Statistics of heat transport across a capacitively coupled double quantum dot circuit

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We study heat current and the full statistics of heat fluctuations in a capacitively coupled double quantum dot system. This work is motivated by recent theoretical studies and experimental works on heat currents in quantum dot circuits. As expected intuitively, within the (static) mean-field approximation, the system at steady state decouples into two single-dot equilibrium systems with renormalized dot energies, leading to zero average heat flux and fluctuations. This reveals that dynamic correlations induced between electrons on the dots are solely responsible for the heat transport between the two reservoirs. To study heat current fluctuations, we compute the steady-state cumulant generating function for heat exchanged between reservoirs using two approaches: the Lindblad quantum master equation approach, which is valid for arbitrary Coulomb interaction strength but weak system-reservoir coupling strength, and the saddle point approximation for the Schwinger-Keldysh coherent-state path integral, which is valid for arbitrary system-reservoir coupling strength but weak Coulomb interaction strength. Using thus obtained generating functions, we verify the steady-state fluctuation theorem for stochastic heat flux and study the average heat current and its fluctuations. We find that the heat current and its fluctuations change nonmonotonically with the Coulomb interaction strength (U) and system-reservoir coupling strength (Γ) and are suppressed for large values of U and Γ.

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I. INTRODUCTION

Studying transport processes in nanosized electronic quantum dot junctions has been an active research area for past two decades [1–7]. The motivation is twofold: the desire to design more efficient electronic devices and heat engines [8–12] and also the need for a platform for testing fundamental principles. From the technological perspective, useful devices have been proposed theoretically, and a few have been tested experimentally. For example, nanodiodes [13], transistors [14], switches [15], and other electronic elements relevant for device applications have been proposed [16]. Understanding charge and heat transport in nanosystems is relevant for these applications. However, due to the small size, fluctuations of fluxes flowing through these systems are not negligible. These fluctuations are not arbitrary but follow universal relations called fluctuation theorems which generalize the second law of thermodynamics to small scale [17–27]. These identities relate the number of microscopic realizations of transport processes which produce a certain amount of entropy to those which annihilate the same amount of entropy. Nanoelectronic devices have served as useful platforms for testing these identities [28–31]. These theorems are not only aesthetically appealing but also are used to gain insights into transport processes. For example, they have been used to characterize efficiency fluctuations [32–34] of nanoscale heat engines, which is an important fundamental generalization of Carnot's analysis [35] of macroscopic heat engines to microscale. Thermoelectric engines, which are of current theoretical and experimental interest, constitute one such class of nanoscale heat engines [8,36–39] that convert heat to electrical work.

Although heat flow plays a central role in determining the efficiencies of these engines, heat currents at nanoelectronic junctions are not as well explored as the charge currents. Recently, there has been some interest in exploring the effects of various many-body interactions in thermoelectric heat engines. For example, effects of electron-phonon and electron-electron interactions on efficiencies of two-terminal and three-terminal thermoelectric engines have been studied [40–51]. Furthermore, important experimental advancements in measuring heat currents in nanoelectric junctions have been achieved recently [39,52,53].

Motivated by these works, we study steady-state heat flux and fluctuations across a capacitively (Coulomb) coupled double quantum dot system. This system is known to act as a heat rectifier in some parameter regime [54]. A similar model with spin-degenerate states on each dot [55–57] is shown to exhibit a quantum phase transition in the Kondo regime at equilibrium. Capacitive coupling has been exploited to probe charge fluctuations in nanojunctions [58–60] to understand Coulomb drag effects, where electric charge flux in a circuit induces a charge flux in another capacitively coupled circuit [61–69]. In this work we are interested in the study of heat flux that is induced due to Coulomb interactions between electrons in a capacitively coupled double quantum dot system. To study heat flux and its fluctuations, we calculate its cumulant generating function, defined using the two-point measurement scheme, using two different approaches valid in different parameter regimes (for temperatures above the Kondo temperature [70,71]): the Lindblad quantum master equation [72–75], which is valid at high temperatures, weak system reservoir coupling strength, and arbitrary Coulomb interaction strength, and the Schwinger-Keldysh [76–79] saddle

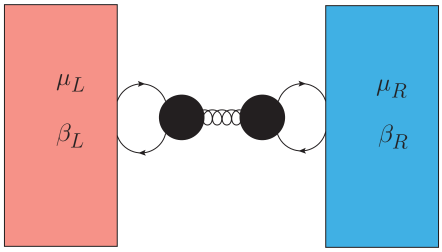

FIG. 1. Schematic of the model considered.

point method (random-phase approximation with mean-field dressed propagators) [77,78,80–82], which is valid for weak Coulomb interaction strength and arbitrary system reservoir coupling strength. We verify the steady-state heat fluctuation theorem and calculate heat flux and its fluctuations. Heat flux through the same model system [83] and fluctuations of heat flow in a variant of this model were studied recently to understand near-field radiative heat transfer within the bare random-phase approximation [84]. Here we present results that are valid beyond the bare random-phase approximation (the random-phase approximation with mean-field dressed propagators). We find that the steady-state scaled cumulant generating functions obtained using both the approximation schemes satisfy Gallavotti-Cohen symmetry and hence the steady-state fluctuation theorem for the heat fluctuations. Heat flux and its fluctuations are nonmonotonic functions of Coulomb interaction strength and decay exponentially for asymptotically large Coulomb interaction strength. Similar nonmonotonic behavior is seen with respect to system-reservoir coupling strength. The flux and its fluctuations are suppressed as a power law (Γ^{-4}) for large coupling strength.

In Sec. II we introduce the model system and define the moment generating function for stochastic heat transfer using the two-point measurement scheme. In Sec. III we compute this moment generating function using two approximation schemes and discuss heat flux and fluctuations. We conclude in Sec. IV.

Note that in this work we use natural units such that $\hbar = 1$. With this, all quantities are expressed in units of frequency, i.e., $[\epsilon_\alpha] = [\epsilon_{\alpha k}] = [g_{\alpha k}] = [\text{time}^{-1}] = [\mu_\alpha] = [\beta_\alpha^{-1}] = [\Delta Q]$.

II. MODEL SYSTEM AND MOMENT GENERATING FUNCTION

A schematic of the model system considered in this work is shown in Fig. 1. It consists of two capacitively (Coulomb) coupled quantum dots (each having a single orbital) individually coupled to two different fermionic reservoirs. The whole system is described by the following Hamiltonian:

$$\hat{H} = \underbrace{\sum_{\alpha=L,R} \epsilon_\alpha c_\alpha^\dagger c_\alpha + U c_L^\dagger c_L c_R^\dagger c_R}_{H_S} + \sum_{\alpha=L,R} \underbrace{\sum_k \epsilon_{\alpha k} d_{\alpha k}^\dagger d_{\alpha k}}_{H_\alpha} + \sum_{\substack{k\\ \alpha=L,R}} [g_{\alpha k} d_{\alpha k}^\dagger c_\alpha + g_{\alpha k}^* c_\alpha^\dagger d_{\alpha k}]. \quad (1)$$

Here $c_\alpha^\dagger$ (c_α) and $d_{\alpha k}^\dagger$ ($d_{\alpha k}$) stand for the fermionic creation (annihilation) operator for creating (annihilating) an electron in the αth ($\alpha = L, R$) quantum dot and in the state labeled by k in the αth fermionic reservoir, respectively. The first term in Eq. (1) represents the Hamiltonian of two isolated quantum dots, each having a single orbital with energies ϵ_α; the second term represents Coulomb interaction between electrons on the two quantum dots, the third term is the Hamiltonian for the free electrons in the reservoirs, and the last term stands for hybridization between electrons on quantum dots and the reservoirs. Throughout this work we assume wideband approximation; that is, we assume that $g_{\alpha k}$ is independent of k and the density of states of reservoirs is a constant function of energy. We note that the Hamiltonian given in Eq. (1) is a variant of the Anderson Hamiltonian [70,85,86].

When the quantum dots are brought together and are coupled to two reservoirs, energy and particles are exchanged. In this work, we are interested in calculating the statistics of steady-state fluxes flowing through the double quantum dot system. In the long-time limit, only the heat flows between the left and the right reservoirs. The heat fluxes at the left and right interfaces are balanced at steady state, and the particle flux between the system and reservoirs vanishes. The physical reason for this is as follows: (i) since the coupling between the two quantum dots does not change the number of particles on the dots (i.e., the particle exchange between the two dots is not allowed by microscopic dynamics), the net number of particles exchanged between the left (right) dot and the left (right) reservoir is constrained to 0 and 1 and does not grow with time; hence, the particle flux and fluctuations are suppressed at the steady state. (ii) Similarly, energy cannot indefinitely accumulate in the system due to the boundedness of the system energy spectrum, and energy fluxes at the left and right interfaces balance out at long times.

The distribution function $P[\Delta Q; (T - T_0)]$ for heat (ΔQ) flowing from the right reservoir to the left reservoir within a time $T - T_0$ can be obtained using a two-point measurement protocol [25,26] for the observable corresponding to the operator $\frac{1}{2}[(H_L - \mu_L N_L) - (H_R - \mu_R N_R)]$ as

$$P[\Delta Q; T - T_0] = \int_{-\infty}^{+\infty} \frac{d\chi}{2\pi} \mathcal{Z}[\chi; T - T_0] e^{i\chi\Delta Q}, \quad (2)$$

where $\mathcal{Z}[\chi; T - T_0]$ is the moment generating function for ΔQ and is given as

$$\mathcal{Z}[\chi; T - T_0] = \mathrm{Tr}[\mathcal{U}_\chi(T, T_0)\rho(T_0)\mathcal{U}_0(T_0, T)], \quad (3)$$

where $\mathcal{U}_\chi(T_1, T_2) = e^{-i(T_1 - T_2)H_\chi}$, with

$$H_\chi = \sum_{\alpha=L,R} \epsilon_\alpha c_\alpha^\dagger c_\alpha + U c_L^\dagger c_L c_R^\dagger c_R + \sum_{\substack{k\\ \alpha=L,R}} \epsilon_{\alpha k} d_{\alpha k}^\dagger d_{\alpha k} + \sum_{\substack{k\\ \alpha=L,R}} [g_{\alpha k} e^{-i(\epsilon_{\alpha k} - \mu_\alpha)\chi_\alpha} d_{\alpha k}^\dagger c_\alpha + g_{\alpha k}^* e^{i(\epsilon_{\alpha k} - \mu_\alpha)\chi_\alpha} c_\alpha^\dagger d_{\alpha k}] \quad (4)$$

and $\chi_L = -\chi_R = \frac{\chi}{2}$. The trace in Eq. (3) is over the combined Fock space of the system and reservoirs, and $\rho(T_0)$ is the density matrix of the whole system at initial time T_0, taken here as

$$\rho(T_0) = \frac{e^{-\sum_{\alpha=L,R,S}\beta_\alpha[H_\alpha-\mu_\alpha N_\alpha]}}{\mathrm{Tr}\big[e^{-\sum_{\alpha=L,R,S}\beta_\alpha[H_\alpha-\mu_\alpha N_\alpha]}\big]}; \tag{5}$$

that is, system and reservoir initial states are noninteracting equilibrium states with different temperatures and chemical potentials.

