Rapid Communications

Anisotropic exceptional points of arbitrary order

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A pair of anisotropic exceptional points (EPs) of arbitrary order is found in a class of non-Hermitian random systems with asymmetric hoppings. Both eigenvalues and eigenvectors exhibit distinct behaviors when these anisotropic EPs are approached from two orthogonal directions in the parameter space. For an order-N anisotropic EP, the critical exponents ν of phase rigidity are $(N-1)/2$ and $N-1$, respectively. These exponents are universal within the class. The order-N anisotropic EPs split and trace out multiple ellipses of EPs of order 2 in the parameter space. For some particular configurations, all the EP ellipses coalesce and form a ring of EPs of order N. Crossover to the conventional order-N EPs with $\nu = (N-1)/N$ is discussed.

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Introduction. There has been a surge of interest in the physics of non-Hermitian systems [1–49]. For classical wave systems, the attention has been focused on a broad range of systems involving material gain/loss or radiation leaking into open space. A prominent property of non-Hermitian systems is the existence of singularities called exceptional points (EPs), at which the Hamiltonian matrix becomes defective and multiple eigenstates coalesce [28–58]. EPs have been studied in various non-Hermitian systems due to their fascinating properties, which accounts for unusual transmission or reflection [9] and potential applications in lasing [15,19] and sensing [39,41]. Non-Hermiticity induces topological properties with new features that cannot be found in Hermitian systems [25,54,59–64].

The *most* common EPs are "order-2" EPs at which two eigenstates coalesce. An order-2 EP is usually associated with a square-root singularity in the eigenvalues, with $E \propto \pm\sqrt{\delta}$ where δ denotes a small deviation from the EP in the parameter space [55], and is associated with two Riemann sheets in the complex δ plane. It requires encircling two cycles around the order-2 EP ($\delta = 0$) to bring a state back to itself [3,28]. The coalescence of two eigenstates at the EP makes the right and left coalesced eigenvectors orthogonal, i.e., $\langle\psi^L_{\mathrm{EP}}|\psi^R_{\mathrm{EP}}\rangle = 0$, where the superscripts L and R signify left and right eigenvectors [56]. As a result, the so-called phase rigidity $\rho_m = \frac{|\langle\psi^L_m|\psi^R_m\rangle|}{\langle\psi^R_m|\psi^R_m\rangle}$ of each eigenstate m behaves in a power law $\rho_m \propto |\delta|^\nu$ at small $|\delta|$ with $\nu = 1/2$ [35,44,65]. The exponent ν measures the rate that the left and right eigenvectors turn orthogonal when an EP is approached. Phase rigidity is the inverse square-root of the Petermann factor, which underlies the enhancement in intrinsic laser linewidth and spontaneous emission rate due to the nonorthogonality of laser modes [36,66–68]. The coalescence of multiple EPs can lead to a higher-order EP [36]. At an order-N EP, the matrix is highly defective and N eigenstates coalesce. Near an order-N EP, the N eigenvalues generally follow the $\sqrt[N]{\delta}$ singularity with one eigenvalue in each of the N Riemann sheets in the complex δ plane and N rounds of adiabatic encircling are needed to return to the initial sheet [35]. For an order-N EP, the exponent of the phase rigidity becomes $\nu = \frac{N-1}{N}$ [30,69]. Higher-order EPs have been suggested to enhance sensing capabilities [39]. However, it is difficult to achieve a higher-order EP as it requires some detailed tuning of multiple parameters.

Recently, order-2 anisotropic EPs are studied both theoretically and experimentally in different physical systems [35,44,61,57]. An anisotropic EP is associated with different singular behaviors when the EP is approached from two orthogonal directions in a two-parameter space, e.g.,(ξ, η) plane [44]. Anisotropic EPs occur in a two-parameter space when the EPs form a continuous curve [44]. Near an order-2 anisotropic EP ($\xi = \eta = 0$), two eigenvalues follow the behavior of $E \propto \pm\sqrt{\eta^2 + B\xi}$, where B is a constant [44,61]. Along the line $\eta = 0$, the $\pm\sqrt{B\xi}$ singularity is followed. However, along the orthogonal direction, $\xi = 0$, two eigenvalues exhibit linear crossing behavior, namely, $E \propto \pm\eta$, near the anisotropic EP [32,44,45]. The phase rigidity is also anisotropic bearing two different exponents: 1/2 (along $\eta = 0$) and 1 (along $\xi = 0$), i.e., $\rho \propto |\xi|^{1/2}$ and $\rho \propto |\eta|$, respectively.

