Nuclear Polarization in Muonic Atoms of Deformed Nuclei

MIN-YI CHEN

*Physics Department, Columbia University, New York, New York 10027**

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The analysis of muonic x-ray spectra in deformed nuclei was carried out in the past by diagonalizing the electric quadrupole interaction between the spin doublets $(2p_{3/2}, 2p_{1/2})$, $(3d_{5/2}, 3d_{3/2})$ and the lowest rotational band of the nucleus. With the present experimental accuracy, this procedure is no longer adequate. To take into account the muonic and nuclear states not included in the diagonalization, we renormalize the electric quadrupole interaction by virtual excitations into those states. We find that the renormalization correction amounts to a few percent of the quadrupole matrix elements. This explains the systematic tendency for the intrinsic quadrupole moments obtained from the traditional analysis to be a few percent larger than the values deduced from Coulomb excitation experiments. We also find that the inclusion of the renormalization corrections has a very significant effect on the parameters of the charge distribution and the intensity ratios of the hyperfine multiplets.

1. INTRODUCTION

THE electric quadrupole hyperfine spectra of muonic atoms of deformed nuclei have been measured accurately in the last few years. While the experimental accuracy has improved to a fraction of keV at a few MeV, the analysis of the spectra still remains the same as that of the pioneering works of Wilets[1] and Jacobsohn,[2] who diagonalized the electric quadrupole interaction between the lowest rotational band of the nuclei and the spin multiplets: $1s_{1/2}$, $(2p_{1/2}, 2p_{3/2})$, $(3d_{3/2}, 3d_{5/2})$ (called the model space in this paper). It is desirable to make a more careful theoretical analysis in which the configurations of the muon and the nucleus are no longer restricted to the model space.

Up to now all the experimental data have been analyzed by starting from a certain charge distribution, for example, the modified Fermi distribution

$$\rho(r',\theta')=N\left[1+\exp\left(4\ln 3\,\frac{r'-c(1+\beta Y_{20}(\theta'))}{t(1+\beta' Y_{20}(\theta'))}\right)\right]^{-1}.$$

The parameters c, t, β, β' are then varied to get a best fit to the experimental spectra. In more detail,[3] the charge density is expressed in the form

$$\rho(r',\theta')=\rho_0(r')+\rho_2(r')\,Y_{20}(\theta'),$$

where $\mathbf{r}'$ is the position vector for the nuclear charge density. The Dirac equation is solved in the spherical part $\rho_0(r')$ to give the muonic states in the model space. The quadrupole part of the electrostatic interaction H_Q is then diagonalized within the model space, where

$$H_Q=\tfrac{1}{2}eQ_0f(r)P_2(\cos\vartheta),$$

$$Q_0=2\left(\frac{4\pi}{5}\right)^{1/2}\int_0^\infty\rho_2(r')r'^4dr',$$

$$f(r)=\frac{2}{Q_0}\left(\frac{4\pi}{5}\right)^{1/2}\left(r^2\int_r^\infty\rho_2(r')\,\frac{dr'}{r'}+\frac{1}{r^3}\int_0^r\rho_2(r')r'^4dr'\right).$$

The nuclear states are taken to be rotational states $|IK\rangle$. The model space consists of states of the form

$$|IK, nlj, FM\rangle,$$

where F, M are the total and the z component of the angular momentum of the muon-nucleus system. (We will suppress the index n when the state $|nlj\rangle$ is one of the states $1s_{1/2}$, $2p_{3/2}$, $3d_{5/2}$, $3d_{3/2}$.) The matrix element of H_Q has the following form:

$$\langle I_1K_1, l_1j_1, FM\,|\,H_Q\,|\,I_2K_2, l_2j_2, FM\rangle=\alpha_{j_1j_2}A_2(I_1K_1l_1j_1F, I_2l_2j_2), \quad (1.1)$$

where $K_1=K_2$, since the quantum number K is the same within a rotational band. Also,

$$\alpha_{j_1j_2}=-(e^2Q_0/10)\,\langle l_1j_1\,|\,f(r)\,|\,l_2j_2\rangle, \quad (1.2)$$

$$A_2\equiv A_2(I_1K_1l_1j_1F, I_2l_2j_2)$$
$$=(-1)^{I_1+I_2+F-K+(1/2)}5[(2j_1+1)(2j_2+1)(2I_1+1)\times(2I_2+1)]^{1/2}\begin{Bmatrix}F & j_1 & I_1\\ 2 & I_2 & j_2\end{Bmatrix}\begin{pmatrix}I_1 & 2 & I_2\\ -K_1 & 0 & K_1\end{pmatrix}\times\begin{pmatrix}j_1 & 2 & j_2\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}\tfrac{1}{2}[1+(-1)^{l_1+l_2}]. \quad (1.3)$$

The quantity $\alpha_{j_1j_2}$ remains essentially constant in the model space. (The slight differences of the radial wave functions of $2p_{3/2}$, $2p_{1/3}$ will cause $\alpha_{j_1j_2}$ to vary by less

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[1] L. Wilets, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, 3 (1954).

[2] B. A. Jacobsohn, Phys. Rev. **96**, 1637 (1954).

