Rapid Communications Editors' Suggestion

Exciton condensation in quantum Hall bilayers at total filling $\nu_T = 5$

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We study the coupled quantum Hall bilayers each at half-filled first excited Landau levels by varying the layer distance. Based on numerical exact diagonalization on a torus, we identify two distinct phases separated by a critical layer distance d_c. From d_c to an infinite layer distance, the topological phase is smoothly connected to a direct tensor product of two Moore-Read states, while interlayer coherence emerges at $d < d_c$ characterized by xy easy-plane ferromagnetic energy spectra, gapless pseudospin excitations, long-range current-current correlations, and finite exciton superfluid stiffness, corresponding to the exciton superfluid state. More interestingly, the results of the ground-state fidelity, the evolution of the energy spectra, and the superfluid stiffness indicate a possible continuous transition. Theoretically, it can be interpreted as a topological phase transition which simultaneously changes the topology of the ground state and breaks symmetry, providing an interesting example of transitions beyond the Landau paradigm.

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Introduction. Quantum Hall bilayers [1,2], which can be realized in quantum wells [3,4] or bilayer graphene [5–7], have triggered substantial interest in pursuing exotic phenomena such as the Bose-Einstein condensation of excitons [8,9] and anyonic statistics [10–15]. The possible emerging non-Abelian physics and quantum phase transitions remain not well understood [16–18].

In particular, bilayers with half-filled lowest Landau levels (LLs) for each layer have attracted great interest from both experimental measurements [9,19–23] and theoretical investigations [24–49]. The exciton superfluid phase (or Halperin "111 state" [2,50]) was first established experimentally at a layer distance comparable to the magnetic length [9] based on a zero-bias interlayer tunneling conductance [51] and a vanishing Hall counterflow resistance [22,52]. Other phases such as a composite Fermi liquid (CFL) at larger distances [53] and a novel intermediate phase [28–30] have also been extensively investigated. In addition, the nature of quantum phase transitions among various phases is still controversial. Inspired by the rich physics of these $\nu_T = 1$ bilayers with half-filled lowest LLs in each layer, a natural question arises about the quantum phase diagram for electronic systems with fully filled lowest LLs and half-filled first excited LLs, corresponding to bilayers with a total filling $\nu_T = 5$. Each decoupled layer with a filling $\nu = 5/2$ is believed to be a Moore-Read (MR) state [11,54,55]. When the layer distance goes to zero, an interlayer coherent state is theoretically expected, although there is no experimental study presented along this line. By tuning the layer distance, the nature of the possible intermediate phase and the quantum phase transition remain unclear, which motivates our present work.

Previous theoretical studies have not reached a consistent conclusion on this problem. The calculations based on the Hartree-Fock approximation claim a transition from a 111 state to a charge ordered state [40,56], while variational and exact diagonalization (ED) calculations on the sphere geometry found a bilayer phase coherent state at a small layer distance and two uncoupled 5/2 states at large layer separations by Shi *et al.* [57]. Nevertheless, unbiased exact simulations for quantum states at intermediate layer distances and a quantum phase transition for torus geometry are still absent. Different from the sphere geometry, there is no orbital number shift on the torus and competing states with the same filling factor can be compared on an equal footing [58].

In the present Rapid Communication, we use ED to calculate systems with up to 18 electrons on a torus. Based on the energy spectra, pseudospin gap, exciton superfluid stiffness, current-current correlations, the Berry curvature, as well as drag Hall conductance, we identify a direct phase transition at d_c between the exciton superfluid phase with interlayer coherence and a phase with strong intralayer correlations—the latter can be smoothly connected to two decoupled copies of the MR state. Here, the finding of d_c is consistent with a previous variational calculation [57]. Moreover, the calculation of fidelity, the exciton superfluid stiffness, the evolution of the energy spectrum, and the ground-state energy derivatives indicate the transition is continuous, which is beyond the Landau paradigm [16–18,59,60]. Based on an analysis of symmetries and topological orders, we propose theoretical interpretations of such a transition as exciton condensation which simultaneously breaks $U(1) \times U(1)$ symmetry and changes the topology. The exciton condensation leads to a $C = 2$ topologically ordered state in Kitaev's notation [61], which is consistent with the 111 state.

