-potential operator from the one given by Eq. (II.7). In their expansion only nondiagonal matrix elements of $U^{(0)}$ appear in intermediate states. Their second-order term includes all of our higher-order terms which contain elastic scattering in intermediate states. The effect of including $U^{(0)}$ in the propagator has been investigated by Feshbach and Hüfner[18] using several idealized models. Their work indicates that at high energies the effect is small.

In the above discussion we have assumed that the incident particle is distinguishable from the target nucleons. In case the projectile is a nucleon one should consider completely antisymmetrized wave functions of the $(N+1)$ nucleons. Takeda and Watson[19] showed that the main effects of the Pauli principle may be accounted for by using properly antisymmetrized scattering amplitudes for each nucleon-nucleon scattering. Still omitted are processes referred to as "target exchanges." Sawicki[20] has computed these corrections and concludes that at high energies they are negligible. In our work the Pauli principle is simply taken into account following the prescription of Takeda and Watson.[19]

III. FIRST- AND SECOND-ORDER POTENTIALS

A. First-Order Potential

The first-order potential is given by Eq. (II.11). For nonvanishing momentum transfer we require the t matrix off the energy shell, because the kinematics for nucleon-nucleus scattering differ from those for nucleon-nucleon scattering. The off-energy-shell effects can only be calculated if a nucleon-nucleon potential is assumed. This effect has been investigated by Mulligan,[21] and Reading and Mackellar.[22] The t matrix is generally nonlocal and independent of nuclear correlations, and the nonlocal corrections are third order in the nucleon-nucleon interaction. In our work we assume the t matrix to be the same function of the momentum transfer both on and off the energy shell. We are then interested in studying a nonlocality which is very different from the one discussed above. Our nonlocal potential arises from multiple-scattering effects, depends on two-body correlations, and is of second order in t.

The most general form of the nucleon-nucleon scattering amplitude consistent with rotational invariance, parity, time-reversal invariance, and charge independence may be written as[23]

$$\begin{aligned} M(\vec{\Delta}) = {} & A + C(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\hat{n} + B\vec{\sigma}_1\cdot\hat{n}\vec{\sigma}_2\cdot\hat{n} \\ & + \tfrac{1}{2}(G+H)\vec{\sigma}_1\cdot\hat{m}\vec{\sigma}_2\cdot\hat{m} + \tfrac{1}{2}(G-H)\vec{\sigma}_1\cdot\hat{l}\vec{\sigma}_2\cdot\hat{l} \\ & + [A' + C'(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\hat{n} + B'\vec{\sigma}_1\cdot\hat{n}\vec{\sigma}_2\cdot\hat{n} \\ & + \tfrac{1}{2}(G'+H')\vec{\sigma}_1\cdot\hat{m}\vec{\sigma}_2\cdot\hat{m} + \tfrac{1}{2}(G'-H')\vec{\sigma}_1\cdot\hat{l}\vec{\sigma}_2\cdot\hat{l}]\vec{\tau}_1\cdot\vec{\tau}_2, \end{aligned} \tag{III.1}$$

where $\vec{\sigma}$ and $\vec{\tau}$ are the spin and isospin operators, respectively. A, B, C, G, H, A', B', C', G', and H' are functions of the momentum transfer $\vec{\Delta} = \vec{k}' - \vec{k}$. $\hat{n}$, $\hat{l}$, and $\hat{m}$ are three mutually orthonormal vectors defined to be in the directions $(\vec{k}\times\vec{k}')$, $(\vec{k}+\vec{k}')$, and $(\vec{k}'-\vec{k})$, respectively.

$t_{N\,\mathrm{c.m.}}$, the two-body t matrix in the nucleon-nucleus barycentric system, and $M(=f_{2\,\mathrm{c.m.}})$, the corresponding scattering amplitude in the nucleon-nucleon barycentric system, are related by[2]

$$t_{N\,\mathrm{c.m.}} = \bar{\eta} M, \tag{III.2}$$

where

$$\bar{\eta} = -\frac{1}{(2\pi)^2 \epsilon_{1,L}} \left(\frac{k_{1,L}}{k_{2\,\mathrm{c.m.}}}\right) \left(\frac{1 + (m/2\epsilon_{1,L})(N + 1/N)}{(1 + m/N\epsilon_{1,L})(1 + Nm/\epsilon_{1,L})}\right). \tag{III.3}$$

L refers to the lab system, ϵ_1 and k_1 are the energy and momentum of particle 1 which we take to be the projectile, and N is the number of target nucleons.

We now consider even-even nuclei with $J = T = 0$ and assume that the scattering amplitudes are only functions of the momentum transfer. In coordinate space the first-order potential is then given by[24]

$$\langle \vec{r}' \mid V^{(1)} \mid \vec{r} \rangle = V^{(1)}(r)\delta(\vec{r}-\vec{r}')$$
$$= 4\pi(N-1)\bar{\eta}\int q^2 dq\, F(q)A(q)j_0(qr)$$
$$+ 4\pi(N-1)\bar{\eta}\frac{1}{ir}\int q^3 dq\, F(q)\mathcal{C}(q)j_1(qr)\vec{\sigma}_0\cdot\vec{1}, \quad \text{(III.4)}$$

where j_0 and j_1 are spherical Bessel functions, F is the form factor, and

$$\mathcal{C}(q) = \frac{k_{2\ \text{c.m.}}}{k_{N\ \text{c.m.}}}\frac{C(q)}{|\vec{k}'_{2\ \text{c.m.}}\times\vec{k}_{2\ \text{c.m.}}|}. \quad \text{(III.4')}$$

Assuming that the scattering amplitudes are so slowly varying compared to the form factor that their momentum dependence may be neglected, which is valid only for heavy nuclei, it is easily seen that Eq. (III.4) reduces to the simple potential of Eq. (I.1) with $\rho(r)$ being the nuclear density.

A potential of the form given by Eq. (III.4) may violate unitarity because of the imaginary spin-orbit potential. The reason may be that the spin-orbit potential in coordinate space is simply not of the form

$$(1/r)(d\rho/dr)\vec{\sigma}\cdot\vec{1},$$

because to obtain the above form requires integrating beyond the physical region and the off-energy and off-momentum shell spin-orbit scattering amplitudes may differ drastically from the physical ones. However, since the spin-orbit potential is proportional to l, if unitarity is to be violated it is violated in the higher partial waves where there is little scattering, so it need not be a serious problem. A possible way of insuring unitarity is to cut off the spin-orbit potential for high l's.

As already stated, the first-order optical potential does not satisfactorily explain the large-angle scattering off light nuclei, especially ^{4}He. We attribute this discrepancy to multiple-scattering effects and the impulse approximation, and we now fully investigate this problem.

B. Second-Order Potential

We must first evaluate the matrix elements appearing in Eq. (II.12). For simplicity's sake we consider only even-even nuclei with $J=T=0$ and also assume L-S coupling with $L=S=0$. Using the scattering amplitude given by Eq. (III.1), for such nuclei we obtain[14]

$$\langle 0 \mid e^{i\vec{q}\cdot\vec{z}_I}e^{i\vec{q}'\cdot\vec{z}_j}t_I(\vec{q})t_j(\vec{q}') \mid 0\rangle$$
$$- \langle 0 \mid e^{i\vec{q}\cdot\vec{z}_I}t_I(\vec{q}) \mid 0\rangle\langle 0 \mid e^{i\vec{q}'\cdot\vec{z}_j}t_j(\vec{q}') \mid 0\rangle$$
$$= [\rho_2(\vec{q},\vec{q}') - \rho(\vec{q})\rho(\vec{q}')]\bar{\eta}^2\left(\alpha - \frac{3\beta}{N-1}\right)$$
$$- \rho(q)\rho(q')\bar{\eta}^2\left(\frac{3\beta}{N-1}\right), \quad \text{(III.5)}$$

where

$$\rho(\vec{q}) = \langle 0 \mid e^{i\vec{q}\cdot\vec{z}_I} \mid 0\rangle, \quad \rho_2(\vec{q},\vec{q}') = \langle 0 \mid e^{i\vec{q}\cdot\vec{z}_I}e^{i\vec{q}'\cdot\vec{z}_j} \mid 0\rangle,$$

and α and β are given by

$$\alpha = A^2 + C^2 + 2AC\vec{\sigma}_0\cdot\hat{n}, \quad \text{(III.5')}$$
$$\beta = [C^2 + B^2 + \tfrac{1}{2}(G^2+H^2) + 2BC\vec{\sigma}_0\cdot\hat{n}]$$
$$+ A' + C' + 2A'C'\vec{\sigma}_0\cdot\hat{n}$$
$$+ 3[C'^2 + B'^2 + \tfrac{1}{2}(G'^2+H'^2) + 2B'C'\vec{\sigma}_0\cdot\hat{n}].$$

$\vec{\sigma}_0$ operates on the spin of the incident nucleon, $A^2 \equiv A(q)A(q')$, and $2BC \equiv C(q)B(q') + C(q')B(q)$. We observe that we have two types of terms; one proportional to the one-body density squared and the other to the correlation function. However, the term proportional to ρ^2 is of order $(1/N)$, and for heavier nuclei it is consistent with the derivation we have presented to neglect it. As we will see, if $\rho(q)\rho(q')$ is of the "super separable" form, it does not lead to any essential difficulties.

