

# Keldysh field theory of a driven dissipative Mott insulator: Nonequilibrium response and phase transitions

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Understanding strongly correlated systems driven out of equilibrium is a challenging task necessitating the simultaneous treatment of quantum mechanics, dynamical constraints, and strong interactions. A Mott insulator subjected to a uniform and static electric field is prototypical, raising key questions such as the fate of Bloch oscillations with increasing correlation strength, the approach to a steady-state DC transport regime and the role of dissipation in it, and electric-field-driven phase transitions. Despite tremendous efforts over the past decade employing various numerical and analytical approaches, the manner in which a nonequilibrium steady state gets established has remained an unresolved problem. We develop here an effective large- $\mathcal{N}$  Keldysh field theory for studying nonequilibrium transport in a regular one-dimensional dissipative Mott insulator system subjected to a uniform electric field. Upon abruptly turning on the electric field (a quench), a transient oscillatory current response reminiscent of Bloch oscillations is found. In the regime of small tunneling conductance the amplitude of these oscillations, over a large time window, decreases as an inverse square power law in time, ultimately going over to an exponential decay beyond a large characteristic time  $\tau_d$  that increases with  $\mathcal{N}$ . Such a relaxation to a steady-state DC response is absent in the dissipation-free Hubbard chain at half filling. The steady-state current at small fields is governed by large-distance cotunneling, a process absent in the equilibrium counterpart. The low-field DC current has a Landau-Zener-Schwinger form but qualitatively differs from the expression for pair-production probability for the dissipation-free counterpart. The breakdown of perturbation theory in the Mott phase possibly signals a nonequilibrium phase transition to a metallic phase. Our study sheds light on the approach of a driven, dissipative strongly correlated system to a nonequilibrium steady state and also provides a general analytic microscopic framework for understanding other nonequilibrium phenomena in these systems.

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## I. INTRODUCTION

A central challenge in the area of dissipative quantum systems driven far from equilibrium relates to understanding the relaxation of initial conditions and the approach to nonequilibrium steady states. The temporal evolution is governed by the distribution of the initial disturbance over the many-body eigenmodes of the system, the nature of the bath and its coupling to the system, and the driving protocol. Mott insulator systems driven out of equilibrium are particularly interesting as they provide a meeting ground for quantum mechanics, strong interactions, dynamical processes, and constraints. Many recent studies have attacked the problem of the nonequilibrium response of Fermionic [1–23] or Bosonic [24–28] Mott insulator systems subjected to a uniform and static electric field. One of the key questions concerns the fate of Bloch oscillations with increasing correlation strength [5,8,10,12,13,22,24,26]. Another important question is regarding the role played by dissipation in the attenuation of the Bloch oscillations and the eventual approach to a nonequilibrium steady state (DC transport in particular) [6,7,11,14,15,22]. A third crucial issue is related to the nature of nonequilibrium phase transitions in Mott insulator systems [2–4,6,7,16,19–24]. Different techniques have been employed in the literature that address some of these issues—these include numerical approaches such as solving time-dependent Schrödinger

equations [2], nonequilibrium dynamical mean-field theory (NDMFT) [5,6,8,9,11–15,22,23,29], time-dependent density matrix renormalization group (TDMRG) [3,18–20], as well as analytic ones based on the Bethe ansatz [4,21], including the phenomenological generalizations to  $\mathcal{PT}$ -symmetric models [1,7]. In this paper, we develop a new analytic field theoretical approach based on the Keldysh technique and address the above three questions. Our method also provides a general analytic framework to investigate novel and wide variety of nonequilibrium phenomena in strongly correlated systems.

It is long known that a noninteracting particle hopping on a periodic lattice subjected to a uniform electric field exhibits Bloch oscillations—the spectrum is discrete (Wannier-Stark ladder [30,31]), and the particle motion is bounded. Correlations, dissipation, and disorder can all suppress the Bloch oscillations by providing relaxation or breaking lattice translation symmetry. For field strengths such that the potential energy change between neighboring sites far exceeds correlation and other energy scales in the problem, Bloch oscillations have been found to persist [5,8,25,26]. Physically, this can be understood from the fact that the noninteracting Wannier-Stark states are highly localized at the lattice sites at strong fields, and the correlations remain local in the Wannier-Stark basis. At fields where the potential energy drop in a bond is comparable to the interaction strength, study of the Bose-Hubbard model at integer filling establishes that the motion remains finite [24]. Recent numerical studies of fermionic

Mott insulators show that at large fields, the electrons execute Bloch oscillations whose frequency approaches the noninteracting counterpart [5]. At smaller fields, the understanding for a long time was that interactions, through mixing of different momentum modes, attenuate the Bloch oscillations ultimately giving way to a steady-state DC response [5,8]. However, recent work suggests that the apparent steady-state DC behavior is only transient and ultimately gives way to finite (oscillatory) motion with a period different from that of the noninteracting Wannier-Stark states [22]. The current understanding is that dissipation is a necessary ingredient for establishing steady-state DC response.

Bloch oscillations can be suppressed by dissipation through coupling the system to a bath. Earlier literature shows that even at a single-particle level, coupling the system to a phonon bath [32] or a fermionic bath [33] results in a finite DC response at any value of the coupling strength; however, for the case of coupling to a phonon bath, signatures of the Wannier-Stark ladder are still evident in the spectral function, which are found to diminish with increasing electron-phonon coupling [34]. Recent works have also considered the effect of correlations in dissipative models. The dissipation is introduced either by coupling the system to a bath [6,11,14,15] or by phenomenological means, for example, by introducing non-Hermitian terms in Hamiltonians preserving  $\mathcal{PT}$  symmetry [1,7] or using Lindblad formulations [35]. The former (heat bath) case has been studied using a numerical Keldysh DMFT approach [6], while the Bethe ansatz method is usually employed in the latter for one-dimensional systems [7]. Both these approaches yield a steady-state nonequilibrium response and nonequilibrium transitions from the Mott insulator state to a metallic state. In addition, an important observation was made in Ref. [6] that weak dissipation does not completely suppress quantum coherent oscillations—the numerically calculated single-particle spectral function shows “Bloch islands” at beating frequencies involving the noninteracting Bloch oscillations and the Coulomb interaction strength. These features get suppressed as dissipation is increased. Despite these advances in the numerical studies of the microscopic model, many important issues have not yet been addressed; for instance, it is not known how the transient Bloch oscillations decay in time eventually establishing a DC current state and how they get suppressed in the presence of dissipation. Phenomenological models such as the  $\mathcal{PT}$  symmetric Hubbard models are analytically tractable and give valuable insights such as the critical behavior near the nonequilibrium Mott insulator to metal transition; however, relating the model parameters directly to experimentally relevant quantities has proved to be a challenge. Moreover, these models are designed to study the nonequilibrium steady state but not the transient response.

In band insulators, the linear response conductivity vanishes at zero temperature but electronic transport at finite electric fields is possible through the generation of low-energy particle-hole pairs by the Landau-Zener-Schwinger (LZS) mechanism [36–38], with the probability  $P$  of this process related to the electric field measured in terms of the potential energy drop,  $D$ , across a link, and the band-gap  $\Delta$  as  $P \sim \exp[-\Delta^2/cD]$ , where  $c$  is a constant with the dimension of energy. For the fermionic Hubbard chain subjected to

an electric field, a similar expression has been proposed in Ref. [3], with band-gap  $\Delta$  being replaced by the Mott gap. Turning on a finite dissipation (coupling to a fermionic bath) under such nonequilibrium conditions, DMFT calculations of Ref. [6] show that the Hubbard bands leak into the Mott gap, and beyond some value of the dissipation strength, a quasiparticle feature, signaling a bad metallic phase appears, in the spectral function. The crucial question here is whether and under what circumstances this dielectric breakdown becomes a true nonequilibrium phase transition. Analysis of the phenomenological  $\mathcal{PT}$  symmetric fermionic Hubbard chain [7] suggests that this is a true nonequilibrium quantum phase transition and is associated with breaking of  $\mathcal{PT}$  symmetry in the metallic phase.

In this paper we develop an effective Keldysh field theory of a dissipative one-dimensional Mott insulator subjected to a uniform electric field and study it analytically to address the broad questions outlined above. Our microscopic model consists of a one-dimensional array of mesoscopic metallic quantum dots - each of these quantum dots contains a large number of electrons occupying the dot energy levels. The large number of degrees of freedom (DoF) in each mesoscopic dot effectively constitute a fermionic bath and provide a source of dissipation through the Landau damping mechanism. In addition, as we discuss below, the large DoF acts as a large- $\mathcal{N}$  parameter (see also Ref. [39]) and facilitates a tractable analytic treatment of our model. The analytic tractability that our large- $\mathcal{N}$  formulation provides is analogous to that of large dimensionality in the DMFT approach to the Hubbard model. Under equilibrium conditions, the model is described by the following Hubbard-like Hamiltonian with multiple flavors (representing dot energy levels) of electrons at each site (we set electron charge  $e = 1$ , lattice spacing  $a = 1$ ,  $\hbar = 1$ ,  $k_B = 1$ ):

$$\hat{H} = \hat{H}^0 + \hat{H}^C + \hat{H}^T, \text{ where} \quad (1)$$

$$\hat{H}^0 = \sum_{k,\alpha} \xi_\alpha c_{j,\alpha}^\dagger c_{j,\alpha}, \quad (2)$$

$$\hat{H}^C = \sum_k E_C \left[ \left( \sum_\alpha c_{k,\alpha}^\dagger c_{k,\alpha} \right) - N_0 \right]^2, \quad (3)$$

$$\hat{H}^T = \sum_k \sum_{\alpha,\beta} (\tilde{t}_{\alpha\beta}^{k,k+1} c_{k,\alpha}^\dagger c_{k+1,\beta} + \text{H.c.}). \quad (4)$$

Here  $k$  labels the site index,  $\alpha$  represents the different energy levels ( $E_\alpha$ ) within a dot,  $\xi_\alpha = E_\alpha - \mu$  ( $\mu$  being the Fermi level in the dot),  $\tilde{t}_{\alpha\beta}^{k,k+1}$  is the interdot tunneling matrix element connecting levels  $\alpha$  and  $\beta$  on dots labeled  $k$  and  $k+1$ , respectively,  $E_C$  is the Coulomb energy of single-electron charging, and  $N_0$  is the equilibrium charge on a dot. The tunneling between the dots could be through an insulating barrier (as is the case in granular metals) or through ballistic point contacts (as may be the case in artificial quantum dot arrays). The Fermi energy in each dot is assumed to be the largest energy scale. In addition, we also have a small energy scale,  $\delta$ , which is the mean level spacing in the dot and is approximately related to the volume of the dot,  $V$ , and the density of states at the Fermi level,  $\nu(\mu)$  through  $\delta \approx 1/(\nu(\mu)V)$ . Elementary excitations in each isolated dot

are of the low-energy particle-hole kind, which in the limit of large dot size, tend to become gapless. Interestingly, other models such as the Sachdev-Ye-Kitaev (SYK) [40,41] model on a one-dimensional lattice [42,43] interaction share a similar structure, and are also characterized by gapless excitations locally.

We study the model in Eq. (1) in the Mott insulator regime where  $E_C \gg \delta$ ,  $T$  and  $g \lesssim 1$ , where  $T$  is the temperature, and  $g$  is the dimensionless interdot tunneling conductance. For granular metals, the intergrain tunneling conductance is of the form  $g \approx \pi^2 |\tilde{t}_{\alpha,\beta}|^2 [V\nu(\mu)]^2 = \pi^2 |\tilde{t}_{\alpha,\beta}|^2 / \delta^2$ . For ballistic point contacts separating the quantum dots, the transverse (waveguide) momentum  $\mathbf{k}_\perp$  is conserved during tunneling (i.e.,  $\tilde{t}_{\alpha,\beta} \equiv \tilde{t}_{\mathbf{k}_\perp}$ ), but the longitudinal momentum  $k_\parallel$  is not, and  $g$  has the form,  $g \approx \pi^2 \sum_{\mathbf{k}_\perp} |\tilde{t}_{\mathbf{k}_\perp}|^2 (v^{1D}L)^2$ , where  $v^{1D}$  is the one-dimensional density of states associated with the different subbands labeled by  $\mathbf{k}_\perp$  and  $L$  is the dot size. In this Mott insulator regime, a conventional perturbation expansion in the interaction is not possible. We therefore adopt a bosonization scheme well known in the literature as the Ambegaokar-Eckern-Schön (AES) [44,45] model of granular metals—a class of Mott insulators. The AES model is, in effect, a rotor model with the difference that now the phases at each site in the AES model are dual to the total charge in the dot at that site. The AES model consists of a charging part that represents Coulomb blockade effects, and a dissipative tunneling part that describes interdot hopping of electrons. Unlike other dissipative models such as Caldeira-Leggett [46], the tunneling part of the AES model is periodic in the phase fields reflecting charge quantization. The large number of degrees of freedom on each dot makes the model analytically tractable, allowing one to discard terms in the effective action that are higher order than two in the interdot tunneling conductance. The model is tailor-made for studying transport, and consequently, information about the internal low-energy excitations at a site appears only at the level of the tunneling term.