Model and method. We consider $\nu_T = 5/2 + 5/2$ bilayer electronic systems subject to a perpendicular magnetic field. We neglect the width of these two identical layers and put them on a torus spanned by vectors $\mathbf{L_x}$ and $\mathbf{L_y}$. The orbital number (or flux number) in each layer N_ϕ is determined by the area of the torus, i.e., $|\mathbf{L_x} \times \mathbf{L_y}| = 2\pi N_\phi$. In the absence of interlayer tunneling, this system with fully polarized spins can be described by the projected Coulomb interaction, which reads

$$V = \frac{1}{N_\phi} \sum_{i<j,\alpha,\beta} \sum_{\mathbf{q},\mathbf{q}\neq\mathbf{0}} V_{\alpha\beta}(q) e^{-\frac{q^2}{2}} L_n^2\left[\frac{q^2}{2}\right] e^{i\mathbf{q}\cdot(\mathbf{R}_{\alpha,i}-\mathbf{R}_{\beta,j})}. \quad (1)$$

Here, $\alpha(\beta) = 1, 2$ denote two layers or, equivalently, two components of a pseudospin-1/2. $q = |\mathbf{q}| = \sqrt{q_x^2 + q_y^2}$, $V_{11}(q) = V_{22}(q) = e^2/(\varepsilon q)$, and $V_{12}(q) = V_{21}(q) = e^2/(\varepsilon q) \cdot e^{-qd}$ are the Fourier transformations of the intralayer and interlayer Coulomb interactions, respectively. d represents the distance between two layers in the unit of magnetic length l_B. $L_n(x)$ is the Laguerre polynomial with a Landau level index n and $\mathbf{R}_{\alpha,i}$ is the guiding center coordinate of the ith electron in layer α. Here, we consider rectangular unit cells with $L_x = L_y = L$ [62].

Energy spectra. Without interlayer tunneling, the bilayer system has separate conservations for the electron number in each layer, which allows us to label eigenstates by pseudospin S_z defined as $S_z \equiv (N^\uparrow - N^\downarrow)/2$, where $N^\uparrow$ and $N^\downarrow$ denote the number of electrons for the top and bottom layers, respectively. Then we can study the energy spectra by targeting different pseudospin sectors. Here, the energy shift $d \cdot S_z^2/N_\phi$ induced by an imbalance of charge in two layers [63] has been considered. When the layer distance goes to zero, as shown in Fig. 1(a), the lowest energies in each pseudospin S_z sector are exactly degenerate, indicating that we have not only conserved S_z but also full SU(2) symmetry. This

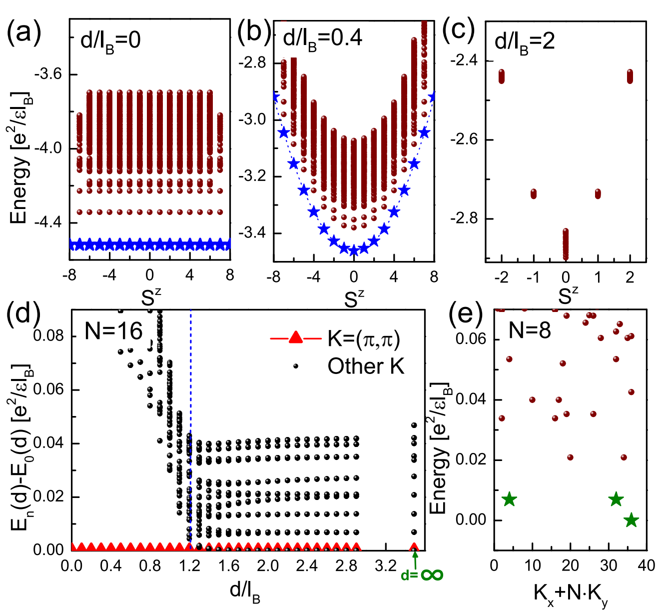

FIG. 1. The energy spectra of different pseudospin S_z sectors at layer distance (a) $d/l_B = 0$, (b) $d/l_B = 0.4$, and (c) $d/l_B = 2$. (d) Low-lying energy spectra as a function of layer distance d/l_B. Here, the total electron number$N = 16$ and each layer has an equal number of electrons. (e) The energy spectrum of a single-layer $N = 8$ system at $n = 1$ LL; the green stars highlight the topological sectors of the MR state.

