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Generalized Variational Bounds for Multichannel Scatterings

YUKAP HAHN

Laboratories of Theoretical Studies, Goddard Space Flight Center, NASA, Greenbelt, Maryland 20771 and*
Department of Physics, University of Connecticut, Storrs, Connecticut† 06268

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The variational bounds on scattering parameters are reformulated without the explicit use of multichannel projection operators, thus avoiding completely the difficulty of nonorthogonality in rearrangement and exchange collisions. Resonance-energy calculations can be carried out using the operator which is effectively projected on the closed channels.

I. INTRODUCTION AND SUMMARY

THE variational bound (VB) on scattering parameters, as originally formulated,[1] employs in an essential way the projection operator technique of Feshbach.[2] Consequently, the difficulty of explicitly deriving such channel projection operators for reaction in which particle exchanges and rearrangements take place severely limits the applicability of VB method.[3] There have been given several alternate derivations[4–8] of VB which eliminate various practical difficulties of carrying out actual calculations and further clarify the basis of the theory.

In this paper, we generalize the previous result by *eliminating* the multichannel projection operators P and Q *completely*, so that the method can be applied readily to more complex scattering problems involving exchanges and rearrangements. It is shown that the correct asymptotic boundary conditions alone are *sufficient* to derive the bound. In this connection, the observation of Burke and Taylor[8] in their formulation of VB, that the short-range correlation function need not be orthogonalized to the open channel functions, is most important. The generalization of VB discussed here is equivalent to their point of view, but the present form seems to bring out more clearly, the bound property and the range of applicability. The proof given here is also more direct and explicit.

The generalized VB should then be useful for many interesting scattering systems, such as e^+H above the pickup threshold, e^-He and e^- atom scatterings, and also *nd* and *pd* scatterings. Thus, so long as the asymptotic conditions (2.2), which involve only P_i with i running over all the open channels, can be specified, the result (2.23) is applicable to all possible rearrangement collisions, including the exchanges in case of identical particles. For reactions involving many-particle clusters, P_i may not be readily available. But this problem is not directly related to the nonorthogonality of the rearrangement channel wave functions, and has to be solved before the scattering problem is considered.

Alternative procedures in which G^P need not be explicitly evaluated have also been discussed, and certain orthogonality properties of the operators which appear in (2.23) brought out. Finally, a possible way of estimating resonance energies is considered in which the Q operator is not required.

II. GENERALIZED VARIATIONAL BOUNDS

For simplicity, we consider explicitly a two-channel problem

$$1+(2+c)\rightarrow 2+(1+c), \qquad (2.1)$$

where the particles 1 and 2 are distinguishable and c has no internal structure, but not necessarily an infinite mass. This model contains all the essential properties of more general scattering systems of interest here.

The asymptotic boundary conditions on Ψ are given by

$$\Psi\rightarrow P_i\Psi_i \qquad (2.2)$$

as the channel coordinates $R_i\rightarrow\infty$, with $i=1, 2$, and where

$$P_i=\psi_0^{(i)}(\mathbf{r}_{jc})\rangle\langle\psi_0^{(i)}(\mathbf{r}_{jc}')^*,$$

$$P_i^2=P_i=P_i^\dagger, \qquad (2.3)$$

$$Q_i=1-P_i, \qquad Q_iP_i=0,$$

$$[P_1, P_2]\neq 0. \qquad (2.4)$$

The variational bound as originally formulated[1] assumes an explicit form of the open channel operator P such that[2]

$$P\Psi=P_1\Psi_1+P_2\Psi_2 \qquad (2.5)$$

and

$$P^2=P=P^\dagger. \qquad (2.6)$$

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* ASEE-NASA Summer Faculty Fellow.

† Permanent address.

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The knowledge of P then immediately leads to its complement Q, given by

$$Q=1-P,$$
$$Q^2=Q=Q^\dagger, \qquad QP=0. \tag{2.7}$$

The physical reason for the bound is essentially that the virtual excitations have an *attractive effect* on scatterings, or, mathematically,

$$Q(H-E)Q>0 \tag{2.8}$$

for a given total energy E. Therefore, it seems from (2.8) that the formulation of VB requires the Q operator in an essential way. However, it is in general extremely difficult to construct such operators explicitly, and P_i are not simply related to P due to complicated overlaps.

