Rapid Communications   Editors' Suggestion

Discrete time quasicrystals

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Between space crystals and amorphous materials there exists a third class of aperiodic structures which lack translational symmetry but reveal long-range order. They are dubbed quasicrystals and their formation, similar to the formation of space crystals, is related to spontaneous breaking of translational symmetry of underlying Hamiltonians. Here, we investigate spontaneous emergence of quasicrystals in periodically driven systems. We consider a quantum many-body system which is driven by a harmonically oscillating force and show that interactions between particles result in spontaneous self-reorganization of the motion of a quantum many-body system and in the formation of a quasicrystal structure in time.

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Quasicrystals are related to spatial structures which cannot be reproduced by translation of an elementary cell but reveal long-range order [1–3]. Quasicrystals are a subject of research in solid state physics but also in optics [4–7] and ultracold atomic gases [8].

Recently research of crystalline structures has migrated to the time domain [9] (for phase-space crystals see Refs. [10–13]). Indeed, a quantum many-body system can spontaneously self-organize its motion and start moving periodically forming a crystalline structure in the time domain. While the first idea of such time crystals turned out to be impossible for the realization [14–17] another type of spontaneous formation of crystalline structures in time was proposed. These are the so-called discrete time crystals that are periodically driven quantum many-body systems which break spontaneously discrete time translation symmetry of Hamiltonians and start moving with a period different from the driving period [18–21]. Discrete time crystals have already been realized in laboratories [22–26] and they draw considerable attention in the literature [27–47] (see also Refs. [48–53] for a classical version of time crystals). In the field of time crystals, quasicrystal structures have been investigated in classical systems [54], quantum systems [40,55,56], and in an experiment on magnon condensation [57]. In Refs. [58,59] quasicrystal response of systems which are driven quasiperiodically in time was demonstrated. Quasiperiodic response of a periodically driven many-body system was analyzed in Ref. [60] but with no spontaneous time translational symmetry breaking process involved. In the present Rapid Communication we analyze how a quasicrystal structure forms due to spontaneous breaking of discrete time translation symmetry of a many-body time-periodic Hamiltonian.

One-dimensional (1D) quasicrystal sequence can be generated by a cut of a square lattice with the help of a line whose gradient is an irrational number [61–63]. For the Fibonacci quasicrystal the gradient is the golden ratio and the successive cuts of vertical and horizontal lines of the square lattice produce a sequence *LRLLRLRL*... of two elementary cells which we denote by *L* and *R*, see Fig. 1. The sequence corresponds to a quasicrystal structure where there is no translation symmetry but two elementary cells are not distributed randomly so that the sequence reveals long-range order [1]. A finite fragment of the Fibonacci quasicrystal sequence can be obtained by cutting the square lattice with a line whose gradient is a rational number that approximates the golden ratio, see Fig. 1. In the following we show how any finite fragment of the Fibonacci quasicrystal structure can spontaneously emerge in the time evolution of a periodically driven many-body system if interactions between particles are sufficiently strong.

We focus on ultracold atoms bouncing between two orthogonal harmonically oscillating mirrors in a 2D model. Such a system can be realized experimentally [64] (for the stationary mirror experiments see Refs. [65–72]). A single atom bouncing between the mirrors is described, in the frame oscillating with the mirrors [73,74], by the Hamiltonian [75]

$$H = \frac{p_x^2 + p_y^2}{2} + x + y + \lambda_x x\cos(\omega t + \Delta\phi) + \lambda_y y\cos(\omega t), \tag{1}$$

where ω is the frequency of the mirrors' oscillations, $\Delta\phi$ the relative phase, and $\lambda_{x,y}$ the amplitudes of the oscillations. The mirrors are located at $x = 0$ and at $y = 0$ and the gravitational force $\vec{F}_g$ points in the $-(\mathbf{x} + \mathbf{y})$ direction, see inset of Fig. 1. We assume that in the many-body case, N bosons are bouncing between the mirrors and interact via Dirac-delta potential $g_0\delta(\mathbf{r})$ [76]. Such contact interactions are determined by the s-wave scattering length of atoms which is assumed to be negative $g_0 < 0$. The system is periodically driven, thus, we may look for a kind of stationary states which evolve periodically in time. They are eigenstates of the Floquet Hamiltonian $\hat{\mathcal{H}}(t) = \hat{H} - i\partial_t$, where $\hat{H}$ is a many-body version of (1) with the contact interactions between