For the sake of clarity and with little loss of generality we now consider the following simplified scattering amplitude:

$$t_{N\ \text{c.m.}}(\vec{k}',\vec{k}) = \bar{\eta}A + \bar{\eta}\,\mathcal{C}(\vec{\sigma}_0+\vec{\sigma}_I)\cdot(\vec{k}\times\vec{k}'), \quad \text{(III.6)}$$

where $\bar{\eta}$ and $\mathcal{C}$ are as defined by Eqs. (III.3) and (III.4'). For scattering of nucleons by spin-0 nuclei, the above amplitude gives rise to the only two possibly nonvanishing types of terms: central and spin-orbit. In evaluating the second-order potential let us make further use of the forward-scattering approximation[13,15] neglecting terms of θ^2 and higher. From the symmetry of the integrand in Eq. (II.12) we expect the dominant contributions to this integral to come from $\vec{k}'' = \frac{1}{2}(\vec{k}+\vec{k}')$, and we approximate $(\vec{k}\times\vec{k}'')$ by $\frac{1}{2}(\vec{k}\times\vec{k}')$. Making use of the scattering amplitude given by Eq. (III.6) and the forward-scattering approximation, the second-order potential in momentum space as given by Eq. (II.12) may be rewritten as

$$\langle \vec{k}' \mid V^{(2)} \mid \vec{k}\rangle = \langle \vec{k}' \mid V_C^{(2)} \mid \vec{k}\rangle + \langle \vec{k}' \mid V_S^{(2)} \mid \vec{k}\rangle, \quad \text{(III.7a)}$$

where

$$\langle \vec{k}' \mid V_C^{(2)} \mid \vec{k}\rangle = (N-1)^2\bar{\eta}^2A^2(0)\int d\vec{k}''\frac{\tilde{C}(\vec{k}'-\vec{k}'',\vec{k}''-\vec{k})}{\epsilon-\epsilon(k'')+i\delta}, \quad \text{(III.7b)}$$

$$\langle \vec{k}' \mid V_S^{(2)} \mid \vec{k}\rangle = (N-1)^2\bar{\eta}^2A(0)\mathcal{C}(0)\vec{\sigma}\cdot(\vec{k}\times\vec{k}')$$
$$\times\int\frac{d\vec{k}''\,\tilde{C}(\vec{k}'-\vec{k}'',\vec{k}''-\vec{k})}{\epsilon-\epsilon(k'')+i\delta}, \quad \text{(III.7c)}$$

$$\tilde{C}(\vec{q},\vec{q}') \equiv \rho_2(\vec{q},\vec{q}') - \rho(\vec{q})\rho(\vec{q}').$$

$\tilde{C}$ is referred to as the pair-correlation function

and arises because the nucleons inside the nucleus are correlated as a result of the Pauli principle and dynamical effects which depend on the nucleon-nucleon interaction. This is to be further discussed in Sec. V. We remark that we have assumed a simplified pair-correlation function in so far as we have neglected its possible spin and isospin dependence.

We now wish to obtain the coordinate-space representation of the above potential. The above integration turns out to be quite manageable if one expresses the correlation function $\tilde{C}$ in terms of the Fourier transform $C(\vec{x}, \vec{x}')$. Proceeding as indicated, we find that in coordinate space the second-order central potential assumes the following attractive form:

$$\langle \vec{x}' | V_C^{(2)} | \vec{x} \rangle = (2\pi)^3 (N-1)^2 \bar{\eta}^2 A^2(0) C(\vec{x}, \vec{x}') \times \int \frac{d\vec{k}\, e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}}{\epsilon - \epsilon(k) + i\delta} . \tag{III.8}$$

The remaining integral is simply the well-known free Green's function for a particle of mass μ and energy ϵ. In our work we restrict ourselves to the nonrelativistic limit, $\epsilon(k) = k^2/2\mu$.

We now proceed to obtain the second-order spin-dependent potential in coordinate space:

$$\langle \vec{x}' | V_S^{(2)} | \vec{x} \rangle = \mathfrak{N} \vec{\sigma} \cdot \int d\vec{k}\, d\vec{k}'\, d\vec{k}''\, (\vec{k} \times \vec{k}') \times \frac{\tilde{C}(\vec{k}' - \vec{k}'', \vec{k}'' - \vec{k})}{\epsilon - \epsilon(k'') + i\delta} e^{i(\vec{k}'\cdot\vec{x}' - \vec{k}\cdot\vec{x})} , \tag{III.9}$$

where

$$\mathfrak{N} \equiv [(N-1)^2/(2\pi)^3] \bar{\eta}^2 A(0) \mathfrak{C}(0) .$$

We eliminate $\vec{k}$ and $\vec{k}'$ in favor of $\vec{\nabla}_x$ and $\vec{\nabla}_{x'}$ and are then left with an integral similar to the one studied for the second-order central potential. The second-order spin-dependent potential then reduces to

$$\langle \vec{x}' | V_S^{(2)} | \vec{x} \rangle = \vec{\sigma} \cdot [\vec{\nabla}_x \times \vec{\nabla}_{x'} \bar{V}_S(\vec{x}, \vec{x}')] , \tag{III.10a}$$

where

$$\bar{V}_S(\vec{x}, \vec{x}') = (N-1)^2 (2\pi)^3 \bar{\eta}^2 A(0) \mathfrak{C}(0) C(\vec{x}, \vec{x}') \times \int \frac{d\vec{k}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\epsilon - \epsilon(k) + i\delta} . \tag{III.10b}$$

The second-order potential as given by Eqs. (III.8) and (III.10) involves the closure and forward-scattering approximations. The closure approximation has been discussed in the previous section. The forward-scattering approximation overestimates the second-order potential for scattering at finite angles, the corrections being most important at large angles.[16] If t depends only on the momentum transfer, it is trivial to take into account its momentum dependence; however, the second-order potential would no longer be simply related to the nuclear densities. Since the second-order potential is most strongly felt at large momentum transfer (especially for the diffraction structure which we are interested in), a simple way of reducing this overestimate is to evaluate t in Eqs. (III.8) and (III.10) at one-half the momentum transfer at the minimum. Thereby a slight momentum dependence is introduced, the importance of which depends on the momentum dependence of the scattering amplitude and the correlation function.

IV. EQUIVALENT EFFECTIVE LOCAL POTENTIAL

The Schrödinger equation for a nonlocal potential is essentially an integrodifferential equation. Because of the large number of partial waves scattered at the energies considered, solving the above equation is at present somewhat impracticable. Furthermore, we are interested in seeing whether the data can be understood in terms of a local potential. We then proceed to obtain an effective equivalent local potential using a method presented by Mulligan.[21]

For a nonlocal potential $V(\vec{r}, \vec{r}')$, the Schrödinger equation assumes the form

$$[(1/2\mu)\nabla^2 + \epsilon]\psi(\vec{r}) = \int V(\vec{r}, \vec{r}')\psi(\vec{r}')\, d\vec{r}' = \tilde{V}(\vec{r} \mid \vec{\nabla}/i)\psi(\vec{r}) , \tag{IV.1}$$

where

$$\tilde{V}(\vec{r} \mid \vec{\nabla}/i) = \int V(\vec{r}, \vec{r} + \vec{x}) e^{i\vec{x}\cdot(\vec{\nabla}/i)}\, d\vec{x} .$$

In our work we consider only correlation functions which give rise to nonlocal potentials of the "super separable form"[25]:

$$V(\vec{r}, \vec{r}') = v(\tfrac{1}{2}(\vec{r} + \vec{r}'))u(\vec{r} - \vec{r}') . \tag{IV.2}$$

Short-range correlations and even Pauli correlations in heavy nuclei are such that R_c, the range of nonlocality u, is much smaller than R, the range of the nucleus as measured by v: $R_c \ll R$. Under the above conditions we may then write

$$V(\vec{r}, \vec{r}') = \tfrac{1}{2}[v(\vec{r})u(\vec{r} - \vec{r}') + v(\vec{r}')u(\vec{r} - \vec{r}')] . \tag{IV.3}$$

The momentum-dependent potential equivalent to the nonlocal potential of Eq. (IV.3) is given by

$$\tilde{V}(\vec{r}\,|\,\vec{\nabla}/i) = \tfrac{1}{2}[v(\vec{r})\tilde{u}(\vec{\nabla}/i) + \tilde{u}(\vec{\nabla}/i)v(\vec{r})], \qquad \text{(IV.3')}$$

where $\tilde{u}$ is the Fourier transform of u as given by Eq. (IV.1). For the purpose of the following presentation we consider the unsymmetrized potential $v(\vec{r})\tilde{u}(\vec{\nabla}/i)$ which differs from the one given by Eq. (IV.3') simply by $\frac{1}{2}[\tilde{u}(\vec{\nabla}/i)v(\vec{r})]$, where the gradient operates only on v.