In this Rapid Communication, we show that a class of non-Hermitian random systems with asymmetric hoppings can exhibit two anisotropic EPs of arbitrarily high order. Their occurrence entails a two-parameter space, in contrast to high-order EPs reported earlier [48,49] which involve only one parameter. Near the order-N anisotropic EP, the eigenvalues have the form $A_i\sqrt{\delta}$ and $B_i\epsilon$ in two orthogonal directions, respectively, where A_i and B_i are constants and δ and ϵ denote small deviations from the anisotropic EP. The phase rigidity shows a power-law behavior with two universal exponents, $(N-1)/2$ and $N-1$, which reduces to the exponents of 1/2 and 1 found recently for $N = 2$ anisotropic EPs [44]. Hence, phase rigidity not only describes the evolution of a system from

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being Hermitian to bearing an EP, but also identifies the different orders of EPs. Its value can be obtained by measuring the wave functions. Interestingly, the two order-N anisotropic EPs split and give rise to multiple elliptical trajectories of order-2 EPs in the parameter space, and these EP trajectories always merge at the two order-N anisotropic EPs, which act as their common vertices. In a particular configuration, corresponding to the non-Hermitian Hamiltonian being linearly spanned by three spin-j matrices, all the order-2 exceptional ellipses coalesce and form a ring of EPs of order $N = 2j + 1$, which is the generalization of the ring of order-2 EPs found recently in photonic crystal slabs [32] to arbitrary higher order.

Two anisotropic EPs of arbitrary order. To put our discussion in perspective, we start with a simple system that admits closed-form solutions: an ordered asymmetric nearest-neighbor hopping model with N sites [63,70]. The non-Hermitian Hamiltonian is an $N \times N$ tridiagonal matrix:

$$H = \begin{pmatrix} 0 & t_1 & 0 & 0 & 0 \\ t_2 & 0 & t_1 & 0 & 0 \\ 0 & t_2 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & t_1 \\ 0 & 0 & 0 & t_2 & 0 \end{pmatrix}, \tag{1}$$

where $t_{1,2} = t \mp \gamma$ with t and γ being real. The system is nonreciprocal if $\gamma \neq 0$. For convenience, we normalized the energy scale by setting $\gamma = 1$ in this work. The physical implementation of such nonreciprocal hoppings was discussed originally in the context of magnetic flux lines in type-II superconductors [70]. The nonreciprocal hoppings can in principle be realized using coupled ring resonators [71,72]. Alternatively, the asymmetric hopping model can be implemented with ultracold atoms in optical lattices [63] or electronic circuits [73,74]. The eigenvalues of H have the square-root form

$$E_p = 2\sqrt{t^2 - 1}\cos\left(\frac{p\pi}{N+1}\right), \qquad p = 1, 2, \ldots, N. \tag{2}$$

There are two EPs at $t = \pm 1$, at which Eq. (1) reduces to a Jordan block form or its transpose, and all N eigenstates coalesce to a single one with eigenvalue $E_{\mathrm{EP}} = 0$, leading to an EP of order N.

We now introduce diagonal entries (on-site terms) through a new parameter ϵ to the Hamiltonian. For example, we consider $H' = H + \epsilon s_z^j$, where s_z^j is related to the z component of the spin-j operator by $s_z^j = S_z^j / j\hbar$ with $j = \frac{N-1}{2}$. Explicitly, we have $(S_z^j)_{pq} = (j + 1 - p)\hbar\delta_{pq}$, where δ_{pq} is the Kronecker delta function and $p, q = 1, 2, \ldots, N$. In the (t, ϵ) plane, $(\pm 1, 0)$ are two EPs of order N for any N. These two EPs exhibit anisotropic behaviors. Taking $j = 3/2$ as an example, the eigenvalues of the 4×4 Hamiltonian H' have the form