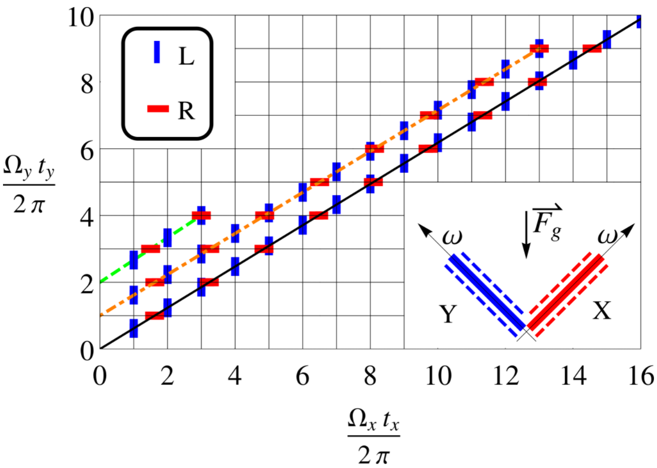

FIG. 1. Generation of the one-dimensional Fibonacci quasicrystal. The solid black line cuts the square lattice. The tangent of the angle that the line forms with the vertical axis is equal to the golden ratio. Consecutive cuts of the line with the vertical (L) and horizontal (R) lines of the lattice form the Fibonacci quasicrystal sequence, $LRLLRLRL\ldots$. Dashed green and dotted-dashed orange lines correspond to rational approximation of the golden ratio, 3/2 and 13/8, respectively. In the case of ultracold atoms bouncing between two mirrors which oscillate with frequency ω (see the schematic plot in the inset where $\vec{F}_g$ denotes the gravitational force), the green and orange lines are related to spontaneous formation of finite fragments of the Fibonacci quasicrystal in the time domain with $\Omega_x/\Omega_y = s_y/s_x = 3/2$ and 13/8, respectively. That is, bounces of atoms off the left L and right R mirrors form a sequence of events that reproduces a fragment of the Fibonacci quasicrystal of length $s_x + s_y$. Parameters Ω_x and Ω_y are frequencies of bouncing of atoms off the left and right mirrors, respectively, see text. The axes of the main figure can be considered as two independent time axes [54], $t_{x,y}$, related to periodic motion along the x and y directions.

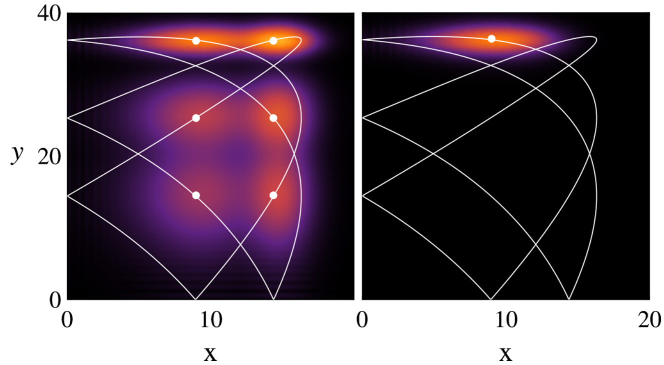

FIG. 2. Density of atoms bouncing between two orthogonal oscillating mirrors at $t = 2\pi/3\omega$. The left (L) mirror is located at $x = 0$ and the right (R) mirror at $y = 0$ and the gravitational force $\vec{F}_g$ points in the $-(\mathbf{x}+\mathbf{y})$ direction, see inset of Fig. 1. Left panel is related to the symmetry preserving state which evolves periodically with the driving period $2\pi/\omega$—the left and right mirrors are visited by atoms alternately: $LRLRLR$. The presented density consists of $s_x s_y$ localized Wannier-like wavepackets ($s_x = 2$, $s_y = 3$). The trajectory the Wannier wavepackets are moving along is drawn in the panels. Right panel corresponds to a symmetry broken state where interactions between atoms result in spontaneous breaking of discrete time translation symmetry of the Hamiltonian and emergence of a quasicrystal structure in time. Atoms are visiting the left and right mirrors in an order that matches the sequence $LRLLR$ i.e. a finite fragment of the Fibonacci quasicristal is reproduced because the golden ratio gradient of the line in Fig. 1 is approximated by the rational number $\Omega_x/\Omega_y = s_y/s_x = 3/2$. The parameters of the system are: $\lambda_x = 0.094$, $\lambda_y = 0.030$, $\omega = 1.1$, $\Delta\phi = 2\pi/3$, $g_0 N = 0$ (left) and $g_0 N = -0.022$ (right). The latter results in $U_{\mathbf{ii}}N/J = -81$, with $J = 4.8 \times 10^{-6}$, in the Hamiltonian (2) that describes an effective $s_x \times s_y$ lattice. The results are obtained within the quantum secular approach [79].