In equilibrium or linear response situations, the AES model appears in diverse contexts including unusual transport phenomena in granular Mott insulators such as cotunneling dominated variable-range hopping [45,47] and breakdown of the Wiedemann-Franz law by emergent Bosonic modes [48] and the Kondo effect in quantum critical metals [49,50]. A Bosonic channel for thermal transport analogous to that in the AES model [48] has recently been reported for the SYK model [43]. It is also well-known that even in the regime of metal-like conduction ( $g \gg 1$ ,  $T \gg g\delta$ ), the low-energy excitations of the AES model are not quasiparticle-like, i.e., are not characterized by their momenta and spin, a property shared with the SYK model [43].

We generalize the AES model to the nonequilibrium case using the Keldysh formalism. For the case of a single mesoscopic quantum dot connected to noninteracting leads, a similar Keldysh generalization has been studied in the literature (see, e.g., Ref. [51]). The granular chain, as we shall see, has significantly different physics from the single dot problem arising from the periodicity of the lattice and also the relevance of long-range tunneling processes since potential energy gain from cotunneling over multiple dots can offset the Coulomb blockade effects. In the equilibrium (Matsubara) treatment of the AES model, to properly treat

charge quantization effects, essential in Coulomb blockade, finite winding numbers of the phase fields must be taken into account. In the real time Keldysh case, this is achieved by going to a mixed phase-charge representation (instead of a pure phase-only representation) and restricting the path integral over the classical component of the charge field to integer values.

We calculate the current response of our Keldysh AES model for the granular Mott insulator subjected to a uniform electric field at temperatures much smaller than  $D$  and  $E_C$ , and we further assume the mesoscopic dots are sufficiently large so that the temperature greatly exceeds the mean level spacing  $\delta$ . After the electric field is switched on, the leading-order (in  $g$ ) current response shows an oscillatory transient response whose primary components are the two beat frequencies,  $\omega_\pm = |D \pm 2E_C|$ ,

$$J_{\text{tr}} \approx -\frac{4g\Theta(\tau)}{(2\pi)^2 E_C} \frac{1}{\tau^2} \left\{ \left( \frac{D-2E_C}{D+2E_C} \right) \sin[(D+2E_C)\tau] + \left( \frac{D+2E_C}{D-2E_C} \right) \sin[(D-2E_C)\tau] \right\}. \quad (5)$$

These oscillations arise, as we shall show in the paper, from a combination of the periodicity of the lattice, Coulomb correlations, and charge quantization. These beat frequencies have also been observed [6] in DMFT calculations of the dissipative Hubbard model in the form of “island” features in the spectral function, and in the dissipationless Bose-Hubbard model [25]. In the absence of correlations ( $E_C = 0$ ), these oscillations would correspond to the Bloch oscillation frequency  $\omega_B = |D|$ . However, the  $1/\tau^2$  decay of the amplitude of the current oscillations does not persist indefinitely, and we show that it crosses over to an exponential decay to the steady state beyond a characteristic time  $\tau_d \sim 1/T_D$ , where  $T_D$  is the effective electron temperature in the dots in the nonequilibrium steady state. We find that the temperature  $T_D$  decreases with  $\mathcal{N}$ , and vanishes as  $\mathcal{N} \rightarrow \infty$ .

Apart from these oscillations, the current also has a finite DC component for  $|D| > 2E_C$ ,

$$J_{\text{dc}} = \frac{g\Theta(\tau)}{\pi} [(D-2E_C)\Theta(D-2E_C) + (D+2E_C) \times \Theta(-2E_C-D)], \quad (6)$$

and is a direct consequence of the presence of dissipation.

Next, to understand the nature of the DC response at small fields,  $|D| < 2E_C$ , we consider the long time limit of the current response. For this purpose, we take into account higher-order cotunneling processes over multiple dots such that the Coulomb blockade is offset by the extra potential energy gain. We provide analytic expressions for the field dependence of current up to  $O(g^2)$ . The analysis of higher-order terms at arbitrary field strengths rapidly becomes very complicated; however we infer some general features. In the zero temperature limit, there is a hierarchy of thresholds,  $D_{\text{th}}^{(n)} = 2E_C/n$ , with the  $n$ th-order current corresponding to the matching of the Coulomb scale with the electrostatic potential energy gain from cotunneling over  $n$  successive dots. The leading order in  $g$  contributions to the current near these

thresholds has the form

$$j^{(n)}(D) \sim nDg^n(1 - 2E_C/nD)^{2n-1}\Theta(nD - 2E_C), \quad (7)$$

where  $\Theta$  is the Heaviside step function. Based on this expression, we show that at low fields and small  $g$ , the field dependence of the current has the LZS form,  $j(D) \sim D[g/\ln^2(1/g)]^{2E_C/D}$ , but with qualitative differences from the LZS particle-hole pair production probability  $P \sim e^{-E_C^2/cD}$  for the nondissipative Hubbard chain at half filling [4] deep in the Mott insulator phases.

An important question relates to the nature of the transition from the Mott insulating state to a conducting state as a function of the field. In the dissipation-free case, it is evident from the expression for the LZS pair production probability that it is a crossover, howsoever sharp, and not a true phase transition. A true phase transition to a metallic state is indicated if the perturbation expansion for the current made from within the Mott insulator phase diverges as a function of  $g(\lesssim 1)$  or  $D(< 2E_C)$ . If the form of the current is assumed to have the form shown in Eq. (7) for a finite but small field strength away from the thresholds, then the criterion for divergence of the perturbation expansion for the current is

$$g \exp[D/E_C] \sim 1. \quad (8)$$

However, as we have already mentioned above, the field dependence of high- $n$ th-order terms is complicated for fields away from the respective thresholds  $D^{(n)} = 2E_C/n$ , and it is not currently clear to us how the above criterion would change.

The rest of the paper is organized as follows. In Sec. II, beginning with the microscopic model of Eq. (1), we outline the derivation of our effective Keldysh-AES action. The electric field is introduced through a time-dependent vector potential. We also present the functional representation of the charge current in terms of the correlation functions of the phase fields. In Sec. III, we analyze the leading-order contribution to the current from the time the electric field is turned on. We show that there are Bloch-like oscillations whose amplitudes decay as a power-law in time upto a large time  $\tau_d$ . Further, the existence of a finite DC response at long times is also established. Section IV is devoted to the analysis of the long-time DC behavior for small field strengths. For this purpose, the higher-order cotunneling processes over multiple dots are considered in a perturbative expansion in small  $g$ , around the ‘‘atomic limit’’ of isolated dots. We discuss the LZS form of the current response at small fields, and the possible nonequilibrium phase transition to a metallic state. Finally, in Sec. V we conclude with a discussion of our results and open questions.

## II. KELDYSH-AES ACTION

In this section, we obtain the effective Keldysh-AES action from the microscopic Hamiltonian introduced in Eq. (1) and also provide functional representation of the charge current that will be used throughout. Our derivation of the effective Keldysh-AES action parallels the one in Ref. [51] for the case of a single quantum dot connected to noninteracting leads.

The first step consists of Hubbard-Stratonovich decoupling of the part of the action corresponding to Eq. (1) that contains the Coulomb interaction term:

$$\begin{aligned} e^{-i \int_t H_C} &= \exp \left[ -i \sum_k \int_t E_C \left( \sum_\alpha \bar{\psi}_{k,\alpha} \psi_{k,\alpha} - N_0 \right) \left( \sum_\alpha \bar{\psi}_{k,\alpha} \psi_{k,\alpha} - N_0 \right) \right] \\ &\propto \int DV \exp \left\{ i \sum_k \int_t \frac{1}{4E_C} \left[ V - 2E_C \left( \sum_\alpha \bar{\psi}_{k,\alpha} \psi_{k,\alpha} - N_0 \right) \right]^2 \right\} e^{-i \int_t H_C}. \end{aligned} \quad (9)$$

To study nonequilibrium transport, we put our action on the Keldysh contour and we label the fields with superscripts  $+$  and  $-$  corresponding, respectively, to the forward and backward time parts of the Keldysh contour. For incorporating the initial condition information (i.e., the initial density matrix) it is customary to work with a rotated classical-quantum basis in the Keldysh space:

$$V^c = \frac{1}{2}(V^+ + V^-), \quad V_q = V^+ - V^-, \quad (10)$$

$$\psi^c = \frac{1}{\sqrt{2}}(\psi^+ + \psi^-), \quad \psi^q = \frac{1}{\sqrt{2}}(\psi^+ - \psi^-), \quad (11)$$

$$\bar{\psi}^c = \frac{1}{\sqrt{2}}(\bar{\psi}^+ - \bar{\psi}^-), \quad \bar{\psi}^q = \frac{1}{\sqrt{2}}(\bar{\psi}^+ + \bar{\psi}^-), \quad (12)$$

$$\Psi = \begin{pmatrix} \psi^c \\ \psi^q \end{pmatrix}, \quad \bar{\Psi} = (\bar{\psi}^c \ \bar{\psi}^q). \quad (13)$$

We call the superscripts  $c$  and  $q$  the ‘‘classical’’ and ‘‘quantum’’ components, respectively. The action  $S$  now assumes the form

$$S = S^0 + S^C + S^T, \text{ where}$$

$$S^0 = \sum_{k,\alpha} \int_t \bar{\Psi}_{k,\alpha} \begin{bmatrix} i\partial_t + i\eta + \mu - E_\alpha - V_k^c & -\frac{V_k^q}{2} + 2i\eta F_k \\ -\frac{V_k^q}{2} & i\partial_t - i\eta + \mu - E_\alpha - V_k^c \end{bmatrix} \Psi_{k,\alpha},$$

$$\begin{aligned}
S^C &= \sum_k \int_t \left( \frac{1}{2E_C} V_k^c V_k^q + N_0 V_k^q \right), \\
S^T &= \sum_{k,\alpha,\beta} \int_t \bar{\Psi}_{k\alpha} \begin{bmatrix} \tilde{\tau}_{\alpha,\beta}^{k,k+1} & 0 \\ 0 & \tilde{\tau}_{\alpha,\beta}^{k,k+1} \end{bmatrix} \Psi_{k+1,\beta} + \text{c.c.}
\end{aligned} \tag{14}$$