$$E = \pm\frac{\sqrt{2}}{6}\sqrt{A \pm \sqrt{B}}, \tag{3}$$

where $A = 27(t^2 - 1) + 10\epsilon^2$ and $B = 405(t^2 - 1)^2 + 432(t^2 - 1)\epsilon^2 + 64\epsilon^4$. They reduce to square-root forms with $E = \pm\frac{\sqrt{5}\pm 1}{2}\sqrt{t^2 - 1}$ when $\epsilon = 0$ and to linear forms with $E = \pm\epsilon$ or $E = \pm\frac{1}{3}\epsilon$ when $t = \pm 1$. Thus, the eigenvalues show different dispersion behaviors along different directions in the neighborhood of the EP, which bears the characteristic signature of anisotropic EPs. For general values j, the eigenvalues of H' reduce to square-root forms given by Eq. (2) when $\epsilon = 0$, and linear forms with $E_p = \frac{j+1-p}{j}\epsilon$ when $t = \pm 1$, where $p = 1, 2, \ldots, N$. Thus $(t, \epsilon) = (\pm 1, 0)$ are two anisotropic EPs of any order N for H'. Recent studies of anisotropic EPs involved only the coalescence of two states, i.e., $N = 2$ [44,61]. Here we have shown that a very simple Hamiltonian H' carries anisotropic EPs of arbitrary high order, and the anisotropy can be attributed to the fact that perturbations introduced to different entries (diagonal or superdiagonal/subdiagonal elements) lead to different singular behaviors. To characterize eigenstates near an anisotropic EP, we calculate the phase rigidity near the anisotropic EP at $(1, 0)$ in the (t, ϵ) plane and find $\rho \propto |\delta|^{(N-1)/2}$ with $\delta = t - 1$ when $\epsilon = 0$, and $\rho \propto |\epsilon|^{N-1}$ when $t = 1$ [75]. The exponents of phase rigidity are, respectively, $(N - 1)/2$ and $(N - 1)$ in the two orthogonal directions, also exhibiting anisotropic behaviors.

Universality. Surprisingly, the above EPs and their anisotropic behaviors are universal even in the presence of certain types of disorder. Let us consider the following Hamiltonian with disorder:

$$H_{\mathrm{dis}} = \begin{pmatrix} c_1\epsilon & a_1 t_1 & 0 & 0 & 0 \\ b_1 t_2 & c_2\epsilon & a_2 t_1 & 0 & 0 \\ 0 & b_2 t_2 & c_3\epsilon & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & a_{N-1} t_1 \\ 0 & 0 & 0 & b_{N-1} t_2 & c_N\epsilon \end{pmatrix}, \tag{4}$$

where a_i, b_i, c_i are arbitrary real numbers with $a_i, b_i > 0$ and $t_{1,2} = t \mp 1$. The Hamiltonian H_{dis} in Eq. (4) describes a system with not only asymmetric but also random hoppings. For a given configuration of a_i, b_i, c_i, H_{dis} can be implemented using different platforms such as cold atoms [63], coupled ring resonators [71,72], or electronic circuits [73,74] with hoppings and/or on-site energies fine-tuned [75]. When $a_i = b_i = 1$ and $c_i = (j + 1 - i)/j$, the Hamiltonian H_{dis} reduces to $H' = H + \epsilon s_z^j$ discussed earlier where j is the spin index. It turns out that the points $(\pm 1, 0)$ remain two anisotropic EPs in the presence of randomness, therefore displaying robustness. The combination of chiral symmetry [75] when $\epsilon = 0$ and unidirectional hoppings when $t = \pm 1$ guarantees the occurrence of the two EPs of arbitrary order. It can be shown rigorously that all eigenvalues take the square-root form, namely, $E_i = \pm D_i\sqrt{t_1 t_2}$ when $\epsilon = 0$, where D_i are constants [75]. Near each anisotropic EP, say $(1, 0)$, the eigenvalues of H_{dis} show anisotropic behaviors: $E_i \approx \pm D_i\sqrt{2\delta}$ with $\delta = t - 1$ along $\epsilon = 0$, and $E_i = c_i\epsilon$ along $t = 1$. The critical exponents of the phase rigidity are, respectively, $(N - 1)/2$ and $N - 1$ for all eigenstates, independent of the values of a_i, b_i, c_i [75].