particles included, see Ref. [73]. The corresponding eigenvalues are called quasienergies of the system [75,77]. The discrete time translation symmetry of the time-periodic Hamiltonian, $\hat{\mathcal{H}}(t + 2\pi/\omega) = \hat{\mathcal{H}}(t)$, implies that all Floquet eigenstates must evolve with the driving period $2\pi/\omega$. In the following we show that in the limit when the number of particles $N \to \infty$ but $g_0 N = \text{const.}$ [78], there are subspaces of the Hilbert space of the system where low-lying quasienergy eigenstates are fragile because they form macroscopic superposition. Consequently even an infinitesimally weak perturbation, e.g., a measurement of a position of one atom, is sufficient to induce collapse of the many-body state to one of the superimposed states. It results in breaking of the discrete time translation symmetry of the Hamiltonian [18]. Interestingly an evolving symmetry broken state can reveal a sequence of events (bounces of atoms off the left L and right R mirrors) which forms a finite fragment of the Fibonacci quasicrystal in time.

Let us start with the single-particle problem (1) which consists of the independent motion along x and y directions. We are interested in a resonant driving of the system, i.e., the frequencies Ω_x and Ω_y of the unperturbed particle motion along the x and y directions fulfill $s_x\Omega_x = \omega$ and $s_y\Omega_y = \omega$ with integer s_x and s_y. The description of a resonantly driven particle can be reduced to an effective tight-binding Hamiltonian [28,73,75,80,81]. When we switch from a single particle to many bosons resonantly driven, the single-particle tight-binding Hamiltonian becomes the Bose-Hubbard Hamiltonian,

$$\hat{\mathcal{H}}_{\text{eff}} = -\frac{1}{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle} J_{\mathbf{ij}}\, \hat{a}_{\mathbf{i}}^{\dagger}\hat{a}_{\mathbf{j}} + \frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} U_{\mathbf{ij}}\, \hat{a}_{\mathbf{i}}^{\dagger}\hat{a}_{\mathbf{j}}^{\dagger}\hat{a}_{\mathbf{j}}\hat{a}_{\mathbf{i}}, \tag{2}$$

which is the many-body Floquet Hamiltonian written in a basis of time-periodic functions $W_{\mathbf{i}=(i_x,i_y)}(\mathbf{r},t) = w_{i_x}(x,t)w_{i_y}(y,t)$ which are localized wave packets propagating along the classical resonant orbit with the period $T = s_x s_y 2\pi/\omega$ and which play a role of Wannier functions known in condensed matter physics [82], see Fig. 2. There are $s_x s_y$ Wannier functions $W_{\mathbf{i}}$ which are products of localized wave packets $w_{i_x}(x,t)$ and $w_{i_y}(y,t)$ moving along the x and y directions with the periods $2\pi/\Omega_x$ and $2\pi/\Omega_y$, respectively. In (2), $\hat{a}_{\mathbf{i}}$'s are the standard bosonic annihilation operators, the nearest neighbor tunneling amplitudes $J_{\mathbf{ij}} = -(2/T)\int_0^T dt \int d^2\mathbf{r} W_{\mathbf{i}}^*(\mathbf{r},t)[H - i\partial_t]W_{\mathbf{j}}(\mathbf{r},t)$, and the coefficients of the effective interactions $U_{\mathbf{ij}} = (2/T)\int_0^T dt \int d^2\mathbf{r}\, g_0 |W_{\mathbf{i}}|^2|W_{\mathbf{j}}|^2$ for $\mathbf{i} \neq \mathbf{j}$ and similar $U_{\mathbf{ii}}$ but by factor two smaller [28,40,73]. In the present Rapid Communication we choose the amplitudes of the mirrors' oscillations,

λ_x and λ_y, so that the resulting amplitudes for nearest neighbor tunnelings along the x and y directions are the same, $J \equiv J_{\mathbf{ij}}$. Typically, the coefficient for the on-site interactions $U_{\mathbf{ii}}$ is at least an order of magnitude larger than $U_{\mathbf{ij}}$ for long-range interactions ($\mathbf{i} \neq \mathbf{j}$).