In the next section, we calculate $\mathcal{Z}[\chi;T-T_0]$ approximately using two approaches and study the heat flux and its fluctuations.

III. APPROXIMATE CALCULATION OF THE MOMENT GENERATING FUNCTION

In this section we compute $\mathcal{Z}[\chi;T-T_0]$ defined in Eq. (3) using two approaches approximately: (i) the Lindblad quantum master equation approach in which coupling between the system and reservoirs is assumed to be weak and (ii) the Schwinger-Keldysh path integral approach in which Coulomb interaction strength is assumed to be weak but the system reservoir coupling can be arbitrary.

A. Lindblad quantum master equation approach

$\mathcal{Z}[\chi;T-T_0]$ defined in Eq. (3) can be reexpressed as

$$\mathcal{Z}[\chi;T-T_0] = \mathrm{Tr}_s\big[\rho_s^\chi(T)\big], \tag{6}$$

where $\rho_s^\chi(T) = \mathrm{Tr}_{\mathrm{Res}}[\mathcal{U}_\chi(T,T_0)\rho(T_0)\mathcal{U}_{\{0,0\}}(T_0,T)]$ is the counting-field-dependent reduced system density matrix at time T obtained by tracing out the two reservoirs. Using the standard Born-Markov-Secular approximations, the (counting-field-dependent) Lindblad quantum master equation can be derived for $\rho_s^\chi(T)$ [72–75], which is given as

$$\begin{aligned}\frac{d}{dT}\rho_s^\chi(T) = &-i\sum_{\alpha=L,R}\epsilon_\alpha\big[c_\alpha^\dagger c_\alpha,\rho_s^\chi(T)\big] - \frac{1}{2}\sum_{\alpha=L,R}\big[\Gamma_\alpha f_\alpha(\epsilon_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})\big\{c_\alpha c_\alpha^\dagger,\rho_s^\chi(T)\big\} + \Gamma_\alpha[1-f_\alpha(\epsilon_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})]\big\{c_\alpha^\dagger c_\alpha,\rho_s^\chi(T)\big\}\big]\\ &+\sum_{\alpha=L,R}\big[\Gamma_\alpha f_\alpha(\epsilon_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})e^{i(\epsilon_\alpha-\mu_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})\chi_\alpha}c_\alpha^\dagger\rho_s^\chi(T)c_\alpha + \Gamma_\alpha[1-f_\alpha(\epsilon_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})]e^{-i(\epsilon_\alpha-\mu_\alpha+Uc_{\bar\alpha}^\dagger c_{\bar\alpha})\chi_\alpha}c_\alpha\rho_s^\chi(T)c_\alpha^\dagger\big],\end{aligned} \tag{7}$$

where $\alpha \neq \bar\alpha = L,R$; $\Gamma_\alpha = 2\pi\sum_k|g_{\alpha k}|^2\delta(\epsilon-\epsilon_{\alpha k}) = 2\pi|g_\alpha|^2\rho_\alpha$ [here the wideband approximation is invoked, i.e., coupling between system and reservoirs states, $g_{\alpha k} = g_\alpha$, and the density of states of the reservoirs, $\sum_k\delta(\epsilon-\epsilon_{\alpha k}) = \rho_\alpha$, is assumed to be independent of k and energy, respectively], and $f_\alpha(\hat o) = (e^{\beta_\alpha(\hat o-\mu_\alpha)}+1)^{-1}$, with β_α and μ_α being the temperature and chemical potential of the αth reservoir. It is important to note that if the (static) mean-field approximation is made here, i.e., replacing $c_{\bar\alpha}^\dagger c_{\bar\alpha}$ by $\langle c_{\bar\alpha}^\dagger c_{\bar\alpha}\rangle_S = \lim_{(T-T_0)\to\infty}\frac{\mathrm{Tr}_s[c_{\bar\alpha}^\dagger c_{\bar\alpha}\rho_s^\chi(T)]}{\mathrm{Tr}_s[\rho_s^\chi(T)]}$, the right-hand side of Eq. (7) can be separated into two terms which depend only on the dynamics of individual dots whose energies are renormalized by coupling to the other dot. This results in two decoupled quantum dots which equilibrate with their own reservoirs at long time. Thus, within this approximation, heat flux and fluctuations through the system vanish at steady state. Hence, mean-field approximation leads to no steady-state heat flux and fluctuations. We therefore need to go beyond the mean-field approximation to have nonzero flux and fluctuations at steady state.

By taking matrix elements in the occupation number basis of the two dots, $|N_L,N_R\rangle$ (with $N_\alpha = 0,1$), it can be seen that the populations $[\langle N_L,N_R|\rho_s^\chi(T)|N_L,N_R\rangle]$ are decoupled from the coherences $[\langle N_L,N_R|\rho_s^\chi(T)|N_L',N_R'\rangle]$, which die out exponentially fast with time. Further, we restrict ourselves to the parameter regime $\epsilon_\alpha = (\mu_\alpha - \frac{U}{2})$, which simplifies the analysis. In this regime, we need to solve only the following 2×2 matrix equation:

$$\frac{d}{dT}\big|P_s^\chi(T)\big\rangle = \mathcal{L}^\chi\big|P_s^\chi(T)\big\rangle, \tag{8}$$

where

$$\big|P_s^\chi(T)\big\rangle = \begin{bmatrix}\langle 1,0|\rho_s^\chi(T)|1,0\rangle + \langle 0,1|\rho_s^\chi(T)|0,1\rangle\\ \langle 0,0|\rho_s^\chi(T)|0,0\rangle + \langle 1,1|\rho_s^\chi(T)|1,1\rangle\end{bmatrix} \tag{9}$$

and the Liouvillian $\mathcal{L}^\chi$ is given as

$$\mathcal{L}^\chi = \sum_{\alpha=L,R}\begin{bmatrix}-\Gamma_\alpha\bar f_\alpha & \Gamma_\alpha[1-\bar f_\alpha]e^{-i\frac{U}{2}\chi_\alpha}\\ \Gamma_\alpha\bar f_\alpha e^{i\frac{U}{2}\chi_\alpha} & -\Gamma_\alpha[1-\bar f_\alpha]\end{bmatrix}, \tag{10}$$

where $\bar f_\alpha = (e^{\beta_\alpha U/2}+1)^{-1}$. Note that the structure of the Liouvillian given in Eq. (8) is very similar to the case of charge transport through a resonant level system [87] when the two many-body states of the level are identified with the singly occupied and doubly (un)occupied states of the double quantum dot system.

Using the solution of Eq. (8) in Eq. (6) (equivalent to $\mathcal{Z}[\chi;T-T_0]=\langle\mathcal{I}|e^{\mathcal{L}^{\chi}(T-T_0)}|P_s^0(T_0)\rangle$, with $\langle\mathcal{I}|=[1\quad 1]$), $\mathcal{Z}[\chi;T-T_0]$ is obtained as

$$\mathcal{Z}[\chi;T-T_0]=e^{-\frac{(\Gamma_L+\Gamma_R)}{2}(T-T_0)}\Bigg\{\cosh\{\Lambda[\chi](T-T_0)\}+\frac{\sinh\{\Lambda[\chi](T-T_0)\}}{\Lambda[\chi]}\Bigg[\sum_{\alpha=L,R}\Gamma_\alpha\bigg(\frac{1}{2}+[f_\alpha[f_{SL}(1-f_{SR})+f_{SR}(1-f_{SL})]$$
$$\times(e^{iU\chi_\alpha/2}-1)+(1-f_\alpha)[1-f_{SL}(1-f_{SR})-f_{SR}(1-f_{SL})](e^{-iU\chi_\alpha/2}-1)]\bigg)\Bigg]\Bigg\}. \tag{11}$$

To arrive at this explicit expression for $\mathcal{Z}[\chi;T-T_0]$, we have used the initial condition, i.e.,

$$\left|P_s^0(T_0)\right\rangle=\begin{bmatrix} f_{SL}(1-f_{SR})+f_{SR}(1-f_{SL}) \\ 1-f_{SL}(1-f_{SR})-f_{SR}(1-f_{SL})\end{bmatrix}, \tag{12}$$

which is equivalent to $\rho_s^\chi(T_0)=\frac{e^{-\beta_S[H_S-\mu_S N_S]}}{\mathrm{Tr}[e^{-\beta_S[H_S-\mu_S N_S]}]}$. Further, $f_{S\alpha}=(e^{\beta_S(\epsilon_\alpha-\mu_S)}+1)^{-1}$ represents the initial occupation of the two isolated dots, with β_S and μ_S being the temperature and chemical potential of the dots. The function $\Lambda[\chi]$ is

$$\Lambda[\chi]=\sqrt{\left(\frac{\Gamma_L+\Gamma_R}{2}\right)^2+\Gamma_L\Gamma_R\{\bar{f}_L[1-\bar{f}_R](e^{iU\chi/2}-1)+\bar{f}_R[1-\bar{f}_L](e^{-iU\chi/2}-1)\}}. \tag{13}$$

In the long-time limit (i.e., $T-T_0\to\infty$), the scaled cumulant generating function, defined as

$$\mathcal{F}[\chi]=\lim_{(T-T_0)\to\infty}\frac{\ln\mathcal{Z}[\chi;T-T_0]}{(T-T_0)}, \tag{14}$$

is given by

$$\mathcal{F}[\chi]=-\frac{(\Gamma_L+\Gamma_R)}{2}+\Lambda[\chi]. \tag{15}$$

This scaled cumulant generating function has the same form as that of charge transport through a resonant level model [87]. This is due to the mapping between the two models, as discussed earlier. It is straightforward to see that the cumulant generating function ($\mathcal{F}[\chi]$) satisfies Gallavotti-Cohen symmetry: $\mathcal{F}[-\chi-i(\beta_L-\beta_R)]=\mathcal{F}[\chi]$. This symmetry leads to the detailed steady-state fluctuation theorem for the distribution function for heat flow: $\lim_{(T-T_0)\to\infty}\frac{P[+\Delta Q;T-T_0]}{P[-\Delta Q;T-T_0]}=e^{(\beta_L-\beta_R)\Delta Q}$. Note that the last two terms of Eq. (11), which depend on the initial condition, do not have the required symmetry and the fluctuation theorem is not satisfied at short times. This is because the moment generating function defined in Eq. (3) keeps track of only the net heat flowing between the left and right reservoirs and does not keep track of the energy change in the system, which also contributes to the total entropy production at short times.