[3] H. L. Acker, Nucl. Phys. **78**, 153 (1966).

than 10% within the model space.) We have used the notations of Edmonds.[4]

When the matrix $\langle H_Q \rangle$ is diagonalized, a complicated spectrum results. The wave functions $|\, \nu, FM\rangle$ are linear combinations of the unperturbed states in the model space

$$|\, \nu, FM\rangle = \sum_{I,j} C(I, j, \nu, F)\, |\, IK, lj, FM\rangle,$$

where the quantities $C(I, j, \nu, F)$ are obtained by diagonalization. The index ν is used to specify one particular state. The observed K and L x-ray intensities then depend upon the $E1$ reduced matrix element between the initial state $|\, \nu_i, F_i M_i\rangle$ and the final state $|\, \nu_f, F_f M_f\rangle$:

$$\langle \nu_f, F_f \,||\, M(E1) \,||\, \nu_i, F_i\rangle = \sum_{I_f j_f I_i j_i} C^*(I_f j_f \nu_f F_f) \times C(I_i j_i \nu_i F_i)\, \langle I_f K_1, l_f j_f, F_f \,||\, M(E1) \,||\, I_i K_1, l_i j_i, F_i\rangle,$$

where the reduced matrix elements between the unperturbed states can be written as

$$\langle I_f K_1, l_f j_f, F_f \,||\, M(E1) \,||\, I_i K_1, l_i j_i, F_i\rangle = M_1 B_1(I_f K_1 l_f j_f F_f, I_i l_i j_i F_i\rangle, \quad (1.4)$$

and

$$M_1 = \langle l_f j_f \,|\, r_\mu \,|\, l_i j_i\rangle,$$

$$B_1(I_f K_1 l_f j_f F_f, I_i l_i j_i F_i) = (-1)^{I_i + j_f + j_i + F_f + (1/2)} \delta_{I_f I_i} \times [(2j_f+1)(2j_i+1)(2F_f+1)]^{1/2} \begin{Bmatrix} j_f & F_f & I_f \\ F_i & j_i & 1 \end{Bmatrix} \times \begin{pmatrix} j_f & 1 & j_i \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \tfrac{1}{2}[1+(-1)^{l_f+l_i+1}]. \quad (1.5)$$

When the experimental spectra are analyzed by the method sketched above, the intrinsic quadrupole moments obtained are somewhat larger than the values deduced from Coulomb excitation experiments, the values of the skin thickness come out to be smaller than would be expected and the relative intensities are not so well reproduced by the assumption of the statistical filling of the $4f$ levels.[5,6,7] Therefore, more nuclear and muonic states other than those in the model space are taken into account in the following analysis.

2. THEORY

In heavy deformed nuclei, the energy differences of the unperturbed states in the model space, and the electric quadrupole matrix element $\langle H_Q \rangle$, are all of the order of 50 keV, while configurations not included in the model space are at least several MeV away. Therefore, to take into account the effects of configurations outside of the model space, it is sufficient to use perturbation theory. We shall basically follow the approach of Eden and Francis.[8,9] Suppose the eigenvalue problem to be solved is $(H_0+V)\Psi = E\Psi$. The unperturbed wave functions ϕ_i are defined by $H_0\phi_i = \epsilon_i\phi_i$. The model space is spanned by ϕ_i, $i=1, m$. The eigenfunction in the model space is denoted by

$$\Psi_M = \sum_{i=1}^{m} a_i \phi_i,$$

while the complete wave function is an infinite sum

$$\Psi = \sum_{i=1}^{\infty} a_i \phi_i.$$

The wave functions Ψ and Ψ_M are formally related by

$$\Psi = \Omega_M \Psi_M,$$

with

$$\Omega_M = 1 - [Q_M/(H_0 - E)]V\Omega_M,$$

where

$$Q_M = \sum_{i=m+1}^{\infty} |\, \phi_i\rangle\langle \phi_i \,|.$$

The infinite-dimensional problem $(H+V)\Psi = E\Psi$ is formally reduced to a finite-dimensional problem

$$(H_0 + V\Omega_M)\Psi_M = E\Psi_M$$

in the model space. We should diagonalize the model interaction $V\Omega_M$ instead of V in the finite-dimensional model space. In calculating the transition matrix elements between the final state Ψ_f and the initial state Ψ_i, we use the following relation:

$$\langle \Psi_f \,|\, M(E1) \,|\, \Psi_i\rangle = \langle \Psi_{fM} \Omega_M{}^\dagger \,|\, M(E1) \,|\, \Omega_M \Psi_{iM}\rangle,$$

where Ψ_{fM}, Ψ_{iM} are the eigenfunctions in the model space. Since all the states which are not included in the model space are very far away, it is sufficient to keep only the first term in the iteration expansion of Ω_M:

$$\Omega_M \approx 1 - [Q_M/(H_0 - E)]V.$$

Now it is clear that in order to improve the analysis outlined in Sec. 1, we have to replace the matrix ele-

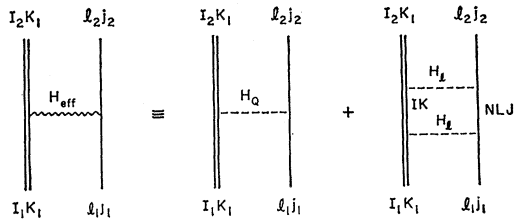

FIG. 1. Effective quadrupole matrix element.