To conclude this part, the description of the resonantly driven many-body system is reduced, in the time-periodic basis $W_{\mathbf{i}}(\mathbf{r}, t)$, to the Bose-Hubbard Hamiltonian (2) [73,83]. The resonant driving is related to nonlinear classical resonances where a particle cannot absorb an unlimited amount of energy because transfer of the energy changes a period of motion of the system, a particle goes out of the resonance and the transfer stops [73].

For negligible interactions between particles the ground state of $\hat{\mathcal{H}}_{\rm eff}$ is a Bose-Einstein condensate $\Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = \prod_{j=1}^N \psi(\mathbf{r}_j, t)$, i.e., all atoms occupy a condensate wave function $\psi(\mathbf{r}, t) \propto \sum_{\mathbf{i}} W_{\mathbf{i}}(\mathbf{r}, t)$ which evolves with the driving period $2\pi/\omega$. Indeed, despite the fact that each $W_{\mathbf{i}}$ evolves with the period $T = s_x s_y 2\pi/\omega$, after each period $2\pi/\omega$, the Wannier wave functions $W_{\mathbf{i}}$ exchange their positions so that the condensate wave function $\psi(\mathbf{r}, t)$ propagates with the driving period, see Fig. 2. When the interactions between atoms are attractive and sufficiently strong it is energetically favorable to group all atoms in a single localized wave packet $W_{\mathbf{i}}(\mathbf{r}, t)$ [18]. Then, we expect the ground state of $\hat{\mathcal{H}}_{\rm eff}$ to be of the form $\Psi_0 = \prod_{j=1}^N W_{\mathbf{i}}(\mathbf{r}_j, t)$ where $\mathbf{i}$ can be arbitrary. However, such a state cannot be a Floquet eigenstate of the system because it evolves with the period $T = s_x s_y 2\pi/\omega$ while the discrete time translation symmetry of the Hamiltonian requires that all Floquet eigenstates must evolve with the period of the driving $2\pi/\omega$. In order to reconcile the energy and symmetry requirements, the ground state of $\hat{\mathcal{H}}_{\rm eff}$ takes the form $\Psi_0 \propto \sum_{\mathbf{i}} \prod_{j=1}^N W_{\mathbf{i}}(\mathbf{r}_j, t)$ which is a macroscopic superposition of Bose-Einstein condensates [84,85]. However, such a macroscopic superposition is extremely fragile and it is sufficient, e.g., to measure the position of one atom and the ground state collapses to one of the Bose-Einstein condensates which form the macroscopic superposition, $\Psi_0 \to \Psi \approx \prod_{j=1}^N W_{\mathbf{i}}(\mathbf{r}_j, t)$ [85,86]—which $W_{\mathbf{i}}$ is chosen depends on a result of the measurement. In the limit when $N \to \infty$ but $U_{\mathbf{ii}} N = \text{const.}$, the latter state is robust and evolves with the period $T = s_x s_y 2\pi/\omega$ and thus breaks time translation symmetry of the many-body Hamiltonian [18]. The described scenario is an example of a process of spontaneous breaking of time translation symmetry in the quantum many-body system. Similar spontaneous symmetry breaking phenomenon is not present in [60] because Floquet states are related to single Fock states in the position representation.