The variational bound on λ is given by[1]

$$\lambda\leq\lambda_U'=\lambda^P+\Delta_U', \tag{2.9}$$

where

$$\lambda=2\pi\hbar^2\mathbf{a}\cdot K^{-1}\mathbf{a},$$
$$\Delta_U'=2(Q\Psi_t\mid QHP\mid P\Psi^P)+(Q\Psi_t\mid\mathcal{H}'-E\mid Q\Psi_t). \tag{2.10}$$

In (2.9), λ^P is the scattering parameter corresponding to the "static" approximation, and, in (2.10), the vector $\mathbf{a}$ specifies the initial conditions, K^{-1} is the inverse reactance matrix, and $Q\Psi_t$ is a trial function in the Q space. We also have

$$P(H-E)P\Psi^P=0, \qquad P\Psi^P\equiv\Psi^P, \tag{2.11a}$$
$$P(H-E)PG^PP=-P, \qquad PG^PP\equiv G^P, \tag{2.11b}$$
$$\mathcal{H}'=Q[H+HG^PH]Q=Q\mathcal{H}'Q. \tag{2.11c}$$

Possible subtraction terms in (2.10) when E lies above a finite number of eigenstates of $\mathcal{H}'$ have been neglected.

Now, we examine (2.10) more closely to see if P and Q can be eliminated completely. For this purpose, we first look at the $P\Psi^P$ Eq. (2.11a) which gives λ^P. Since the explicit form of P_i is known from the boundary conditions (2.2), we can write, using (2.5),

$$P\Psi^P\equiv P_1\Psi_1^P+P_2\Psi_2^P. \tag{2.12}$$

We also note that P has the important property (Appendix B)

$$P_iP=PP_i=P_i, \tag{2.13}$$
$$P_iQ=0, \tag{2.14}$$

although P is not, in general, a simple linear combination of P_i. Using (2.13) and (2.14) in (2.11a), we get for $P\Psi^P$ a coupled set of equations

$$P_1(H-E)P_1\Psi_1^P=-P_1(H-E)P_2\Psi_2^P, \tag{2.15a}$$
$$P_2(H-E)P_2\Psi_2^P=-P_2(H-E)P_1\Psi_1^P. \tag{2.15b}$$

Therefore (2.11a) *can* be replaced by (2.15), i.e.,

$$P_i(H-E)(P_1\Psi_1^P+P_2\Psi_2^P)=0 \tag{2.15$'$}$$

for $i=1$, 2, where P does not come in. Similarly, we have, for G^P,

$$P_i(H-E)PG^PP=-P_i. \tag{2.16}$$

Solving (2.16) explicitly, we have

$$G^P=G_{11}^P+G_{12}^P+G_{21}^P+G_{22}^P, \tag{2.17}$$

where

$$G_{ij}^P\equiv P_iG_{ij}^PP_j$$

and

$$G_{11}^P=[P_1\{E-H-(H-E)g_2^P(H-E)\}P_1]^{-1},$$
$$G_{12}^P=G_{11}^P(H-E)g_2^P,\ \text{etc.}, \tag{2.18}$$
$$g_i^P=[P_i(E-H)P_i]^{-1}=P_ig_i^PP_i. \tag{2.19}$$

(A simple derivation of G^P is given in Appendix A.) Using the properties (2.7), (2.13), (2.15), and (2.16), we can immediately rewrite (2.10) in the form

$$\begin{aligned}\Delta_U&=2(Q\Psi_t\mid H-E\mid P\Psi^P)\\&\quad+(Q\Psi_t\mid H-E+(H-E)G^P(H-E)\mid Q\Psi_t)\\&=2(X_t\mid H-E\mid P\Psi^P)\\&\quad+(X_t\mid H-E+(H-E)G^P(H-E)\mid X_t),\end{aligned} \tag{2.20}$$

where X_t is an arbitrary square-integrable function which satisfies the boundary conditions at the origin, but is *not* necessarily orthogonal to P. The last equality in (2.20) comes about essentially because M defined by

$$M\equiv\mathcal{H}-E=H-E+(H-E)G^P(H-E) \tag{2.21}$$

is an operator in the Q space, and

$$N\equiv(H-E)P\Psi^P \tag{2.21$'$}$$

is also a function in the Q space. That is,

$$P_iM=MP_i=0, \qquad M=QMQ,$$
$$P_iN=0, \qquad N=QN. \tag{2.22}$$

Therefore, we finally have the generalized variational bound (GVB) given by

$$\begin{aligned}\lambda&\leq\lambda^P+2(X_t\mid H-E\mid\Psi^P)\\&\quad+(X_t\mid H-E+(H-E)G^P(H-E)\mid X_t)\\&\equiv\lambda^P+\Delta_U=\lambda_U,\end{aligned} \tag{2.23}$$