spectrum is consistent with the exciton condensed 111 state, with spontaneous ferromagnetization which can be seen from the ground-state spin degeneracy. However, when the layer distance is finite but small enough, as shown in Fig. 1(b), our data show the nondegenerate ground state located in the $S_z = 0$ sector, and the low-energy excitations are pseudospin excitations among different S_z sectors, which can be fitted into $\Delta E = E(S_z) - E(S_z = 0) = \alpha S_z^2$. These facts indicate that the ground state is an xy easy-plane ferromagnet instead of an Ising ferromagnet, and the interlayer correlations dominate the low-energy physics for small d. Physically, an electron in one layer is bound to a hole in the other layer forming an exciton at $d = 0$, and then the bilayer system can be mapped into a monolayer at $\nu = 1$ for the first excited Landau level. When d is finite but smaller than a critical value, the difference between the interlayer and intralayer Coulomb interaction breaks the pseudospin invariance down to $U(1)$, leading to the xy easy-plane pseudospin ferromagnet as indicated in Figs. 1(a) and 1(b). However, for a larger layer distance $d = 2.0 l_B$, the lowest-energy excitations exist within the same pseudospin S_z sector [see Fig. 1(c)], indicating the low-lying excitations are dominated by intralayer correlations. These results indicate there are two distinct phases as the layer distance d is varied.

The flow of low-lying energies with d/l_B indeed indicates a direct transition at $d_c/l_B \approx 1.2$ from an exciton superfluid phase ($d < d_c$) to a phase with a distinct structure of the spectra ($d > d_c$) which can be smoothly connected to the two decoupled copies of the MR state at $d/l_B = +\infty$ [see Fig. 1(d)]. Figure 1(e) shows the energy spectrum of each decoupled layer with eight electrons, where the threefold degeneracy (in addition to the twofold center-of-mass degeneracy) in the momentum sectors $(K_x, K_y)/(2\pi/N) = (N/2, N/2), (0, N/2), (N/2, 0)$ occurs, supporting the idea that each decoupled layer is indeed in the MR state. When coupling two layers together, we identified a 36-fold near degeneracy of two copies of the MR state at the $d > d_c$ side.

Pseudospin excitations. From the energy spectra we identify a single-phase transition at $d_c/l_B \approx 1.2$ without energy level crossing. Below, we characterize the transition from the perspective of low-energy excitations. We directly calculate the pseudospin excitation gap, which measures the energy cost when flipping the pseudospin of one particle. The pseudospin gap is defined as $\Delta_{\text{ps}}(d) \equiv E_0(N_\uparrow, N_\downarrow, d) - E_0(N/2, N/2, d) + d \cdot S_z^2/N_\phi$ [64], where $N_\uparrow = N/2 + S_z$ and $N_\downarrow = N/2 - S_z$. As shown in Figs. 2(a) and 2(c), the finite-size scaling of $\Delta_{\text{ps}}(d)$ for $S_z = 1$ and $S_z = 2$ shows the excitation gap goes to zero in the thermodynamic limit for $d/l_B \lesssim 1.2$. However, for $d/l_B \gtrsim 1.2$, the $S_z = 1$ pseudospin excitation displays a significant even-odd effect determined by the electron number in each layer, as shown in Fig. 2(b). For systems with an even number of electrons in each layer, flipping a single pseudospin costs finite energy [see the inset of Fig. 2(b)] while the energy cost vanishes when the electron number in each layer is odd. This even-odd effect disappears in the $S_z = 2$ pseudospin gap [see Fig. 2(c)]. The distinct behavior of the $S_z = 1$ and $S_z = 2$ pseudospin gap is consistent with the picture of the existence of intralayer paired composite fermions. Furthermore, we will show below that interlayer coherence immediately