The Born approximation suggests that we make a Taylor series expansion of $\tilde{u}(\vec{\nabla}/i)$ about $\vec{k}_0$, where for convenience we take $\vec{k}_0$ to be the incident momentum:

$$v(\vec{r})\tilde{u}(\vec{\nabla}/i)\psi(\vec{r}) = v(\vec{r})\tilde{u}(\vec{k}_0)\psi(\vec{r}) + [v(\vec{r})\vec{\nabla}_k\tilde{u}(\vec{k})]_{\vec{k}=\vec{k}_0} \times (\vec{\nabla}/i - \vec{k}_0)\psi(\vec{r}) + \cdots . \qquad \text{(IV.4)}$$

The above series converges quite rapidly for the cases of interest, and in most of our work we retain only the first two terms. The problem is thereby reduced to that of scattering from two potentials V_1 and V_2, where:

$$V_1\psi(\vec{r}) = v(r)\tilde{u}(k_0)\psi(\vec{r}),$$

$$V_2\psi(\vec{r}) \simeq [v(r)\vec{\nabla}_k\tilde{u}(k)]_{k_0}\cdot(\vec{\nabla}/i - \vec{k}_0)\psi(\vec{r}),$$

with $V_1 \gg V_2$.

The problem is then best studied in terms of the "two-potential formula."[26] We approximate ψ, the solution to scattering from the two potentials V_1 and V_2, by ϕ which is obtained as if only V_1 were present. Once ϕ is obtained, we may study the expansion for the nonlocal potential given by Eq. (IV.4) and thereby obtain an effective equivalent local potential. In spirit, the above approach is essentially the distorted-wave Born approximation, and we thereby improve on the Born approximation.

We now apply the above technique to a Schrödinger equation with a spin-independent nonlocal potential:

$$[-\nabla^2/2\mu + v_S(\vec{r})\vec{\sigma}\cdot\vec{1} + \sum_{j=1} v_0^{(j)}(\vec{r})\tilde{u}_C^{(j)}(\vec{\nabla}/i) - \epsilon]\psi(\vec{r}) = 0. \qquad \text{(IV.5)}$$

We expand $\tilde{u}_C^{(j)}(\vec{\nabla}/i)$ about $\vec{k}_0$ and keep only the first two terms in the series given by Eq. (IV.4). We then replace $\psi(\vec{r})$ by $\phi(\vec{r})$ which, because we are considering high-energy and small-angle scattering, may be calculated using the eikonal approximation[8]:

$$\phi(\vec{r}) \simeq \exp\Big\{i\vec{k}_0\cdot\vec{r} - \frac{ik_0}{2\epsilon}\int_{-\infty}^{z}[v_C(\vec{b}+\hat{k}_0z') + v_S(\vec{b}+\hat{k}_0z')\vec{\sigma}\cdot\vec{1}]dz'\Big\}\chi_i, \qquad \text{(IV.6)}$$

where

$$v_C(\vec{r}) = \sum_{j=1} v_C^{(j)}(\vec{r})\tilde{u}_C^{(j)}(\vec{k}_0),$$

$\vec{b}$ lies in the plane perpendicular to $\vec{k}_0$, and χ_i is the initial spinor. Proceeding as indicated, we find that Eq. (IV.5) is thereby reduced to a simple Schrödinger equation with an equivalent effective local potential given by

$$V_{\text{eff}}(\vec{r}) = V_{\text{eff}}^{(C)}(\vec{r}) + V_{\text{eff}}^{(S)}(\vec{r})\vec{\sigma}\cdot\vec{1}, \qquad \text{(IV.7a)}$$

where

$$V_{\text{eff}}^{(C)}(\vec{r}) = \sum_{j=1} v_C^{(j)}(\vec{r})\tilde{u}_C^{(j)}(\vec{k}_0)\left[1 - \frac{k_0}{2\epsilon}\sum_{j=1} v_C^{(j)}(\vec{r})\frac{d\tilde{u}_C^{(j)}(k_0)}{dk}\right], \qquad \text{(IV.7b)}$$

$$V_{\text{eff}}^{(S)}(\vec{r}) = v_S(\vec{r})\left[1 - \frac{k_0}{2\epsilon}\sum_{j=1} v_C^{(j)}(\vec{r})\frac{d\tilde{u}_C^{(j)}(k_0)}{dk}\right]. \qquad \text{(IV.7c)}$$

We remark that the validity of the above approximations depends only on the weakness of the nonlocality.

We now apply the same technique to obtain a local potential effectively equivalent to the spin-dependent nonlocal potential:

$$\langle\vec{x}'\,|\,V_S^{(2)}\,|\,\vec{x}\rangle = \vec{\sigma}\cdot[\vec{\nabla}_x\times\vec{\nabla}_{x'}v_S(\vec{x})u_S(|\vec{x}-\vec{x}'|)]. \qquad \text{(IV.8)}$$

The above potential gives rise to the following term in the Schrödinger equation

$$-\vec{\sigma}\cdot[\vec{\nabla}_x v_S(\vec{x})]\times\vec{\nabla}_x[\tilde{u}_S(\vec{\nabla}/i)\psi(\vec{x})],$$

where the first gradient operates on v_S only. We treat the spin-dependent nonlocal potential in a "Born-like approximation," retaining only the first term in the expansion given by Eq. (IV.4). If v_S depends only on the magnitude of $\vec{x}$, to lowest order the spin-dependent nonlocal potential may be approximated by a spin-dependent local potential:

$$\vec{\sigma}\cdot[\vec{\nabla}_x\times\vec{\nabla}_{x'}v_S(x)u_S(|\vec{x}-\vec{x}'|)] \to \tilde{u}_S(k_0)\frac{1}{ix}\frac{dv_S(x)}{dx}\vec{\sigma}\cdot\vec{1}\delta(\vec{x}-\vec{x}'). \qquad \text{(IV.9)}$$

It is difficult to determine how consistent it is to use the potential so obtained in calculations which exactly integrate the Schrödinger equation. However, the approximations involve only the nonlocality of the second-order potential, which is usually of much less importance than the first-order potential and the zeroth moment of the second-order potential which are treated exactly. The convergence of the expansion given in Eq. (IV.4) can be systematically investigated.

The potential so obtained is then used in a Schrödinger equation in which the kinematics of the projectile are treated relativistically[26]:

$$-\frac{1}{2m}\left(1+\frac{\epsilon}{M_t}\right)\nabla^2\psi + \frac{\epsilon}{m}V\psi = \frac{1}{2m}(\epsilon^2 - m^2)\psi, \qquad \text{(IV.10)}$$

where m and M_t are the masses of the projectile

and target, respectively; ϵ is the total energy of the projectile; and V is the potential.

V. GENERAL PROPERTIES OF THE NUCLEON-NUCLEON INTERACTION AND NUCLEAR STRUCTURE

A. Nucleon-Nucleon Interaction

1. 630 MeV

To determine the scattering amplitude given by Eq. (III.1) in terms of a phase-shift analysis is very complicated at this energy because it requires a large number of partial waves, and also because we are above the threshold for pion production. In our work we make use of three solutions presented by MacGregor, Arndt, and Wright.[27] Following their notation we refer to them as A2, B3, and C5. They differ essentially in the treatment of the inelastic parameters. For specific details we refer the reader to the above papers.

The scattering amplitudes corresponding to the above sets of phase shifts were calculated using what is commonly referred to as a modified phase-shift analysis. In our work we attempt to discriminate among the above sets of nucleon-nucleon phase shifts by means of high-energy elastic nucleon-nucleus scattering.