where Ψ^P is given by (2.12), satisfying (2.15); G^P is given by (2.17), satisfying (2.16); and X_t is the square-integrable trial function. In the form (2.23), we have eliminated P and Q completely, and the only knowledge of the open channels is contained in P_i. Possible modification due to the presence of a finite number of subtraction terms can be made on (2.23), if necessary.[1]

In order to bring out more closely the effect of nonorthogonal part of X_t on the bound property of (2.23), we carry out explicitly the proof of the bound using the Kohn variational principle. It has been shown[1] that (2.10) is completely *equivalent* to the Kohn principle

$$[\lambda]=\lambda_t+(\Psi_t\,|\,H-E\,|\,\Psi_t),\qquad \delta[\lambda]=0 \tag{2.24}$$

if the trial function Ψ_t in (2.24) is of the specific form

$$\begin{aligned}\Psi_t&=P\Psi^P+G^PPHQ\Psi_t+Q\Psi_t\\ &\equiv P\Psi_t+Q\Psi_t.\end{aligned} \tag{2.25}$$

The error function Ω is then given by

$$\begin{aligned}\Omega&\equiv\Psi-\Psi_t\\ &=G^PPHQ\Omega+Q\Omega.\end{aligned} \tag{2.26}$$

Using the Kato identity, we have

$$\begin{aligned}\lambda-\lambda_t&=(\Psi\,|\,H-E\,|\,\Psi_t)\\ &=(\Psi_t\,|\,H-E\,|\,\Psi_t)-\tau',\end{aligned} \tag{2.27}$$

where

$$\tau'=(\Omega\,|\,H-E\,|\,\Omega). \tag{2.28}$$

Substitution of (2.26) into (2.27) and (2.28) gives, after some algebra,

$$\lambda=\lambda^P+\Delta_U'-\tau', \tag{2.29}$$

where

$$\tau'=(Q\Omega\,|\,\mathcal{H}'-E\,|\,Q\Omega), \tag{2.30}$$

and where we have also used

$$\lambda_t=\lambda^P+(Q\Psi_t\,|\,QHP\,|\,P\Psi^P).$$

Thus, we have

$$\lambda\leq\lambda^P+\Delta_U'=\lambda_U', \tag{2.31}$$

if

$$\mathcal{H}'-E\geq0. \tag{2.32}$$

The situation with (2.23) is little more involved; we have, instead of (2.25),

$$\Psi_t=P\Psi^P+FX_t, \tag{2.33}$$

where

$$F=1+G^PP(H-E). \tag{2.34}$$

[The form (2.33) is equivalent to requiring that $P(H-E)\Psi_t=0$ is solved *exactly* for Ψ_t in the form $\psi_t=P\Psi_t+X_t$. Details are given in Sec. III.] Substitution of (2.33) into (2.24) again gives

$$\lambda=\lambda^P+\Delta_U-\tau, \tag{2.35}$$

where Δ_U is given by (2.20) and

$$\tau=(\Omega\,|\,H-E\,|\,\Omega)=(\omega\,|\,\mathcal{H}-E\,|\,\omega) \tag{2.36}$$

with

$$\Omega\equiv\Psi-\Psi_t=F(X-X_t)\equiv F\omega. \tag{2.37}$$

It is clear that F of (2.34) is an operator which is nonzero only when operated on a Q-space function, and thus

$$FP=0,\qquad FQ=F. \tag{2.38}$$

Therefore, if we assume that

$$\mathcal{H}-E=M\geq0, \tag{2.39}$$

then

$$\tau\geq0 \tag{2.40}$$

and

$$\lambda\leq\lambda^P+\Delta_U=\lambda_U. \tag{2.41}$$

We again stress the fact that (2.23) is completely equivalent to (2.24) with (2.33) under the condition (2.39), as was the case with (2.9) being equivalent to (2.24) with (2.25).