Further, using the above long-time-limit scaled cumulant generating function ($\mathcal{F}[\chi]$), cumulants of heat flux can be obtained as $C_n=i^n\frac{d}{d\chi^n}\mathcal{F}[\chi]$ (given in Appendix A). Figure 2 shows the four cumulants as a function of Coulomb interaction strength. It is clear from Fig. 2 that heat flux and fluctuations are suppressed exponentially for large U. This is due to the exponential dependence of Fermi functions on U. Physically, the transition from the singly occupied states to the doubly (un)occupied state becomes exponentially less probable as U is increased [54]. Further, we note that for intermediate values of U, fluctuations of heat are enhanced.

Since $\mathcal{F}[\chi]$ is a periodic function of χ with period $\frac{4\pi}{U}$, in the long time limit $P[\Delta Q;T-T_0]$ acquires the Dirac comb structure: $\lim_{(T-T_0)\to\infty}P[\Delta Q;T-T_0]=\sum_{n=-\infty}^{+\infty}p[n;T-T_0]\delta[\Delta Q-\frac{nU}{2}]$, with $p[n;T-T_0]=\int_0^{2\pi}\frac{d\chi}{2\pi}e^{\mathcal{F}[\frac{2\chi}{U}](T-T_0)}e^{i\chi n}$.

$p[n;T-T_0]$ is computed numerically and is shown in Fig. 3 along with $\ln\frac{p[n,T-T_0]}{p[-n,T-T_0]}$ in the inset, demonstrating the validity of the steady-state Gallavotti-Cohen fluctuation theorem for the stochastic heat flow.

In the next sub-section we present results obtained within the saddle point approximation for the path integral formulation of the Schwinger-Keldysh technique.

B. Schwinger-Keldysh path integral approach

We compute the moment generating function $\mathcal{Z}[\chi;(T-T_0)]$ using the path integral on the Schwinger-Keldysh contour. The results obtained are valid for arbitrary dot-reservoir coupling strength. However, the effect of the Coulomb interaction is treated approximately.

$\mathcal{Z}[\chi;T-T_0]$, defined in Eq. (3), can be expressed as

$$\mathcal{Z}[\chi;T-T_0]=\mathrm{Tr}[\mathcal{T}_c e^{-i\int_c d\tau H_{\chi(\tau)}(\tau)}\rho(T_0)], \tag{16}$$

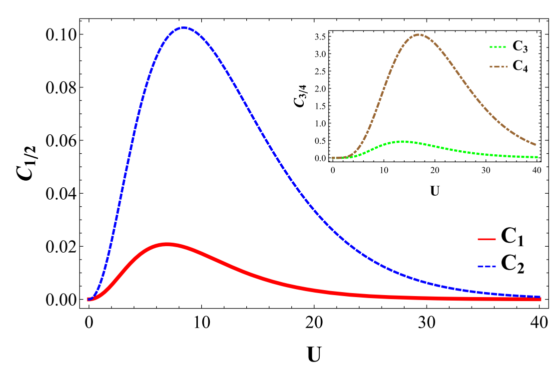

FIG. 2. First and second cumulants of heat transferred from the right to left reservoir as a function of Coulomb interaction strength U (in units of β_L^{-1}) for the parameters $\beta_R=0.5\beta_L$, $\Gamma_L=\Gamma_R=0.1\beta_L^{-1}$. The inset shows the third and fourth cumulants. Cumulants (C_n) are plotted in units of $\beta_L^{-(n+1)}$.

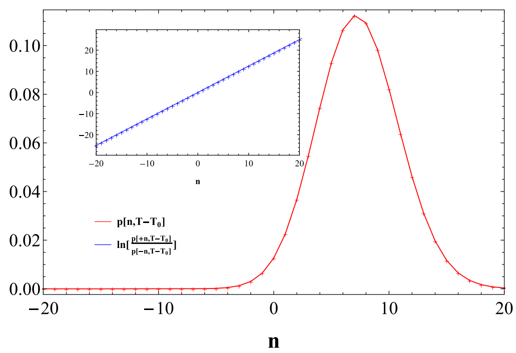

FIG. 3. Plot of $p[n, T - T_0]$ vs n for $U = 5.0\beta_L^{-1}$, $T - T_0 = 100.0(\frac{\Gamma_L+\Gamma_R}{2})^{-1}$, with all other parameters being the same as in Fig. 2. A plot of $\ln \frac{p[n,T-T_0]}{p[-n,T-T_0]}$ vs n is shown in the inset.

where $\mathcal{T}_c e^{-i\int_c d\tau H_{\chi(\tau)}(\tau)}$ is the evolution operator defined on the Schwinger-Keldysh contour [76,78,79,88,89] shown in Fig. 4, going from T_0 to T and back to T_0. Here $H_{\chi(\tau)}(\tau) = H_\chi$ on the forward contour, and $H_{\chi(\tau)}(\tau) = H_{\chi=0}$ on the backward contour.

$\mathcal{Z}[\chi; T - T_0]$ can be expressed as a functional integral using Grassmann field variables. The resultant path integral over Grassmann fields cannot be evaluated exactly due to the presence of the quartic term (with coupling constant U). It can be evaluated approximately by bosonizing the action using Hubbard-Stratonovich decoupling [77,78,90–93] and evaluating the resulting action using the saddle point approximation [77,78,80–82]. Details of the calculation are provided in Appendix B.

The final expression for the long-time-limit scaled cumulant generating function is obtained as

$$\mathcal{F}[\chi] = -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln[1 - \mathbb{T}(\omega)\{n_L(\omega)[1+n_R(\omega)] \times [e^{i\chi\omega} - 1] + n_R(\omega)[1+n_L(\omega)][e^{-i\chi\omega} - 1]\}], \tag{17}$$

with the bosonic distribution function given by $n_\alpha(\omega) = (e^{\beta_\alpha\omega} - 1)^{-1}$ and the transmission function (an even function of ω) given as

$$\mathbb{T}(\omega) = \frac{4U^2\mathrm{Re}\{[\tilde{P}^0_{LL}]^R(\omega)\}\mathrm{Re}\{[\tilde{P}^0_{RR}]^R(\omega)\}}{|1 + U^2[\tilde{P}^0_{LL}]^R(\omega)[\tilde{P}^0_{RR}]^R(\omega)|^2}. \tag{18}$$

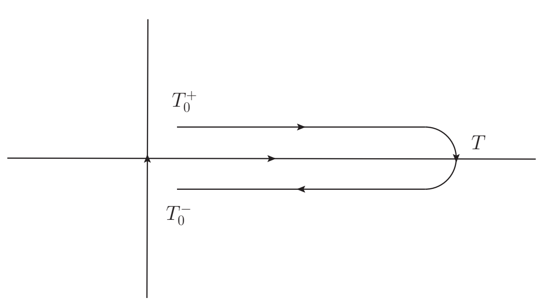

FIG. 4. Schematic of the Schwinger-Keldysh contour.

The general expression for $[\tilde{P}^0_{\alpha\alpha}]^R(\omega)$ (χ independent) is given in Eq. (B30), where $\{\phi^0_\alpha\}$ have to be determined self-consistently by solving Eqs. (B24). For the case $\epsilon_\alpha = \mu_\alpha - \frac{U}{2}$, Eqs. (B24) have a unique stable solution, $\phi^0_\alpha = \frac{U}{2}$, in the regime

$$\sqrt{\frac{\beta_L U}{2}}\sqrt{\frac{\beta_R U}{2}} < \frac{\pi}{\sqrt{\Psi'[\frac{1}{2} + \frac{\beta_L\Gamma_L}{4\pi}]}}\frac{\pi}{\sqrt{\Psi'[\frac{1}{2} + \frac{\beta_R\Gamma_R}{4\pi}]}}. \tag{19}$$

Using this in Eq. (B30), we get a simple-looking expression for $[\tilde{P}^0_{\alpha\alpha}]^R(\omega)$, which is given as

$$[\tilde{P}^0_{\alpha\alpha}]^R(\omega) = \frac{i}{\pi}\frac{\Gamma_\alpha}{(\omega + i\Gamma_\alpha)} \times \left[\frac{\Psi[\frac{1}{2} + \frac{\beta_\alpha\Gamma_\alpha}{4\pi} - i\frac{\beta_\alpha\omega}{2\pi}] - \Psi[\frac{1}{2} + \frac{\beta_\alpha\Gamma_\alpha}{4\pi}]}{\omega}\right]. \tag{20}$$

As discussed in Appendix B, the Fourier transform of $[\tilde{P}^0_{\alpha\alpha}]^R(\omega)$ in the time domain decays exponentially with rates linear in Γ_α and β_α^{-1}. As $[\tilde{P}^0_{\alpha\alpha}]^R$ is directly related to charge density fluctuations on the quantum dots [78], charge density fluctuations on quantum dots are exponentially suppressed in time for large Γ_α and β_α^{-1}.