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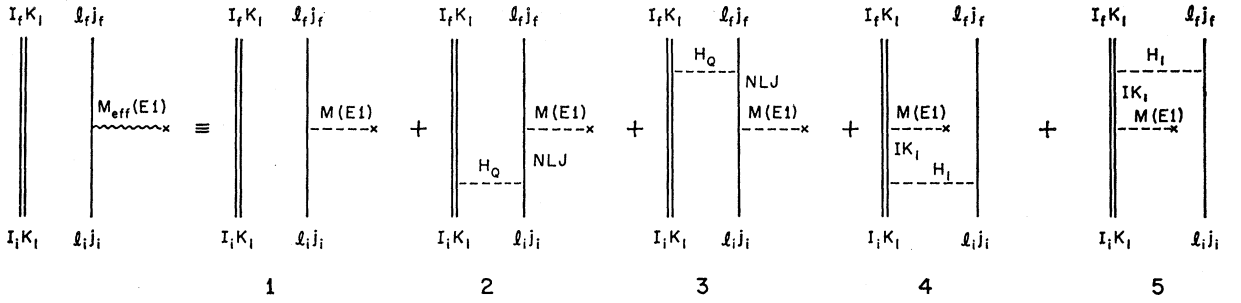

FIG. 2. Effective $E1$ transition matrix element.

ments of H_Q and $M(E1)$, Eqs. (1.1) and (1.4), by effective matrix elements H_{eff} and $M_{\text{eff}}(E1)$ defined by the graphs in Figs. 1 and 2, respectively.

A. Effective Quadrupole Interaction

It is convenient to separate the second-order graph in Fig. 1 into two parts: (i) The nuclear intermediate states I remain in the model space and the muonic intermediate states $|\, NLJ\rangle$ are outside the model space, and (ii) the nuclear intermediate states $|\, I\rangle$ are outside the model space.

When the nuclear states remain in the lowest rotational band, it is straightforward to calculate the second-order correction. Representing the correction to the matrix elements by $\langle H_Q^2\rangle$, we get

$$\langle H_Q^2\rangle=\sum_{I,N,L,J}\frac{\langle I_1K_1, l_1j_1, FM\,|\,H_Q\,|\,IK_1, NLJ, FM\rangle\langle IK_1, NLJ, FM\,|\,H_Q\,|\,I_2K_1, l_2j_2, FM\rangle}{E_{I_2K_1}+E_{l_2j_2}-E_{IK_1}-E_{NLJ}}.$$

The angular part can be integrated first with the help of Eq. (1.1):

$$\langle H_Q^2\rangle=[(e^2Q_0)^2/20]\sum_L A_{2,L}\sum_N\frac{\langle R_1\,|\,f(r_\mu)\,|\,R_{NL}\rangle\langle R_{NL}\,|\,f(r_\mu)\,|\,R_2\rangle}{E_{I_2K_1}+E_{l_2j_2}-E_{IK_1}-E_{NLJ}},$$

where R_1, R_2, and R_{NL} are radial wave functions, and

$$A_{l,L}=A_{l,L}(I_1K_1l_1j_1F;I_2l_2j_2)$$
$$=\sum_I\sum_{J=L-(1/2)}^{L+(1/2)}(-1)^{I_1+I_2}(2l+1)(2J+1)(2I+1)[(2j_1+1)(2j_2+1)(2I_1+1)(2I_2+1)]^{1/2}$$
$$\times\begin{Bmatrix}F & j_1 & I_1\\ l & I & J\end{Bmatrix}\begin{Bmatrix}F & j_2 & I_2\\ l & I & J\end{Bmatrix}\begin{pmatrix}I_1 & l & I\\ -K_1 & K_1-K & K\end{pmatrix}\begin{pmatrix}I_2 & l & I\\ -K_1 & K_1-K & K\end{pmatrix}\begin{pmatrix}j_1 & l & J\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}j_2 & l & J\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$$
$$\times\tfrac{1}{2}[1+(-1)^{L+l+l_1}]\tfrac{1}{2}[1+(-1)^{L+l+l_2}]. \quad (2.1)$$

The sum $\sum_I$ extends over the rotational band. The sum over the complete set of munoic states, both discrete and continuous, can be carried out exactly by the reference spectrum method.[10,11] Since the second-order corrections are small, the nonrelativistic theory should be sufficient. That is, we calculate the correction to the wave function

$$|\, X\rangle=\sum_N\frac{|\, R_{NL}\rangle\langle R_{NL}\,|\,f(r_\mu)\,|\,R_2\rangle}{E_{I_2K_1}+E_{l_2j_2}-E_{IK}-E_{NLJ}}$$

by solving the inhomogeneous equation

$$(E_{I_2K}+E_{l_2j_2}-E_{IK}-H_{L\mu})\,|\,X\rangle=g(r_\mu),$$

where

$$H_{L\mu}=-\frac{\hbar^2}{2m_\mu}\frac{d^2}{dr_\mu^2}+\frac{\hbar^2}{2m_\mu}\frac{L(L+1)}{r_\mu^2}+V(r_\mu),$$
$$g(r_\mu)=f(r_\mu)\,|\,R_2\rangle.$$