2. 1 GeV

At 1 GeV the scattering matrices are as yet essentially undetermined in terms of phase shifts. However, high-energy scattering at small momentum transfer has been fairly successfully described by Regge-pole theory. Assuming only Regge poles, the scattering amplitude is simply given by Eq. (III.6). The amplitudes were parametrized in the following convenient form[28]:

$$A_{NN}(\Delta)=[(i+\alpha_A^{NN})D_A k\sigma_{NN}/4\pi]e^{-b_A^2\Delta^2},$$

$$C_{NN}(\Delta)=i(\Delta^2/4m^2)^{1/2}C'_{NN}(\Delta),$$

where

$$C'_{NN}(\Delta)=[(i+\alpha_C^{NN})D_C k\sigma_{NN}/4\pi]e^{-b_C^2\Delta^2},$$

k is the center-of-mass momentum, and $\vec{\Delta}$ is the momentum transfer. The parameters were obtained by adjusting the Regge parametrization of Rarita *et al.*[29] so as to make them consistent with the experimental data of Dutton *et al.*[30]:

$$\alpha_A^{pn}=0.1\pm0.16,\quad \alpha_A^{pp}=-0.5\pm0.15,$$

$$D_A=1,\quad b_A^2=4\ \text{GeV}^{-2},$$

$$\alpha_C^{pn}=\alpha_C^{pp}=-1.1,\quad D_C=2.8,\quad b_C{}^2=5\ \text{GeV}^{-2}.$$

The above scattering amplitudes describe the elastic scattering of two free nucleons. As discussed in Sec. III, we need the scattering amplitudes off the energy shell since the kinematics are those of the nucleon-nucleus system. In our work we assume that the scattering amplitudes are functions only of the momentum transfer, and we simply relate the t matrix in the N-c.m. and 2-c.m. systems according to Eq. (III.2).

B. Nuclear Structure

1. *One-Body Density*

For even-even nuclei with $Z=N/2$ and $T=0$, the charge and body form factors are simply related by

$$F_{ch}=F_b(F_{ch}^p+F_{ch}^n), \tag{V.1}$$

where F_{ch}^p and F_{ch}^n are the normalized proton and neutron form factors for which we used the experimental measurements of Janssens *et al.*[31]

The charge form factor of ^{4}He has been measured up to large momentum transfer ($q^2\sim 20$ fm^{-2}) by Frosch *et al.*[5] using elastic electron scattering. They found a sharp minimum at $q^2\sim 10$ fm^{-2}:

$$F_{ch}(q^2)=[1-(a^2q^2)^n]e^{-b^2q^2},$$

with $n=6$, $a=0.316\pm0.001$ fm, $b=0.681\pm0.002$ fm. This diffraction structure occurs at a larger momentum transfer than the one measured by Palevsky *et al.*[4] using elastic scattering of 1-GeV protons. The p-^{4}He data cannot be fitted using the first-order potential determined by the above density including the experimental uncertainties in a and b.

The charge form factors of ^{12}C and ^{16}O are for small momentum transfer consistent with the shell model using harmonic-oscillator wave functions[32]:

$$F_{ch}(q^2)=[1-\alpha(qa_{ch})^2/(4+6\alpha)]e^{-q^2a_{ch}^2/4},$$

where $\alpha=(Z-2)/3$, and

$$a_{ch}=1.71\ \text{fm}\quad \text{for } ^{12}\text{C},$$

$$=1.82\ \text{fm}\quad \text{for } ^{16}\text{O}.$$

The above form factors provide good fits up to the first minimum, and beyond it they have a tendency to be too small.

Once more we would like to emphasize that a given one-body density can always be obtained from a simple Slater determinant, and there is no need to introduce short-range nucleon-nucleon correlations. However, a knowledge of the two-body density would provide a way of discriminating among the many wave functions giving rise to the same one-body density and would also permit us to determine the pair-correlation function. The one-body density reflects only indirectly on the

two-body density in so far as one assumes a reference frame for uncorrelated wave functions. In elastic scattering a one-body operator measures only the one-body density which is essentially determined by elastic electron scattering, and to detect correlations one needs a two-body operator such as the second-order optical-potential operator. Combining elastic electron and high-energy nucleon scattering, we then have a consistent criterion for detecting correlations.

2. *Two-Body Density*

We present a parametrization for the two-body density which is automatically consistent with the one-body density. This is to be contrasted with the usual way of introducing correlations by modifying the radial functions of the relative motion.

The two-body correlation function in momentum and coordinate space is respectively given by

$$\tilde{C}(\vec{q},\vec{q}') = \tilde{\rho}_2(\vec{q},\vec{q}') - \tilde{\rho}(\vec{q})\tilde{\rho}(\vec{q}'), \tag{V.2a}$$

$$C(\vec{x},\vec{x}') = \rho_2(\vec{x},\vec{x}') - \rho(\vec{x})\rho(\vec{x}'). \tag{V.2b}$$

The two are related by a Fourier transformation. It may be easily shown that a necessary and sufficient condition for the two-body to give rise to the correct one-body density is

$$\tilde{C}(\vec{q},0) = \tilde{C}(0,\vec{q}') = 0. \tag{V.3}$$

The above requirement can be trivially satisfied; however, in order to obtain a reasonable form for the two-body density one must make sure that it describes hard-core or Pauli correlations. This is no longer trivial.

We now apply the above ideas to ^{4}He, ^{12}C, and ^{16}O.

(i) ^{4}He

In ^{4}He there are no correlations due to the Pauli principle, and we need consider only short-range correlations. A convenient parametrization for the two-body correlation function satisfying the requirements stated above and dependent upon one parameter λ, identifiable as a correlation range, is provided for by

$$C_S(\vec{x},\vec{x}') = \bar{\rho}\left\{\exp\left[-\frac{(\vec{x}+\vec{x}')^2}{8b^2}\right]\exp\left[-\frac{(\vec{x}-\vec{x}')^2}{16\lambda^2}\right] - \left(\frac{2}{1+\lambda^2/2b^2}\right)^{3/2}\exp\left[-\frac{(\vec{x}+\vec{x}')^2}{2(b^2+\lambda^2/2)}\right]\times\exp\left[-\frac{(\vec{x}-\vec{x}')^2}{8\lambda^2}\right]\right\}, \tag{V.4}$$

where

$$\bar{\rho} = \rho^2(0)[2^{3/2}(1+\lambda^2/2b^2)^{-3/2} - 1]^{-1}.$$

The correlation length λ is determined as a result of trying to fit the 1-GeV p-^{4}He elastic-scattering data. From our work we have determined λ^2 to be approximately 0.3 fm^2, which is of the order of the characteristic length associated with the repulsive core.

(ii) ^{12}C *and* ^{16}O

In order to obtain information concerning the correlations due to dynamics we must estimate the relative importance of the correlations due to the Pauli principle. To calculate the Pauli correlations we assume the shell model, according to which ^{12}C and ^{16}O are closed-j-shell nuclei. For small momentum transfer the dominant term is the one proportional to $\vec{q}\cdot\vec{q}'$, and in coordinate space it gives rise to a "superseparable nonlocality":

$$C_P(\vec{x},\vec{x}') = -D(\nu^4/\pi^3)[(\vec{x}+\vec{x}')^2 - (\vec{x}-\vec{x}')^2]\times\exp\{-(\nu/2)[(\vec{x}+\vec{x}')^2 + (\vec{x}-\vec{x}')^2]\}, \tag{V.5}$$

where ν is the harmonic-oscillator parameter, and

$$D = 2/99 \quad \text{for } ^{12}\text{C},$$
$$= 1/60 \quad \text{for } ^{16}\text{O}.$$

To study the effects of the hard core we introduce dynamic correlations in the Jastrow manner.[33] We calculate the two-body correlation function corresponding to the Jastrow wave function by making a cluster expansion, and we neglect contributions involving more than one correlated pair. This is a reasonable approximation for large nuclei only. In any case, this calculation suggests that we take the correlation function to be the sum of Pauli and short-range correlations:

$$C(\vec{x},\vec{x}') = C_S(\vec{x},\vec{x}') + C_P(\vec{x},\vec{x}'). \tag{V.6}$$

The above pair-correlation function is such that the two-body density is consistent with the one-body density and contains both short-range and Pauli correlations.