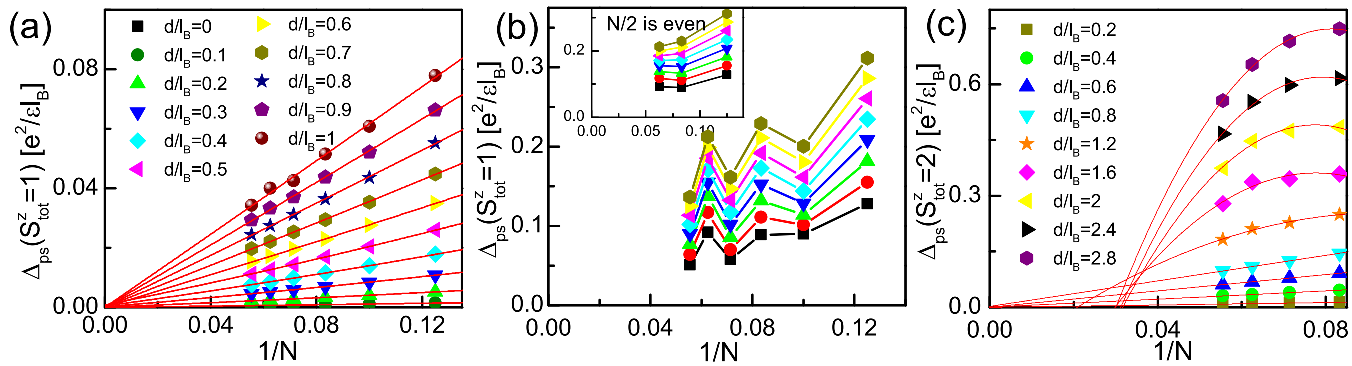

FIG. 2. (a)–(c) show the pseudospin excitation gap $\Delta_{\rm ps}$ for (a) $S_z = 1$ and $d < d_c$, (b) $S_z = 1$ and $d > d_c$, and (c) $S_z = 2$. The finite-size scaling of $\Delta_{\rm ps}$ using a parabolic function indicates the gapless nature at $d < d_c$ [(a) and (c)]. The inset of (b) shows $\Delta_{\rm ps}$ for systems with an even number of particles in each layer. The even-odd effect disappears at $d > d_c$ for (c) $S_z = 2$.

establishes in the gapless phase at $d \leqslant d_c$, leading to exciton superfluidity.

Exciton superfluidity. To study the exciton superfluidity, we calculate both the current-current correlations and superfluid stiffness. We define the interlayer current operator $J_m \equiv i(c^\dagger_{m\uparrow} c_{m\downarrow} - \text{H.c.})$ and probe the interlayer coherence by studying current-current correlations $\langle J_m J_n \rangle$, where $m, n = 1, \dots, N_\phi$ are orbital indices and the corresponding distance is $|n - m| L / N_\phi$. As shown in the inset of Fig. 3(a), $\langle J_m J_n \rangle$ decays very slowly and saturates to a finite value when $d < d_c$ while it becomes vanishingly small when $d > d_c$, which directly proves the existence of interlayer coherence in the 111 state. To keep track of such a property when tuning the layer distance d, we choose the value of $\langle J_m J_n \rangle$ at the largest distance $|n - m| = N_\phi/2$ and study its value versus d; as shown in Fig. 3(a), the interlayer coherence is softened with the increase of layer distances and finally disappears after the smooth transition at d_c.

In order to get the exciton superfluid stiffness ρ_s, we add twisted boundary phases $0 \leqslant \theta^\alpha_\lambda \leqslant 2\pi$ along the λ direction ($\lambda = x$ or y) in the layer α, and study the energy evolution. Physically imposing opposite boundary phases for two layers plays a similar role as the counterflow experiments, where the longitudinal counterflow conductivity indicates superfluidity. Figure 3(b) shows the energy flow of the two lowest states in the same momentum sector $(K_x, K_y) = (\pi, \pi)$ with twisted phases. The exciton superfluid stiffness ρ_s can be obtained by fitting the energy flows according to [26]

$$E(\theta_t)/|\mathbf{L_x} \times \mathbf{L_y}| = E(\theta_t = 0)/A + \tfrac{1}{2}\rho_s \theta_t^2 + O(\theta_t^4), \qquad (2)$$

where $E(\theta_t)$ is the ground-state energy with twisted (opposite) boundary phases θ_t between two layers. As shown in Fig. 3(b), we have a finite exciton superfluid stiffness when $d < d_c$, while $\rho_s = 0$ at the $d > d_c$ side due to the totally flat energy curve against the twisted phases, indicating the disappearance of superfluidity. The quantitative evolution of the superfluid stiffness $\rho_s > 0$ with the layer distance will be discussed later in Fig. 4(b) to address the precise nature of the quantum phase transition.