The correction to the matrix element is then obtained by integration. If $L=l_2$, we have to replace the expression for $g(r_\mu)$ by

$$g(r_\mu)=f(r_\mu)\,|\,R_2\rangle-|\,R_2\rangle\langle R_2\,|\,f(r)\,|\,R_2\rangle.$$

It is necessary to do the projection, since in summing

[10] M. Y. Chen, preceding paper Phys. Rev. C **1**, 1167 (1970).
[11] M. Y. Chen, thesis, Princeton University, 1968 (unpublished).

over N, L, J we have to exclude the states already in the model space. Thus the contribution to the effective matrix element can be written in the form

$$\langle H_Q^2\rangle = \sum_L \beta_L A_{2L},$$
$$\beta_L = [(e^2Q_0)^2/20]\langle g(r_\mu) \mid X\rangle. \quad (2.2)$$

The second-order terms in which the nucleus is scattered into levels beyond the lowest rotational band are much more difficult to evaluate. We represent the corrections to the matrix elements by $\sum_l \langle H_l^2\rangle$, where $\langle H_0^2\rangle$ is the contribution from virtual excitations of monopole vibrations, $\langle H_1^2\rangle$ is the contribution from excitations of giant dipole states, etc., where

$$H_l = -[4\pi e^2/(2l+1)]\sum_{p_i} (r_<^l/r_>^{l+1})\mathbf{Y}_l(\Omega_{p_i})\cdot\mathbf{Y}_l(\Omega_\mu).$$

The sum $\sum_{p_i}$ extends over the protons in the nucleus. (The quantity $\sum_l H_l$ is the electrostatic interaction between the muon and the protons.) We note here that if the nuclear intermediate states $\mid IK\rangle$ are also deformed, and if we can write the reduced matrix elements $(I_1K_1 \mid\mid \mathbf{Y}_l(\mathbf{r}_{p_i}) \mid\mid IK)$ in the form

$$(I_1K_1 \mid\mid \mathbf{Y}_l \mid\mid IK) = (-1)^{I_1-K_1}[(2I_1+1)(2I+1)]^{1/2} \times \begin{pmatrix} I_1 & l & I \\ -K_1 & K_1-K & K \end{pmatrix} \langle \chi_1 \mid Y_{l,K_1-K} \mid \chi\rangle,$$

where $\mid \chi_1\rangle$, $\mid \chi\rangle$ are the intrinsic states of $\mid I_1K\rangle$, $\mid IK\rangle$, respectively, then the angular part of $\langle H_l^2\rangle$ can be integrated out first. In effect,

$$\langle H_l^2\rangle = \sum_{I,K;N,L,J} \frac{\langle I_1K_1, l_1j_1, FM \mid H_l \mid IK, NLJ, FM\rangle\langle IK, NLJ, FM \mid H_l \mid I_2K_1, l_2j_2, FM\rangle}{E_{I_2K_1}+E_{l_2j_2}-E_{IK}-E_{NLJ}}$$
$$= \sum_L A_{lL}\gamma_{lL}.$$

The matrix A_{lL} is given by (2.1) and

$$\gamma_{lL} = \frac{4\pi e^4}{(2l+1)^2}\sum_{\chi,N} \frac{\langle \chi_1R_1 \mid \sum_{p_i} (r_<^l/r_>^{l+1}) Y_{l,K_1-K} \mid \chi R_{NL}\rangle\langle \chi R_{NL} \mid \sum_{p_i} (r_<^l/r_>^{l+1}) Y_{l,K_1-K} \mid \chi_1R_2\rangle}{E_{\chi_1}+E_{l_2}-E_\chi-E_{NL}}, \quad (2.3)$$

where R_1, R_{NL} are radial munoic wave functions.

To summarize, the effective matrix element can be written as

$$\langle H_{\text{eff}}\rangle = \alpha_{j_1j_2}A_2 + \sum_L \beta_L A_{2L} + \sum_{lL} \gamma_{lL}A_{lL}. \quad (2.4)$$

The quantities α, β_L, γ_{lL} remain almost constant in the model space while A_2, A_{lL} are different matrices. It is of interest to note that the matrices A_{lL} can be written as the sum of two parts:

$$A_{lL} = C_{lL}A_2 + d_{lL}1,$$

where 1 is the unit matrix, and C_{lL}, d_{lL} are constants within the model space. From (2.4), we obtain the following expression:

$$\langle H_{\text{eff}}\rangle = (\alpha_{j_1j_2} + \sum_L \beta_L C_{2L} + \sum_{lL} \gamma_{lL}C_{lL})A_2 + (\sum_L \beta_L d_{2L} + \sum_{lL} \gamma_{lL}d_{lL})1. \quad (2.5)$$

Therefore, the effects of the inclusion of the second-order matrix elements are very simple: first, a renormalization of the coefficients $\alpha_{j_1j_2}$, or equivalently, a renormalization of the intrinsic quadrupole moment Q_0, by a factor η, where

$$\eta = 1 + (\sum_L \beta_L C_{2L} + \sum_{lL} \gamma_{lL}C_{lL})/\alpha,$$

and second, an increase of binding of all hyperfine components by a constant amount ΔE, where

$$\Delta E = \sum_L \beta_L d_{2L} + \sum_{lL} \gamma_{lL}d_{lL}.$$

The intrinsic quadrupole moment determined in the past without taking into account the above corrections is really an effective moment $Q_{\text{eff}} = \eta Q_0$. The calculated values of η are of the order of 1.03—1.05, which explains the systematic deviation[6,7] of Q_0 obtained from the earlier analysis from that calculated from Coulomb excitation measurements. The constant term, $\Delta E \approx 10$ keV for $1s$ level and ≈ 2 keV for $2p$ level, is the usual nuclear polarization correction, which has a very significant effect on the parameters of the charge distribution, especially the skin thickness.[7,12]

[12] D. Hitlin *et al.*, following paper, Phys. Rev. C **1**, 1184 (1970).