Note that $\mathcal{F}[-\chi - i(\beta_L - \beta_R)] = \mathcal{F}[\chi]$, which is the steady-state Gallavotti-Cohen fluctuation symmetry; this symmetry leads to the standard steady-state fluctuation theorem for the stochastic heat flux flowing from the right reservoir to the left reservoir ($P[\Delta Q; T - T_0]$) i.e., $\lim_{(T-T_0)\to\infty}\frac{P[+\Delta Q; T-T_0]}{P[-\Delta Q; T-T_0]} = e^{(\beta_L-\beta_R)\Delta Q}$.

The algebraic form of the scaled cumulant generating function given in Eq. (17) is similar to that of heat transport across two-terminal harmonic oscillator junctions [94–96]. However, unlike harmonic oscillator junctions, the transmission function given in Eq. (18) depends on temperatures of the reservoirs. We note that a similar expression for the scaled cumulant generating function [Eq. (17)] for a variant of the model considered here [84] and the transmission function [Eq. (18)] for the same model [83] were obtained using the bare random-phase approximation recently. In contrast to these works, analytical expressions for polarization functions could be obtained here by invoking the wideband assumption. Using $P^0_{\alpha\alpha}(\omega)$ as defined in Eq. (B30) with $\phi^0_L = \phi^0_R = 0$ in Eq. (18) gives the transmission function obtained in Ref. [83] for the wideband reservoir case.

In Fig. 5, transmission functions $\mathbb{T}(\omega)$ obtained within the bare random-phase approximation [Bare-RPA; Eq. (18) along with Eq. (B30) for $\phi_\alpha = 0$] and the random-phase approximation [RPA; Eq. (18) along with Eq. (20)] for the case $\epsilon_\alpha = \mu_\alpha - \frac{U}{2}$ [within the regime defined by Eq. (19)] are plotted as a function of ω for comparison. The difference between transmission functions obtained within Bare-RPA and RPA is through an extra factor of $\pm\frac{\beta_\alpha U}{4\pi}$ in the argument of digamma functions in the expressions for bare polarization functions $[\tilde{P}^0_{\alpha\alpha}]^R(\omega)$ [Eq. (B30)] compared to mean-field polarization functions [Eq. (20)]. Hence, the difference between the two transmission functions vanishes for high temperatures or small Coulomb interaction strengths ($\beta_\alpha U \ll 1$), which can clearly be seen in Fig. 5.

Figure 6 shows long-time limit of the first four (scaled) cumulants ($C_n = i^n\frac{d^n}{d\chi^n}\mathcal{F}[\chi]$ given in Appendix C) of heat

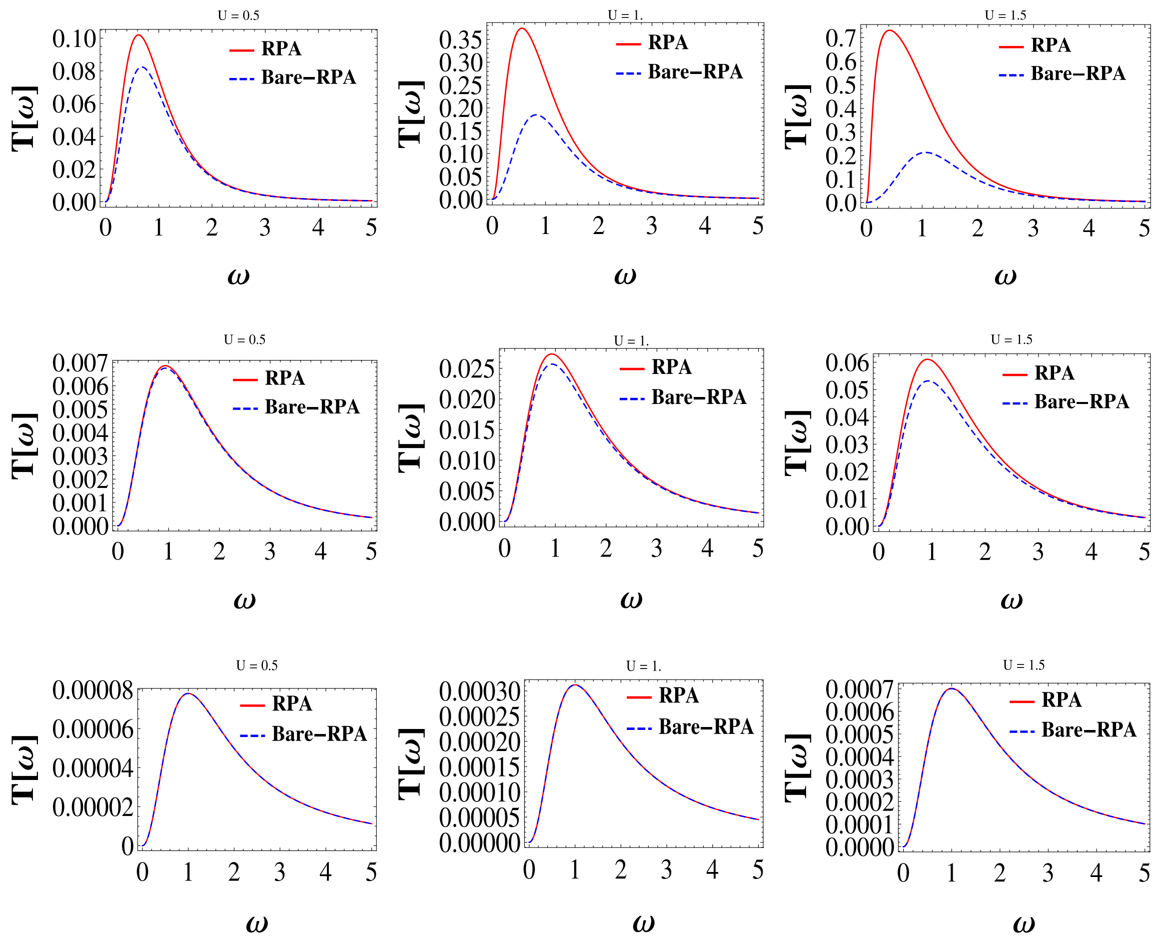

FIG. 5. Plots of transmission functions $\mathbb{T}(\omega)$ obtained using the bare random-phase approximation (Bare-RPA) and random-phase approximation (RPA) as a function of ω (in units of Γ_L) for $\epsilon_\alpha = \mu_\alpha - \frac{U}{2}$ [in the parameter regime defined by Eq. (19)] for different values of U (in units of Γ_L) for $\Gamma_R = 1.0\Gamma_L$ and $\beta_L = 10.0\Gamma_L^{-1}$, $\beta_R = 5.0\Gamma_L^{-1}$ (first row); $\beta_L = 1.0\Gamma_L^{-1}$, $\beta_R = 0.5\Gamma_L^{-1}$ (second row); and $\beta_L = 0.1\Gamma_L^{-1}$, $\beta_R = 0.05\Gamma_L^{-1}$ (third row). As the transmission functions are even functions of ω, here we plot only positive values of ω.

flux as a function of the system reservoir coupling strength ($\Gamma_L = \Gamma_R = \Gamma$). Heat flux and its fluctuations exhibit nonmonotonic behavior as a function of Γ. This can be understood

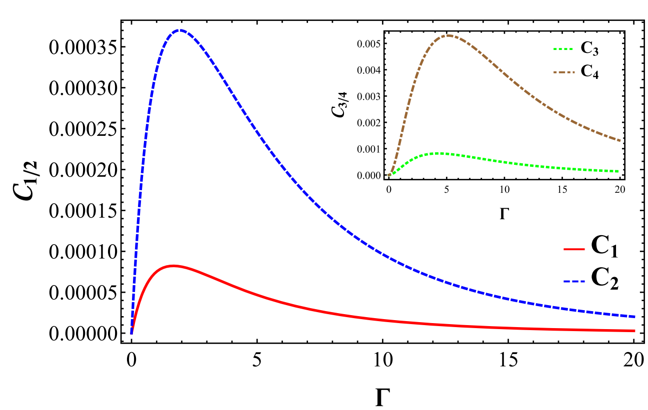

FIG. 6. First and second cumulants of heat transferred from the right to left reservoir as a function of system reservoir coupling strength ($\Gamma_L = \Gamma_R = \Gamma$ in units of β_L^{-1}) for the parameters $\beta_R = 0.5\beta_L$, $U = 0.1\beta_L^{-1}$. The inset shows the third and fourth cumulants. Cumulants (C_n) are plotted in units of $\beta_L^{-(n+1)}$.

as follows: as noted in the Appendix B, electron density fluctuations on dots are solely responsible for heat flux, which are exponentially suppressed in time with the rate depending linearly on Γ. Hence, the heat flux is suppressed for large Γ. At low temperatures ($\beta_L^{-1} \approx 0$, $\beta_R^{-1} \approx 0$), only low-frequency behavior of the transmission function is important, and for small ω, $\mathbb{T}(\omega) \approx \frac{16U^2}{\pi^2\Gamma^4}\omega^2$. Hence, at low temperatures and large system-reservoir coupling strength Γ, heat flux and its fluctuations decay as a power law ($\approx \Gamma^{-4}$) with Γ. Similar nonmonotonic behavior of the particle flux through the double quantum dot system was reported recently [97]. If we use this approximate expression for the transmission function in the expression for heat flux [C_1 given in Eq. (C1)], we get the result [98] $C_1 = \frac{16\pi^2U^2}{15\Gamma^4}(\beta_R^{-4} - \beta_L^{-4})$ (the Stefan-Boltzmann law [99]), as discussed in Ref. [83].

Figure 7 shows $P[\Delta Q; T - T_0]$, with the inset plot of $\ln \frac{P[\Delta Q; T-T_0]}{P[-\Delta Q; T-T_0]}$ showing the validity of the steady-state Gallavotti-Cohen fluctuation theorem.