The interlayer correlations can also be detected by the drag Hall conductance, which can be calculated by integrating the Berry curvature $F(\theta^\alpha_x, \theta^\beta_y)$. Physically, a gapped state has a well-defined Berry curvature and thus a well-defined Chern number, while a gapless state has singularities in the Berry curvature due to the energy level crossing. As shown in Fig. 3(c) for the energy gap $E_1 - E_0$ as a function of twisted phases, one can only get a well-defined Berry curvature or Chern number at the $d > d_c$ side since there is always a finite gap between the ground state and the first excited state. When $d < d_c$, the gap closes near the twisted phase point $(2\pi, 2\pi)$, indicating singularities in the Berry curvature. At the $d > d_c$ side, we find the Berry curvature is nearly flat without any singularity (see the Supplemental Material) and its integral leads to a drag Hall conductance $\sigma^d_{xy} = 0$, indicating the

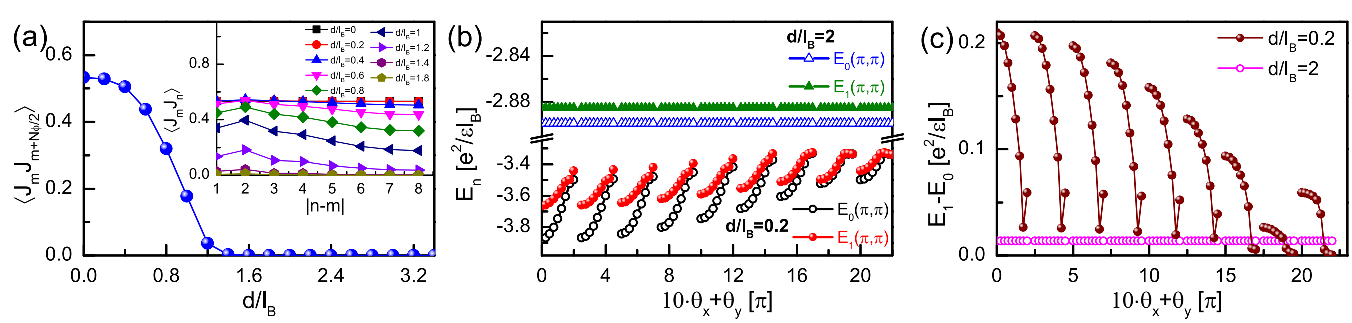

FIG. 3. (a) The values of current-current correlations $\langle J_m J_{m+N_\phi/2} \rangle$ vs layer distances d. The inset shows $\langle J_m J_n \rangle$ as a function of orbital distance. (b) The energy flow with twisted boundary phases for $d/l_B = 0.2$ and $d/l_B = 2$. (c) The energy gap $E_1 - E_0$ as a function of twisted phases for $d/l_B = 0.2$ and $d/l_B = 2$. Here, the systems have $N = 16$ electrons.