B. Effective $E1$ Transition Matrix Elements

The corrections in the above section are second-order corrections to the energy. In calculating the intensity ratios of the hyperfine multiplets, we should also include the first-order correction to the wave function. We calculate the five diagrams in Fig. 2, labeled 1 to 5 from left to right, to get the effective transition matrix element $M_{\text{eff}}(E1)$. Following the procedure of the previous section, we can first integrate out the angular factors and then sum over the complete set of munoic intermediate states by the reference spectrum method. The results are given in the following equation:

$$\langle I_f K_1, l_f j_f, F_f \,||\, M_{\text{eff}}(E1) \,||\, I_i K_1, l_i j_i, F_f\rangle = M_1 B_1(I_f K_1 l_f j_f F_f, I_i l_i j_i F_i) + \sum_L M_{2L}B_{2L} + \sum_L M_{3L}B_{3L} + M_4 B_4 + M_5 B_5,$$

where M_1, B are given by (1.5), $M_{2L}B_{2L}$, $M_{3L}B_{3L}$, M_4B_4, M_5B_5 are the contributions that come from diagrams 2 to 5 in Fig. 2. The quantities B_1, B_{2L}, B_{3L}, B_4, B_5 are matrices, while M_1, M_{2L}, M_{3L}, M_4, M_5 are constants in the model space. Also,

$$B_{2L}=\sum_{J=L-(1/2)}^{L+(1/2)}(-1)^{-I_i+j_f+J+F_f+F_i-K_1}(2J+1)[(2I_f+1)(2I_i+1)(2j_f+1)(2j_i+1)(2F_f+1)]^{1/2} \times\begin{Bmatrix}F_i & J & I_f\\ 2 & I_i & j_i\end{Bmatrix}\begin{pmatrix}I_f & 2 & I_i\\ -K_1 & 0 & K_1\end{pmatrix}\begin{pmatrix}J & 2 & j_i\\ -\frac12 & 0 & \frac12\end{pmatrix}\tfrac12[1+(-1)^{L+l_i}]\begin{Bmatrix}j_f & F_f & I_f\\ F_i & J & 1\end{Bmatrix}\begin{pmatrix}j_f & 1 & J\\ -\frac12 & 0 & \frac12\end{pmatrix}\tfrac12[1+(-1)^{l_f+L+1}],$$

$$B_{3L}=\sum_{J=L-(1/2)}^{L+(1/2)}(-1)^{-I_f+j_i+J+2F_f-K_1}(2J+1)[(2I_f+1)(2I_i+1)(2j_f+1)(2j_i+1)(2F_f+1)]^{1/2} \times\begin{Bmatrix}F_f & j_f & I_f\\ 2 & I_i & J\end{Bmatrix}\begin{pmatrix}I_f & 2 & I_i\\ -K_1 & 0 & K_1\end{pmatrix}\begin{pmatrix}j_f & 2 & J\\ -\frac12 & 0 & \frac12\end{pmatrix}\tfrac12[1+(-1)^{l_f+L}]\begin{Bmatrix}J & F_f & I_i\\ F_i & j_i & 1\end{Bmatrix}\begin{pmatrix}J & 1 & j_i\\ -\frac12 & 0 & \frac12\end{pmatrix}\tfrac12[1+(-1)^{L+l_i+1}],$$

$$B_4=\sum_I(-1)^{I-I_i-j_f+(1/2)}(2I+1)[(2I_f+1)(2I_i+1)(2j_f+1)(2j_i+1)(2F_f+1)]^{1/2} \times\begin{Bmatrix}I_f & F_f & j_f\\ F_i & I & 1\end{Bmatrix}\begin{Bmatrix}F_i & j_f & I\\ 1 & I_i & j_i\end{Bmatrix}\begin{pmatrix}I_f & 1 & I\\ -K_1 & 0 & K_1\end{pmatrix}\begin{pmatrix}I & 1 & I_i\\ -K_1 & 0 & K_1\end{pmatrix}\begin{pmatrix}j_f & 1 & j_i\\ -\frac12 & 0 & \frac12\end{pmatrix}\tfrac12[1+(-1)^{l_f+l_i+1}],$$

$B_5=B_4$.