In order to describe the system we apply the mean-field approach [18,44,73,87]. The mean-field approximation is valid because the ground state of (2) for negligible interactions and also symmetry broken states, $\Psi \approx \prod_{j=1}^N W_{\mathbf{i}}(\mathbf{r}_j, t)$, in the regime of the quasicrystal formation are Bose-Einstein condensates. The mean-field energy of the system per particle reads $E = -(J/2) \sum_{\langle \mathbf{i},\mathbf{j} \rangle} a_{\mathbf{i}}^* a_{\mathbf{j}} + (N/2) \sum_{\mathbf{ij}} U_{\mathbf{ij}} |a_{\mathbf{i}}|^2 |a_{\mathbf{j}}|^2$ [44,73] and we are looking for a condensate wave function $\psi(\mathbf{r}, t) = \sum_{\mathbf{i}} a_{\mathbf{i}} W_{\mathbf{i}}(\mathbf{r}, t)$ which minimizes E [78,87]. In the left panel of Fig. 2 we show such a wave function $\psi(\mathbf{r}, t)$ obtained for negligible interactions and for $\Omega_x = \omega/2$ and $\Omega_y = \omega/3$ (i.e., $s_x = 2$, $s_y = 3$). The wave function $\psi(\mathbf{r}, t)$ is a uniform superposition of $s_x s_y = 6$ localized Wannier wave packets, it evolves with the period $2\pi/\omega$ and describes atoms bouncing alternately off the left (L) and right (R) mirrors. If we plot probabilities for the measurement of atoms close to the left mirror, $\rho_L(t) = \int dy |\psi(x \approx 0, y, t)|^2$, and close to the right mirror, $\rho_R(t) = \int dx |\psi(x, y \approx 0, t)|^2$, we can see that maximal values of $\rho_{L,R}(t)$ appear alternately and form a periodic sequence of events $LRLR\ldots$, see Fig. 3. However, if the interactions are sufficiently strong, i.e., $U_{\mathbf{ii}} N/J \lesssim -6.5$, the system chooses spontaneously motion with the period $T = s_x s_y 2\pi/\omega$. That is, the mean-field approach shows that the ground state energy is degenerate and the corresponding wave functions are not uniform superposition of $W_{\mathbf{i}}$. The system prepared in a lowest energy mean-field state breaks discrete time translation symmetry of the many-body Hamiltonian because it evolves with the period different from the driving period. For $U_{\mathbf{ii}} N/J \lesssim -25$ the symmetry broken degenerate ground states reduce to $\psi(\mathbf{r}, t) \approx W_{\mathbf{i}}(r, t)$ with accuracy better than 99%—which $W_{\mathbf{i}}$ is chosen by the system is determined in a spontaneous symmetry breaking process. In Fig. 2 we show an example of such a ground state wave function $\psi(\mathbf{r}, t)$ where a single localized wave packet bouncing between the mirrors is visible. The corresponding probabilities $\rho_{L,R}(t)$ form a sequence of events $LRLLR$, whose length is $s_x + s_y = 5$, which is repeated with the period T, see Fig. 3. The sequence is a fragment of the Fibonacci quasicrystal. The time quasicrystal states predicted by the mean-field approach lives forever. The predictions are valid in the limit when $N \to \infty$ but $g_0 N = \text{const}$ because then the corresponding symmetry preserving eigenstates of the quantum many-body model (2) are degenerate and their superpositions, that form the symmetry-broken states, do not decay [18].

It now becomes clear how to realize conditions where any finite fragment of the Fibonacci quasicrystal emerges due to spontaneous breaking of discrete time translation symmetry of the Hamiltonian: (i) One has to choose a rational number s_y/s_x which approximates the golden ratio and reproduces a given fragment of the Fibonacci quasicrystal sequence when it is taken as the gradient of the line in Fig. 1. (ii) Then, we know which resonant subspace of the periodically driven many-body system is able to realize such a quasicrystal, i.e., the subspace corresponding to the frequencies of unperturbed single-particle motion $\Omega_x = \omega/s_x$ and $\Omega_y = \omega/s_y$. (iii) If the many-body system is prepared in a low-lying eigenstate within this subspace, then either atoms are bouncing off the left and right mirrors in the alternate way (if the interactions are negligible) or the bounces on the mirrors form a sequence of events that reproduces a finite fragment of the Fibonacci quasicrystal (if the interactions are sufficiently strong). In the right panels of Fig. 3 we illustrate these two situations for $s_x = 8$ and $s_y = 13$. In the symmetry preserving case, the probabilities for detection atoms close to the left and right mirrors, $\rho_{L,R}(t)$, show a periodic sequence of maxima $LRLR\ldots$. However, when the attractive interactions are sufficiently strong, the discrete time translation symmetry is spontaneously broken and the Fibonacci quasicrystal $LRLLRLRL\ldots$ is formed [54]. We would like to stress that the quasicrystal structure formed