The approximate positions of resonances, without the energy-shift operator $(H-E)G^P(H-E)$, is often estimated by the Ritz-type variational calculation[9] for the operator QHQ. Obviously, such a procedure is no longer available since Q is assumed unknown explicitly. However, from the discussion given above, it is clear that $M=(\mathcal{H}-E)$ of (2.21) should be a good operator to use for such purpose. For E below the lowest eigenstate of $\mathcal{H}$, M is positive definite and thus essentially replaces QHQ in the resonance energy calculation. The resulting eigenvalues M_μ give

$$\mathcal{E}_\mu{}^Q=M_\mu+E, \tag{2.42}$$

where the shift effect is taken into account approximately so long as $G^P(E)\approx G^P(\mathcal{E}_\mu{}^Q)$ and G^P is a slowly varying function of E near $E\approx\mathcal{E}_\mu{}^Q$. It is important to keep in mind that E which appears in the operator M should all be the *same* in order to stay in the Q space so that (2.22) is valid. The operator M should give a better estimate of $\mathcal{E}_\mu{}^Q$ than QHQ especially when the shifts are large. However, G^P is in general difficult to construct, although not impossible, and an iterative method of getting $\mathcal{E}_\mu{}^Q$ without an explicit use of G^P is considered in the next section.

III. ITERATION PROCEDURE FOR G^P

The generalized variational bound (2.23) has been shown to be equivalent to the Kohn variational principle (2.24) with (2.33). However, for practical purposes, both procedures involve G^P which could be rather involved. In Ref. 5, we have discussed an iterative procedure of avoiding the explicit use of G^P by replacing (2.25) with an *exact* solution of the differential equations

$$P(H-E)P\Psi_t=-PHQ\Psi_t \tag{3.1}$$

for a given $Q\Psi_t$. It is required that $Q\Psi_t$ is optimized *variationally* by minimizing the expression Δ_U. For the case of e^-H scattering, Burke and Taylor[8] made an important observation that even (3.1) is unnecessarily strong and in fact an *exact* solution of the equation

$$P(H-E)\Psi_t=0 \tag{3.2}$$

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[9] A. K. Bhatia, A. Temkin, and J. F. Perkins, Phys. Rev. **153**, 177 (1967), where references to earlier work can be found.

is sufficient, without separating Ψ_t into two parts which requires the explicit form of P and Q. Their argument is that one *can* always write (3.1) from (3.2) since $P+Q=1$. Of course this is a trivial point if such P and Q are explicitly available. However, when they are not, as in most cases of interest, then the advantage of (3.2) over (3.1) in practical calculations is more than a simple identity as it seems. Using (2.13), we can in fact show explicitly below that the exact solution of a set of coupled equations

$$P_i(H-E)\Psi_t=0, \qquad i=1,\, 2 \tag{3.3}$$

combined with the Kohn variational principle (2.24) is completely equivalent to (2.23). To see this, we formally solve (3.3) as

$$\Psi_t=P_1f_1+P_2f_2+X_t, \tag{3.4}$$

where

$$P_if_i=P_i\Psi_i{}^P+(G_{ii}{}^P+G_{ij}{}^P)(H-E)X_t, \qquad i\neq j \tag{3.5}$$

and X_t is again a square-integrable function but *not* necessarily orthogonal to both P_1 and P_2. Thus, Ψ_t of (3.4) is exactly the form (2.33), and the proof of the bound is then identical to the one given in Sec. II; we obtain the same result (2.23) as expected. In so far as the practical procedure is concerned, (3.3) combined with (2.24) may be much simpler than (2.23) since we do not have to solve for G^P explicitly. We let

$$X_t=\sum_{i=1}^{N} c_iX_{it}(\alpha), \tag{3.6}$$

where c_i are the linear parameters and α is one of the nonlinear parameters. For a fixed value of α, c_i can be determined from the equation

$$\sum_j (H_{ij}-EN_{ij})c_j=-R_{1i}-R_{2i}, \tag{3.7}$$

where

$$H_{ij}=(X_{it}, HX_{jt}), \qquad N_{ij}=(X_{it}, X_{jt}),$$

$$R_{1i}=(X_{it}, [H-E]P_1f_1).$$

By simultaneously diagonalizing the energy matrix and the normalization matrix, we get

$$X_t=\sum_{\mu=1}^{N}\frac{|X_{\mu t})(X_{\mu t}{}^*|}{E-E_{\mu t}(\alpha)}[H-E](P_1f_1+P_2f_2). \tag{3.8}$$