Here  $F_k$  is related to the distribution function for *noninteracting* electrons in the  $k$ th dot and is, in general, a function of two time arguments, i.e.,  $F_k(t, t')$ . For the case of thermal equilibrium,  $F_k$  depends only on the difference  $t - t'$ , and in frequency space, it has the form  $F(\omega) \equiv 1 - 2f(\omega) = \tanh(\omega/2T)$ , where  $f(\omega)$  is the Fermi-Dirac distribution function and  $T$  is the temperature. The infinitesimally small positive constant,  $\eta$ , ensures the theory has the proper causal structure. At this stage, it would seem natural to integrate out the noninteracting fermions, and expand the resulting determinant to obtain an effective field theory for the Hubbard-Stratonovich fields. However, the Hubbard-Stratonovich fields effectively shift the entire band of electrons and, in fact, the shifts are large ( $\sim E_C$ ) whenever tunneling events occur. We therefore perform a gauge transformation to eliminate the fluctuating Hubbard-Stratonovich fields that appear in  $S^0$ ,

$$\Psi_{k,\alpha} \rightarrow e^{-i\hat{\phi}_k} \Psi_{k,\alpha}, \bar{\Psi}_{k,\alpha} \rightarrow \bar{\Psi}_{k,\alpha} e^{i\hat{\phi}_k}, \tag{15}$$

where

$$\hat{\phi}_k = \phi_k^c + \phi_k^q \frac{\sigma_1}{2}, \tag{16}$$

and the phase fields  $\hat{\phi}_k$  are chosen such that their classical and quantum components obey

$$\partial_t \phi_k^{c,q} = V_k^{c,q}. \tag{17}$$

After the above gauge transformation, we have

$$\begin{aligned}
S^0 &= \sum_{k,\alpha} \int_t \bar{\Psi}_{k,\alpha} \begin{bmatrix} i\partial_t + i\eta + \mu - E_\alpha & 2i\eta F_k \\ 0 & i\partial_t - i\eta + \mu - E_\alpha \end{bmatrix} \\
&\quad \times \Psi_{k,\alpha},
\end{aligned} \tag{18}$$

$$S^C = \sum_k \int_t \left( \frac{1}{2E_C} \partial_t \phi_k^c \partial_t \phi_k^q + N_0 \partial_t \phi_k^q \right), \tag{19}$$

$$S^T = \sum_{k,\alpha,\beta} \int_t \left( \tilde{\tau}_{\alpha,\beta}^{k,k+1} \bar{\Psi}_{k\alpha} \exp(-i\hat{\phi}_{k,1}) \Psi_{k+1,\beta} + \text{c.c.} \right),$$

$$\hat{\phi}_{k,1} = \hat{\phi}_{k+1} - \hat{\phi}_k. \tag{20}$$

The term in Eq. (19) proportional to  $N_0$  is a Berry phase term. Our next step is to integrate out the fermions to obtain an effective action in terms of the phase fields. We denote the fermion-bilinear part of the action as  $S_F = S^0 + S^T = \hat{\Psi} \hat{G}^{-1} \hat{\Psi}$ , with

$$\hat{G}^{-1} = \hat{G}_0^{-1} + \hat{T}, \tag{21}$$

where

$$(\hat{G}_0)_{k,\alpha;k,\alpha}^{-1} = \begin{bmatrix} (g_{k,\alpha}^R)^{-1} & 2i\eta F_k \\ 0 & (g_{k,\alpha}^A)^{-1} \end{bmatrix}, \tag{22}$$

$$\hat{T}_{k,\alpha;k+1,\beta} = \tilde{\tau}_{\alpha,\beta}^{k,k+1} \exp(-i\hat{\phi}_{k,1}). \tag{23}$$

In Eq. (22), the diagonal elements are the usual inverse retarded and advanced Green functions,

$$(g_{k,\alpha}^{R,A})^{-1} = i\partial_t \pm i\delta + \epsilon_F - E_\alpha. \tag{24}$$

The interdot hopping matrix  $\hat{T}$  is diagonal in Keldysh space as well as in the time indices. Integrating out the fermions gives us  $Z = \int D\phi \exp[\iota S^C[\phi] + \text{tr} \ln(\iota \hat{G}^{-1})]$ , and we use Eq. (21) to re-express the fermionic determinant as

$$\ln(\hat{G}^{-1}) = \ln(1 + \hat{G}_0 \hat{T}) + \ln(\hat{G}_0^{-1}). \tag{25}$$

To obtain the effective action in terms of the phase fields, we discard the  $\phi$ -independent  $\ln(\hat{G}_0^{-1})$  make a Taylor expansion of  $\ln(1 + \hat{G}_0 \hat{T})$ . The first-order term vanishes since  $\text{tr}(\hat{G}_0 \hat{T}) = 0$  as  $\hat{G}_0$  is diagonal in  $k$  and  $T_{k,k} = 0$ . Then, up to second order in  $\hat{T}$  we have

$$Z = \int D\phi \exp[\iota S^C[\phi] + \iota S^{\text{tun}}[\phi]],$$

$$S^{\text{tun}}[\phi] = \frac{\iota}{2} \text{tr}(\hat{G}_0 \hat{T} \hat{G}_0 \hat{T}). \tag{26}$$

Here  $\hat{G}_0$  has the following structure in Keldysh space:

$$(\hat{G}_0)_{k,\alpha;k,\alpha}(t, t') = \begin{bmatrix} g_{k,\alpha}^R & F_k(g_{k,\alpha}^R - g_{k,\alpha}^A) \\ 0 & g_{k,\alpha}^A \end{bmatrix}(t, t'), \tag{27}$$

where

$$\begin{aligned}
g_{k,\alpha}^{R,A}(t, t') &= \frac{1}{2\pi} \int_\omega g_{k,\alpha}^{R,A}(\omega) \exp[-i\omega(t - t')] \\
&= \int_\omega \frac{\exp[-i\omega(t - t')]}{\omega \pm i\delta + \mu - E_\alpha}.
\end{aligned} \tag{28}$$

We assume that the matrix elements of  $\hat{T}$  are independent of the energy indices and also replace summation over the discrete states by corresponding integrals,  $\sum_\alpha \leftrightarrow V \int_\epsilon d\epsilon v(\epsilon)$ , with  $v(\epsilon) = \frac{1}{V} \sum_\alpha \delta(\epsilon - E_\alpha)$  the density of states in a dot. The summations over the energy indices gives quantities of the form  $\sum_\alpha g_{k,\alpha}^{R,A}(\omega) = V \int_\epsilon v(\epsilon) g_{k,\alpha}^{R,A}(\omega) \approx \mp(\pi\iota)V v(\omega + \mu) \approx \mp(\pi\iota)V v(\mu)$ . With these approximations, we arrive at

$$\text{tr}(\hat{G}_0 \hat{T} \hat{G}_0 \hat{T}) \approx -2\pi^2 |\tilde{\tau}|^2 (V v(\mu))^2 \int_{t,t'} \sum_k \text{tr}\{\Lambda_k(t - t') \exp[-i\hat{\phi}_{k,1}(t')] \Lambda_{k+1}(t' - t) \exp[i\hat{\phi}_{k,1}(t)]\}, \tag{29}$$

where

$$\Lambda_k(\omega) = (2t) \begin{bmatrix} G^R(\omega) & F_k(\omega)[G^R - G^A] \\ 0 & G^A(\omega) \end{bmatrix}, \quad G^{R,A}(\omega) = \frac{1}{2\pi} \int_{\epsilon} g_{k,\epsilon}^{R,A}(\omega). \quad (30)$$

Thus,

$$S_{\text{tun}} \approx -t g \int_{t,t'} \sum_k \text{tr} \{ \Lambda_k(t-t') \exp[-t \hat{\phi}_{k,1}(t')] \Lambda_{k+1}(t'-t) \exp[t \hat{\phi}_{k,1}(t)] \}. \quad (31)$$

For a granular metal, we assume that the tunneling matrix connects any pair of levels in the neighboring grains with characteristic magnitude  $|\tilde{t}|$ , in which case,  $g = \pi^2 [V v(\mu)]^2 |\tilde{t}|^2 \sim |\tilde{t}|^2 (\mathcal{N}/\mu)^2$ . Here  $g$  is the dimensionless interdot tunneling conductance and  $\mathcal{N}$  the total number of electrons in a dot. To give an estimate of the largeness of  $\mathcal{N}$ , for a 10-nm metallic dot with conduction electron density of  $\sim 10^{28} \text{ m}^{-3}$ , we have  $\mathcal{N} \sim 10^4$ . Our regime of interest is  $g \lesssim 1$ , independent of the number of electrons in the dot. Thus, for the granular metal we require the tunneling amplitudes to scale as  $|\tilde{t}| \sim 1/\mathcal{N}$ . Physically, this means that as the number of transmission channels increases, the individual tunneling amplitudes should scale inversely so as to keep  $g$  unchanged.

For the case of ballistic point contacts, we label the energy levels by transverse and longitudinal momenta,  $\mathbf{k}_{\perp}$  and  $k_{\parallel}$ , respectively. The transverse momentum is conserved during tunneling but the longitudinal momentum is not. The tunneling matrix element thus connects any pair of longitudinal momenta, and we assume they all have a characteristic magnitude  $|\tilde{t}|$ . In this case, the dimensionless conductance  $g = \pi^2 \sum_{\mathbf{k}_{\perp}} |\tilde{t}|^2 (v^{1D} L)^2 \sim |\tilde{t}|^2 N_{\text{ch}} (\mathcal{N}_{1D}/\mu)^2$ , where  $N_{\text{ch}}$  is the total number of transverse channels and  $\mathcal{N}_{1D}$  is the typical number of electrons having the same transverse momentum. To keep  $g \lesssim 1$ , we require the tunneling amplitude to scale as  $|\tilde{t}| \sim 1/(\sqrt{N_{\text{ch}}} \mathcal{N}_{1D})$ , and we show below that the large- $\mathcal{N}$  parameter in this case is  $\mathcal{N} = N_{\text{ch}}$ .

We will present below a large- $\mathcal{N}$  justification for dropping higher-order terms in the tunneling action.

### A. Consequences of large- $\mathcal{N}$

Let us now discuss a couple of crucial consequences of having a large number of electrons in each dot. Consider first the  $O(\tilde{t}^4)$  term in the tunneling action for the granular metal. The basic argument for disregarding such contributions has been presented in Ref. [45]. Here we show that this is essentially a large- $\mathcal{N}$  argument. The fourth-order tunneling terms are of the form  $\text{tr}(\hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T})$ . These processes involves two or three dots. Consider for example the three dot term (with consecutive dots labeled  $i, j, k$ ),

$$\begin{aligned} & \text{tr}(\hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T}) \\ &= \sum_{\substack{ijk \\ \alpha_1, \dots, \alpha_4}} (\hat{G}_0)_{i,\alpha_1} \hat{T}_{\alpha_1\alpha_2}^{ij} (\hat{G}_0)_{j,\alpha_2} \hat{T}_{\alpha_2\alpha_3}^{jk} (\hat{G}_0)_{k,\alpha_3} \hat{T}_{\alpha_3\alpha_4}^{kj} (\hat{G}_0)_{j,\alpha_4} \hat{T}_{\alpha_4\alpha_1}^{ji}. \end{aligned}$$

Now the tunneling amplitudes  $\tilde{t}$  are of the form  $\tilde{t}_{\alpha\beta}^{ij} = |\tilde{t}| e^{i\chi_{\alpha\beta}^{ij}}$ , where  $\chi_{\alpha\beta}^{ij}$  is a phase associated with the link  $ij$  and energy levels  $\alpha, \beta$ . The key point is that for irregular dots, the phases  $\chi_{\alpha\beta}^{ij}$  are random. For the case of a large number of levels, the

random phases cause the vanishing of all terms except for the case  $\alpha_4 = \alpha_2$ , where the random phases cancel exactly. Thus, there are only three independent energy indices to be summed over resulting in a factor of  $\mathcal{N}^3$ . However, since the  $\tilde{t}$  scale as  $1/\mathcal{N}$ , it is evident that the overall scaling of this term is  $1/\mathcal{N}$ . In general, the number of independent energy indices in the perturbative expansion of the tunneling action equals the number of dots involved in that term.

We now discuss the case of ballistic point contacts. The fourth-order three-dot term can be written as

$$\begin{aligned} & \text{tr}(\hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \hat{G}_0 \hat{T}) \\ &= \sum_{\substack{ijl, \mathbf{k}_{\perp} \\ k_1, \dots, k_4}} (\hat{G}_0)_{i,k_1} \hat{T}_{k_1 k_2}^{ij} (\hat{G}_0)_{j,k_2} \hat{T}_{k_2 k_3}^{jl} (\hat{G}_0)_{l,k_3} \hat{T}_{k_3 k_4}^{lj} (\hat{G}_0)_{j,k_4} \hat{T}_{k_4 k_1}^{ji}, \end{aligned}$$

where  $k_1, \dots, k_4$  are longitudinal momenta and we have suppressed the transverse momentum label  $\mathbf{k}_{\perp}$  for brevity. Since the tunneling elements scale as  $|\tilde{t}| \sim 1/(\sqrt{N_{\text{ch}}} \mathcal{N}_{1D})$ , each term in the above sum scales as  $1/(N_{\text{ch}}^2 \mathcal{N}_{1D}^4)$ . Now the sum over the four longitudinal momenta brings a factor of  $\mathcal{N}_{1D}^4$ , and the sum over the transverse momentum gives a factor  $N_{\text{ch}}$ . Thus, we find that the above fourth-order contribution scales as  $1/N_{\text{ch}}$ . To be able to neglect this fourth-order term, we require  $N_{\text{ch}} \gg 1$ ; i.e., the width of the point contact should be much larger than the Fermi wavelength.