For illustration, we consider three disordered systems of different dimensions: $N = 2, 3, 4$. One eigenstate is taken as a representative state to calculate the phase rigidity for each system. The phase rigidity for the three systems is shown in Figs. 1(a) and 1(b) for the two orthogonal trajectories in the parameter space. Both trajectories go through the anisotropic EP at $(t, \epsilon) = (1, 0)$. The values of a_i, b_i, c_i for each case

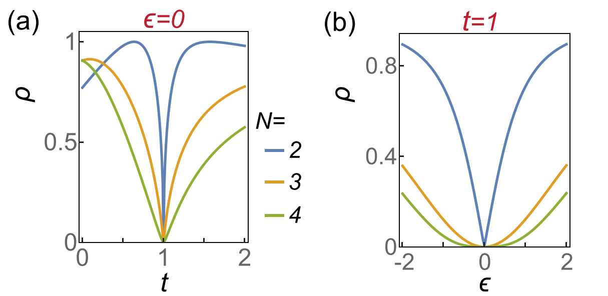

FIG. 1. Phase rigidity for three disordered systems $H_{\rm dis}$ of different sizes, $N = 2, 3, 4$, along two orthogonal trajectories in the(t, ϵ) plane: (a) $\epsilon = 0$; (b) $t = 1$.

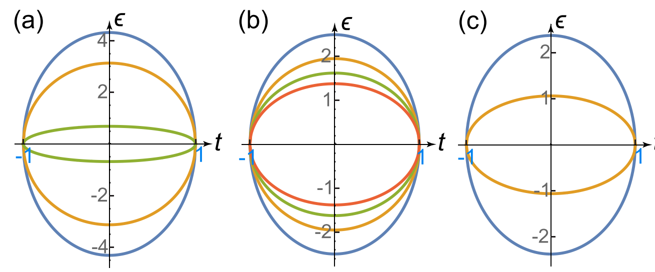

FIG. 2. Order-2 EPs form ellipses in the (t, ϵ) plane. (a) An $N = 6$ system. (b) An $N = 8$ system. The values of a_i, b_i and c_i are randomly chosen in (a) and (b) [75]. (c) The $N = 4$ ordered system $H' = H + \epsilon s_z^j$ with $j = 3/2$.

are given in the Supplemental Material [75]. Along $\epsilon = 0$ trajectory, the critical exponents of phase rigidity near the anisotropic EP are $1/2, 1, 3/2$ for $N = 2, 3, 4$, respectively, conforming to the $(N-1)/2$ rule. Along the other trajectory $(t = 1)$, the critical exponents are $1, 2, 3$, respectively, agreeing with the $N - 1$ rule.

Ellipses of anisotropic EPs. Intriguingly, we find that the two order-N anisotropic EPs at $(t, \epsilon) = (\pm 1, 0)$ split and trace out multiple elliptical trajectories formed by order-2 EPs in the form of $t^2 + \alpha_i \epsilon^2 = 1$ in the (t, ϵ) plane, with the points $(\pm 1, 0)$ being the common vertices of the ellipses, where α_i are some positive numbers depending on a_i, b_i, c_i. For the case of $N = 2$, the coalescence of two eigenvalues occurs on the ellipse $t^2 + \frac{(c_1-c_2)^2}{4a_1b_1}\epsilon^2 = 1$, on which $E = \beta\epsilon$ with $\beta = (c_1 + c_2)/2$. For $N = 3$, it can be analytically shown that there always exists an ellipse of order-2 EPs in the form of $t^2 + \alpha\epsilon^2 = 1$, on which the coalesced eigenvalue is also proportional to ϵ, i.e., $E = \beta\epsilon$. Here both α and β depend on the values of all a_i, b_i, and c_i [75]. Analytically we found $\alpha = -\frac{(c_1-\beta)(c_2-\beta)(c_3-\beta)}{a_1b_1(c_3-\beta)+a_2b_2(c_1-\beta)}$ for the $N = 3$ scenario [75]. Here we conjecture that the proportionality between the coalesced eigenvalue on each ellipse and ϵ holds for any N. With the ansatz $E = \beta\epsilon$, it can be argued from the characteristic equation $f_N(E) \equiv \det(H_{\rm dis} - EI) = 0$ that the relation $t_1t_2 = -\alpha\epsilon^2$ holds with α to be determined. Furthermore, at an EP, where two eigenvalues coalesce, we have $f'_N(E) = 0$. Thus, we can determine several solutions of β_i and α_i by solving $f_N(E) = 0$ and $f'_N(E) = 0$, noting $\alpha_i = -t_1t_2/\epsilon^2$ should be real. Therefore the order-2 EPs ellipses are $t^2 + \alpha_i\epsilon^2 = 1$ [75]. Here we would like to point out that all points on each ellipse are also anisotropic EPs [75].