Substitution of (3.8) into (3.3) gives the coupled equations for P_1f_1 and P_2f_2 which are to be solved *exactly*. We have explicitly included the α dependence of $E_{\mu t}$ in (3.8). Thus, we have

$$P_i(H-E)P_if_i+\sum_{\mu=1}^{N}\frac{P_i(H-E)|X_{\mu t})(X_{\mu t}|(H-E)}{E-E_{\mu t}(\alpha)}P_if_i$$
$$=-P_i\left[H-E+\sum_{\mu=1}^{N}(H-E)\frac{|X_{\mu t})(X_{\mu t}|}{E-E_{\mu t}(\alpha)}(H-E)\right]P_jf_j, \tag{3.9}$$

where $i\neq j$. For each value of α, the diagonalization of H_{ij} has to be repeated, and (3.9) solved *exactly*. The resulting parameters then are a bound as in (2.23), essentially due to (2.38). This iteration procedure should converge for most cases.[5] It requires some caution, however, in applying (3.9) in an extensive search for the "best" α which gives the minimum λ_U. Since X_{it} are *not* preorthogonalized to the P space, $E-E_{\mu t}(\alpha)$ may vanish occasionally as α is swept over a range of values. So long as such spurious singular points are avoided, the procedure outlined above is the same as (2.23). In this connection, we mention a procedure proposed recently by Harris,[10] precisely to overcome such singularities. The main point is to choose

$$E=E_{\mu t}, \qquad \mu=1, \cdots, N \tag{3.10}$$

and writing

$$X_t=\sum_{\mu=1}^{N} a_\mu X_{\mu t}, \tag{3.11}$$

we obtain, for the original equation $(H-E)\Psi_t=0$,

$$\langle X_{\mu t}\,|\,H-E\,|\,P_1f_1\rangle+\langle X_{\mu t}\,|\,H-E\,|\,P_2f_2\rangle=0. \tag{3.12}$$

Equation (3.12) is sufficient to estimate the scattering parameters. Of course, the bound property is *lost* since P_1f_1 and P_2f_2 are *not obtained exactly*, i.e., the P equations are not solved exactly for P_if_i.

Finally, we consider a possible way of eliminating G^P in the resonance energy calculation. If we define Py_{it}, for each X_{it}, by the equations

$$P(H-E)Py_{it}=-P(H-E)X_{it} \tag{3.13}$$

exactly such that

$$Py_{it}\rightarrow(\text{const})_iG^P, \tag{3.14}$$

then the matrix elements needed to evaluate the resonance energies are given by

$$M_{ij}=(X_{it}\,|\,M\,|\,X_{jt})$$
$$=(X_{it}\,|\,H-E\,|\,X_{jt})+(X_{it}\,|\,H-E\,|\,Py_{jt}). \tag{3.15}$$

It is now a simple matter to diagonalize M_{ij}, with the added feature that M_{ij} never vanishes (all the eigenvalues are positive definite). This is perhaps an improvement over the procedure of Ref. 8.

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[10] F. E. Harris, Phys. Rev. Letters **19**, 173 (1967).

APPENDIX A: STATIC GREEN'S FUNCTION P^P

The static Green's function G^P which satisfies (2.11b) assumes the form (2.7) if there exists[2] an operator P which satisfies (2.12)–(2.14). Equation (2.16) can be written as

$$P_1(H-E)(G_{11}{}^P+G_{21}{}^P)+P_1(H-E)(G_{12}{}^P+G_{22}{}^P)=-P_1, \tag{A1}$$

$$P_2(H-E)(G_{22}{}^P+G_{12}{}^P)+P_2(H-E)(G_{21}{}^P+G_{11}{}^P)=-P_2, \tag{A2}$$

where $G_{ij}{}^P$ are yet undefined. Since, for example, $G_{12}{}^P$ and $G_{22}{}^P$ in (A1) involve P_2 on their right, we rewrite (A1) and (A2) in the form

$$P_1(H-E)G_{11}+P_1(H-E)G_{21}=-P_1, \tag{A3}$$

$$P_1(H-E)G_{12}+P_1(H-E)G_{22}=0, \tag{A4}$$

and

$$P_2(H-E)G_{22}+P_2(H-E)G_{12}=-P_2, \tag{A5}$$

$$P_2(H-E)G_{21}+P_2(H-E)G_{11}=0. \tag{A6}$$

The above separation is justified since, from (2.13), the P_2 part of P can always be incorporated in (A5) and not in (A4), and similarly for P_1. Now (A3) is coupled to (A6) and (A5) is coupled to (A4). The solution is trivially obtained to be the form (2.18). The step leading to (A3)–(A6) from (A1) and (A2) can be shown to be correct in another way by solving a coupled inhomogeneous equation