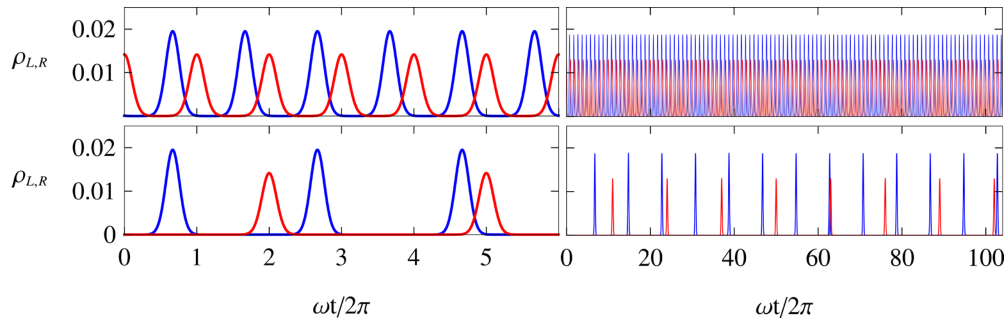

FIG. 3. Scaled probabilities for the detection of atoms close to the left mirror $\rho_L(t)$ (blue lines) and close to the right mirror $\rho_R(t)$ (red lines), where $\rho_L(t) = \int dy|\psi(x \approx 0, y, t)|^2$ and there is an analogous expression for $\rho_R(t)$. Top panels are related to symmetry preserving states while bottom panels to states where the discrete time translation symmetry of the Hamiltonian is spontaneously broken. Left panels correspond to $\Omega_x/\Omega_y = s_y/s_x = 3/2$ while in the right panels such ratios are equal 13/8. Symmetry preserving states form periodic sequences of the elementary cells L and R associated with the alternate appearance of maxima of $\rho_L(t)$ and $\rho_R(t)$. In the symmetry broken case, bounces of atoms off the left and right mirrors form a sequence of events that reproduces a finite fragment (of length $s_x + s_y$) of the Fibonacci quasicrystal which is repeated in the time evolution of the system with the period $T = s_x s_y 2\pi/\omega$. The results shown in the left panels correspond to the same parameters as in Fig. 2, while in the right panels: $\lambda_x = 0.087$, $\lambda_y = 0.026$, $\omega = 1.77$, $\Delta\phi = \pi/2$, $s_y = 13$, $s_x = 8$, $g_0 N = 0$ (top right panel) and $g_0 N = -0.029$ (bottom right panel). The latter results in $U_{\mathbf{ii}}N/J = -80$ and $J = 2.3 \times 10^{-6}$ in the Bose-Hubbard Hamiltonian that describes an effective $s_x \times s_y$ lattice.

by the bouncing atoms is related to the sequence of bounces not to the sequence of time intervals between the bounces—the latter can be different, see Fig. 3. In the experiment, the time intervals which are very small can be disrupted due to imperfections of the motion of the mirrors which can result in defects in the Fibonacci quasicrystal. Long time stability of our phenomenon resulting from the coupling of the system to the subspace complementary to the resonant subspace requires further investigation but we expect that the considered quasicrystal is a prethermal state.

To conclude, quasicrystal structures can emerge in the time domain due to spontaneous breaking of discrete time translation symmetry of the time-periodic many-body Hamiltonian. They can be realized in ultracold atoms bouncing between oscillating atom mirrors if atoms are loaded to a resonant classical orbit. The latter can be done if an atomic Bose-Einstein condensate is prepared in a trap located at a classical turning point of a resonant trajectory and afterwards the trapping potential is turned off at a proper moment of time [44]—the mirrors can be realized by two blue-detuned repulsive light sheets formed by focusing laser beams with cylindrical lenses. It results in a quantum state where all atoms occupy a single localized Wannier-like wave packet that evolves along a resonant orbit. For sufficiently strong attractive interactions between atoms, the localized atomic wave packet will perform evolution with a quasicrystal structure in time and will not decay. In contrast, for negligible interactions, atoms will tunnel to other localized wave packets evolving along the orbit which indicates decay of the quasicrystal.

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[73] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.99.220303 for the detailed description of the resonant motion of atoms bouncing on oscillating atom mirrors and for the description of the approach used in the analysis of the spontaneous emergence of quasicrystals in time. A short discussion of experimental aspects and stability of resonantly driven systems is also included.
[74] In order to switch from the laboratory frame (where one mirror oscillates like $-\frac{\lambda_x}{\omega^2}\cos(\omega t+\Delta\phi)$ along the x direction and the other like $-\frac{\lambda_y}{\omega^2}\cos\omega t$ along the y direction) to the coordinate frame where the mirrors do not move, the unitary transformation $U_y = e^{iy\frac{\lambda_y}{\omega}\sin\omega t}e^{ip_y\frac{\lambda_y}{\omega^2}\cos\omega t}$, and a similar one for the motion along x, has been applied. We use the gravitational units but assume that the gravitational acceleration is given by $g/\sqrt{2}$.

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