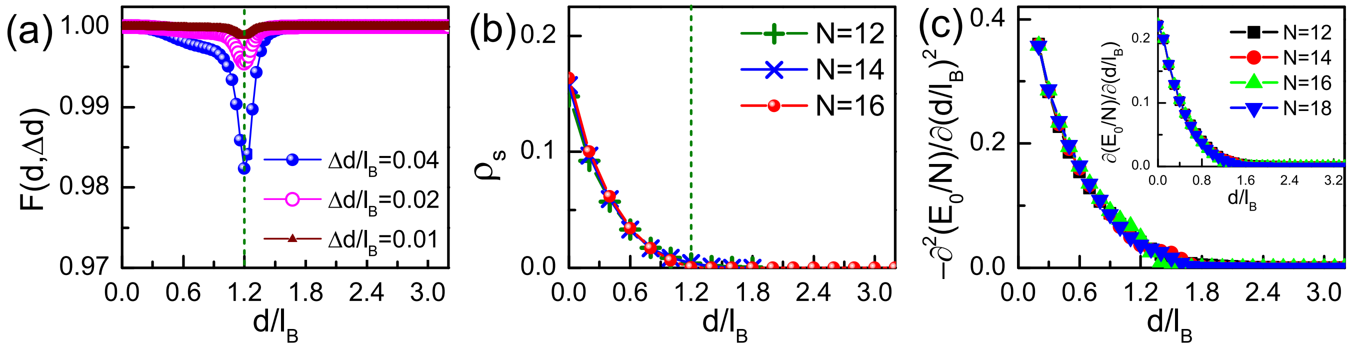

FIG. 4. (a) The fidelity of $N = 16$ systems with layer distances and different intervals of parameters Δd. (b) The exciton superfluid stiffness ρ_s as a function of layer distance d/l_B. (c) The first-order (inset) and second-order derivative curves of the ground-state energy E_0/N as a function of layer distance d/l_B.

Hall conductances are equal in both layer symmetric and antisymmetric channels [62].

Continuous phase transition. Since the exciton superfluid phase and two copies of the MR phase have different symmetries and topological orders, a direct continuous transition is beyond the Landau paradigm. From the energy spectra in Fig. 1(d), the level crossing is absent in the vicinity of the critical distance d_c, indicating a continuous transition. We further probe the nature of such a transition by calculating the ground-state fidelity, superfluid stiffness, as well as the ground-state energy derivatives. The fidelity is defined by the wave-function overlap between the ground state at $d - \Delta d$ and d, i.e., $F(d, \Delta d) = |\langle\Psi(d - \Delta d)|\Psi(d)\rangle|$, which has been shown to be a good indicator to distinguish a continuous transition from a first-order transition for both symmetry-breaking and topological phase transitions [71,72]. As shown in Fig. 4(a), we find the ground-state fidelity displays a single weak dip at the critical distance d_c instead of showing a sudden jump. In addition, as shown in Fig. 4(b), the exciton superfluid stiffness ρ_s is finite at $d < d_c$, but smoothly decreases with an increase of the layer distance, and becomes vanishingly small after the transition. Figure 4(c) shows the first-order and second-order derivatives of the ground-state energy, which are both smooth functions of layer distances. Thus the numerical evidence indicates the direct transition between these two phases might be continuous.

Field theory of transition and exciton condensation. Here, we provide a possible scenario of the observed transition. We consider the electron to be fractionalized into a boson and a fermion with an emergent $u(1)_i$ gauge field at each layer, i.e., $c_i = b_i\psi_i$, where $i =\uparrow, \downarrow$ denotes two layers. While ψ_i only carries $u(1)_i$ charge, b_i carries both emergent $u(1)_i$ and global $U(1)_i$ charge (corresponding to the charge conservation at each layer). To obtain a MR state at each layer, pairs of fermions form a $p + ip$ superconductor [11,54], while pairs of bosons form a $\nu = 1/8$ state called the $u(1)_8$ state [73–75]. The effective theory is

$$\mathcal{L} = \sum_{i=\uparrow,\downarrow}\left(\frac{8}{4\pi}\alpha_i d\alpha_i + \frac{2}{2\pi}(eA_i + a_i)d\alpha_i - \frac{2e^2}{4\pi}A_i dA_i + \Psi_i^\dagger[i\partial_0 - a_{i,0} + h_i(\vec{p} + \vec{a}_i)]\Psi_i\right), \tag{3}$$

where $a_{i,\mu}$ is the emergent gauge field from fractionalization, and $\alpha_{i,\mu}$ characterizes the $u(1)_8$ state at the ith layer. ada is shorthand notation of the Chern-Simons term $\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$ [76,77]. The first two terms correspond to the $u(1)_8$ state, and the third term characterizes the Hall response of the filled lowest LL. Integrating out the α_i field gives rise to the quantized Hall conductivity $\sigma_{xy} = \frac{5}{2}\frac{e^2}{h}$ for each layer. In the second line, $\Psi_i(p) = [\psi_i(p), \psi_i^\dagger(-p)]^T$ is the Nambu spinor, $h_i(\vec{p}) = (\frac{p_x^2+p_y^2}{2m} - \mu)\sigma^z + \Delta_i(p_x\sigma^x + p_y\sigma^y)$ is the Bogoliubov–de Gennes (BdG) Hamiltonian of $p + ip$ superconductors (SCs) at the ith layer with the Pauli matrix σ acting on Nambu space, and Δ_i denotes the pairing condensate. $m > 0$ and μ are the effective mass and chemical potential of a fractionalized fermion, respectively. When $\mu > 0$, the $p + ip$ SC is in the topological phase with the BdG Chern number $C = 1$ at each layer [54,61,78].