There is a second very important consequence of large- $\mathcal{N}$  that provides a crucial simplification in nonequilibrium situations and which has not been appreciated in the literature. This relates to the temporal variation of the the  $F_k$  under general nonequilibrium conditions. It is convenient to work with the Wigner representation,  $F_k(t, t') \equiv \int (d\epsilon) F_k(\epsilon, \tau) e^{-ie\tau(t-t')}$ , where  $\tau = (t + t')/2$ , and the relation with the time-dependent distribution function is  $F_k(\epsilon, t) = 1 - 2f_k(\epsilon, t)$ . The total number of electrons in the  $k$ th dot is  $N_0 + n_k^c(t) = \int d\epsilon v(\epsilon) f(\epsilon, t)$ , where  $n_k^c(t)$  is the classical component of the number field conjugate to the quantum component of the phase,  $\phi_k^q$ . In the rest of the paper, we will be specifically interested in the case of constant  $N_0$ . More general, time-dependent  $N_0$  can if a time-dependent gate voltage is applied to the quantum dots. Thus, in our case we have

$$\frac{dn_k^c}{dt} = V \int d\epsilon v(\epsilon) \frac{df_k(\epsilon, t)}{dt}. \quad (32)$$

The right-hand side of Eq. (32) is, by using the continuity equation, simply the net current into the dot, and is given by the functional derivative  $\langle \delta S / \delta \phi_k^q(t) \rangle_{\phi}$ , which has the form  $g \int d\epsilon h(\epsilon, t) \equiv j_{k-1,k}(t) - j_{k,k+1}(t)$ . Consequently, the continuity equation leads us to a kinetic equation for the distribution  $f_k(\epsilon, t)$  of the form  $V v(\mu) df_k/dt + gh(\epsilon, t) = 0$ .

The quantity  $h$  is a functional of the distributions  $\{f_k\}$  and also depends on the tunneling conductance and electric field. Recognizing  $V\nu(\mu) = 1/\delta$ , we find that the distribution function evolves with a large characteristic timescale that is proportional to  $1/g\delta$  and increases linearly with the total number of electrons in the grain ( $\delta \sim 1/\mathcal{N}$ ). We now assume that the grains are coupled to an external thermal bath, whose effect we model by an additional relaxation term in the kinetic equation, i.e.,

$$\frac{df_k}{dt} = -g\delta h[f] + \frac{f_k - f_k^{\text{eq}}}{\tau_{eb}}, \quad (33)$$

where  $f_k^{\text{eq}}$  is the equilibrium Fermi-Dirac distribution function and  $\tau_{eb}$  is the electron-bath relaxation time. If  $1/\tau_{eb} \ll g\delta$ , then the distribution functions  $f_k$  may be approximated by their equilibrium values. We will now proceed with this, and hence  $F_k(\epsilon) = \tanh(\epsilon/2T)$ . In contrast, in the usual Hubbard models, the electron distribution function at every site is a time-dependent quantity under general nonequilibrium conditions since in that case there is no large- $\mathcal{N}$  mitigating factor.

### B. Keldysh-AES action

We resume our derivation of the effective Keldysh AES action. Henceforth, we will describe tunneling in both the granular metal as well as the point contact cases by the action in Eq. (31) and note that  $g$  can have different forms for the two cases. Now let us manipulate  $S_{\text{tun}}$  to a more dealable form. We introduce new fields  $C$  and  $S$  defined as

$$C = \exp(i\phi^c) \cos\left(\frac{\phi_q}{2}\right), \quad S = \exp(i\phi^c) \sin\left(\frac{\phi_q}{2}\right). \quad (34)$$

These are related to the  $\hat{\phi}$  fields in Eq. (16) through

$$\exp(i\hat{\phi}) = C + iS\sigma_1, \quad \exp(-i\hat{\phi}) = \bar{C} - i\bar{S}\sigma_1. \quad (35)$$

The tunneling action under equilibrium conditions then takes the form

$$S_{\text{tun}} = 4g \sum_k \int_{t,t'} [\bar{C}_{k,1} - i\bar{S}_{k,1}]_t \begin{bmatrix} 0 & \Sigma_{k,1}^A \\ \Sigma_{k,1}^R & \Sigma_{k,1}^K \end{bmatrix}_{t-t'} \times \begin{bmatrix} C_{k,1} \\ iS_{k,1} \end{bmatrix}_{t'}, \quad (36)$$

where

$$\Sigma_{k,1}^{R(A)}(t) = i[G_k^{R(A)}(t)G_k^K(-t) + G_{k+1}^K(t)G_k^{A(R)}(-t)], \quad (37)$$

$$\Sigma_{k,1}^K(t) = i[G_k^K(-t)G_{k+1}^K(t) - (G^R - G^A)_t(G^R - G^A)_{-t}], \quad (38)$$

with  $G_k^K = F_k(G^R - G^A)$ . It is evident from Eq. (37) that  $\Sigma^{R(A)}$  also have a causal structure, i.e.,  $\Sigma^R(t) \propto \Theta(t)$ , etc. Under general nonequilibrium conditions, the quantities  $\Sigma^{R,A,K}(t, t')$  describing particle-hole excitations in the dots depend on both the time arguments and not just their difference.

Let  $F_b(\epsilon) = \coth(\epsilon/2T) = 1 + 2f_b$ , where  $f_b$  is the equilibrium Bose distribution function. We make use of the

following identities:

$$(G^R - G^A)_\epsilon = -i, \quad (39)$$

$$\int_\epsilon \frac{1}{2\pi} [F(\epsilon + \omega) - F(\epsilon)] = \frac{\omega}{\pi}, \quad (40)$$

$$\int_\epsilon \frac{1}{2\pi} [1 - F(\epsilon - \omega)F(\epsilon)] = \frac{\omega}{\pi} F_b(\omega), \quad (41)$$

to obtain

$$(\Sigma_{k,1}^R - \Sigma_{k,1}^A)_\omega = i \int_\epsilon \frac{1}{2\pi} [F_{k+1}(\epsilon) - F_k(\epsilon - \omega)] = \frac{i}{\pi} \omega, \quad (42)$$

$$(\Sigma_{k,1}^K)_\omega = i \int_\epsilon \frac{1}{2\pi} [1 - F_{k+1}(\epsilon)F_k(\epsilon - \omega)] = \frac{i}{\pi} \omega F_b(\omega). \quad (43)$$

We will later find it convenient to work in the  $\pm$  Keldysh contour. Hence, we re-express our phase action in this contour. We ignore  $N_0$  by assuming that it can be set to zero by some gate voltage. We have

$$S_C[n, \phi] = \frac{1}{E_C} \sum_k \int_t [(\partial_t \phi_k^+)^2 - (\partial_t \phi_k^-)^2], \quad (44)$$

$$S_{\text{tun}}[\phi] = g \sum_k \int_{t,t'} (\exp(-i\phi_{k,1}^+) \exp(-i\phi_{k,1}^-))_t L_{k,1}(t - t') \times \begin{pmatrix} \exp(i\phi_{k,1}^+) \\ \exp(i\phi_{k,1}^-) \end{pmatrix}_{t'}, \quad (45)$$

$$L = \frac{1}{4} \begin{pmatrix} \Sigma^R + \Sigma^A + \Sigma^K & \Sigma^R - \Sigma^A - \Sigma^K \\ -\Sigma^R + \Sigma^A - \Sigma^K & -\Sigma^R - \Sigma^A + \Sigma^K \end{pmatrix}. \quad (46)$$

Note that the diagonal elements of the matrix  $L$  written in the  $\pm$  basis contain the combination  $\Sigma^R + \Sigma^A$  and the off-diagonal elements contain  $\Sigma^R - \Sigma^A$ . In the (equilibrium) Matsubara formalism, finite winding numbers of the phase fields must be considered to bring out the charge quantization effects. In our continuous time formalism, the charge quantization effects are brought out by a procedure discussed, for example, in Ref. [51] that we briefly describe below.

### C. Phase windings and charge quantization

We are interested in the small tunneling regime,  $g \lesssim 1$ . In this regime, the phases in each dot fluctuate strongly and hence we represent the action in terms of the conjugate variables, i.e., the number fields. For this, we first perform a Hubbard-Stratanovich decoupling of the charging term, which leads to the following action in the phase-charge representation:

$$S[n, \phi] = \sum_k \int_t ([n_k^c + N_0] \partial_t \phi_k^q + n_k^q \partial_t \phi_k^c - 2E_C n_k^c n_k^q) + S_{\text{tun}}[\phi]. \quad (47)$$

To properly understand the quantization of the charge degrees of freedom, we first work in a contour,  $t \in [0, P]$ . The requirement that  $\phi^-(0) = \phi^+(0) + 2\pi W$  ( $W$  is an integer)

leads us to an unconstrained field,  $\phi^c$ , and

$$\phi_q(t) = \tilde{\phi}_q(t) + \frac{2\pi W}{P}(t - P), \quad (48)$$

with Dirichlet conditions,  $\tilde{\phi}^q(0) = \tilde{\phi}^q(P) = 0$ . Consider first the situation where tunneling is absent. Using Eq. (48) in the first term of Eq. (47), we see that the partition function has contributions of the form  $\sum_W e^{i2\pi(n^c + N_0)W}$ , which vanishes unless  $n^c + N_0$  is an integer. Writing  $N_0 = [N_0] + n_g$ , where  $[N_0]$  is the integer part of  $N_0$  and  $n_g \in [0, 1)$  is the residual ‘‘gate charge’’ on a dot, the integration over the Hubbard-Stratonovich field  $n^c$  is equivalent to a sum over integers,  $\sum_{[n^c] - n_g}$ , where  $[n^c]$  is the integer part of  $n^c$ . Making a change of variables,  $n^c \rightarrow n^c - n_g$ , the sum becomes one over integer values of  $n^c$ . Now the part of the action containing the time derivative of the classical phase field is a function only of the boundary values of the field. Performing the path integral over the boundary fields gives us the constraint that  $n^q = 0$  at the boundaries. Let us now imagine turning on the tunneling at some time. From the structure of the tunneling action, Eq. (45), it is clear that  $n^+$  and  $n^-$  can change only in integer steps. This quantization condition is independent of the time boundary or the length of the time interval. Translated back in the language of the Keldysh closed-time contour, the condition that the *initial* values of  $n^c$  can only take integer values together with the fact that boundary values of  $n^q$  are zero, one concludes that  $n^+(-\infty) = n^-(-\infty) \in \mathbb{Z}$ , and both change in only in integer steps during tunneling events. In this paper, we are interested in the Mott insulator regime with zero gate charge, i.e.,  $n_g = 0$  (or integer  $N_0$ ) and therefore we drop the  $N_0 \partial_t \phi^q$  term in the action. The point  $n_g = 1/2$  is special due to degeneracy between  $n^c = 0, 1$ . The gate charge,  $n_g$ , can also be made to fluctuate in time by using a time-dependent gate voltage. These different scenarios can also be studied using our formalism and will be taken up elsewhere.

#### D. Functional representation of charge current

Here we obtain the functional representation for the charge current in the presence of a constant electric field. The electric field is introduced in the form of a time-dependent vector potential that is turned on at some instant of time, say  $t = 0$ . In every link, the classical component of the vector potential has the form

$$A_{k,1}^c(t) = \Theta(t)Dt, \quad (49)$$

where  $D$  is the potential energy change across a link as already mentioned in Sec. I. This changes the tunneling part of the action by incorporating the Peierls shifts in the phase differences,  $\phi_{k,1}^{c,q}(t) \rightarrow \phi_{k,1}^{c,q}(t) + A_{k,1}^c(t)$ . The tunneling part of the action now has the form

$$S_{\text{tun}}[\phi, A^c, A^q] = g \sum_k \int_{t,t'} [(e_{k,1}^+(t))^* (e_{k,1}^-(t'))^*] L(t - t') \times \begin{bmatrix} e_{k,1}^+(t') \\ e_{k,1}^-(t') \end{bmatrix}, \quad (50)$$

where,  $e_{k,1}^\pm(t) = \exp[i\phi_{k,1}^\pm(t) - iA_{k,1}^\pm(t)]$ . The functional representation of the classical component of the charge current in a link,  $\hat{J}_{k,1}[A^c(t)]$ , is obtained by taking the functional

derivative with respect to  $A_{k,1}^q(t)$ , and setting this quantum source term to zero,

$$\hat{J}_{k,1}(\tau) = -i g \int_t [(e_\tau^+)^* L_{\tau t}^{++} e_\tau^+ - (e_\tau^+)^* L_{\tau t}^{+-} e_\tau^+ + (e_\tau^+)^* L_{\tau t}^{-+} e_\tau^- + (e_\tau^+)^* L_{\tau t}^{--} e_\tau^- - (e_\tau^-)^* L_{\tau t}^{++} e_\tau^+ - (e_\tau^-)^* L_{\tau t}^{+-} e_\tau^+ - (e_\tau^-)^* L_{\tau t}^{-+} e_\tau^- - (e_\tau^-)^* L_{\tau t}^{--} e_\tau^- + (e_\tau^-)^* L_{\tau t}^{--} e_\tau^-]. \quad (51)$$

Here we have suppressed the site indices and written the time arguments as subscripts for brevity.