$$P(H-E)Pf=-PR. \tag{A7}$$

Using the fact that

$$Pf\equiv P_1f_1+P_2f_2,$$

$$PR=P_1R_1+P_2R_2, \tag{A8}$$

we have

$$P_1\bar{\Psi}f_1=P^1\Psi^P+G_{11}{}^P(H-E)P_2\Psi^{P2}+(G_{11}{}^P+G_{12}{}^P)PR, \tag{A9}$$

where

$$[G_{11}{}^P]^{-1}P_1\bar{\Psi}^P=0, \qquad [g_1{}^P]P_1\Psi^{P1}=0.$$

On the other hand, we formally have

$$Pf=P\Psi^P+G^PPR \tag{A10}$$

with

$$P_1\Psi_1{}^P=P_1\bar{\Psi}^P+G_{11}{}^P(H-E)P_2\Psi^{P_2},$$

$$P\Psi^P\equiv P_1\Psi_1{}^P+P_2\Psi_2{}^P.$$

Comparing (A9) with (A10), we recover (2.17).

APPENDIX B: PROJECTION OPERATORS P_i AND P

The crucial properties of the projection operators which were used in the derivation of the generalized variational bound (2.23) are (2.13) and (2.14), and we give here a brief discussion of their validity.

From one of the requirements that P must satisfy in constructing it from P_i [Eq. (2.5)], we have, in the case of two channels, for example,

$$(P_i,[\Psi-(P_1\Psi_1+P_2\Psi_2)])=0, \tag{B1}$$

or

$$(P_i,Q\Psi)=0, \tag{B2}$$

which is simply (2.14). Now, (2.14) in turn gives

$$P_iQ=P_i(1-P)=0, \tag{B3}$$

or simply

$$P_iP=P_i, \tag{B4}$$

as it should. Using the Hermiticity property (2.3), we also have

$$PP_i=P_i. \tag{B5}$$

There are several cases for which P and P_i are explicitly available. First, we have the e^-H elastic scattering in which[1]

$$P=P_1+P_2-P_1P_2=P_1+P_2-P_2P_1,$$

$$Q=1-P=Q_1Q_2=Q_2Q_1. \tag{B6}$$

Obviously, both (B4) and (B5) are satisfied.

Second, we have the explicit P for the e^+H elastic and pickup collisions given by Mittleman,[11] as derived by the procedure worked out by Feshbach,[2]

$$P=\psi_0\rangle\left[\delta(\mathbf{r}-\mathbf{r}')+\sum_\lambda\frac{u_\lambda\rangle\langle u_\lambda{}^*}{\lambda^2-1}\right]\langle\psi_0{}^*+\sum_\lambda\psi_0u_\lambda\rangle\frac{\lambda}{\lambda^2-1}$$

$$\times\langle v_\lambda{}^*\phi_0{}^*+\sum_\lambda\phi_0v_\lambda\rangle\frac{\lambda}{\lambda^2-1}\langle u_\lambda{}^*\psi_0{}^*+\phi_0\rangle$$

$$\times\left[\delta(\mathbf{p}-\mathbf{p}')+\sum_\lambda\frac{v_\lambda\rangle\langle v_\lambda{}^*}{\lambda^2-1}\right]\langle\phi_0{}^*, \tag{B7}$$

where $\psi_0(\mathbf{r})$ is the hydrogenic ground-state wave function and $\phi_0(\mathbf{p})$ is for the positronium. u_λ and v_λ are solutions of the coupled equations

$$u_\lambda+\lambda K_1v_\lambda=0,$$

$$v_\lambda+\lambda K_2u_\lambda=0, \tag{B8}$$

with

$$K_1=8\psi_0{}^*, \qquad \phi_0=K_2{}^+=-\sum_\lambda\lambda^{-1}u_\lambda v_\lambda{}^*. \tag{B9}$$

Defining the channel operators P_i, $i=1$, 2, as

$$P_1=\psi_0\rangle\langle\psi_0{}^*, \qquad P_2=\phi_0\rangle\langle\phi_0{}^*, \tag{B10}$$

we can show, after some algebra using (B8)–(B10), that (B4) and (B5) are again satisfied.

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