The transition to the 111 state is described by interlayer exciton condensation $\langle c_\uparrow c_\downarrow^\dagger\rangle = \langle b_\uparrow b_\downarrow^\dagger\rangle\langle\psi_\uparrow\psi_\downarrow^\dagger\rangle \neq 0$, which simultaneously breaks S_z conservation and leads to $C = 2$ topological order [61]. It is possible that $\langle\psi_\uparrow\psi_\downarrow^\dagger\rangle$ becomes nonzero, breaking the residue $Z_2^\uparrow \times Z_2^\downarrow$ of the emergent $u(1)_\uparrow \times u(1)_\downarrow$ symmetry before exciton condensation, but the exciton inducing the condensation of $\langle\psi_\uparrow\psi_\downarrow^\dagger\rangle$ is not a fine-tuned result, and indeed the numerical results show a single transition. In the presence of interlayer coherence, the Hamiltonian of the fractionalized fermion is $H = \begin{pmatrix} h_\uparrow & h_{\uparrow\downarrow} \\ h_{\uparrow\downarrow}^\dagger & h_\downarrow \end{pmatrix}$, where $h_{\uparrow\downarrow} = \mathrm{diag}(\Phi, -\Phi^*)$ with $\Phi = \langle\psi_\uparrow\psi_\downarrow^\dagger\rangle$. The BdG Chern number is the sum of two layers $C = 1 + 1 = 2$. The topological order of the $C = 2$ state is Abelian, which can be captured by a Chern-Simons term [45,61],

$$\mathcal{L} = \sum_i\left[\frac{8}{4\pi}\alpha_i d\alpha_i + \frac{2}{2\pi}(eA_i + a_i)d\alpha_i + \frac{1}{2\pi}\beta_i da_i\right] - \frac{4}{4\pi}\beta_- d\beta_-, \tag{4}$$

where $\beta_- = \frac{\beta_\uparrow - \beta_\downarrow}{2}$, β_i is the dual theory [75,79,80] of the Higgs field Δ_i that breaks $u(1)_i$ to Z_2^i, and the gapped fermion part is neglected. The last term is the forecasted Chern-Simons term to capture the quartonic statistics of the $C = 2$ topological order. Now we can integrate out a_i since they are linear in the Lagrangian to get $\mathcal{L} = \sum_{ij} \beta_i K_{ij} d\beta_j + \sum_i \frac{e}{2\pi} A_i d\beta_i$, where $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ corresponding to the 111 state [32]. The K matrix indicates a gapless gauge field β_- dual to the Goldstone bosons originating from S^z symmetry breaking.

Concluding remarks. We have shown a direct continuous transition in $\nu_T = 5$ quantum Hall bilayers based on both ED calculations on a torus. Moreover, we proposed an exotic scenario of such a transition, where the topology changing and symmetry breaking take place simultaneously. The $\nu_T = 5$ bilayers host significantly different physics from $\nu_T = 1$, which can be seen more clearly when coupling two independent layers by changing the layer distances. When $\nu_T = 1$, each decoupled layer is a gapless CFL state, while for $\nu_T = 5$, each decoupled layer is a fully gapped MR state. Previous studies indicate the $\nu_T = 1$ system has an intermediate phase when tuning the layer distance [28–30], while it is absent when $\nu_T = 5$ based on this work. We propose that our findings of an exotic topological quantum transition and exciton superfluid at $\nu_T = 5$ can be detected in quantum Hall bilayers composed of double-well GaAs heterostructures or bilayer graphene, which has been successfully engineered to probe the $\nu_T = 1$ bilayer system.

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