We found that $N/2$ and $(N-1)/2$ order-2 EPs ellipses, respectively, occur for even and odd N. For illustration, we show the numerically determined order-2 EP ellipses for $N = 6$ and $N = 8$ systems with random a_i, b_i, c_i in Figs. 2(a) and 2(b), respectively. There are three ellipses for $N = 6$ and four for $N = 8$. All ellipses have $(\pm 1, 0)$ as its vertices, giving two anisotropic EPs of order N. We note that closed-form expressions for order-2 EPs ellipses can be found for the particular case $H' = H + \epsilon s_z^j$ with $j = 3/2$ considered previously, where H is given by Eq. (1). Its eigenvalues are given by Eq. (3), which shows that the eigenvalues coalesce pairwise when $B = 0$. This condition gives two ellipses: $t^2 + \frac{8}{45}\epsilon^2 = 1$ and $t^2 + \frac{8}{9}\epsilon^2 = 1$ as depicted in Fig. 2(c). We emphasize that the ellipses of order-2 EPs are always pinned by the two order-N EPs at $(\pm 1, 0)$, with their shapes determined by the specific values of a_i, b_i, c_i.

A ring of anisotropic EPs of arbitrary order. In the following, we show that when the coefficients a_i, b_i, c_i correspond to some linear combination of spin-j matrices s_τ^j, $\tau = x, y, z$, all ellipses of order-2 EPs will coincide and form a ring of order-N EPs. In this case, the Hamiltonian takes the form

$$H^j = ts_x^j - is_y^j + \epsilon s_z^j, \tag{5}$$

where $s_\tau^j, \tau = x, y, z$ are spin-j matrices defined by $s_\tau^j = S_\tau^j/j\hbar$, with S_τ^j being the spin-j operators. The dimensions of the matrix H^j are $N \times N$, where $N = 2j + 1$. Explicitly, we have $s_x^j = \frac{s_+^j + s_-^j}{2}$ and $s_y^j = \frac{s_+^j - s_-^j}{2i}$, where $(s_+^j)_{pq} = \frac{\sqrt{(q-1)(2j+2-q)}}{j}\delta_{p,q-1}$ and $(s_-^j)_{pq} = \frac{\sqrt{(2j+1-q)q}}{j}\delta_{p,q+1}$ with $p, q = 1, 2, \ldots, N$ [76]. We note that H^j here and $H' = H + \epsilon s_z^j$ have the same form only when $j = \frac{1}{2}$ and $j = 1$, but not so for larger values of j.

Since Eq. (5) formally resembles the Hamiltonian of a spin in an artificial "magnetic field" $\vec{B} = (t, -i, \epsilon)$, we can always rotate the spin space to make the rotated z' axis parallel to $\vec{B}$. Such a rotation transforms the Hamiltonian H^j into $\mathcal{H}^j = R^{-1}H^jR = \sqrt{t^2 + \epsilon^2 - 1}s_z^j$ with $R = e^{-is_z^j j\phi_0}e^{-is_y^j j\theta_0}$, where $\theta_0 = \cos^{-1}\frac{\epsilon}{\sqrt{t^2+\epsilon^2-1}}$ and $\phi_0 = \cos^{-1}\frac{t}{\sqrt{t^2-1}}$ [77]. Thus, the eigenvalues of $\mathcal{H}^j$ can be derived immediately as

$$E_q = \frac{q - j - 1}{j}\sqrt{t^2 + \epsilon^2 - 1}, \tag{6}$$

where $q = 1, 2, \ldots, N$. Clearly the equation $t^2 + \epsilon^2 - 1 = 0$ represents a ring of order-N EPs in the (t, ϵ) plane, which has unity radius. Each EP on the ring is actually an anisotropic EP [75]. To show this, we consider an arbitrary point $\vec{a} = (\cos\theta, \sin\theta)$ on the ring as depicted in Fig. 3(a). We introduce $\vec{d}_\perp = \delta(\cos\theta, \sin\theta)$ and $\vec{d}_\parallel = \delta(\sin\theta, -\cos\theta)$ with $0 < \delta \ll 1$ to represent small displacements from $\vec{a}$ in the radial and tangential directions of the ring, respectively, which are indicated by a gray and a blue arrow in Fig. 3(a). All N eigenvalues exhibit a square-root form with $E_\perp \propto \sqrt{2\delta + \delta^2} \approx \sqrt{2\delta}$ at $\vec{a} + \vec{d}_\perp$. However, the N eigenvalues are linear with respect to δ, namely, $E_\parallel \propto \delta$, at $\vec{a} + \vec{d}_\parallel$. Specifically, we show the real and imaginary parts of eigenvalue surfaces of the $j = 3/2$ case with $H^{3/2} = ts_x^{3/2} - is_y^{3/2} + \epsilon s_z^{3/2}$