### III. TRANSIENT CURRENT RESPONSE

In this section, we obtain the current response to leading order (in  $g$ ) upon turning on the uniform electric field by performing the average of the current functional in Eq. (51) over the phase fields. This primarily involves a calculation of the bond correlators defined as

$$\Pi_{\sigma\sigma'}(\tau, \tau') = \langle \exp[-i\phi_{j,1}^\sigma(\tau) + i\phi_{j,1}^{\sigma'}(\tau')] \rangle. \quad (52)$$

Here  $\langle \dots \rangle$  denotes averaging with the full action,  $S[\phi, \phi]$ .

We calculate the bond correlators as a perturbation series in the tunneling conductance  $g$ , by treating the charging action as the bare action and expanding the tunneling part in the exponential to various orders in  $g$ . We denote  $\langle \dots \rangle_0$  to represent averaging with the bare action. The bare bond correlator,  $\Pi_{\sigma\sigma'}^{(0)}$ , factorizes into a product of two single site correlators,

$$\Pi_{\sigma\sigma'}^{(0)}(\tau, \tau') = C_{\sigma\sigma'}(\tau, \tau') C_{\sigma'\sigma}(\tau', \tau), \quad (53)$$

where

$$C_{\sigma\sigma'}(\tau, \tau') = \langle e^{-i(\phi^\sigma(\tau) - \phi^{\sigma'}(\tau'))} \rangle_0. \quad (54)$$

Let us first consider  $C_{+-}(\tau - \tau')$ . Performing the functional integral over the phase fields  $\phi^\pm$  we get the equations

$$\partial_t n^+ = -\delta(t - \tau), \quad \partial_t n^- = -\delta(t - \tau'). \quad (55)$$

The solution depends on the boundary conditions at  $t = -\infty$ . We assume that in the remote past, the system is in thermal equilibrium, and hence the probability distribution for  $n^c$  is  $P(n^c) = \exp(-\beta(n^c)^2 E_C) / \sum_{n=-\infty}^{\infty} \exp(-\beta E_C n^2)$ . In the zero temperature limit,  $P(n^c) = \delta_{n^c, 0}$ . Furthermore since  $n^q(-\infty) = 0$ , we have  $n^+(-\infty) = n^-(-\infty) = 0$ . Thus, the solution to Eq. (55) is

$$n^+(t) = -\Theta(t - \tau), \quad n^-(t) = -\Theta(t - \tau'). \quad (56)$$

Plugging this back, we get

$$C_{+-}(\tau, \tau') = \exp[iE_C(\tau - \tau')]. \quad (57)$$

Similarly,

$$C_{-+}(\tau, \tau') = \exp[-iE_C(\tau - \tau')], \quad (58)$$

$$C_{\pm\pm}(\tau, \tau') = \exp(\mp iE_C|\tau - \tau'|). \quad (59)$$

Using these site correlators in Eq. (52) for the bond correlators in Eq. (51), and using the causal structure of  $\Sigma^{R(A)}$ , we obtain the following expression for the leading-order nonequilibrium

current:

$$J(\tau) = \frac{g}{2\pi} \int_{-\infty}^{\tau} dt (e^{iD(\tau\Theta(\tau)-t\Theta(t))}) \{2\Sigma^R(\tau-t) \times \cos[2E_C(\tau-t)] - 2i\Sigma^R(\tau-t) \cos[2E_C(\tau-t)]\} + \text{c.c.}. \quad (60)$$

Since the upper limit of the integral is  $t = \tau$  and  $\Sigma^A(t)$  has a  $\Theta(-t)$  structure, we can replace

$$\Sigma^R(\tau-t) \rightarrow \Sigma^R(\tau-t) - \Sigma^A(\tau-t) \quad (61)$$

and use the relation for the Fourier transform, Eq. (42). For  $\tau < 0$ , the average current clearly vanishes. Let us split the

$$J_{<}(\tau) = -\frac{4tge^{iD\tau}\Theta(\tau)}{(2\pi)^2} \int_0^{\infty} d\omega \frac{\omega \cos((\omega + 2E_C)\tau)}{\omega + 2E_C} + \text{c.c.},$$

$$J_{>}(\tau) = \frac{2tg\Theta(\tau)}{(2\pi)^2} \left[ \int_0^{\infty} d\omega \frac{\omega e^{i(\omega+2E_C+D)\tau}}{\omega + 2E_C + D} + \int_0^{\infty} d\omega \frac{\omega e^{-i(\omega+2E_C-D)\tau}}{\omega + 2E_C - D} - \int_0^{\infty} d\omega \frac{\omega(4E_C + 2\omega)}{(2E_C + \omega)^2 - D^2} \right] + \text{c.c.} \quad (63)$$

Now, using

$$\int_0^{\infty} d\omega \frac{\omega e^{i\omega\tau}}{\omega + x} = \frac{i}{\tau} - x \int_x^{\infty} du \frac{e^{i(u-x)\tau}}{u},$$

$$= \frac{i}{\tau} - xe^{-ix\tau} [i\pi\Theta(x) - \text{Ei}(ix\tau)], \quad (64)$$

the expression for the current simplifies to

$$J(\tau) = \frac{2tg\Theta(\tau)}{(2\pi)^2} ((2E_C + D)\{\text{Ei}[i(2E_C + D)\tau] - \text{Ei}[-i(2E_C + D)\tau]\} - (2E_C - D)\{\text{Ei}[i(2E_C - D)\tau] - \text{Ei}[-i(2E_C - D)\tau]\}) - 2iE_C \sin(D\tau) [\text{Ei}(i2E_C\tau) + \text{Ei}(-i2E_C\tau)] - 2\pi i(2E_C + D)\Theta(2E_C + D) + 2\pi i(2E_C - D)\Theta(2E_C - D). \quad (65)$$

The current response at long times  $\tau \gg \tau_+ = \max[|D + 2E_C|^{-1}, |D - 2E_C|^{-1}]$  has two components ( $J(\tau \gg \tau_0) = J_{\text{dc}} + J_{\text{tr}}$ ): a dc part,

$$J_{\text{dc}} = \frac{g\Theta(\tau)}{\pi} [(D - 2E_C)\Theta(D - 2E_C) + (D + 2E_C) \times \Theta(-2E_C - D)] \quad (66)$$

and a transient part,

$$J_{\text{tr}} \approx -\frac{4g\Theta(\tau)}{(2\pi)^2 E_C} \frac{1}{\tau^2} \left\{ \left( \frac{D - 2E_C}{D + 2E_C} \right) \sin[(D + 2E_C)\tau] + \left( \frac{D + 2E_C}{D - 2E_C} \right) \sin[(D - 2E_C)\tau] \right\}, \quad (67)$$

that oscillates with the two beat frequencies  $\omega_{\pm} = |D \pm 2E_C|$  and slowly decays in accordance with an inverse square law in time. Such oscillations are absent in classical RC networks subjected to a constant electric field, where only exponential relaxation may occur. The amplitudes of the two oscillation frequencies are inversely related. Close to a resonance,  $D = \pm 2E_C$ , the amplitude of the faster mode tends to vanish and the slower mode dominates. At high fields,  $|D| \gg 2E_C$ , the beat frequencies are approximately  $\omega_{\pm} \approx |D| = \omega_B$ , where  $\omega_B$  is the Bloch oscillation frequency for noninteracting electrons. It is instructive to compare with the fermionic Hubbard chain at half-filling—a quantum model that is the

integral in Eq. (60) into two parts,  $J = J_{<} + J_{>}$ , where  $J_{<}$  involves integration from  $t = -\infty$  to 0 and in  $J_{>}$ ,  $t = 0$  to  $\tau$ :

$$J_{<}(\tau) = \frac{ge^{iD\tau}\Theta(\tau)}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^0 dt [e^{i(2E_C-\omega)(\tau-t)}(\omega - |\omega|) + e^{-i(2E_C+\omega)(\tau-t)}(\omega + |\omega|) + \text{c.c.}],$$

$$J_{>}(\tau) = \frac{g\Theta(\tau)}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega \int_0^{\tau} dt [e^{i(2E_C-\omega+D)(\tau-t)}(\omega - |\omega|) + e^{-i(2E_C+\omega-D)(\tau-t)}(\omega + |\omega|) + \text{c.c.}]. \quad (62)$$

After performing the time integration and some simple manipulations, we get

dissipation-free counterpart of ours. At strong electric fields, the Bloch oscillations in this model also occur [5] at  $\omega_B$ , and which has a simple physical explanation. Consider a noninteracting model of fermions hopping on a one-dimensional lattice:

$$H_{\text{el}}^{(0)} = -t \sum_{\langle ij \rangle \sigma} [c_{i\sigma}^{\dagger} c_{j\sigma} + \text{H.c.}] + \sum_{i\sigma} \epsilon_j n_{i\sigma}, \quad (68)$$

where  $\epsilon_j = Dj$  is the linearly varying potential energy in the presence of a constant electric field. As is well-known (see, e.g., Ref. [32]), the above Hamiltonian is easily diagonalized by the transformation

$$f_n = \sum_i J_{i-n}(2t/D) c_i, \quad (69)$$

which gives us a discrete spectrum, the Wannier-Stark ladder, with energies  $E_n = nD$ , with  $n$  an integer. The wave function corresponding to  $E_n$  is localized, centered around the site  $n$ , and with a spatial extent of the order of  $L = 2t/D$ . Since there is no matrix element connecting different Wannier-Stark levels, no net current flows in the system. If the gain in potential energy across a link,  $D$ , greatly exceeds the tight binding hopping energy, then the Wannier-Stark states are highly localized. Introducing now a small local Hubbard repulsion term of strength  $E_C$  in Eq. (68), we find that the interaction remains approximately local even in the

