Anomalous Hall viscosity at the Weyl-semimetal–insulator transition

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(Received 6 February 2019; revised manuscript received 29 March 2019; published 24 May 2019)

We show that three-dimensional Lifshitz fermions arising as the critical theory at the Weyl-semimetal–insulator transition naturally develop an anomalous Hall viscosity at finite temperature. We discuss how to couple the system to nonrelativistic background sources for stress-tensor and momentum currents via a form of Newton-Cartan geometry with torsion and derive the Kubo formulas for the Hall viscosities. While the Lifshitz system that arises most naturally has scaling exponent $z = \frac{1}{2}$, we also generalize the theory for arbitrary Lifshitz scaling z and show that, in the limit $z \to 0$, it may be given a Chern-Simons interpretation by dimensionally reducing along the anisotropic direction. The Hall viscosities are expressed in terms of Dirichlet eta functions and their temperature dependence is dictated by the scaling exponent.

DOI: 10.1103/PhysRevB.99.195146

I. INTRODUCTION

In recent years, an increasing amount of interest has been devoted to understand the emergence of nondissipative (Hall) viscosity in quantum many systems [1,2].[1] In particular, two-dimensional Lifshitz fermions have been shown to possess a nonvanishing Hall viscosity both at finite temperature and at finite magnetic field [4,5], while a similar analysis in the case of the three-dimensional (3D) fermions in magnetic field has been carried out in [6]. However it is not clear whether such features are a universal property of critical Lifshitz theories or not. In the latter case, the presence of Hall viscosity may be a definite macroscopic signature of the quantum critical point. In parallel, a considerable amount of effort has been devoted to the formulation of effective field theory of nonrelativistic quantum systems. The most prominent example of this is the use of Newton-Cartan geometry [7,8] to construct the effective action for quantum Hall systems [9,10]. Even in the absence of the full Galilei group, Lifshitz and anisotropic theories have been extensively investigated [11–13].

We will study 3D critical Lifshitz fermions with broken time-reversal symmetry. By Lifshitz fermions we mean a fermionic system with anisotropic scaling in one spatial direction. The scaling exponent will be denoted by z. This is in slight contrast to the usual nomenclature in which the time direction scales with a different exponent. Recent studies using AdS/CFT [14] have suggested that such systems develop a finite Hall viscosity in a thermal bath characterizing the quantum critical region. In the AdS/CFT theory the Hall viscosity is proportional to the mixed gauge/gravitational anomaly of the high-energy fermionic theory. While this is intriguing, it is hard to explain from quantum field-theoretical considerations since at the critical point, no obvious notion of chiral symmetry is present. In particular, we will be interested in the $z = \frac{1}{2}$ theory, which is expected to describe the quantum critical point of a Weyl-semimetal–insulator transition [15].[2] We also study the $z \to 0$ limit, which is amenable to some extent to an effective field-theory treatment. The Weyl-semimetal–insulator transition and the critical point can be described by starting from the UV Dirac-type Lagrangian [17]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m + \gamma^\mu\gamma_5 b_\mu)\psi\,. \tag{1}$$

This simple model is known to have two quantum phases. When $b^2 + m^2 < 0$, the low-energy physics is described by two Weyl nodes displaced in momentum space, whereas for $b^2 + m^2 > 0$ a gap is present.[3] At the critical point, the system develops a quadratic energy dispersion in the b direction, and its low-energy physics may be described by an anisotropic two-component fermionic Hamiltonian. This can be seen explicitly by choosing a convenient basis of gamma matrices $\gamma^\mu = \{\tau_3 \otimes \sigma_3, i\tau_3 \otimes \sigma_2, -i\tau_3 \otimes \sigma_1, i\tau_2 \otimes \mathbb{1}\}$ with τ_i and σ_i denoting two copies of the standard Pauli matrices. Assuming b_μ to be spacelike, we chose coordinates such that it points in the 3 direction. The Dirac-type Hamiltonian is then

$$H = \begin{pmatrix} \sigma_\perp k_\perp + (b+m)\sigma_3 & \sigma_3 k_3 \\ \sigma_3 k_3 & \sigma_\perp k_\perp + (b-m)\sigma_3 \end{pmatrix}. \tag{2}$$