With this approach there is no need to worry about the center-of-mass motion since we do not make explicit use of a wave function, but instead we work directly with the one- and two-body densities. The expectation value of operators in the ground state involves only the internal wave function for which only $(N-1)$ of the coordinates are independent:

$$\rho_N(\vec{r}_1,\ldots,\vec{r}_N) = |\psi_0(\vec{r}_1,\ldots,\vec{r}_N)|^2\delta^{(3)}\left(\sum_{i=1}^{N}\vec{r}_N\right). \tag{V.7a}$$

For one- and two-body operators only the one- and two-body densities are required:

$$\rho_2(\vec{x},\vec{y}) = \int\cdots\int d\vec{r}_3,\ldots,d\vec{r}_N\rho_N(\vec{x},\vec{y},\vec{r}_3,\ldots,\vec{r}_N), \tag{V.7b}$$

$$\rho(\vec{x}) = \int \rho_2(\vec{x}, \vec{y}) d\vec{y} \,. \tag{V.7c}$$

All of our previous results therefore remain unchanged. We remark that even simple product wave functions are then correlated due to the center-of-mass restriction, but as pointed out by Foldy and Walecka[17] this is simply an N-particle correlation condition. This is being further investigated by Gal and Hüfner.[34]

VI. DESCRIPTION OF THE POTENTIAL

A. Evaluation of the Optical Potential

We first give a brief recapitulation of the previous sections and list the main results. In Sec. III we showed that the first-order potential is closely related to the one-body density [Eq. (III.4)], and that the second-order potential is nonlocal and directly related to the two-body correlations [Eqs. (III.8) and (III.10)]. In Sec. IV we presented a technique for obtaining a local potential which is effectively equivalent to a spin-dependent nonlocal potential [Eqs. (IV.7) and (IV.9)]. The pair-correlation functions are given by Eqs. (V.4), (V.5), and (V.6).

We now illustrate how to apply the above method to the case of ^{4}He for which we need only consider the short-range correlations. To directly make use of Eqs. (IV.7) and (IV.9) we need the following identifications:

$$v_C^{(1)}(x) = V_C^{(1)}(x)\,, \quad u_C(|\vec{x} - \vec{x}'|) = \delta(\vec{x} - \vec{x}')\,, \quad \tilde{u}_C^{(1)}(k) = 1\,,$$

$$v_C^{(2)}(x) = c_2 e^{-x^2/2b^2}\,,$$

$$v_C^{(3)}(x) = c_3 e^{-x^2/2(b^2 + \lambda^2/2)}\,,$$

$$u_C^{(2)}(r) = e^{-r^2/16\lambda^2} e^{i(2\mu\epsilon)^{1/2} r}/r\,,$$

$$u_C^{(3)}(r) = e^{-r^2/8\lambda^2} e^{i(2\mu\epsilon)^{1/2} r}/r\,,$$

where the constants c_2 and c_3 are easily identifiable in terms of the previous equations. Similar expressions are obtained for the spin-dependent potential. Since all the Fourier transforms involved are essentially alike, we only give a typical one, namely $\tilde{u}_C^{(2)}(k)$:

$$\tilde{u}_C^{(2)}(k) = \frac{4\pi^{3/2}\lambda}{ik}\{e^{-4\lambda^2[(2\mu\epsilon)^{1/2}+k]^2}\mathrm{erfc}[-i2\lambda(\sqrt{2\mu\epsilon}+k)] + e^{-4\lambda^2[(2\mu\epsilon)^{1/2}-k]^2}\mathrm{erfc}[-i2\lambda(\sqrt{2\mu\epsilon}-k)]\}\,. \tag{VI.1}$$

$\mathrm{erfc}(z) \equiv -\mathrm{erf}(z) + 1$, where erf is the error function.[35] The above expressions are then evaluated numerically.

In addition to the energy dependence of the scattering amplitudes, the equivalent local potential has an energy dependence which arises from the nonlocality. In the next section we show that the above Fourier transforms are closely related to the correlation length introduced by Johnston and Watson.[13]

For ^{12}C and ^{16}O one needs, in addition, to consider Pauli correlations, and except for some minor modifications the work proceeds in the same manner.

B. Comparison with Other Works

Of the works discussed in the introduction, that of Johnston and Watson[13] is of most interest to us since in it the second-order potential is approximated by a local potential, and we now compare it with our work. Our work differs from theirs in three basic respects: (i) The derivation of the optical-potential operator; (ii) the parametrization of the correlation function; and (iii) the evaluation of the second-order potential and its reduction to a local potential.

The first difference has been discussed in Sec. II. As far as the potential is concerned it involves constant factors of $[(N-1)/N]$, and it does not affect the shape of the first- and second-order potentials.

Unlike our correlation functions, the correlation functions they used are not consistent with the one-body density. For short-range correlations they assume a two-body distribution of the Brueckner-Gammel type:

$$C(\vec{x}, \vec{x}') \simeq \rho^2(\vec{x}) f_B(|\vec{x} - \vec{x}'|)\,, \tag{VI.2a}$$

where

$$f_B(r) = -e^{-r^2/b^2}(1 - \gamma r^2/b^2)\,, \tag{VI.2b}$$

with $\gamma = \frac{2}{3}$ and $b = 1.42$ fm. For the Pauli correlations they used the expressions obtained from the Fermi-gas model.

Johnston and Watson[13] and Johnston[14] give essentially the following result for the second-order potential:

$$V_{JW,C}^{(2)}(r) = N^2 \frac{(2\pi)^6 \epsilon_{N\ \mathrm{c.m.}}}{ik_{N\ \mathrm{c.m.}}} \bar{\eta}^2 A^2(0) \frac{R_B}{\mathcal{V}_N} \rho(r)\,, \tag{VI.3a}$$

$$V_{JW,S}^{(2)}(r) = N^2 \frac{(2\pi)^6 \epsilon_{N\ \mathrm{c.m.}}}{k_{N\ \mathrm{c.m.}}} \bar{\eta}^2 A(0) \mathcal{C}(0) \frac{R_B}{\mathcal{V}_N} \frac{1}{r} \frac{d\rho(r)}{dr}\,. \tag{VI.3b}$$

$\epsilon_{N\ \mathrm{c.m.}}$ is the energy of the incident nucleon with effective reduced mass μ in the nucleon-nucleus barycentric system, $\epsilon_{N\ \mathrm{c.m.}} = (k_{N\ \mathrm{c.m.}}^2 + \mu^2)^{1/2}$; $\bar{\eta}$ is given in Eq. (III.3); $A(0)$ and $\mathcal{C}(0)$ are defined in Eq. (III.6); $\mathcal{V}_N$ is the nuclear volume; and R_B is an estimate of an integral involving the correlation function and is known as the correlation length:

$$R_B = \int_0^\infty f_B(r) dr\,.$$

We now show that in terms of our work the second potential given in Eq. (VI.3) corresponds to making the following three approximations: (i) As-

suming the validity of the Born approximation for the nonlocality; (ii) using ultrarelativistic kinematics in the Green's function, $\epsilon(k) \simeq k$; and (iii) assuming the range of the correlation to be much smaller than the radius so that the correlation function given by Eq. (VI.2) is valid. We only examine the second-order central potential since for the spin-dependent contribution one would proceed in exactly the same fashion.

The nonlocal second-order central potential in coordinate space is given by Eq. (III.8). Using the ultrarelativistic Green's function $[\epsilon(k) \simeq k]$ and the correlation function given by Eq. (VI.2) we obtain for Eq. (III.8)

$$\langle \vec{x}' | V_C^{(2)} | \vec{x} \rangle \simeq v(x) u(|\vec{x} - \vec{x}'|), \tag{VI.4a}$$

where

$$v(x) = -(2\pi)^5 (N-1)^2 \epsilon_{N\ \mathrm{c.m.}} \bar{\eta}^2 A^2(0) \rho^2(x), \tag{VI.4b}$$

$$u(\vec{x} - \vec{x}') = f_B(|\vec{x} - \vec{x}'|) \frac{e^{i\epsilon_{N\ \mathrm{c.m.}}|\vec{x}-\vec{x}'|}}{|\vec{x} - \vec{x}'|}. \tag{VI.4c}$$

Assuming the validity of the Born approximation for the nonlocality, we determine the equivalent local potential by keeping just the first term in the expansion given by Eq. (IV.4):

$$\tilde{u}(k) = \int e^{i\vec{k}\cdot\vec{r}} f_B(r) (e^{i\epsilon r}/r) d\vec{r}$$

$$\simeq -(2\pi/ik) \int_0^\infty f_B(r) dr \equiv -(2\pi/ik) R_B. \tag{VI.5}$$

With the approximation indicated above one readily obtains

$$[V_C^{(2)}(x)]_{\mathrm{Born}} = (N-1)^2 \frac{(2\pi)^6 \epsilon_{N\ \mathrm{c.m.}}}{ik_{N\ \mathrm{c.m.}}} \bar{\eta}^2 A^2(0) R_B \rho^2(x). \tag{VI.6}$$

This is essentially the result of Johnston and Watson as given by Eq. (VI.3), except that the above authors have further approximated ρ^2 by (ρ/v_N). The above equation shows that the radial dependence of the second-order potential is sharper than that of the first-order potential since it is roughly proportional to ρ^2.