Let L_{L,k_α} be the operator in

$$L_{L,k_\alpha}=d^2/dr_\mu^2-L(L+1)(1/r_\mu^2)-(2m_\mu/\hbar^2)V(r_\mu)-k_\alpha^2,$$

and let $h_\alpha(r_\mu)$ be the solution of

$$L_{L,k_\alpha}h_\alpha(r_\mu)=g_\alpha(r_\mu),\qquad \alpha=2,3.$$

Then

$$M_{2,L}=(m_\mu e^2Q_0/\hbar^2)\int G_2(r_\mu)h_2(r_\mu)\,dr_\mu,$$

where

$$L=l_f\pm1,$$

$$k_2^2=-2m_\mu E_{l_ij_i}/\hbar^2,$$

$$g_2(r_\mu)=r_\mu R_i(r_\mu)f(r_\mu) \quad \text{if } L\neq l_i$$

$$=r_\mu R_i(r_\mu)-r_\mu R_i(r_\mu)\int\xi^2R_i^2(\xi)f(\xi)\,d\xi, \quad \text{if } L=l_i$$

$$G_2(r_\mu)=r_\mu^2R_f(r_\mu) \quad \text{if } L\neq l_i$$

$$=r_\mu^2R_f(r_\mu)-r_\mu R_i(r_\mu)\int\xi^3R_i(\xi)R_f(\xi)\,d\xi, \quad \text{if } L\neq l_i$$

and $R_f(r_\mu)$, $R_i(r_\mu)$ are the muonic radial wave functions, $f(r_\mu)$ is given in Sec. 1. For M_{3L} we have

$$M_{3,L}=(m_\mu e^2Q_0/\hbar^2)\int G_3(r_\mu)h_3(r_\mu)\,dr_\mu,$$

where

$$L=l_i\pm1,$$

$$k_3^2=-2m_\mu E_{l_fj_f}/\hbar^2,$$

$$g_3(r_\mu)=r_\mu R_f(r_\mu)f(r_\mu) \quad \text{if } L\neq l_f$$

$$=r_\mu R_f(r_\mu)f(r_\mu)-r_\mu R_f(r_\mu)\int\xi^2R_f^2(\xi)f(\xi)\,d\xi \quad \text{if } L=l_f,$$

$$G_3(r_\mu)=r_\mu^2R_i(r_\mu) \quad \text{if } L\neq l_f$$

$$=r_\mu^2R_i(r_\mu)-r_\mu R_f(r_\mu)\int\xi^3R_f(\xi)R_i(\xi)\,d\xi \quad \text{if } L=l_f.$$

For M_4 and M_5, we have

$$M_4=\langle R_f\,|\,\mathfrak{F}_4(r_\mu)\,|\,R_i\rangle,$$

$$M_5=\langle R_f\,|\,\mathfrak{F}_5(r_\mu)\,|\,R_i\rangle,$$

where

$$\mathfrak{F}_4(r_\mu)=-\frac{4\pi e^2}{3}\sum_\chi\frac{\langle\chi_1\,|\,\sum_{p_i}(r_</r_>^2)Y_{10}(\Omega_{p_i})\,|\,\chi\rangle\langle\chi\,|\,\sum_{p_i}r_{p_i}Y_{10}(\Omega_{p_i})\,|\,\chi_1\rangle}{E_{\chi_1}-E_\chi+E_{l_ij_i}-E_{l_fj_f}},$$

$$\mathfrak{F}_5(r_\mu)=-\frac{4\pi e^2}{3}\sum_\chi\frac{\langle\chi_1\,|\,\sum_{p_i}(r_</r_>^2)Y_{10}(\Omega_{p_i})\,|\,\chi\rangle\langle\chi\,|\,\sum_{p_i}r_{p_i}Y_{10}(\Omega_{p_i})\,|\,\chi_1\rangle}{E_{\chi_1}-E_\chi-E_{l_ij_i}+E_{l_fj_f}}.$$

TABLE I. The quantities β_L in keV.

Isotope	β	$1s$ $L=2$	$2p$ $L=1$	$L=3$	$3d$ $L=0$	$L=2$	$L=4$
^{150}Nd	0.289	−3.12	−1.69	−0.43	+0.05	−0.06	−0.02
^{152}Sm	0.310	−4.07	−2.32	−0.61	+0.06	−0.09	−0.02
^{162}Dy	0.331	−5.50	−3.48	−0.98	+0.11	−0.16	−0.04
^{164}Dy	0.342	−5.94	−3.80	−1.07	+0.12	−0.18	−0.04
^{168}Er	0.338	−6.33	−4.18	−1.23	+0.14	−0.22	−0.06
^{170}Er	0.324	−5.76	−3.81	−1.13	+0.13	−0.20	−0.04
^{182}W	0.258	−4.13	−2.93	−0.97	+0.12	−0.20	−0.05
^{184}W	0.239	−3.71	−2.62	−0.87	+0.11	−0.17	−0.05
^{186}W	0.236	−3.65	−2.57	−0.86	+0.11	−0.17	−0.05

The notation of the intrinsic wave functions χ is the same as in Sec. 1. The inclusion of M_4B_4 and M_5B_5 in the $M(E1)$ matrix element takes into account that the nucleus spends a fraction of the time in the excited dipole states, which are connected to the ground state by $E1$ transitions.