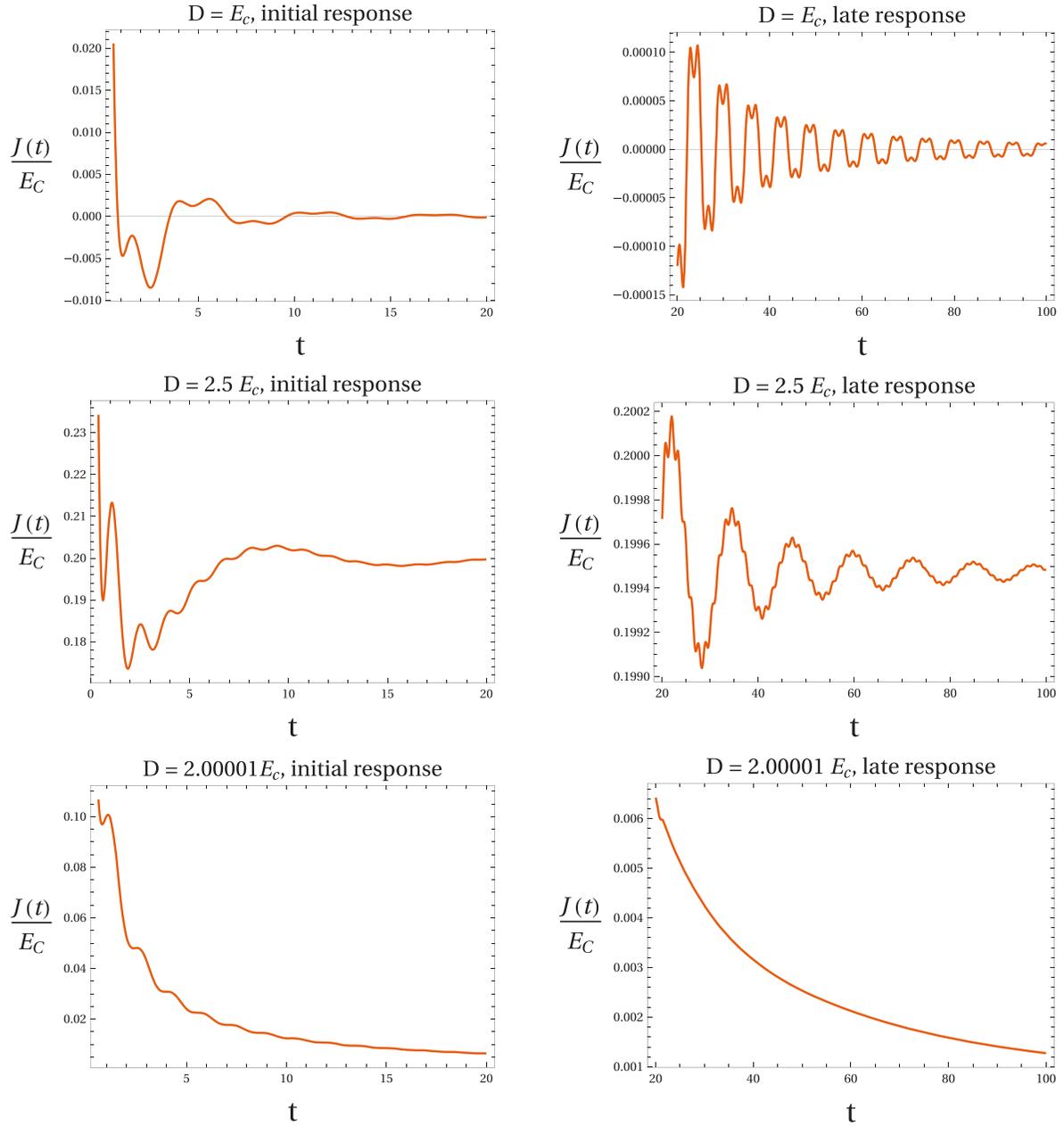


FIG. 1. The current response to leading order in  $g$  after an electric field is turned on as described by Eq. (65). The plots to the left show the initial time response and those to the right show the late time response where the power law decay of the oscillatory behavior is seen. The effect of correlations in the late time response is seen in the form of beating frequencies. A finite steady-state DC response exists only for  $D > 2E_C$

Wannier-Stark basis. For  $D \gg E_C$ , the energy levels are approximately  $nD$ , which leads to Bloch oscillations at frequency  $\omega_B$ .

Consider now the short-time current response. Above the threshold field,  $D_T > 2E_C$ , a finite dc response exists unlike the dissipationless Hubbard chain at half filling. However, the Bloch-like oscillations are present both above and below the threshold field. At short times  $\tau \ll \tau_- = \min[|D + 2E_C|^{-1}, |D - 2E_C|^{-1}]$ , the current response is

$$J(\tau) \approx \frac{g\Theta(\tau)}{\pi} D - \frac{8g\Theta(\tau)}{(2\pi)^2} D(2E_C\tau) [\ln(1/2E_C\tau) + 2 - \gamma], \quad (70)$$

where  $\gamma \approx 0.577$  is the Euler-Mascheroni constant. Remarkably, the initial current response,  $J = gD/\pi$  is independent of the charging energy,  $E_C$ , and appears to be physically related to the fact that sudden changes in the potential effectively short-circuit a capacitor. Plots of the current response for different applied electric field strengths are given in Fig. 1. The transient current response is a central result of this paper.

#### Long time response: The effect of dissipation

The  $1/\tau^2$  decay of the amplitude of current oscillations arises from the ohmic dissipation kernel ( $\Sigma^K(\omega) \sim |\omega|$  in the zero temperature limit which implies  $\Sigma^K(t) \sim 1/t^2$ ).

Higher-order corrections (in  $g$ ) will similarly decay as  $1/\tau^{2n}$ , with  $n > 1$ . Thus, for small  $g$ , one clearly expects that at long times, the transient part of the current response will be dominated by the leading-order term and hence a  $1/\tau^2$  decay of the oscillatory response. The dc part of the current response, however, is not necessarily dominated by the leading order in  $g$  term. For instance in our case, the leading-order dc response vanishes for  $D < 2E_C$ . However, we will show in the following section that a finite dc current exists even for small values of  $D$ , and is dominated by higher order in  $g$  terms.

We now argue that the Bloch-like oscillations and power-law decay of the transient response, a manifestation of charge quantization, are not expected to hold for arbitrary long times. Physically, the finite dc response is a consequence of dissipation, which in turn should ultimately introduce a timescale beyond which an exponential decay rather than a power-law decay should occur. To resolve this, we examine the validity of approximating the ohmic kernel by its zero temperature limit.

The existence of a finite dc current component in response to a dc driving field (see Sec. IV below) implies a finite power dissipation,  $W \sim J_{\text{dc}}D$ , where  $J_{\text{dc}}$  is the dc current. The coupling to an external heat bath is necessary for a steady dc response, for it ensures that the electron distribution in a dot does not run off to infinite temperature as a result of this Joule heating. We assumed earlier that the coupling with the external bath is weak in the sense that the energy relaxation time with the bath,  $\tau_{eb}$  is much greater than the typical electron energy relaxation time within a grain,  $\tau_R \sim \delta/U^2 \sim O(1/\mathcal{N})$ , where  $U$  is the matrix element for electron-electron coupling in the grain. The separation of these time scales makes the electron distribution thermal even when  $W$  is finite. The excess thermal energy in a grain is  $W\tau_{eb}$ , and this is shared by the  $\mathcal{N}$  electrons in the grain, implying a finite temperature  $T_D \sim J_{\text{dc}}D\tau_{eb}/\mathcal{N}$ . The electron-bath realization time  $\tau_{eb}$  will generally decrease with increasing  $\mathcal{N}$ , and we expect  $\tau_{eb} \sim \mathcal{N}^{-2/3}$  if we assume the bath degrees of freedom essentially interact with the surface of the grain. Further, if  $\tau_{eb}$  is due to electron-phonon coupling, then it may also have a temperature dependence; i.e.,

$$\tau_{eb} \sim (1/\mathcal{N})^{2/3}(1/T_D)^n, \quad (71)$$

where  $n \geq 0$  and is model dependent. The temperature  $T_D$  is then

$$T_D \sim \left[ \frac{J_{\text{dc}}D}{\mathcal{N}^{5/3}} \right]^{1/(n+1)}. \quad (72)$$

In the large- $\mathcal{N}$  limit the temperature in the nonequilibrium steady state approaches zero.

The presence of a nonzero  $T_D$  results in an exponential decay for the current oscillations at large times, for in that case the ohmic kernel is  $\Sigma^K(\tau) \sim \pi^2 T_D^2 / \sinh^2(\pi T_D \tau)$ . This goes like  $1/\tau^2$  for  $\tau \ll 1/T_D$  but decays exponentially as  $T_D^2 e^{-2\pi T_D \tau}$  for  $\tau \gg 1/T_D$ . The decay time for the oscillations is large due to large  $\mathcal{N}$ , which offers a rather large time window where the  $1/\tau^2$  decay of the oscillations can be observed. However, ultimately for  $\tau \gg 1/T_D$ , the current oscillations will decay exponentially.

In the following section, we study the effect of higher-order (in  $g$ ) processes on the steady-state part of the current. These higher-order processes govern the dc current response at small values of  $D$ .

#### IV. DC CURRENT AT LOW FIELDS: HIGHER-ORDER PROCESSES

Here we are interested in the long-time steady-state response here, for which we turn on the electric field at  $t = -\infty$  and for all later times, the vector potential is simply  $A_{k,1}(t) = Dt$  (i.e., without the theta function in time). In this case, the expression for current given in Eq. (51) assumes a simpler form,

$$J = 2\iota g \int d\tau [e^{-iD\tau} \Pi^{+-}(\tau)L^{+-}(\tau) - e^{iD\tau} \Pi^{-+}(\tau)L^{-+}(\tau)], \quad (73)$$

since the terms involving Eq. (51) involving the bond correlators  $\Pi^{++}$  and  $\Pi^{--}$  cancel out. Furthermore, for a given sign of  $D$ , only one of the two terms in the integrand contributes. In the rest of the paper, we will assume  $D > 0$  unless otherwise stated, and in this case, only the first term in the integrand in Eq. (73) needs to be calculated. The perturbative expansion of  $J$  is now obtained by expanding the bond correlators in increasing orders in  $g$ ,

$$\Pi_{\sigma,\sigma'} = \Pi_{\sigma,\sigma'}^{(0)} + \Pi_{\sigma,\sigma'}^{(1)} + \dots$$

From Sec. III, we have the leading-order contribution to current as  $J^{(1)} = (g/\pi)[(D - 2E_C)\Theta(D - 2E_C) - (D + 2E_C)\Theta(-D - 2E_C)]$ . We now consider the contribution to the current in the second order in the tunneling conductance  $g$ .

##### A. Second-order steady-state response

The first-order correction to the bare bond correlator of the link labeled  $(k, 1)$  is

$$\Pi_{\mu,\mu'}^{(1)}(\tau, \tau') = \iota g \sum_{n,\sigma\sigma'} \int_{t,t'} W_{\mu\mu'\sigma\sigma'}^{k,n}(\tau, \tau', t, t') L^{\sigma\sigma'}(t - t') e^{-iD(t-t')}, \quad (74)$$

where

$$W_{\mu\mu'\sigma\sigma'}^{k,n}(\tau, \tau', t, t') = \langle \exp[-\iota\phi_{k,1}^\mu(\tau) + \iota\phi_{k,1}^{\mu'}(\tau') - \iota\phi_{n,1}^\sigma(t) + \iota\phi_{n,1}^{\sigma'}(t')] \rangle_{S_C}. \quad (75)$$

Let us define the four-point site correlators,

$$C_{\mu\mu'\sigma\sigma'}(\tau, \tau', t, t') = \langle \exp[-\iota\phi^\mu(\tau) + \iota\phi^{\mu'}(\tau') - \iota\phi^\sigma(t) + \iota\phi^{\sigma'}(t')] \rangle_{S_C}. \quad (76)$$

We now express the function  $W_{\mu\mu'\sigma\sigma'}^{k,n}(\tau, \tau', t, t')$  in terms of the two and four point site correlators. For  $n = k \pm 1$ ,

$$W_{\mu\mu'\sigma\sigma'}^{k,n}(\tau, \tau', t, t') = C_{\sigma'\sigma}(t' - t)C_{\mu'\mu\sigma\sigma'}(\tau', \tau, t, t')C_{\mu\mu'}(\tau - \tau'), \quad (77)$$

while for  $n = k$ ,

$$W_{\mu\mu'\sigma\sigma'}^{k,n}(\tau, \tau', t, t') = C_{\mu'\mu\sigma'\sigma}(\tau', \tau, t', t)C_{\mu\mu'\sigma\sigma'}(\tau, \tau', t, t'). \quad (78)$$

The correlator  $W$  is nonzero only for  $n = k \pm 1$ . For the calculation of current we only need the  $W_{\mu\mu'\sigma\sigma'}$  with  $\mu, \mu' = \{+, -\}$ . These involve the following four-point site correlators:

$$C_{+---}(\tau, \tau', t, t') = \exp[-\iota E_C(-|t - \tau| + |t' - \tau| + |t - t'| - t - \tau + t' + \tau')], \quad (79)$$

$$C_{+-+-}(\tau, \tau', t, t') = \exp[-\iota E_C(-|t - \tau| + |t' - \tau'| - 2(t - t' + \tau - \tau'))], \quad (80)$$

$$C_{+--+}(\tau, \tau', t, t') = \exp[-\iota E_C(|t' - \tau| - |t - \tau'|)], \quad (81)$$

$$C_{+----}(\tau, \tau', t, t') = \exp[-\iota E_C(-|t - \tau'| + |t' - \tau'| - |t - t'| - t - \tau + t' + \tau')], \quad (82)$$

$$C_{-+++}(\tau, \tau', t, t') = \exp[-\iota E_C(|t - \tau'| - |t' - \tau'| + |t - t'| + t + \tau - t' - \tau')], \quad (83)$$

$$C_{-++-}(\tau, \tau', t, t') = \exp[-\iota E_C(|t - \tau'| - |t' - \tau|)], \quad (84)$$

$$C_{-+--}(\tau, \tau', t, t') = \exp[-\iota E_C(|t - \tau| - |t' - \tau'| + 2(t - t' + \tau - \tau'))], \quad (85)$$

$$C_{-+---}(\tau, \tau', t, t') = \exp[-\iota E_C(|t - \tau| - |t' - \tau| - |t - t'| + t + \tau - t' - \tau')]. \quad (86)$$

The four-point site correlators clearly satisfy the identities

$$\begin{aligned} C_{\mu\mu'\sigma\sigma'}(\tau, \tau', t, t') &= C_{\sigma\sigma'\mu\mu'}(t, t', \tau, \tau'), \\ C_{\mu\mu'\sigma\sigma'}(\tau, \tau', t, t') &= C_{\bar{\mu}\bar{\mu}'\bar{\sigma}\bar{\sigma}'}(\tau, \tau', t, t'), \end{aligned} \quad (87)$$

where the bar on the subscripts interchanges the + and - indices.