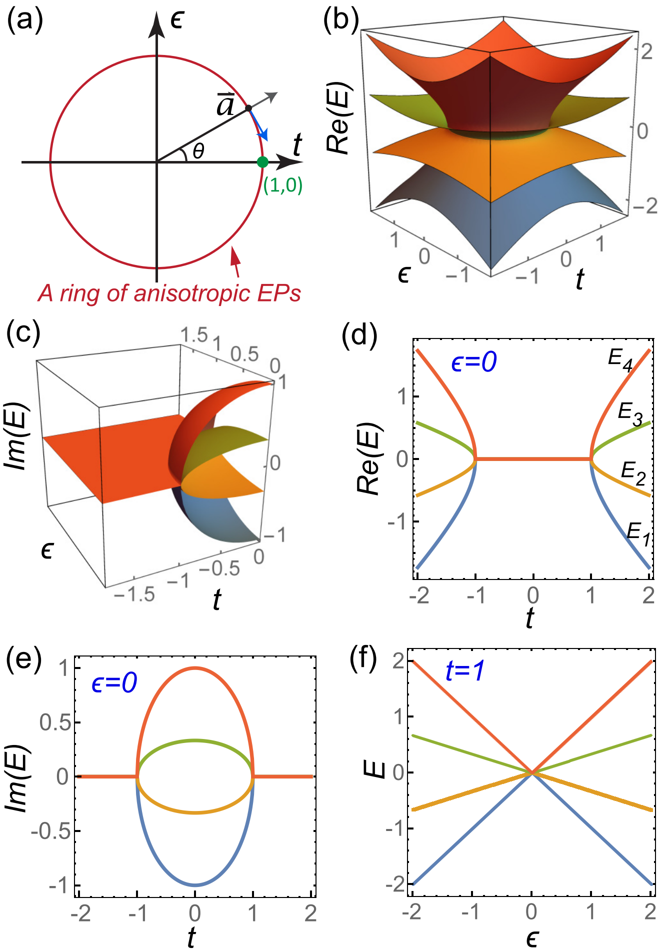

FIG. 3. (a) A ring of anisotropic exceptional points described by $t^2+\epsilon^2=1$ for the Hamiltonian H^j. (b) The real part and (c) imaginary part of the eigenenergy surfaces for the case of $j=3/2$. (d) and (e) The real and imaginary parts of the eigenenergies as functions of t when fixing $\epsilon=0$. (f) The eigenenergies as functions of ϵ when fixing $t=1$.

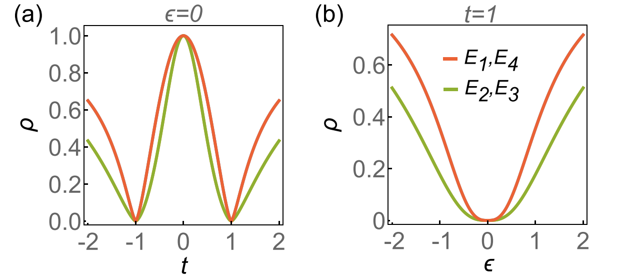

FIG. 4. The phase rigidity for spin-3/2 Hamiltonian $H^{3/2}$ along two trajectories: (a) the radial trajectory $\epsilon=0$, and (b) the tangential trajectory $t=1$ with respect to the EP at $(t,\epsilon)=(1,0)$ on the ring shown in Fig. 3(a). The critical exponents near the EP at $(t,\epsilon)=(1,0)$ are (a) 3/2 and (b) 3, respectively.