IV. CONCLUSION

In this work we studied the heat flux and fluctuations of heat flowing across a capacitively coupled double quantum

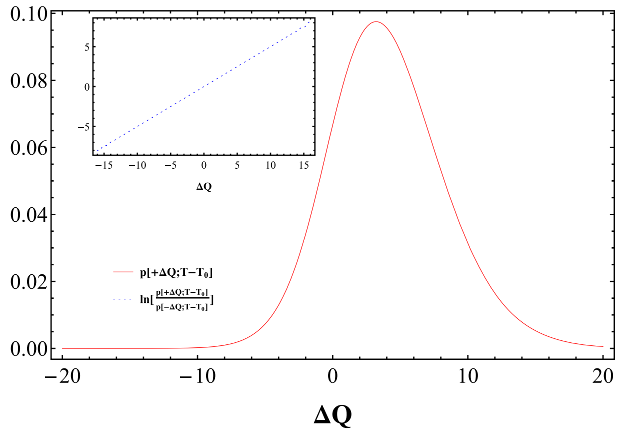

FIG. 7. Plot of $P[\Delta Q; T-T_0]$ vs ΔQ (in units of β_L^{-1}) for $\Gamma_L = \Gamma_R = 2.0\beta_L^{-1}$, $T-T_0 = 10^5(\frac{\Gamma_L+\Gamma_R}{2})^{-1}$, with all other parameters being the same as in Fig. 6. A plot of $\ln\frac{P[\Delta Q;T-T_0]}{P[-\Delta Q;T-T_0]}$ vs ΔQ (in units of β_L^{-1}) is shown in the inset.

dot circuit. We calculated the moment generating function using two theoretical approaches valid in different parameter regimes. Heat flux and its fluctuations (calculated using the Lindblad quantum master equation) exhibit nonmonotonic behavior as a function of Coulomb interaction strength and decay exponentially for asymptotically large Coulomb interaction strengths. Similarly, using the saddle point approximation scheme for the bosonized Schwinger-Keldysh path integral, heat flux and its fluctuations were found to exhibit nonmonotonic behavior as a function of system reservoir coupling strength and to decay as the inverse fourth power of the system-reservoir coupling strength for an asymptotically large system-reservoir coupling strength. A comparison of transmission functions for heat flux obtained using the random-phase approximation (with mean-field dressed propagators) and the bare random-phase approximation showed that although the obtained transmission functions show similar qualitative behavior, they differ quantitatively for stronger Coulomb interaction strength. Further, we have verified that the scaled cumulant generating function obtained using both the approximation schemes has Gallavotti-Cohen symmetry and hence the steady-state fluctuation theorem for the fluctuating heat flux is satisfied.

ACKNOWLEDGMENT

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APPENDIX A: CUMULANTS OF HEAT FLUX USING THE LINDBLAD MASTER EQUATION APPROACH

The analytical expressions for the first four (scaled) cumulants, i.e., heat flux (C_1), heat noise (C_2), the third cumulant (C_3), and the fourth cumulant (C_4), as obtained using the Lindblad quantum master equation are given as

$$
\begin{aligned}
C_1 &= -\frac{U}{2}\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}[\bar{f}_L-\bar{f}_R],\\
C_2 &= \left(\frac{U}{2}\right)^2\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\left[[\bar{f}_L(1-\bar{f}_R)+\bar{f}_R(1-\bar{f}_L)]-2\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}[\bar{f}_L-\bar{f}_R]^2\right],\\
C_3 &= -\left(\frac{U}{2}\right)^3\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)}\left[(\bar{f}_L-\bar{f}_R)-6\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}(\bar{f}_L-\bar{f}_R)[\bar{f}_L(1-\bar{f}_R)+\bar{f}_R(1-\bar{f}_L)]+12\frac{\Gamma_L^2\Gamma_R^2}{(\Gamma_L+\Gamma_R)^4}(\bar{f}_L-\bar{f}_R)^3\right],\\
C_4 &= \left(\frac{U}{2}\right)^4\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\bigg[[\bar{f}_L(1-\bar{f}_R)+\bar{f}_R(1-\bar{f}_L)]-8\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}[\bar{f}_L-\bar{f}_R]^2-120\frac{\Gamma_L^3\Gamma_R^3}{(\Gamma_L+\Gamma_R)^6}[\bar{f}_L-\bar{f}_R]^4\\
&\quad -6\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}[\bar{f}_L(1-\bar{f}_R)+\bar{f}_R(1-\bar{f}_L)]^2+72\frac{\Gamma_L^2\Gamma_R^2}{(\Gamma_L+\Gamma_R)^4}[\bar{f}_L-\bar{f}_R]^2[\bar{f}_L(1-\bar{f}_R)+\bar{f}_R(1-\bar{f}_L)]\bigg].
\end{aligned}
\tag{A1}
$$

APPENDIX B: SADDLE POINT APPROXIMATION FOR THE BOSONIZED SCHWINGER-KELDYSH PATH INTEGRAL

$\mathcal{Z}[\chi; T-T_0]$ given in Eq. (16) can be expressed as a functional integral using Grassmann field variables [76–78,93], $\{\psi_\alpha^\dagger(\tau), \psi_\alpha(\tau)\}$ for the system and $\{\psi_{\alpha k}^\dagger(\tau), \psi_{\alpha k}(\tau)\}$ for the reservoirs. This gives

$$
\mathcal{Z}[\chi; T-T_0] = \frac{1}{\mathcal{N}}\int \mathcal{D}[\{\psi_\alpha^\dagger(\tau), \psi_\alpha(\tau)\}]\int \mathcal{D}[\{\psi_{\alpha k}^\dagger(\tau), \psi_{\alpha k}(\tau)\}]e^{iS^\chi[\{\psi_\alpha^\dagger(\tau),\psi_\alpha(\tau)\},\{\psi_{\alpha k}^\dagger(\tau),\psi_{\alpha k}(\tau)\}]},
\tag{B1}
$$

where $\mathcal{N}$ is the normalization constant (independent of χ) such that $\mathcal{Z}[\chi; T-T_0]|_{\chi=0} = 1$. Here we do not compute $\mathcal{N}$ explicitly and modify it at intermediate steps by absorbing all constants (χ independent). Its value is determined finally by imposing

$\mathcal{Z}[\chi;T-T_0]|_{\chi=0}=1$. $S^\chi[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\},\{\psi^\dagger_{\alpha k}(\tau),\psi_{\alpha k}(\tau)\}]$ is the action of the whole system, given as

$$\begin{aligned}S^\chi[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\},\{\psi^\dagger_{\alpha k}(\tau),\psi_{\alpha k}(\tau)\}]\\ =\sum_{\alpha,\alpha'=L,R}\int_c d\tau\int_c d\tau'\left[\psi^\dagger_\alpha(\tau)\big[G^0_{\rm sys}\big]^{-1}_{\alpha\alpha'}(\tau,\tau')\psi_{\alpha'}(\tau')+\sum_{k,k'}\psi^\dagger_{\alpha k}(\tau)\big[G^0_{\rm res}\big]^{-1}_{\alpha k\alpha'k'}(\tau,\tau')\psi_{\alpha'k'}(\tau')\right]\\ -\sum_{\alpha=L,R}\sum_k\int_c d\tau[g_{\alpha k}e^{-i(\epsilon_{\alpha k}-\mu_\alpha)\chi_\alpha(\tau)}\psi^\dagger_{\alpha k}(\tau)\psi_\alpha(\tau)+g^*_{\alpha k}e^{i(\epsilon_{\alpha k}-\mu_\alpha)\chi_\alpha(\tau)}\psi^\dagger_\alpha(\tau)\psi_{\alpha k}(\tau)]-\int_c d\tau U\psi^\dagger_L(\tau)\psi_L(\tau)\psi^\dagger_R(\tau)\psi_R(\tau),\end{aligned}\tag{B2}$$

where $\chi_L(\tau)=-\chi_R(\tau)=\frac{\chi}{2}$ on the forward contour and $\chi_L(\tau)=\chi_R(\tau)=0$ on the backward contour. Further, $[G^0_{\rm sys}]^{-1}_{\alpha\alpha'}(\tau,\tau')$ and $[G^0_{\rm res}]^{-1}_{\alpha k\alpha'k'}(\tau,\tau')$ are matrix elements (with indices spanning state labels and contour times) of inverse of matrices with elements satisfying the following Schwinger-Dyson or Kadanoff-Baym equations on the Schwinger-Keldysh contour:

$$\begin{aligned}\sum_{\alpha_1=L,R}\int_c d\tau_1\left[\left(i\frac{\partial}{\partial\tau}-\epsilon_\alpha\right)\delta_{\alpha\alpha_1}\delta^c(\tau,\tau_1)\right]\big[G^0_{\rm sys}\big]_{\alpha_1\alpha'}(\tau_1,\tau')&=\delta_{\alpha\alpha'}\delta^c(\tau,\tau'),\\ \sum_{\alpha_1=L,R}\int_c d\tau_1\left[\left(-i\frac{\partial}{\partial\tau'}-\epsilon_{\alpha'}\right)\delta_{\alpha_1\alpha'}\delta^c(\tau_1,\tau')\right]\big[G^0_{\rm sys}\big]_{\alpha\alpha_1}(\tau,\tau_1)&=\delta_{\alpha\alpha'}\delta^c(\tau,\tau'),\\ \sum_{\alpha_1=L,R}\sum_{k_1}\int_c d\tau_1\left[\left(i\frac{\partial}{\partial\tau}-\epsilon_{\alpha k}\right)\delta_{\alpha\alpha_1}\delta_{kk_1}\delta^c(\tau,\tau_1)\right]\big[G^0_{\rm res}\big]_{\alpha_1k_1\alpha'k'}(\tau_1,\tau')&=\delta_{\alpha\alpha'}\delta_{kk'}\delta^c(\tau,\tau'),\\ \sum_{\alpha_1=L,R}\sum_{k_1}\int_c d\tau_1\left[\left(-i\frac{\partial}{\partial\tau'}-\epsilon_{\alpha'k'}\right)\delta_{\alpha_1\alpha'}\delta_{k_1k'}\delta^c(\tau_1,\tau')\right]\big[G^0_{\rm res}\big]_{\alpha k\alpha_1k_1}(\tau,\tau_1)&=\delta_{\alpha\alpha'}\delta_{kk'}\delta^c(\tau,\tau'),\end{aligned}\tag{B3}$$