The result of Eq. (VI.5) may also be obtained using the correlation function of Eq. (V.4) and approximating $\tilde{u}(k)$, which is given by Eq. (VI.1). Assuming the high-energy limit $\lambda\epsilon \gg 1$ and making use of the properties of the error function we obtain

$$\tilde{u}(k) \simeq -4\pi^{3/2}\lambda/ik. \tag{VI.7}$$

Once more we see that $\tilde{u}$ is closely related to the correlation length R_B.

We thus conclude that the result of Johnston and Watson should be valid for heavy nuclei, short-range correlations, and high energies. For ^{4}He and the energies considered in this work, we expect our potential to be more appropriate. This is substantiated by the fits to the experimental data.

VII. COMPARISON OF CALCULATED AND EXPERIMENTAL RESULTS

Using the optical potential obtained as described in the preceding sections we now analyze the elastic scattering of nucleons by ^{4}He at 1 GeV and 630 MeV, and ^{12}C and ^{16}O at 1 GeV. The scattering of 1-GeV protons off ^{4}He is used to determine the parameter λ^2 which enters in the parametrization of the short-range correlations. With the nuclear structure so determined we calculate the scattering of nucleons by ^{4}He at 630 MeV in an attempt to discriminate among several sets of phase shifts. Finally, the scattering of nucleons by ^{12}C and ^{16}O is investigated in order to study the relative importance of Pauli and short-range correlations.

A. Optical Potential

The approximations in order of increasing complexity are: (i) First-order potential with forward-scattering approximation; (ii) first-order potential including the momentum-transfer dependence of the nucleon-nucleon scattering amplitude; (iii) inclusion of the second-order potential, but use of the Born approximation in obtaining the equivalent local potential; (iv) inclusion of the second-order potential, retaining the first two terms in the expansion given by Eq. (IV.4).

We have checked numerically the validity of truncating the expansion given by Eq. (IV.4) by comparing the differential cross sections under stages

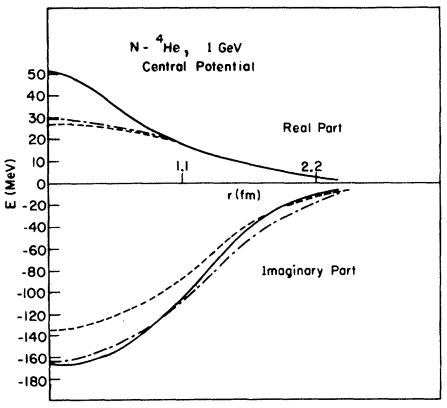

FIG. 1. Central potentials for p-^{4}He scattering at 1 GeV for various approximations. – – – first-order potential only, including its q dependence; – · – · – potential including second-order potential with $\lambda^2 = 0.1$ fm^2; —— potential including second-order potential with $\lambda^2 = 0.3$ fm^2.

(iii) and (iv). The two results are such that we are confident about truncating this series after the first two terms. Throughout our work we retain the first two terms, thereby improving on the Born approximation.

1. *Central Potential*

In Fig. 1 we show the central potentials for nucleon-^{4}He scattering at 1 GeV; comparing the first-order potential including its q dependence and the potentials including the second-order corrections using the parametrization given by Eq. (V.5) with $\lambda^2 = 0.1$ and 0.3 fm^2, respectively. We notice that the depths of both the real and imaginary central potentials are deepened near the center resulting in a sharper transition region at the surface than expected for the first-order potential.

In Fig. 2 we present the first-order potential including its q dependence and the potential including the second-order corrections with $\lambda^2 = 0.3$ fm^2 for p-^{4}He scattering at 630 MeV using the scattering amplitudes corresponding to the phase shifts of solution A2. We also remark that the potentials arising from the different scattering amplitudes (A2, B3, and C5) differ substantially from one another; this emphasizes the importance of including the angular dependence of the scattering amplitudes if they are rapidly varying, especially for light nuclei.

As can be seen from these curves, the first- and second-order potentials have quite different radial dependences, the second-order potential being sharper at the surface. The potentials corresponding to the different approximations then differ mainly at small distances while their tails coincide. The interior of the potential affects most strongly large-momentum-transfer scattering so that the effects of the second-order potential should be most marked there. We thus conclude that for light nuclei the central potential is in no simple manner related to the one-body density because of the importance of taking into account the momentum dependence of the scattering amplitudes and the second-order potential, which depends on nuclear correlations.

The effects of including the q dependence of the scattering amplitudes and the second-order potential are not as marked for ^{12}C and ^{16}O, and we do not present the corresponding potentials. This is in agreement with the prediction that these corrections decrease in importance as the radius of the nucleus increases and also explains the results of Palevsky *et al.*[4]

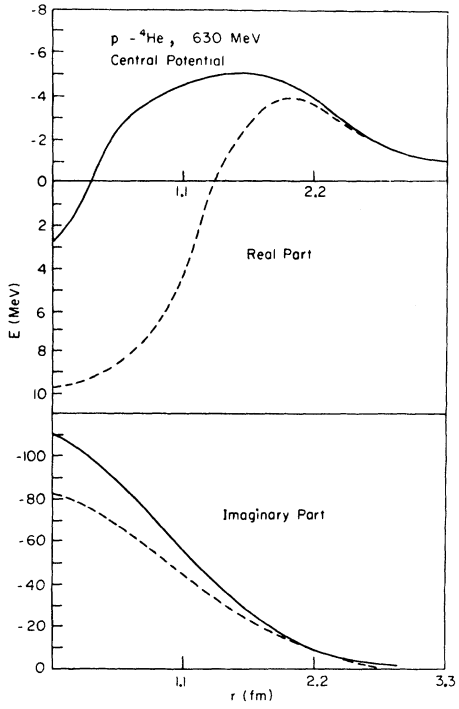

FIG. 2. Central potentials for p-^{4}He scattering at 630 MeV using the phase shifts of solution A2. – – – first-order potential only, including its q dependence; —— potential including second-order potential with $\lambda^2 = 0.3$ fm^2.

2. *Spin-Orbit Potential*

In our scheme the total spin-orbit potential differs from the first-order spin-orbit potential mainly for two reasons: (i) The first-order spin-orbit potential gets modified by the central nonlocal potential as shown in Eq. (IV.7c); (ii) there is also a spin-orbit contribution arising from the

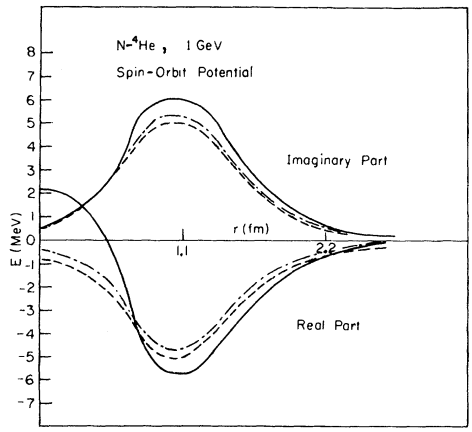

FIG. 3. Spin-orbit potentials for p-^{4}He scattering at 1 GeV for various approximations. – – – first-order spin-orbit potential, including its q dependence; – · – · – first-order potential modified by central nonlocal potential ($\lambda^2 = 0.3$ fm^2); —— total spin-orbit potential including second-order spin-dependent contributions.

spin dependence of the second-order potential as given by Eq. (IV.9).

In Fig. 3 we illustrate the importance of each contribution for N-^{4}He scattering at 1 GeV by showing: (a) the first-order spin-orbit potential including the q dependence of the scattering amplitude, (b) the first-order spin-orbit potential, modified as indicated in approximation (i), (c) the total spin-orbit potential including second-order contributions as indicated by (i) plus (ii). The 1-GeV spin-dependent scattering amplitude was obtained as discussed in Sec. V so that its magnitude is somewhat uncertain.

In Fig. 4 we illustrate the spin-orbit potentials with and without the total second-order contribution as given by (i) plus (ii) for p-^{4}He scattering at 630 MeV, using the scattering amplitudes corresponding to the phase shifts of set A2.

The effects of the multiple-scattering corrections for the spin-orbit potential are not as marked as for the central potential, and there does not seem to be any general trend to the corrections.