3. NUMERICAL RESULTS

A. Calculation of β_L

Once a definite charge distribution is chosen, it is straight forward to evaluate β_L. We choose a uniformly deformed ellipsoid,[3] with deformation parameter β chosen to fit the experimental $B(E2)$ values. The results are listed in Table I. We note here that if the particular subset of the corrections in which the nucleus remains is the ground-state band is summed to all orders, the result should be the same as the exact numerical solution of the coupled Dirac equations in a deformed electrostatic field. The numerical solution was carried out by McKinley.[13] Therefore, the values in Table I can be checked with his results to justify our earlier argument that the third- and higher-order terms can be neglected. He used the deformed charge distribution of Sec. 1 with parameters $c=6.94$ fm, $t=2.34$ fm, $\beta=0.179$, $Q_0=11.0886$ for ^{238}U. The shift in the $1s$ level energy is then -7.46 keV, which should be compared with β_L for the $1s$ level. Although this work did not consider the element ^{238}U, the same method was used by the author previously[11] to calculate β_L for the $1s$ level of ^{238}U. With a uniformly deformed ellipsoid, $R_0=7.43$ fm, $\beta=0.25$, $Q_0=10.554$ b, the result was -6.47 keV. When the calculation is repeated, with the parameters changed so as to force $Q_0=11.088$ b, the shift is then -7.08 keV. This is in good agreement with McKinley's result of -7.46 keV.

B. Calculation of γ_{lL}

In the Paper I,[10,11] it is found that the closure approximation, with an exact handling of the muonic energy denominator, gives reliable results for the

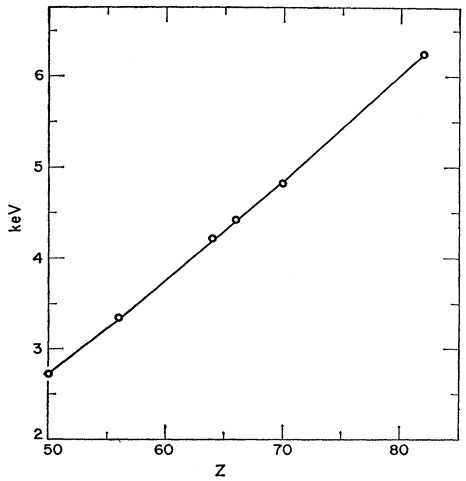

FIG. 3. The quantity $|\sum_{l=0}^{4}\gamma_{l,l}|$ in keV for $1s$ level. All values of $\gamma_{l,l}$ are negative.

[13] J. M. McKinley, Phys. Rev. (to be published).

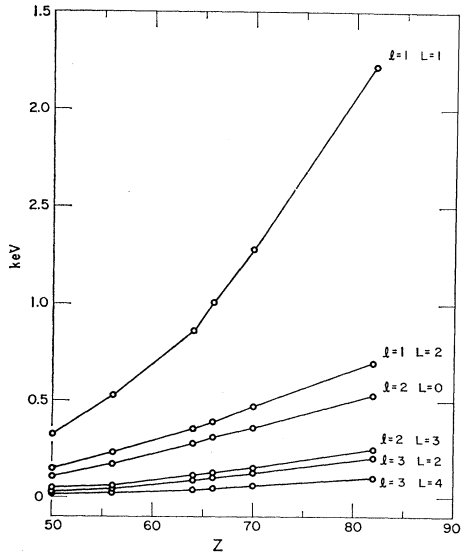

FIG. 4. The quantities $|\gamma_{lL}|$ in keV for $2p$ states. All values of γ_{lL} are negative.

TABLE II. Parameters used in calculating γ_{lL}.[a]

Z	N	Oscillator strength	Average nuclear excitation energy $l=0$	$l=1$	$l\geq 2$
50	70	7.8	30.0	16.2	15.0
56	82	7.5	30.0	15.5	15.0
64	92	7.1	30.0	14.9	15.0
66	100	6.9	30.0	14.6	15.0
70	106	6.7	30.0	14.3	15.0
82	126	6.4	30.0	14.0	15.0

[a] The average nuclear excitation energies used in the closure approximation. All the energies are in MeV. The harmonic-oscillator levels filled in succession are $0g_{9/2}$, $1d_{5/2}$, $0g_{7/2}$, $2s_{1/2}$, $1d_{3/2}$, $0h_{11/2}$.

spherical nuclei. As described in the paper, the giant dipole states are the most important intermediate states. The nuclear polarization depends only on the energy and the dipole strength of the states. So long as the energy and the dipole strength are correct, the detailed structure of those states is not very important. Since the energy of giant dipole states is known experimentally, and the summed dipole strength is determined by NZ/A (with a correction which comes from the exchange potential), the methods developed in Refs. 10 and 11 for spherical nuclei can also be applied here to calculate γ_{lL}. The results are plotted in Figs. 3 and 4. For the $1s$ level, all the matrices A_{lL} are unit matrices, and all γ_{lL} for $l=0$, 1, 2, 3, 4 are added together and plotted on Fig. 3 against Z. For the $2p$ states, the various γ_{lL} are plotted separately on Fig 4. The six points with which the curves are plotted are calculated with harmonic-oscillator states. The parameters are listed in Table II. The calculation does take into account the more important factors; however, the detailed structure of the excited states of each individual nucleus has not been taken into consideration. The uncertainty in the calculation of γ_{lL} should be $\approx 30\%$.

TABLE III. The quantities M for p-s transition.

	M_{21} (fm)	M_{32} (fm)	M_4+M_5 (fm)
^{150}Nd	−0.28	−0.26	0.33
^{152}Sm	−0.31	−0.29	0.36
^{162}Dy	−0.34	−0.34	0.47
^{164}Dy	−0.35	−0.36	0.47
^{168}Er	−0.34	−0.37	0.48
^{170}Er	−0.33	−0.35	0.48
^{182}W	−0.25	−0.29	0.60
^{184}W	−0.23	−0.28	0.60
^{186}W	−0.23	−0.28	0.60

TABLE IV. The quantities M for d-p transitions.