From the structure of the four-point site correlators, we see that the expression for the bond correlators has nonanalytic terms of the type  $e^{\iota E_C|t_1 - t_2|}$ . To deal with these, we make use of the identity,

$$e^{-\iota E_C|t|} = \lim_{\eta \rightarrow 0} \frac{\iota E_C}{\pi} \int_{-\infty}^{\infty} \frac{d\omega e^{-\omega t}}{(\omega - E_C + \iota\eta)(\omega + E_C - \iota\eta)}.$$

We then express  $L^{\sigma\sigma'}(t - t')$  in the Fourier basis and then perform the  $t, t'$  integrals in Eq.(74). After some effort we get the following expression for  $\Pi_{+-}^{(1)}$ :

$$\begin{aligned} \Pi_{+-}^{(1)}(\tau) &= \frac{4\iota E_C^2 g}{\pi} \lim_{\eta \rightarrow 0} \int d\omega \left\{ \frac{L^{+-}(\omega - D)e^{\iota 2E_C\tau}(e^{-\omega\tau} - 1)}{(\omega^2 + \eta^2)((\omega - 2E_C)^2 + \eta^2)} + \frac{H^{+-}(\omega - D)e^{\iota 2E_C\tau}(1 - e^{\iota(4E_C + \omega)\tau})}{[(\omega + 4E_C)^2 + \eta^2][(\omega + 2E_C)^2 + \eta^2]} \right. \\ &\quad \left. + \frac{2L^{+-}(\omega - D)e^{\iota 2E_C\tau}(e^{-\iota(\omega - 6E_C)\tau} - 1)}{[(\omega - 6E_C)^2 + \eta^2][(\omega - 2E_C)^2 + \eta^2]} + \frac{2H^{+-}(\omega - D)(e^{\iota 2E_C\tau} - e^{\iota\omega\tau})}{[(\omega - 2E_C)^2 + \eta^2][(\omega + 2E_C)^2 + \eta^2]} \right\}, \end{aligned} \quad (88)$$

where  $H^{+-}(\omega) = \Sigma^+(\omega) - \Sigma^-(\omega) + \Sigma^K(\omega)$ . Using Eq. (88) in Eq. (73), we obtain the second-order contribution to the current:

$$\begin{aligned} J^{(2)} &= -\frac{8g^2 E_C^2}{\pi} \lim_{\eta \rightarrow 0} \int d\omega \left\{ \frac{L^{+-}(\omega - D)[L^{+-}(2E_C - D - \omega) - L^{+-}(2E_C - D)]}{(\omega^2 + \eta^2)[(\omega - 2E_C)^2 + \eta^2]} \right. \\ &\quad + \frac{H^{+-}(\omega - D)[L^{+-}(2E_C - D) - L^{+-}(\omega + 6E_C - D)]}{[(\omega + 4E_C)^2 + \eta^2][(\omega + 2E_C)^2 + \eta^2]} + 2 \frac{L^{+-}(\omega - D)[L^{+-}(8E_C - D - \omega) - L^{+-}(2E_C - D)]}{[(\omega - 6E_C)^2 + \eta^2][(\omega - 2E_C)^2 + \eta^2]} \\ &\quad \left. + 2 \frac{H^{+-}(\omega - D)[L^{+-}(2E_C - D) - L^{+-}(\omega - D)]}{[(\omega - 2E_C)^2 + \eta^2][(\omega + 2E_C)^2 + \eta^2]} \right\}. \end{aligned} \quad (89)$$

From the steplike structure of the  $L^{+-}$  and  $H^{+-}$  functions, we find that  $J^{(2)} = 0$  for  $D < E_C$ ; thus,  $J^{(2)}$  has a smaller threshold compared to  $J^{(1)}$ , which vanishes below  $2E_C$ . For  $E_C \leq D < 2E_C$ , the calculation of the current simplifies considerably since only one term makes a nonzero contribution in Eq. (89), and we have

$$J^{(2)} = -\frac{8g^2 E_C^2}{\pi} \lim_{\eta \rightarrow 0} \int d\omega \frac{L^{+-}(\omega - D)L^{+-}(2E_C - D - \omega)}{(\omega^2 + \eta^2)[(\omega - 2E_C)^2 + \eta^2]}, \quad E_C \leq D < 2E_C, \quad (90)$$

and upon performing the integration, we arrive at

$$J^{(2)} = \frac{2g^2}{\pi^3 E_C} [(D - E_C)^2 + E_C^2] \log \left[ \frac{D^2}{(D - 2E_C)^2} \right] - \frac{8g^2}{\pi^3} (D - E_C), \quad E_C \leq D < 2E_C. \quad (91)$$

Just above the threshold for  $J^{(2)}$ ,  $D = E_C$ , the current has a power-law behavior,

$$J^{(2)} \approx \frac{8g^2 E_C}{\pi^3} \left( \frac{D}{E_C} - 1 \right)^3, \quad (92)$$

which is to be contrasted with the linear behavior of  $J^{(1)}$  above its threshold. At the other end,  $D = 2E_C$ , the expression for  $J^{(2)}$  has a logarithmic divergence. Physically, this is a manifestation of a resonance:  $D = 2E_C$  is the condition for creating a particle-hole dipole excitation in neighboring grains. For higher fields,  $D > 2E_C$ , more terms in Eq. (89) will now contribute to  $J^{(2)}$ ; however, none of these terms eliminate the logarithmic singularity.

The second-order perturbation correction to the current is justified provided one does not get too close to the singular point, i.e.,

$$g \ln \left| \frac{D}{2E_C - D} \right| \lesssim 1. \quad (93)$$

Similar logarithmic divergence is also evident in  $\Pi^{(1)}(\tau)$ . However, the bond correlator,  $\Pi = \Pi^{(0)} + \Pi^{(1)} + \dots$ , by definition is bounded by  $\pm 1$ . This clearly shows that the divergence in current at the resonance is the result of a perturbative treatment about the bare charging action. The region of validity of the perturbative treatment could be increased in principle by a resummation of the leading singular terms to all orders in  $g$ . Unfortunately, the number of processes contributing to current in higher orders increases rapidly with the order, rendering the calculation of the current at intermediate fields (sufficiently larger than the lowest threshold) quite complicated. The other possibility is a phase transition from the Mott phase to a conducting, metallic phase whose

boundary is given by the condition  $g \ln(2E_C/\epsilon) = 1$ , with  $\epsilon = 2E_C - D \ll E_C$ . The resummation and possible phase transition will be studied in detail elsewhere. Incidentally, the energy scale  $\epsilon = E_C e^{-1/g}$  also appears in the scaling analysis of the single site equilibrium AES model close to the degeneracy point,  $n_g = 1/2$  [52]. Below this scale, phase fluctuations renormalize the gate charge to the fixed point value,  $n_g = 1/2$ , which corresponds to resonant transmission. Finally, for very small values of  $2E_C - D$ , we expect that the energy level discreteness of the dots will begin to matter, and at resonance, the lower cutoff for  $|2E_C - D|$  should at least be of the order of the mean level spacing  $\delta$ , i.e., we need  $g < 1/\ln(2E_C/\delta)$ .

## B. Higher-order contributions and current response at low fields

At low fields, finite contributions to the current appear only at higher orders. An order- $n$  process has a threshold field  $D_{\text{th}}^{(n)} = 2E_C/n$ . Physically, a large-distance cotunneling process provides the potential energy gain required to overcome Coulomb blockade. During the cotunneling process between sites labeled  $i$  and  $i + n$ , the classical charges,  $n^c$ , at the  $n - 1$  intermediate sites only have virtual transitions and thus the only Coulomb blockade cost appears at the sites  $i$  and  $i + n$ . The pure cotunneling process gives the lowest threshold value,  $D_{\text{th}}^{(n)}$ , at any order. The contribution to the current from this process can be shown to be

$$J^{(n)} = \iota 2ng \left( \frac{\iota 2gE_C^2}{\pi} \right)^{n-1} K^{(n)}, \quad (94)$$

where

$$K^{(n)} = \int \prod_{i=1}^{n-1} d\omega_i \left[ \prod_{j=1}^{n-1} \frac{L^{+-}(\omega_j - D)}{\omega_j^2 (\omega_j - 2E_C)^2} \right] \times L^{+-} \left( 2E_C - D - \sum_{p=1}^{n-1} \omega_p \right). \quad (95)$$

The  $L^{+-}$  functions constrain the frequency integration and we have

$$K^{(n)} = \left( \frac{\iota}{2\pi} \right)^n \int_{2E_C - (n-1)D}^D d\omega_1 \int_{2E_C - (n-2)D - \omega_1}^D d\omega_2 \cdots \int_{2E_C - D - \sum_{p=1}^{n-2} \omega_p}^D d\omega_{n-1} \times \frac{(\omega_1 - D)(\omega_2 - D) \cdots (\omega_{n-1} - D)(2E_C - D - \sum_{p=1}^{n-1} \omega_p)}{\omega_1^2 \cdots \omega_{n-1}^2 (\omega_1 - 2E_C)^2 \cdots (\omega_{n-1} - 2E_C)^2}. \quad (96)$$

The integral gets the dominant contribution from the vicinity of  $\omega_i = D$  and is approximately

$$K^{(n)} \approx \left( -\frac{\iota}{2\pi} \right)^n \frac{n^{(2n-1)}}{(2n-1)!} \frac{(D - D_{\text{th}}^{(n)})^{(2n-1)}}{D^{2(n-1)}(2E_C - D)^{2(n-1)}} \Theta(D - D_{\text{th}}^{(n)}), \quad \frac{D - D_{\text{th}}^{(n)}}{D_{\text{th}}^{(n)}} \ll 1. \quad (97)$$

Combining Eqs. (94) and (97), and making the Stirling approximation for factorials, we obtain, for large  $n$ ,

$$J^{(n)} \sim ng^n \left( \frac{e}{2\pi} \right)^{2n-1} \left[ \frac{2E_C}{D(2E_C - D)} \right]^{2(n-1)} [D - D_{\text{th}}^{(n)}]^{2n-1} \Theta[D - D_{\text{th}}^{(n)}] \quad (98)$$

$$\approx anb^n D \left( 1 - \frac{n_D}{n} \right)^{2n-1} \Theta \left( 1 - \frac{n_D}{n} \right), \quad (99)$$

where

$$\begin{aligned} a &= \frac{2\pi}{e} \left(1 - \frac{1}{n_D}\right)^2, \\ b &= g \left(\frac{e}{2\pi}\right)^2 \left(\frac{1}{1 - n_D^{-1}}\right)^2, \\ n_D &= \frac{2E_C}{D}. \end{aligned} \quad (100)$$

Denoting  $[n_D]$  to be the least integer  $\geq n_D$ , the expression for the total current is given by

$$J = \sum_{n=[n_D]}^{\infty} J^{(n)}. \quad (101)$$

For  $D \ll E_C$ , from the large  $n$  form of  $J^{(n)}$  in Eq. (99), we see that the expression for the total current is divergent for  $b \geq 1$ . We identify the onset of this divergence as the breakdown of our perturbation theory which is developed to work in the Mott phase and thus signals the nonequilibrium phase transition to a metallic phase. Thus, for small values of the electric field, the phase boundary for the nonequilibrium phase transition to this metallic phase is given by setting  $b = 1$ :

$$g = g_0 \left[1 - \frac{D}{2E_C}\right]^2, \quad D \ll 2E_C, \quad (102)$$

with  $g_0$  a constant of order one. For given  $g$  and  $E_C$ , the critical electric field is

$$D_c = 2E_C(1 - \sqrt{g/g_0}). \quad (103)$$

Let us now look into the form of current within the Mott phase for small  $D$ . From Eq. (99), we see that the expression for  $J$  in Eq. (101) can be approximated by a saddle point approximation if  $b \ll 1$ . For this we first rewrite Eq. (101) as

$$J = aD \sum_{n=[n_D]}^{\infty} \exp \left[ \ln n + n \ln b + (2n - 1) \ln \left(1 - \frac{n_D}{n}\right) \right]. \quad (104)$$