B. Differential Cross Section and Polarization

1. p-^{4}He *Scattering at* 1 GeV

There are two ways of interpreting the results of Palevsky *et al.*[4] in terms of an optical potential: (i) The spin-dependence of the nucleon-nucleon

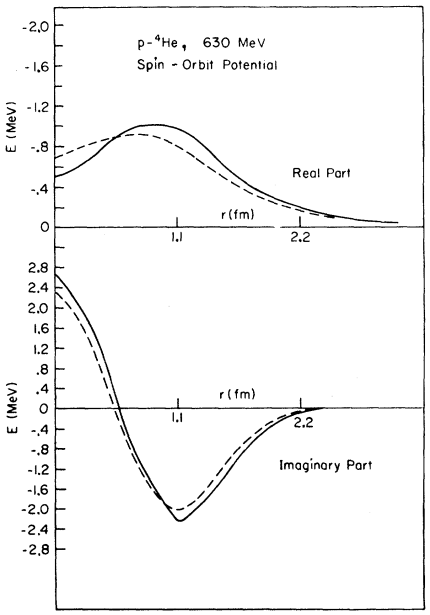

FIG. 4. Spin-orbit potentials for p-^{4}He scattering at 630 MeV using the phase shifts of solution A2. – – – first-order spin-orbit potential, including q dependence; —— total spin-orbit potential including second-order spin-dependent contributions ($\lambda^2 = 0.3$ fm^2).

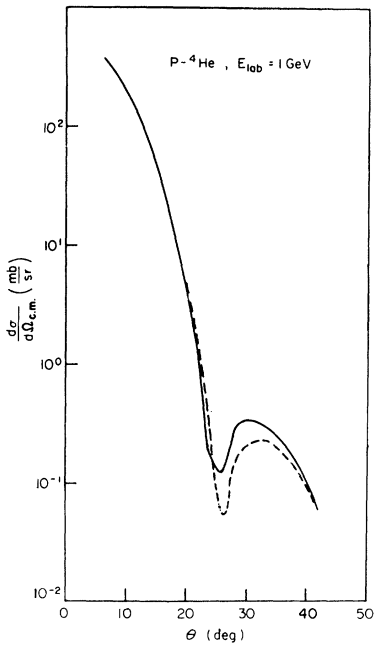

FIG. 5. Differential cross sections for p-^{4}He scattering at 1 GeV for different correlation lengths. – – – $\lambda^2 = 0.1$ fm^2; —— $\lambda^2 = 0.3$ fm^2.

scattering amplitude is non-negligible; (ii) correlation effects are important making it necessary to consider the corrections to the impulse and multiple-scattering approximations. In our work both effects were taken into account, and both turned out to be quite significant. As already discussed, when one uses the form factor measured by Frosch *et al.*[5] and our scattering amplitudes as given in Sec. V, the data cannot be fitted using only the first-order potential.

To show the sensitivity of the results to the two-body correlations, in Fig. 5 we present the differential cross sections corresponding to $\lambda^2 = 0.1$ and 0.3 fm^2, respectively. The effects of the correlations appear particularly marked for ^{4}He, being most significant around the diffraction pattern. We see that the second-order potential is very important in obtaining the diffraction-like structure, and we may then have a possible way of obtaining information about the two-body density.

In Fig. 6 we show the differential cross sections corresponding to: (i) the experimental data of Palevsky *et al.*[4]; (ii) the central potential only with $\lambda^2 = 0.3$ fm^2; (iii) the central plus spin-orbit potentials with $\lambda^2 = 0.3$ fm^2. It is worth noting that we have not tried to get the best possible fit, and that we have made use of only one adjustable parameter, the range of which is quite limited. Concerning the 1-GeV p-^{4}He analysis we come to the following conclusions: (i) Using only the first-order cen-

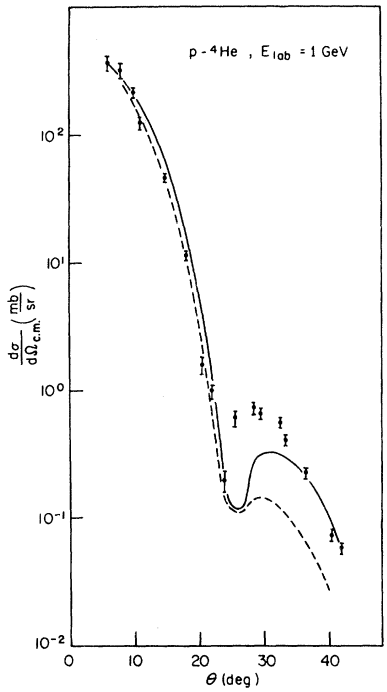

FIG. 6. Differential cross sections for p-^{4}He scattering at 1-GeV lab kinetic energy for various approximations. ⧫ experimental points taken from Ref. 4; – – – only central potential with $\lambda^2 = 0.3$ fm^2; —— central plus total spin-orbit potentials with $\lambda^2 = 0.3$ fm^2.

tral potential gives poor agreement and the diffractionlike structure is not reproduced. This was first noted by Palevsky *et al.*[4] (ii) Adding the spin-orbit term in the first-order potential gives rise to a diffractionlike structure, but the minimum and maximum are too low and the shoulder is too narrow. Spin effects are important. (iii) Putting in the complete second-order potential with $\lambda^2 = 0.3$ fm^2, which corresponds to a reasonable correlation length, improves the fit substantially. However, the rise of the shoulder is still not steep enough. Except for the angular range from 24 to 34° we have good agreement with the data.

Figure 7 shows the polarization corresponding to the first-order potential and the first- plus second-order potentials for $\lambda^2 = 0.3$ fm^2 with the spin-orbit potential given by (i) and (i) plus (ii), as previously discussed. The second-order potential affects the polarization so that it may constitute an additional test for two-body correlations. However, there are no experimental data available and the spin-dependent scattering amplitude is poorly known.

2. p-^{4}He *at* 630 MeV

Gotow *et al.*[36] have measured the elastic-scattering differential cross section for p-^{4}He at 587.5 MeV and the corresponding polarization at 540 MeV. The differential cross sections corresponding to the phase shifts of solutions A2, B3, C5, and the experimental data are presented in Fig. 8. We remark that we have no free parameter since the nuclear structure is as determined at 1 GeV. Assuming that the data at 587.5 and 630 MeV do not differ substantially, we conclude that sets A2 and C5 are favored over B3. Neglecting the difference in energy, the fit, especially at the diffraction pattern is not as good as at 1 GeV. This may be due to several reasons. The second-order potential increases in importance with decreasing energy, and it is also more involved than at 1 GeV because of the complications arising from the nucleon-nucleon scattering amplitude. Furthermore, the diffraction minimum now occurs at a larger angle and the scattering amplitudes are rapidly varying with momentum transfer so that our model may require some refinements, especially as far as the angular approximation for the second-order potential is concerned.

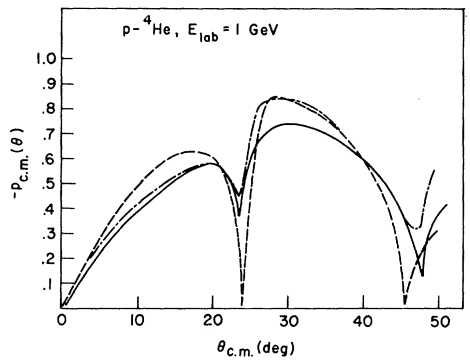

FIG. 7. Polarization for p-^{4}He scattering at 1 GeV. —— first-order potential only; – · – · – first-order spin-orbit potential as modified by nonlocal central potential with $\lambda^2 = 0.3$ fm^2; – – – complete second-order potential with $\lambda^2 = 0.3$ fm^2.

The polarizations corresponding to phase shifts A2, B3, and C5 are presented in Fig. 9. Because our calculations correspond to an energy of 630 MeV while the experimental data are at 540 MeV, it is somewhat meaningless to compare the two. It can however be pointed out that the three results differ significantly from one another so that the polarization data may be of use in selecting among different phase shifts.

3. ^{12}C *and* ^{16}O *at* 1 GeV

The elastic scattering of 1-GeV protons by ^{12}C and ^{16}O has been measured by Palevsky *et al.*[4] We investigate it to study the relative importance of Pauli and short-range correlations. For the correlation functions we used the expressions given by Eqs. (V.4), (V.5), and (V.6).