with the following Kubo-Martin-Schwinger boundary conditions [79,100,101] enforcing the information of the initial state of the system and reservoirs:

$$\begin{aligned}\big[G^0_{\rm sys}\big]_{\alpha\alpha'}(T_0^-,\tau')&=-e^{\beta_S(\epsilon_\alpha-\mu_S)}\big[G^0_{\rm sys}\big]_{\alpha\alpha'}(T_0^+,\tau'),\\ \big[G^0_{\rm sys}\big]_{\alpha\alpha'}(\tau,T_0^-)&=-e^{-\beta_S(\epsilon_{\alpha'}-\mu_S)}\big[G^0_{\rm sys}\big]_{\alpha\alpha'}(\tau,T_0^+),\\ \big[G^0_{\rm res}\big]_{\alpha k\alpha'k'}(T_0^-,\tau')&=-e^{\beta_\alpha(\epsilon_{\alpha k}-\mu_\alpha)}\big[G^0_{\rm res}\big]_{\alpha k\alpha'k'}(T_0^+,\tau'),\\ \big[G^0_{\rm res}\big]_{\alpha k\alpha'k'}(\tau,T_0^-)&=-e^{-\beta_{\alpha'}(\epsilon_{\alpha'k'}-\mu_{\alpha'})}\big[G^0_{\rm res}\big]_{\alpha k\alpha'k'}(\tau,T_0^+).\end{aligned}\tag{B4}$$

Equations (B4) are one of the ways to take care of the initial-state information in the Schwinger-Keldysh path integral formalism [77,78,102,103]. The solution of Eqs. (B3) along with the boundary conditions, Eqs. (B4), is

$$\begin{aligned}\big[G^0_{\rm sys}\big]_{\alpha\alpha'}(\tau,\tau')&=-ie^{-i\epsilon_\alpha(\tau-\tau')}\delta_{\alpha\alpha'}\{\Theta(\tau,\tau')[1-f_S(\epsilon_\alpha)]-\Theta(\tau',\tau)f_S(\epsilon_\alpha)\},\\ \big[G^0_{\rm res}\big]_{\alpha k\alpha'k'}(\tau,\tau')&=-ie^{-i\epsilon_{\alpha k}(\tau-\tau')}\delta_{\alpha\alpha'}\delta_{kk'}\{\Theta(\tau,\tau')[1-f_\alpha(\epsilon_{\alpha k})]-\Theta(\tau',\tau)f_\alpha(\epsilon_{\alpha k})\},\end{aligned}\tag{B5}$$

where $f_X(x)=(e^{\beta_X(x-\mu_X)}+1)^{-1}$ $(X=L,R,S)$. We integrate over the reservoir Grassmann fields in Eq. (B1) to get $\mathcal{Z}[\chi;T-T_0]$ as a path integral only over the system Grassmann fields as

$$\mathcal{Z}[\chi;T-T_0]=\frac{1}{\mathcal{N}}\int\mathcal{D}[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\}]e^{iS^\chi_{\rm sys}[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\}]},\tag{B6}$$

with

$$S^\chi_{\rm sys}[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\}]=\sum_{\alpha,\alpha'=L,R}\int_c d\tau\int_c d\tau'\{\psi^\dagger_\alpha(\tau)[G^0]^{-1}_{\alpha\alpha'}(\tau,\tau')\psi_{\alpha'}(\tau')\}-\int_c d\tau U\psi^\dagger_L(\tau)\psi_L(\tau)\psi^\dagger_R(\tau)\psi_R(\tau),\tag{B7}$$

where

$$\sum_{\alpha_1=L,R}\int_c d\tau_1\{\big[G^0_{\rm sys}\big]^{-1}_{\alpha\alpha_1}(\tau,\tau_1)-\Sigma^c_{\alpha\alpha_1}(\tau,\tau_1)\}[G^0]_{\alpha_1\alpha'}(\tau_1,\tau')=\delta_{\alpha\alpha'}\delta^c(\tau,\tau'),\tag{B8}$$

with the self-energy acquired by the system due to the coupling with the reservoirs given as

$$\Sigma^c_{\alpha\alpha'}(\tau,\tau') = \delta_{\alpha\alpha'}\sum_{k,k'} g^*_{\alpha k} e^{i(\epsilon_{\alpha k}-\mu_\alpha)\chi_\alpha(\tau)} g_{\alpha' k'} e^{-i(\epsilon_{\alpha' k'}-\mu_{\alpha'})\chi_{\alpha'}(\tau')}\big[G^0_{\mathrm{res}}\big]_{\alpha k\alpha k'}(\tau,\tau'). \tag{B9}$$

The Grassmann field path integral given in Eq. (B6) cannot be evaluated exactly due to the quartic term (with coupling constant U). So we evaluate it approximately here. To that end, we decouple the above quartic term by introducing auxiliary real fields (also known as Hubbard-Stratonovich decoupling), which can be interpreted as fluctuating external potentials [77,78,90–93]. This gives

$$\mathcal{Z}[\chi;T-T_0] = \frac{1}{\mathcal{N}}\int \mathcal{D}[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\}]\int \mathcal{D}[\{\phi_\alpha(\tau)\}] e^{iS^\chi[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\},\{\phi_\alpha(\tau)\}]}, \tag{B10}$$

where

$$S^\chi[\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\},\{\phi_\alpha(\tau)\}] = \int_c d\tau\frac{\phi_L(\tau)\phi_R(\tau)}{U} + \sum_{\alpha,\alpha'=L,R}\int_c d\tau\int_c d\tau'\big\{\psi^\dagger_\alpha(\tau)\big[G^c_\phi\big]^{-1}_{\alpha\alpha'}(\tau,\tau')\psi_{\alpha'}(\tau')\big\}, \tag{B11}$$

with $[G^c_\phi]^{-1}(\tau,\tau')$ being the inverse of $[G^c_\phi](\tau,\tau')$, which satisfies the following equation:

$$\sum_{\alpha_1=L,R}\int_c d\tau_1\big\{\big[G^0_{\mathrm{sys}}\big]^{-1}_{\alpha\alpha_1}(\tau,\tau_1) - \phi_\alpha(\tau)\delta_{\alpha\alpha_1}\delta^c(\tau,\tau_1) - \Sigma^c_{\alpha\alpha_1}(\tau,\tau_1)\big\}\big[G^c_\phi\big]_{\alpha_1\alpha'}(\tau_1,\tau') = \delta_{\alpha\alpha'}\delta^c(\tau,\tau'). \tag{B12}$$

Since the Grassmann path integral in Eq. (B10) is quadratic in terms of system fields $\{\psi^\dagger_\alpha(\tau),\psi_\alpha(\tau)\}$, it can be performed exactly (here we have used the identity $\ln\det[\mathbb{A}] = \mathrm{Tr}[\ln\mathbb{A}]$) to get

$$\mathcal{Z}[\chi;T-T_0] = \frac{1}{\mathcal{N}}\int \mathcal{D}[\{\phi_\alpha(\tau)\}] e^{iS^\chi[\{\phi_\alpha(\tau)\}]}, \tag{B13}$$

with

$$S^\chi[\{\psi_\alpha(\tau)\},\{\phi_\alpha(\tau)\}] = \int_c d\tau\frac{\phi_L(\tau)\phi_R(\tau)}{U} - i\mathrm{Tr}\ln\big(\big[G^c_\phi\big]^{-1}\big), \tag{B14}$$

where Tr stands for the trace over contour time and orbital indices. The above algebraic gymnastics does not solve the problem as the final path integral, Eq. (B13), has an action, Eq. (B14), which is highly nonlinear; nevertheless, it is a bosonic path integral, which can be approximately evaluated using the saddle point/stationary-phase method. Within the saddle point approximation, the action, $S^\chi[\{\phi_\alpha(\tau)\}]$, is functional Taylor expanded around the path $\{\phi^0_\alpha(\tau)\}$, which makes the action stationary, i.e.,

$$\frac{\delta}{\delta\phi_\alpha(\tau)}S^\chi[\{\phi_\alpha(\tau)\}]\bigg|_{\{\phi_\alpha(\tau)\}=\{\phi^0_\alpha(\tau)\}} = 0. \tag{B15}$$

Further, the action is approximated by retaining terms in the functional Taylor expansions up to quadratic order, making the action functional, a quadratic form. This quadratic functional integral can be analytically evaluated to get a functional Fredholm determinant multiplied by the exponential of the action evaluated at the stationary path. With this cursory description of the saddle point/stationary-phase approximation, we move ahead.