The saddle point condition is (neglecting some small terms)

$$\ln b + 2 \ln \left(1 - \frac{n_D}{n}\right) + \frac{2n_D}{n - n_D} = 0. \quad (105)$$

In terms of  $x = n_D/n$ , an approximate solution of the above equation can be written as

$$x = x^* - (1 - x^*)^2 \ln(1 - x^*), \quad (106)$$

where  $x^* = (1 - \frac{2}{\ln b})^{-1}$ . The form of current then turns out to be (for  $D < D_c$ )

$$J \sim a \begin{cases} D \exp \left\{ - (4E_C/D) \ln \left[ \sqrt{\frac{g_0}{g}} \left(1 - \frac{D}{2E_C}\right) \right] \right\}, & D \ll D_c \\ D_c \left(\frac{D_c - D}{D_c}\right)^2, & \frac{D_c - D}{D_c} \ll 1. \end{cases} \quad (107)$$

Thus, as the critical field  $D_c$  is approached, the perturbation series for the current diverges, signaling the breakdown of the Mott insulator state. Farther away from the critical field, the form of the current resembles an activated behavior, with

the driving field  $D$  taking the role of the temperature, and the Arrhenius cost changed from the bare value  $E_C$  to an effectively (field-dependent) lower value,  $E_C^{\text{eff}} = E_C \ln[\sqrt{\frac{g_0}{g}} (1 - \frac{D}{2E_C})]^4$ . The similarity with thermal activation is not surprising since the constant and uniform electric field also generates free particles across the excitation gap, albeit through the Landau-Zener-Schwinger mechanism. Closer to the transition field  $D_c$ , we expect the divergence of the current response in Eq. (107) to ultimately get cut off by processes we did not take into account in our perturbation series that consisted, at every order, of only the respective threshold contributions. Further work needs to be done to establish if there is any nonanalyticity in the current response across the transition field, for that would imply a true nonequilibrium phase transition and not a crossover between the Mott insulator and bad metal phases.

## V. DISCUSSION

In summary, we developed an effective Keldysh field theory for studying the nonequilibrium response of dissipative Mott insulator systems and used it to study the nonequilibrium current response to a uniform electric field switched on at some instant of time. Our model, a Keldysh generalization of the AES model for Mott insulators, is in effect a bosonization of the Hubbard model with a large number ( $\mathcal{N}$ ) of electron flavors at the lattice sites. The effective degrees of freedom are the excess charges at the sites and the phases conjugate to these. The large- $\mathcal{N}$  is simultaneously a source of dissipation through the Landau damping mechanism and also affords significant simplification of the effective action (in comparison with the usual Hubbard model) by suppressing all terms that are higher than second order in the interdot tunneling amplitude.

The quantum effect that survives in the large- $\mathcal{N}$  limit is charge quantization, which is respected at every stage in the analysis of our problem. The charge quantization is reflected in sustained Bloch-like oscillations that decay as an inverse square power-law in time up to a large timescale  $\tau_D \sim 1/T_D \sim \mathcal{N}^{-\alpha}$ ,  $\alpha > 0$ . The effect of correlations is to split the Bloch oscillation frequency into two beating frequencies whose difference is of the order of the Coulomb repulsion scale.

The power-law decay of the current oscillations signifies the persistence of Coulomb blockade or charge quantization effects. At small values of tunneling  $g$ , Coulomb blockade effects dominate and the dot charge fluctuations are weak. The presence of a large number of energy levels in the dots may scramble the phase of the electronic states but is not able to erase charge quantization effects at weak tunneling. In the effective action,  $\Sigma^K(t, t')$  that represents correlations between tunneling events at time  $t$  and  $t'$ , decays as a power-law in the zero temperature limit,  $1/(t - t')^2$ , which is essentially why our current oscillations at weak tunneling obey the same power-law decay. But the energy dissipation that is responsible for a dc component in the current, also results in a finite system temperature ( $T_D$ ), which is determined by a combination of the power dissipation and the (weak) coupling to the external heat bath. Note that  $T_D$  is very small due to the effect of large- $\mathcal{N}$ . The nonzero  $T_D$  causes the  $1/t^2$  decay to crossover to an exponential decay after  $t \sim 1/T_D$ .

When  $g$  is large, the behavior is very much like a classical RC circuit where charge can take continuous values. In this regime, the charge fluctuations are strong (i.e., charge quantization effects are weak), and the current oscillations decay exponentially. To see this, at large  $g$ , we expand around the saddle point configuration of the effective action [Eq. (45)] and to quadratic order in the phase fields. In the classical-quantum space, the action takes the form

$$S[\phi] = \sum_q \int_{\omega} [\phi^c \phi^q]_{(q,\omega)} \times \begin{bmatrix} 0 & -\frac{\omega^2}{E_C} - i\omega h(q) \\ -\frac{\omega^2}{E_C} + i\omega h(q) & 2ih(q)|\omega| \end{bmatrix} \begin{bmatrix} \phi^c \\ \phi^q \end{bmatrix}_{(-q,-\omega)}, \quad (108)$$

where  $h(q) = 2g(1 - \cos q)$ . The inverse of the matrix is given by

$$\frac{-1}{(h(q))^2\omega^2 + \frac{\omega^4}{E_C^2}} \begin{bmatrix} 2ih(q)|\omega| & \frac{\omega^2}{E_C} + i\omega h(q) \\ \frac{\omega^2}{E_C} - i\omega h(q) & 0 \end{bmatrix}. \quad (109)$$

We thus have

$$G^{\pm}(\omega) = -\frac{1}{\omega \left[ \frac{\omega}{E_C} \mp ih(q) \right]}, \quad (110)$$

$$G^K(\omega) = -\frac{2ih(q)|\omega|}{[h(q)]^2\omega^2 + \frac{\omega^4}{E_C^2}}.$$

The presence of the imaginary pole in the retarded Green's function implies in the time domain it has an exponential decay with a characteristic timescale,  $1/(gE_C)$ , which is equivalent to the time constant,  $\tau = RC$ , of a resistively shunted capacitor. So the oscillations in the current after the source is turned on at  $t = 0$  would also decay exponentially with the same characteristic timescale.

A major challenge in the area has been to demonstrate a DC current response in lattice translationally invariant Hubbard models. We identified the role played by dissipation in suppressing the Bloch oscillations (even if as a power law in time) and enabling a finite DC current response. We analyzed the DC current response taking into account higher-order cotunneling processes that allow a trade-off between the reduced probability of a long-distance cotunneling and energy gain from the applied electric field. The response at small electric fields is found to be of the LZS form,  $J \sim D[g/\ln^2(1/g)]^{2E_C/D}$ , although the exponent is proportional to the Mott gap  $E_C$  instead of the usual  $e^{-E_C^2/D}$  expected for pair-production probability in the dissipation-free case [3]. We do not find a threshold field below which DC conduction is absent since at any small field, DC conduction is possible through sufficiently high-order cotunneling. At higher fields, the perturbation expansion of the current in powers of the small tunneling conductance breaks down, and from this we obtain the phase boundary for the electric-field-driven Mott insulator to a conducting state. Both a phase transition and a (rapid) crossover are consistent with our results, since the expression that we have obtained for higher-order contributions to the current is only valid at very small values of the driving field  $D$ ; but the instability at  $D_c = 2E_C(1 - \sqrt{g/g_0})$  in

Eq. (105) suggested by the divergence of perturbation theory occurs at a value of  $D$  that is not necessarily small. Thus, corrections to our expressions might become relevant in actually determining whether there is a real phase transition or not.

The AES model regards the interdot tunneling processes to be of the Fermi Golden-Rule type, which breaks down when the characteristic energies of particle-hole excitations in the dots approach the mean level spacing,  $\delta$ . Therefore, the typical potential drop between neighboring sites or the temperature should exceed  $\delta$ . This imposes a cutoff on the regime of validity of our analysis.

We conclude with a brief discussion of future directions. Our approach can also be useful for the study of other far from equilibrium problems of current interest. For example, it is an interesting question as to how an initial nonthermal distribution of dot charges would evolve with time—in particular, whether the long-time behavior retains any memory of the initial conditions. Similar questions have been posed, for example, in the context of relaxation of initial charge distribution in Bosonic cold atom systems [53] and the approach to thermal equilibrium in fermionic quantum chains [54]. Our Keldysh-AES model can also be used to study the energy transport. The problem we have attacked in our paper is the current response to a uniform DC electric field; however, the approach is readily generalized to problems involving time-dependent drives. In this context, it would be interesting to compare with periodically driven Hubbard chains in the absence of dissipation [55]. As we noted in our paper, there are two special values of the background charge on a dot—integer and half odd integer. The integer case that we studied in detail corresponds to a Mott insulator, while the latter is a correlated “bad” metal. The nonequilibrium response close to half odd integer background charges is an open question. Another interesting direction would be to study the nonequilibrium response of driven Josephson-junction arrays. This direction, especially after taking into account long-range Coulomb interactions, would shed more light to understand the sudden jumps observed in the I-V characteristics of disordered superconductors that are in the insulating side and in the proximity of superconductor to insulator transition [56–59].

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## APPENDIX: NORMALIZATION OF THE PARTITION FUNCTION

A key property of the Keldysh partition function is that in the absence of source fields, the partition function is normalized. Demonstrating this for the Keldysh-AES action requires one to take into account the correct causal structure of the Green functions. We expand  $\exp[\iota S_{\text{un}}[\phi]]$  in powers of  $g$ . To leading order, we get

$$Z^{(0)} = \int [D\phi][Dn] \exp[\iota(S_C[n, \phi])]. \quad (A1)$$

Doing the functional integration over  $\phi$ , we see that the constraints  $\partial_t n^+ = 0$  and  $\partial_t n^- = 0$  are imposed and then it immediately follows from the boundary condition,  $n^+(-\infty) = n^-(-\infty)$ , that  $Z^{(0)} = 1$ .

*Order g.* Consider now the  $O(g)$  term in the expansion of the Keldysh partition function.

$$Z^{(1)} = \iota g \sum_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' L^{\sigma\sigma'}(t-t') \times \langle \exp[-\iota\phi_{k,1}^{\sigma}(t) + \iota\phi_{k,1}^{\sigma'}(t')] \rangle_0, \quad (\text{A2})$$

where  $\langle \rangle_0$  denotes averaging with respect to the bare action. Thus,

$$Z^{(1)} = \iota g \sum_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' L^{\sigma\sigma'}(t-t') \Pi_{\sigma\sigma'}(t-t'). \quad (\text{A3})$$

We have the site correlators,

$$C_{++}(t-t') = \exp[-\iota E_C |t-t'|], \quad (\text{A4})$$

$$C_{+-}(t-t') = \exp[\iota E_C (t-t')], \quad (\text{A5})$$

$$C_{-+}(t-t') = \exp[-\iota E_C (t-t')], \quad (\text{A6})$$

$$C_{--}(t-t') = \exp[\iota E_C |t-t'|]. \quad (\text{A7})$$

Then we get the bond correlators,

$$\Pi_{++}(t-t') = \exp[-2\iota E_C |t-t'|], \quad (\text{A8})$$

$$\Pi_{+-}(t-t') = \exp[2\iota E_C (t-t')], \quad (\text{A9})$$

$$\Pi_{-+}(t-t') = \exp[-2\iota E_C (t-t')], \quad (\text{A10})$$

$$\Pi_{--}(t-t') = \exp[2\iota E_C |t-t'|]. \quad (\text{A11})$$

From the bond correlators we immediately see that the term involving  $\Sigma^K$  vanishes. Now let's look at the term with  $\Sigma^+$ . In the time representation, we have to keep in mind that it comes with the causality factor  $\theta(t)$  and hence we write it as  $\Sigma(t)\theta(t)$ . The term involving this reads as,

$$\Sigma^+(t)\theta(t)[\exp(\iota E_C t) - \exp(-\iota E_C t) - \exp(\iota E_C |t|) + \exp(-\iota E_C |t|)]. \quad (\text{A12})$$

Because of the presence of the  $\theta$  function, we see that we can remove the modulus sign from the last two terms and then clearly this contribution vanishes. Similarly, we see that the contribution from terms involving  $\Sigma^-$  also vanishes. Hence, we see that the order  $g$  contribution to the partition function vanishes. We assume that all the higher-order  $g$  contributions to the partition function vanish too and thus the partition function is truly equal to 1.

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