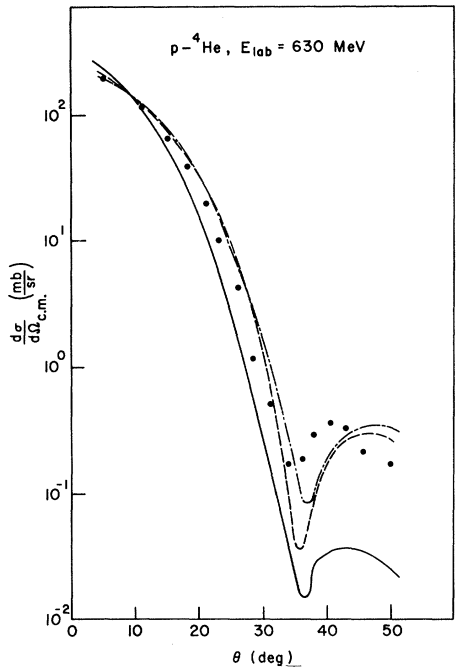

FIG. 8. Calculated differential cross sections for p-^{4}He elastic scattering at 630 MeV for various sets of nucleon-nucleon phase shifts using the complete potential. ⧫ experimental points at $E_{lab} = 587.5$ MeV taken from Ref. 36; – – – phase shifts A2; —— phase shifts B3; –·–·– phase shifts C5.

In Fig. 10 we show the differential cross sections for 1-GeV elastic p-^{12}C scattering using central potentials obtained under the following conditions: (i) first order only, taking into account the q dependence of the scattering amplitude; (ii) first order plus second order for Pauli correlations only; (iii) first order plus second order for short-range correlations only; (iv) first order plus second order for both Pauli and short-range correlations. In Fig. 11 we show the corresponding differential cross sections for ^{16}O.

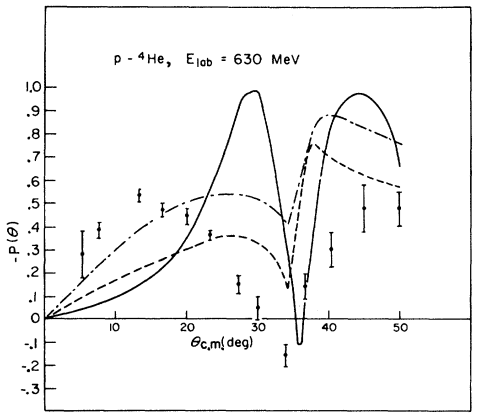

FIG. 9. Calculated polarizations for p-^{4}He at 630 MeV for various sets of nucleon-nucleon phase shifts using the complete potential. ⧫ experimental points at $E_{lab} = 540$ MeV taken from Ref. 36; – – – phase shifts A2; —— phase shifts B3; –·–·– phase shifts C5.

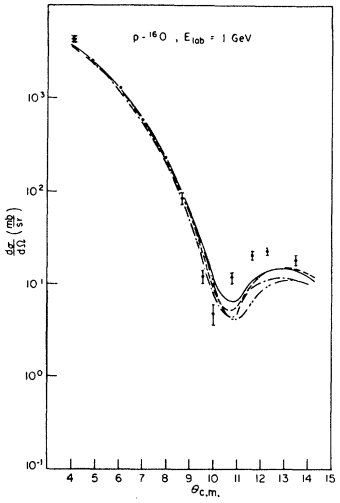

FIG. 10. Differential cross sections for p-^{12}C elastic scattering at 1 GeV using only the total central potential for various two-body correlations. ⧫ experimental points taken from Ref. 4; –·–·– no correlations; – – – short-range correlations with $\lambda^2 = 0.3$ fm^2 only; –··–·· – Pauli correlations only; —— Pauli plus short-range correlations.

We remark that the fit for p-^{12}C is "too good" because when the spin-orbit potential is included the predicted cross section becomes too big while the fit for p-^{16}O improves. Bassel and Wilkin[7] have run into a similar problem. They point out that ^{12}C is a deformed nucleus so that in order to fit the data a "generalized optical potential" may be required.

As far as the correlations are concerned we see once more that they are most significant at large momentum transfers around the diffractionlike structure. The effects of the correlations are quite sizable but not as marked as for ^{4}He. As shown by Glauber[8] and also by this work, the corrections to the first-order potential decrease in importance with heavier nuclei. We remark that there seems to be some interference between the Pauli and the short-range correlations, and this would make it difficult to extract information about dynamical correlations. Our work does not favor either the Pauli or short-range correlations but rather points out the need to include both simultaneously.

VIII. CONCLUSION

We have studied a slightly modified version of Watson's multiple-scattering optical-potential operator in coordinate space including all terms

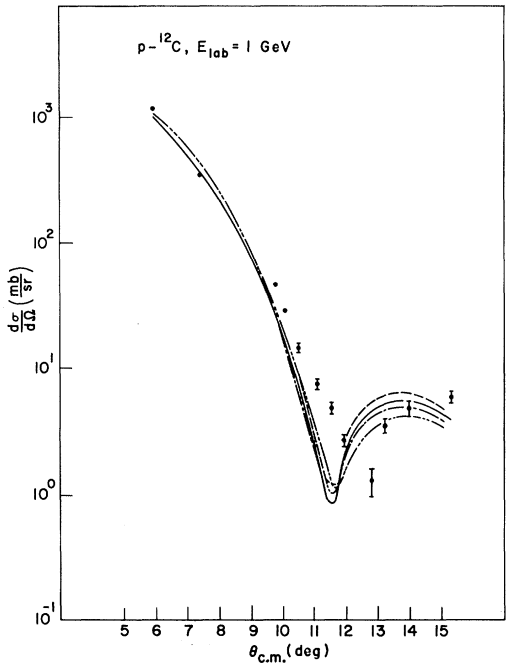

FIG. 11. Differential cross sections for p-^{16}O elastic scattering at 1 GeV using only the total central potential for various two-body correlations. The labeling is as in Fig. 10.

binary in the nucleon-nucleon interaction. We showed that the second-order potential is energy dependent, nonlocal, and directly related to the two-body correlation function. The two-body density was parametrized in a form consistent with the one-body density and dependent upon one parameter identifiable as a correlation length. We have also presented a way of simultaneously including the Pauli and short-range correlations. The nonlocal potential was reduced to an equivalent local potential by using an expansion whose convergence may systematically be investigated.

The correlation effects are especially marked for ^{4}He, and our work helps explain the sharpness of the phenomenological optical potential needed by Palevsky *et al.*[4] to fit the data. Thus high-energy elastic scattering off ^{4}He seems to provide evidence for short-range nucleon-nucleon correlations.

High-energy elastic nucleon-nucleus scattering may also provide information about nucleon-nucleon scattering. We believe that we were partially successful in discriminating among the several sets of phase shifts at 630 MeV calculated by MacGregor *et al.*[23]

Our work on the scattering of 1-GeV protons by ^{12}C and ^{16}O points out the importance of including simultaneously the Pauli and short-range correlations.

The fits we have obtained are not high-quality fits but were never intended to be. It is worth pointing out again that they were obtained using only one adjustable parameter which is related to the correlation length and was fixed by fitting the 1-GeV elastic p-^{4}He scattering data. Better agreement would involve correcting the approximations previously discussed: (i) treating the first-order potential in a less cavalier manner; (ii) taking into account off-energy-shell effects for the nucleon-nucleon scattering amplitude; (iii) using spin- and isospin-dependent correlation functions; (iv) improving the closure approximation; (v) including the nonlocality exactly. These corrections deserve further study.

To conclude, we think that high-energy elastic nucleon-nucleus scattering may provide a useful tool for investigating nuclear structure and possibly the nucleon-nucleon interaction.

ACKNOWLEDGMENTS

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Convenient Expansion for Local Potentials*

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We introduce a new separable expansion for local potentials. The expansion, called the unitary pole expansion, has real, energy-independent form factors, and satisfies the requirements of two-particle unitarity in all orders. The convergence of the expansion is investigated by comparing expanded and exact T matrices for negative energies, and by performing three-body bound-state calculations. In the latter case, a one-term approximation gives energies accurate to within 2% for potentials containing repulsion of the magnitude indicated by two-nucleon data.

I. INTRODUCTION

The solution of the Faddeev equations for systems interacting by means of local two-body potentials is a difficult numerical problem. To solve for the three-body bound state, we must solve a (in general coupled) set of two-dimensional integral equations. For the scattering states, these equations become singular.

Some progress[1] has been made towards the solution of the equations for the bound-state problem. For the scattering states no success has yet been reported.

To avoid the complications of two-dimensional integral equations, a number of authors[2,3] have used separable expansions of the two-body interaction to reduce the problem to a coupled set of one-dimensional integral equations. These equations have been solved both for the three-body bound state and the three-body scattering states. In this paper we propose another expansion of the two-body interaction. The expansion, which we call the unitary pole expansion (UPE), is closely related to the Weinberg[4] series. However, it is considerably easier to use and shows better convergence properties. The form factors for the expansion are real and energy-independent, and the UPE satisfies the requirements of two-particle unitarity