	M_{22} (fm)	M_{20} (fm)	M_{33} (fm)	M_{31} (fm)	M_4+M_5 (fm)
^{150}Nd	−0.25	0.23	−0.19	1.63	0.08
^{152}Sm	−0.25	0.26	−0.22	1.81	0.08
^{162}Dy	−0.32	0.32	−0.26	2.01	0.11
^{164}Dy	−0.34	0.33	−0.27	2.10	0.11
^{168}Er	−0.30	0.35	−0.28	2.10	0.12
^{170}Er	−0.33	0.33	−0.27	2.02	0.12
^{182}W	−0.24	0.29	−0.24	1.56	0.14
^{184}W	−0.23	0.27	−0.22	1.47	0.14
^{186}W	−0.25	0.27	−0.22	1.47	0.14

C. Calculation of M_{2L}, M_{3L}, M_4+M_5

The calculation of the quantities M_{2L}, M_{3L} is similar to the calculation of β_L (Sec. 3A) and the calculation of M_4+M_5 similar to that of γ_{lL} (Sec. 3B). The results are given in Tables III and IV for the K and L x-rays, respectively. Their influence on the relative intensities is appreciable. As an example, the relative intensities of K and L x-rays of ^{182}W with and without the correction are listed in Table V.

4. CONCLUSION

The method developed above is quite suitable for practical analysis of experimental spectra, as it only consumes a negligible amount of computer time. The corrections calculated in the previous section have been applied to the analysis of the muonic dynamic $E2$ hyperfine spectra measured by the Columbia group. The details of the analysis are in Paper III.[12] We note here that without the renormalization corrections, the quadrupole moments obtained are always a few percent larger than the Coulomb excitation measurements. To reduce the quadrupole moment in muonic determination, one attempt was made to introduce a fourth parameter β' in the charge distribu-

TABLE V. Relative intensities of K and L x rays of ^{182}W.

Experimental Energy (MeV)	Intensity	Calculated Energy (MeV)	Intensity With correction	Intensity Without correction
5.19613	0.169	5.19633	0.176	0.186
5.22796	0.310	5.22763	0.314	0.322
5.29586	0.076	5.29640	0.082	0.084
5.31970	0.160	5.31951	0.144	0.143
5.41934	0.285	5.41958	0.283	0.265
2.05036	0.419	2.05031	0.404	0.388
2.17357	0.240	2.17350	0.233	0.248
2.21369	0.341	2.21368	0.362	0.363

TABLE VI. Influence of nuclear polarization on the parameters of the charge distribution.

Isotope	Q_0 C.E.[a] (b)	Without nuclear polarization c (fm)	t (fm)	β	β'	Q_0 (b)	With nuclear polarization[b] c (fm)	t (fm)	β	Q_0 (b)
^{150}Nd	5.17±0.12[b]	6.08	1.68	0.274	−0.0	5.27	5.87	2.34	0.278	5.15
^{152}Sm	5.85±0.15	6.09	1.77	0.302	−0.22	5.85	5.90	2.36	0.296	5.78
^{162}Dy	7.12±0.15	6.26	1.59	0.337	−0.28	7.38	6.01	2.40	0.338	7.36
^{164}Dy	7.50±0.20	6.34	1.30	0.329	−0.12	7.53	6.11	2.19	0.334	7.42
^{168}Er	7.66±0.15	6.34	1.51	0.354	−0.92	7.77	6.17	2.18	0.333	7.77
^{170}Er	7.45±0.13	6.41	1.27	0.341	−0.87	7.80	6.27	1.94	0.326	7.75
^{182}W	6.58±0.06	6.47	1.85	0.272	−0.51	6.57	6.41	2.12	0.248	6.57
^{184}W	6.21±0.06	6.49	1.84	0.269	−0.78	6.19	6.42	2.17	0.237	6.27
^{186}W	5.93±0.05	6.55	1.75	0.243	−0.54	6.01	6.46	2.10	0.222	5.90

[a] Measured by Coulomb excitation experiments.

[b] β' set equal to 0.

tion.[6] However, with the renormalization corrections the agreement with Coulomb excitation measurements is greatly improved and no introduction of the parameter β' is needed. The corrections also have a profound influence on the parameters of the charge distribution.[6,12] The results are listed in Table VI. Without the corrections the values of the skin thickness t are unreasonably small, $\approx$1.6 fm; with the corrections the values of t increase to $\approx$2.2 fm. Finally, the relative intensities of the hyperfine multiplets are calculated from the assumption that the $4f$ levels are statistically populated. This assumption is not expected to be exactly true; therefore, it is no great surprise that the χ^2 of the relative intensities are not nearly as good as the χ^2 of the energies. However, when the intensity correction of Sec. 2B is applied, in at least half of the elements, especially the heavier ones, the intensity χ^2 does improve significantly. The element ^{182}W in Table V is a good example.

To summarize, with the ever improving experimental accuracy, the theoretical analysis of data should also improve. As shown above, reliable values of the quadrupole moment and skin thickness can be obtained only if the higher-order nuclear polarization effect has been taken into account.

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