in Figs. 3(b) and 3(c), where the anisotropic EP ring can be seen. Only a quarter of the ring is shown in Fig. 3(c). At point $\vec{a}$, all four eigenvalues of $H^{3/2}$ coalesce to $E=0$, and all four right (left) eigenvectors also coalesce [75]. The square-root behaviors occurring at $(t,\epsilon)=(\pm1,0)$ are shown in Figs. 3(d) and 3(e) when fixing $\epsilon=0$ and varying t. The linear crossing occurring at $(1,0)$ is shown in Fig. 3(f). We note that the linear crossing point here is intrinsically different from the diabolic-pointlike degeneracies found in Hermitian systems [78], because non-Hermiticity makes all N eigenstates coalesce into a single one at the EP, where the phase rigidity vanishes. The phase rigidity is found to be $\rho_\perp\propto|\delta|^{3/2}$ at $\vec{a}+\vec{d}_\perp$ and $\rho_\parallel\propto|\delta|^3$ at $\vec{a}+\vec{d}_\parallel$ [75]. Thus, similar to the eigenvalues, phase rigidity also exhibits anisotropy: although the phase rigidity for both trajectories follows a power-law behavior, they vanish at the anisotropic EP with different critical exponents. The phase rigidity for all states is calculated numerically and shown in Fig. 4: Fig. 4(a) corresponds to Figs. 3(d) and 3(e), and Fig. 4(b) is for Fig. 3(f). We emphasize that the anisotropic behavior holds for H^j of arbitrary dimensions. It can be shown rigorously that the phase rigidity behaves as $\rho_\perp\propto|\delta|^{(N-1)/2}$ and $\rho_\parallel\propto|\delta|^{N-1}$ in the radial and tangential directions, respectively, for any anisotropic EP on the ring [75].

It is interesting to point out that $H^j=ts_x^j-is_y^j+\epsilon s_z^j$ can be unitarily transformed to a formally PT-symmetric Hamiltonian with complex hoppings $H_{PT}^j=ts_x^j+\epsilon s_y^j+is_z^j$ by a $\pi/2$ rotation of the spin space around the x axis. For the simplest case of $j=1/2$, with the replacements $t\to k_x$ and $\epsilon\to k_y$, we end up with $H_{PT}^{1/2}=k_xs_x^{1/2}+k_ys_y^{1/2}+is_z^{1/2}$, which is similar to the effective Hamiltonian of a non-Hermitian photonic crystal slab (periodic in the xy plane) under symmetric gain and loss, where the ring of order-2 EPs is described by $k_x^2+k_y^2=1$. Such an exceptional ring was experimentally observed [32,45]. EP rings of order 2 can spawn from Weyl points [54,58] when the system becomes non-Hermitian, which is also a special case of our considerations. Apart from engineering hoppings and on-site energies in the coupled ring resonator model according to Eq. (5), the ring of EPs of higher order can be realized by introducing non-Hermiticity into lattice systems bearing general spins, such as stacked triangular lattice layers [79].

Discussion. We note that the relation $t_1t_2=-\alpha\epsilon^2$ for order-2 EP trajectories of $H_{\rm dis}$ is very general. Instead of forming elliptical trajectories, the EPs will form a cluster of parabolas that can be parametrized by $t_1=-\frac{\alpha}{t_2}\epsilon^2$ in the (t_1,ϵ) plane if we set t_2 [rather than the hopping difference $\gamma=(t_2-t_1)/2$] as a constant.

We emphasize that the phase rigidity near the EP $(t,\epsilon)=(1,0)$ in the Hamiltonian in Eq. (1) behaves as $\rho\propto|\delta|^{(N-1)/2}$ with $\delta=t-1$ when $\epsilon=0$. The exponent $\nu=(N-1)/2$ in our class is very different from $\nu=\frac{(N-1)}{N}$ found in the conventional order-N EP. The difference arises from the different singular behaviors in the eigenvalues near an EP. For Eq. (1), different pairs of the square-root singularities, i.e., $\pm A_i\sqrt{\delta}$, share the same EP, whereas for a conventional order-N EP, all the N eigenvalues come from one single Nth root singularity, i.e., $\sqrt[N]{\delta}$. Only two adiabatic encirclings are needed to bring a state back to itself in our case, instead of N encirclings needed for a usual order-N EP. Also, a large value of exponent $\nu=(N-1)/2$, compared with $\nu=\frac{(N-1)}{N}<1$, makes the characteristics of an EP, e.g., orthogonality of left and right eigenvectors, much easier to achieve.

We note that the conventional order-N EP can also be easily obtained from Eq. (1) by setting $t_1 = 0$, i.e., $t = 1$, and at the same time inserting a small perturbation δ to the upper-right corner entry of the Hamiltonian H. In fact, all the intermediate exponents between $\frac{(N-1)}{2}$ and $\frac{(N-1)}{N}$, namely, $\frac{N-1}{n}$ with $2 < n < N$, can also be obtained by introducing perturbations to other entries [75].

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