Using Eq. (B15), the saddle point equations for the action given in Eq. (B14) are obtained as

$$\begin{aligned}\phi^0_L(\tau) &= -iU\big[G^c_{\phi^0}\big]_{RR}(\tau,\tau^+),\\ \phi^0_R(\tau) &= -iU\big[G^c_{\phi^0}\big]_{LL}(\tau,\tau^+),\end{aligned} \tag{B16}$$

where an infinitesimal forward shift (τ^+) of the second argument compared to the first argument of $[G^c_{\phi^0_x}]_{xx}(\tau,\tau^+)$ along the Schwinger-Keldysh contour can be deduced by consistently decoupling the fermionic quartic term using Hubbard-Stratonovich fields in the discretized path integral. Otherwise, there will be an ambiguity, as $[G^c_{\phi^0_x}]_{xx}(\tau,\tau')$ is discontinuous at $\tau=\tau'$ with a jump discontinuity of magnitude i. Equations (B16) for $\{\phi^0_\alpha(\tau)\}$ together with Eqs. (B12) for $[G^c_{\phi^0}]_{\alpha\alpha'}(\tau,\tau')$ constitute a self-consistent system of equations, which may possess more than one solution. When more than one stationary solution exists, then the functional integral is approximated by summing over the result obtained by Gaussian approximating the action around each of the stationary solutions. Expanding the action given in Eq. (B14) around the stationary path and retaining only the quadratic term, we get the approximate expression for $\mathcal{Z}[\chi;T-T_0]$ as (assuming that there is a unique stationary path)

$$\mathcal{Z}[\chi;T-T_0] \approx \frac{1}{\mathcal{N}}\int \mathcal{D}[\{\phi_\alpha(\tau)\}] e^{iS^\chi_{\mathrm{app}}[\{\phi_\alpha(\tau)\}]}, \tag{B17}$$

with $S^{\chi}_{\rm app}[\{\phi_\alpha(\tau)\}]$ representing the approximate action given as

$$S^{\chi}_{\rm app}[\{\psi_\alpha(\tau)\},\{\phi_\alpha(\tau)\}] = \int_c d\tau \frac{\phi^0_L(\tau)\phi^0_R(\tau)}{U} - i{\rm Tr}\ln\left(\left[G^c_{\phi^0}\right]^{-1}\right) + \frac{1}{2}\int_c d\tau\int_c d\tau' \begin{pmatrix}\phi_L(\tau)-\phi^0_L(\tau)\\ \phi_R(\tau)-\phi^0_R(\tau)\end{pmatrix}^T \left[\begin{pmatrix}0 & \frac{\delta^c(\tau,\tau')}{U}\\ \frac{\delta^c(\tau,\tau')}{U} & 0\end{pmatrix}\right. \\ \left. + i\begin{pmatrix}P^0_{LL}(\tau,\tau') & 0\\ 0 & P^0_{RR}(\tau,\tau')\end{pmatrix}\right]\begin{pmatrix}\phi_L(\tau')-\phi^0_L(\tau')\\ \phi_R(\tau')-\phi^0_L(\tau')\end{pmatrix}, \tag{B18}$$

where $P^0_{\alpha\alpha}(\tau,\tau') = [G^c_{\phi^0}]_{\alpha\alpha}(\tau,\tau')[G^c_{\phi^0}]_{\alpha\alpha}(\tau',\tau)$ for $\alpha = L, R$ are the contour-ordered polarization propagators within the random-phase approximation expressed in terms of the mean-field system fermion propagators [solutions of Eqs. (B12) and Eqs. (B16)]. Note that the polarization-dependent terms in Eq. (B18) represent the leading-order correction to the mean-field (saddle point) contribution. After a change of variables (shift transformation $\{\phi_\alpha(\tau)\} \to \{\phi_\alpha(\tau) + \phi^0_\alpha(\tau)\}$), followed by performing the final path integral over $\phi_\alpha(\tau)$ and using the identity $\ln\det[\mathbb{A}] = {\rm Tr}[\ln\mathbb{A}]$, we get

$$\ln\mathcal{Z}[\chi;T-T_0] \approx -\ln\mathcal{N} + i\int_c d\tau\frac{\phi^0_L(\tau)\phi^0_R(\tau)}{U} + {\rm Tr}\ln\left[\left[G^c_{\phi^0}\right]^{-1}\right] - \frac{1}{2}{\rm Tr}\ln\begin{pmatrix} iP^0_{LL}(\tau,\tau') & \frac{\delta^c(\tau,\tau')}{U}\\ \frac{\delta^c(\tau,\tau')}{U} & iP^0_{RR}(\tau,\tau')\end{pmatrix}. \tag{B19}$$

Further extracting

$$-\frac{1}{2}\ln\det\begin{pmatrix}0 & U\delta^c(\tau,\tau')\\ U\delta^c(\tau,\tau') & 0\end{pmatrix}$$

from $\ln\mathcal{N}$ and combining with

$$-\frac{1}{2}\ln\det\begin{pmatrix} iP^0_{LL}(\tau,\tau') & \frac{\delta^c(\tau,\tau')}{U}\\ \frac{\delta^c(\tau,\tau')}{U} & iP^0_{RR}(\tau,\tau')\end{pmatrix}$$

and using the identities

$$\det\begin{pmatrix}\mathbb{I} & \mathbb{A}\\ \mathbb{B} & \mathbb{I}\end{pmatrix} = \det[\mathbb{I} - \mathbb{A}\mathbb{B}]$$

and $\ln\det[\mathbb{A}] = {\rm Tr}[\ln\mathbb{A}]$, we get

$$\ln\mathcal{Z}[\chi;T-T_0] \approx -\ln\mathcal{N} + i\int_c d\tau\frac{\phi^0_L(\tau)\phi^0_R(\tau)}{U} + {\rm Tr}\ln\left(\left[G^c_{\phi^0}\right]^{-1}\right) - \frac{1}{2}{\rm Tr}\ln\left[\delta^c(\tau,\tau') + U^2\int_c d\tau_1 P^0_{RR}(\tau,\tau_1)P^0_{LL}(\tau_1,\tau')\right]. \tag{B20}$$

Tr in the above equation now stands only for the trace over the contour time. The set of approximations made until now can be termed as the mean-field dressed random-phase approximation based on the Feynman diagram representation of the final expression.

The approximate expression for $\ln\mathcal{Z}[\chi;T-T_0]$ given in Eq. (B20) is valid for arbitrary measurement times $(T-T_0)$. But in this work we are interested in only the steady state; hence, we take $(T-T_0)\to\infty$ and neglect information contained in the initial state of the system. To solve the self-consistent system of equations given in Eqs. (B12) and (B16), we approximate $\{\phi^0_\alpha(\tau)\}$ as being independent of contour time ($\{\phi^0_\alpha(\tau)\} = \{\phi_\alpha\}$), meaning we assume that the stationary paths $\{\phi^0_\alpha(\tau)\}$ are independent of time and are the same on the forward and backward branches of the contour. At this level, neglecting the fluctuations of the Hubbard field, or, equivalently, approximating the path integral within the self-consistent Hartree-Fock/mean-field approximation, leads to no heat flux and fluctuations at steady state. This is because within this approximation only the second and the third terms (apart from the normalization factor), which are independent of the counting field, are retained in Eq. (B20). Hence, fluctuations of Hubbard fields (directly related to the charge density fluctuations [78] on the quantum dots) around their mean-field values are necessary to have finite heat flux and fluctuations. Within this approximation, the equation for $[G^c_\phi]_{\alpha\alpha'}(\tau,\tau')$, Eqs. (B12), is solved in the frequency domain by first projecting it onto real times, which gives four Keldysh components for each α,α' (notice that $[G^c_\phi]_{\alpha\alpha'}(\tau,\tau') \propto \delta_{\alpha\alpha'}$ is block diagonal in orbital space) and sending all temporal integrals from $-\infty$ to $+\infty$, followed by Fourier transforming to the frequency domain [76,88,104]. The solution of Eq. (B12) is then given as

$$[G_{\phi^0}]_{\alpha\alpha}(\omega) = \frac{1}{(\omega-\epsilon_\alpha-\phi_\alpha)^2 + \left(\frac{\Gamma_\alpha}{2}\right)^2}\begin{pmatrix}(\omega-\epsilon_\alpha-\phi_\alpha) - i\frac{\Gamma_\alpha}{2}[1-2f_\alpha(\omega)] & i\Gamma_\alpha f_\alpha(\omega)e^{i(\omega-\mu_\alpha)\chi_\alpha}\\ -i\Gamma_\alpha[1-f_\alpha(\omega)]e^{-i(\omega-\mu_\alpha)\chi_\alpha} & -(\omega-\epsilon_\alpha-\phi_\alpha) - i\frac{\Gamma_\alpha}{2}[1-2f_\alpha(\omega)]\end{pmatrix}, \tag{B21}$$

where $\Gamma_\alpha = 2\pi |g_\alpha|^2 \rho_\alpha$. With this, $[G_{\phi^0}]_{\alpha\alpha}(\omega)$ is a function of constant stationary paths $\{\phi^0\}$, which are determined self-consistently using Eq. (B16), which in real time read

$$\begin{aligned}\phi_L^0 &= -iU\big[G^c_{\phi^0}\big]^{++}_{RR}(t,t^+) = -iU\big[G^c_{\phi^0}\big]^{--}_{RR}(t^+,t),\\ \phi_R^0 &= -iU\big[G^c_{\phi^0}\big]^{++}_{LL}(t,t^+) = -iU\big[G^c_{\phi^0}\big]^{--}_{LL}(t^+,t).\end{aligned} \tag{B22}$$

Expressing these equations in the frequency domain using Eq. (B21), we get

$$\begin{aligned}\phi_L^0 &= U\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Gamma_R f_R(\omega)}{\left(\omega-\epsilon_R-\phi_R^0\right)^2+\left(\frac{\Gamma_R}{2}\right)^2},\\ \phi_R^0 &= U\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Gamma_L f_L(\omega)}{\left(\omega-\epsilon_L-\phi_L^0\right)^2+\left(\frac{\Gamma_L}{2}\right)^2}.\end{aligned} \tag{B23}$$

The ω integrals in the above equations can be analytically performed to get

$$\begin{aligned}\phi_L^0 &= U\left\{\frac{1}{2}-\frac{1}{\pi}\mathrm{Im}\Psi\left[\frac{1}{2}+\frac{\beta_R\Gamma_R}{4\pi}+i\frac{\beta_R}{2\pi}\left(\epsilon_R+\phi_R^0-\mu_R\right)\right]\right\},\\ \phi_R^0 &= U\left\{\frac{1}{2}-\frac{1}{\pi}\mathrm{Im}\Psi\left[\frac{1}{2}+\frac{\beta_L\Gamma_L}{4\pi}+i\frac{\beta_L}{2\pi}\left(\epsilon_L+\phi_L^0-\mu_L\right)\right]\right\},\end{aligned} \tag{B24}$$

where $\mathrm{Im}\Psi[z]$ is the imaginary part of the digamma function evaluated at z [98]. Equations (B24) are coupled nonlinear self-consistent equations for $\{\phi_\alpha^0\}$ which are difficult to solve analytically. However, if we specialize to a special parameter regime, $\epsilon_\alpha = \mu_\alpha - \frac{U}{2}$, and noting that $\mathrm{Im}\Psi[z]=0$ for real z, it is clear that $\phi_L^0=\phi_R^0=\frac{U}{2}$ is always a stable solution for Eqs. (B24) if

$$\sqrt{\frac{\beta_L U}{2}}\sqrt{\frac{\beta_R U}{2}} < \frac{\pi}{\sqrt{\Psi'\left[\frac{1}{2}+\frac{\beta_L\Gamma_L}{4\pi}\right]}}\frac{\pi}{\sqrt{\Psi'\left[\frac{1}{2}+\frac{\beta_R\Gamma_R}{4\pi}\right]}}. \tag{B25}$$

From here on we confine ourselves to this regime.

We simplify the expression for $\ln\mathcal{Z}[\chi;T-T_0]$ given in Eq. (B20) using the assumption that $\{\phi_\alpha^0(\tau)\}$ are independent of contour time, hence $\int_c d\tau\,\frac{\phi_L^0(\tau)\phi_R^0(\tau)}{U}=0$, and by absorbing $\mathrm{Tr}\ln([G^c_{\phi^0}]^{-1})$, which is independent of χ into $\ln\mathcal{N}$. Expanding the logarithmic term in the Taylor series, projecting onto real times, and sending intermediate time integrals to $-\infty$ to $+\infty$ and going over to the frequency domain, we get the long-time expression for the scaled cumulant generating function as

$$\mathcal{F}[\chi] \approx -\ln\mathcal{N} - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\det\left[\mathbb{I}_{2\times2}+U^2P^0_{RR}(\omega)P^0_{LL}(\omega)\right]. \tag{B26}$$

Here

$$P^0_{\alpha\alpha}(\omega) = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\begin{pmatrix}[G_\phi]^{++}_{\alpha\alpha}(\omega+\omega')[G_\phi]^{++}_{\alpha\alpha}(\omega') & [G_\phi]^{+-}_{\alpha\alpha}(\omega+\omega')[G_\phi]^{-+}_{\alpha\alpha}(\omega')\\ -[G_\phi]^{-+}_{\alpha\alpha}(\omega+\omega')[G_\phi]^{+-}_{\alpha\alpha}(\omega') & -[G_\phi]^{--}_{\alpha\alpha}(\omega+\omega')[G_\phi]^{--}_{\alpha\alpha}(\omega')\end{pmatrix}. \tag{B27}$$

We notice that

$$P^0_{\alpha\alpha}(\omega) = \Lambda_\alpha(\omega)\tilde{P}^0_{\alpha\alpha}(\omega)\Lambda^\dagger_\alpha(\omega), \tag{B28}$$

where $\tilde{P} = P|_{\chi=0}$ and $\Lambda_\alpha(\omega) = e^{i\frac{\chi_\alpha}{2}\omega\sigma_z}$ (where σ_z is the Pauli matrix). Further, $\tilde{P}^0_{\alpha\alpha}(\omega)$ can be expressed in terms of retarded, advanced, and Keldysh projections of counting-field-independent polarization propagators. After performing the ω' integral analytically in Eq. (B27), we get

$$\tilde{P}^0_{\alpha\alpha}(\omega) = \mathcal{U}^T\begin{pmatrix}[\tilde{P}^0_{\alpha\alpha}]^R(\omega) & [\tilde{P}^0_{\alpha\alpha}]^K(\omega)\\ 0 & [\tilde{P}^0_{\alpha\alpha}]^A(\omega)\end{pmatrix}\mathcal{U}, \tag{B29}$$

with

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Explicit expressions for counting-field-independent Keldysh rotated polarization propagators are given as $[\tilde{P}^0_{\alpha\alpha}]^A(\omega) = -\{[\tilde{P}^0_{\alpha\alpha}]^R(\omega)\}^*$ and $[\tilde{P}^0_{\alpha\alpha}]^K(\omega) = [1+2n_\alpha(\omega)]\{[\tilde{P}^0_{\alpha\alpha}]^R(\omega) - [\tilde{P}^0_{\alpha\alpha}]^A(\omega)\}$ (the bosonic fluctuation dissipation theorem [78]) with the bosonic distribution function given by $n_\alpha(\omega) = (e^{\beta_\alpha\omega}-1)^{-1}$ and

$$\big[\tilde{P}^0_{\alpha\alpha}\big]^R(\omega) = \frac{i}{2\pi}\frac{\Gamma_\alpha}{\omega+i\Gamma_\alpha}\left[\frac{\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}+i\frac{\beta_\alpha}{2\pi}(\epsilon_\alpha+\phi_\alpha-\mu_\alpha-\omega)\big]-\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}+i\frac{\beta_\alpha}{2\pi}(\epsilon_\alpha+\phi_\alpha-\mu_\alpha)\big]}{\omega}\right]$$
$$+\frac{i}{2\pi}\frac{\Gamma_\alpha}{\omega+i\Gamma_\alpha}\left[\frac{\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}-i\frac{\beta_\alpha}{2\pi}(\epsilon_\alpha+\phi_\alpha-\mu_\alpha+\omega)\big]-\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}-i\frac{\beta_\alpha}{2\pi}(\epsilon_\alpha+\phi_\alpha-\mu_\alpha)\big]}{\omega}\right]. \tag{B30}$$

Using $\epsilon_\alpha = \mu_\alpha - \frac{U}{2}$ and $\phi^0_L = \phi^0_R = \frac{U}{2}$ in Eq. (B30) gives

$$\big[\tilde{P}^0_{\alpha\alpha}\big]^R(\omega) = \frac{i}{\pi}\frac{\Gamma_\alpha}{(\omega+i\Gamma_\alpha)}\left[\frac{\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}-i\frac{\beta_\alpha\omega}{2\pi}\big]-\Psi\big[\frac{1}{2}+\frac{\beta_\alpha\Gamma_\alpha}{4\pi}\big]}{\omega}\right]. \tag{B31}$$

We note that $P^0_{\alpha\alpha}(\omega)$ given in Eqs. (B30) and (B31) is a meromorphic function with simple poles in the lower complex plane. Fourier transform (which can easily be obtained) of Eq. (B30) displays oscillations at the characteristic frequency $(\epsilon_\alpha+\phi_\alpha-\mu_\alpha)$ and decays in time with rates depending linearly on Γ_α and β_α^{-1}, whereas Fourier transform of Eq. (B31) displays pure decay behavior, as $(\epsilon_\alpha+\phi_\alpha-\mu_\alpha)=0$. As noted previously, $P^0_{\alpha\alpha}(\omega)$ is related to the charge density fluctuations on the quantum dots; hence, at the mean-field level, depending on the parameter regime, charge density fluctuations on the quantum dots display in the time domain either a simple exponential relaxation or an exponential relaxation with oscillations at the characteristic frequency.

Finally, on using $P^0_{\alpha\alpha}(\omega)$ expressed above Eq. (B28) in terms of $\Lambda_\alpha(\omega)$ and Keldysh rotated quantities [Eq. (B29)] in Eq. (B26) (and fixing $\ln\mathcal{N}$ by imposing the normalization condition $\ln\mathcal{Z}[\chi=0;T-T_0]=0$), the final expression for the long-time-limit scaled cumulant generating function is obtained as

$$\mathcal{F}[\chi] = -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln[1-\mathbb{T}(\omega)\{n_L(\omega)[1+n_R(\omega)][e^{i\chi\omega}-1]+n_R(\omega)[1+n_L(\omega)][e^{-i\chi\omega}-1]\}], \tag{B32}$$

with the transmission function given by

$$\mathbb{T}(\omega) = \frac{4U^2\mathrm{Re}\Big\{\big[\tilde{P}^0_{LL}\big]^R(\omega)\Big\}\mathrm{Re}\Big\{\big[\tilde{P}^0_{RR}\big]^R(\omega)\Big\}}{\Big|1+U^2\big[\tilde{P}^0_{LL}\big]^R(\omega)\big[\tilde{P}^0_{RR}\big]^R(\omega)\Big|^2}, \tag{B33}$$

where $[\tilde{P}^0_{\alpha\alpha}]^R(\omega)$ are given in Eq. (B31) and $n_\alpha(\omega) = (e^{\beta_\alpha\omega}-1)^{-1}$.

APPENDIX C: CUMULANTS OF HEAT FLUX USING THE SCHWINGER-KELDYSH APPROACH

Expressions for the first four long-time-limit scaled cumulants as obtained using the Schwinger-Keldysh saddle point approximation are given as

$$\begin{aligned}
C_1 &= -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega\mathbb{T}(\omega)[n_L(\omega)-n_R(\omega)],\\
C_2 &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^2\mathbb{T}^2(\omega)[n_L(\omega)-n_R(\omega)]^2+\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^2\mathbb{T}(\omega)\{n_L(\omega)[1+n_R(\omega)]+n_R(\omega)[1+n_L(\omega)]\},\\
C_3 &= -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^3\mathbb{T}(\omega)[n_L(\omega)-n_R(\omega)]-\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^3\mathbb{T}^3(\omega)[n_L(\omega)-n_R(\omega)]^3\\
&\quad-\frac{3}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^3\mathbb{T}^2(\omega)[n_L(\omega)-n_R(\omega)]\{n_L(\omega)[1+n_R(\omega)]+n_R(\omega)[1+n_L(\omega)]\},\\
C_4 &= 3\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^4\mathbb{T}^4(\omega)[n_L(\omega)-n_R(\omega)]^4+6\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^4\mathbb{T}^3(\omega)[n_L(\omega)-n_R(\omega)]^2\{n_L(\omega)[1+n_R(\omega)]+n_R(\omega)[1+n_L(\omega)]\}\\
&\quad+\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^4\mathbb{T}(\omega)\{n_L(\omega)[1+n_R(\omega)]+n_R(\omega)[1+n_L(\omega)]\}+\frac{7}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^4\mathbb{T}^2(\omega)[n_L(\omega)-n_R(\omega)]^2\\
&\quad+6\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^4\mathbb{T}^2(\omega)n_L(\omega)n_R(\omega)[1+n_R(\omega)][1+n_L(\omega)].
\end{aligned} \tag{C1}$$

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