For large $|m + b| \gg |k|$, the four-component spinor (ϕ, ψ) can be reduced to a two-component spinor by setting $\phi = -k_3\psi/(m + b)$. In the case $|b - m| \gg |k|$ one solves instead for the spinor components ψ. We note that the charge-conjugation matrix in this representation is $\mathcal{C} = i\mathbb{1} \otimes \sigma_2$. In particular, this means that charge conjugation is a symmetry of the effective two-band Hamiltonian acting on ψ (see Fig. 1):

$$H = \sigma_\perp p_\perp + \sigma_3\left(s p_3^2 + \Delta\right). \tag{3}$$

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[1]This should be distinguished from the appearance of odd viscosity in classical plasma in external magnetic fields [3].

[2]The stability of the Lifshitz point under interactions has been shown in [16].

[3]See Appendix A for further discussion.

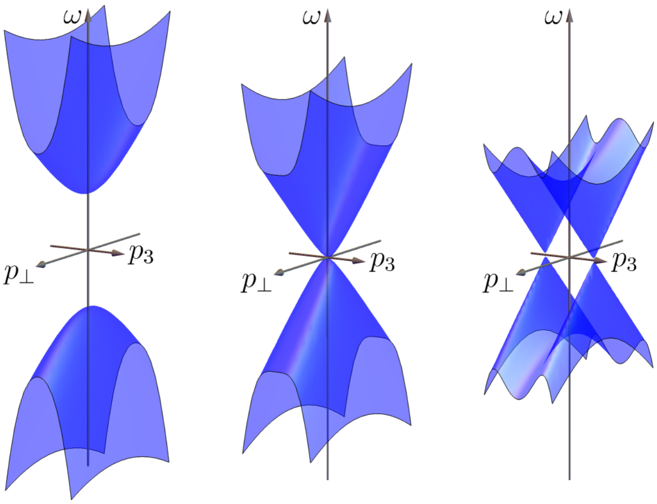

FIG. 1. Phases of the two-band Hamiltonian (3). The figure shows the dispersion relation as function of $p_\perp$ and p_3. The left figure is the insulating phase. The right figure is deep in the Weyl-semimetal phase and the middle figure shows the critical point in-between the two.

Compared to the four-band model, we have rescaled momenta by setting $k_3^2/|b+m| \to p_3^2$ and $k_\perp \to p_\perp$ and wrote $\Delta = b - m$ and $s = -\mathrm{sgn}(b+m)$. The model is gapped for $s\Delta > 0$, in a Weyl-semimetal phase for $s\Delta < 0$. At $\Delta = 0$ there is a critical point with anisotropic Lifshitz scaling symmetry $p_3 \to \lambda^{1/2} p_3$, $(\omega, p_\perp) \to \lambda(\omega, p_\perp)$. Here, $s = \pm 1$ sets the direction of fusion between the chiral Weyl points. One may say that s acts as a remnant of the emergent chiral symmetry of the model. From now on we will study the critical theory at $\Delta = 0$.

To discuss symmetries and coupling to background fields, it is slightly more natural to switch to a Lagrangian formulation. Since the rotation group is broken, we need to work with fermionic degrees of freedom transforming under the reduced rotation group SO(1, 2) only. These are just familiar (2 + 1)-dimensional fermions φ. The appropriate γ matrices γ_A, $A \in \{0, 1, 2\}$, up to unitary equivalence are taken to be $\gamma^A = (\sigma_3, -i\sigma_2, i\sigma_1)$. It is well known that the Lorentzian Clifford algebra in (2 + 1) dimensions allows a Majorana representation consistent with the already stated invariance of the Hamiltonian under charge conjugation with charge-conjugation matrix $C = i\sigma_2$. The Lagrangian is

$$\mathcal{L} = \bar{\varphi}(-p)[\gamma_A p^A + \mu(p)]\varphi(p)\,, \tag{4}$$

where $\bar{\varphi} = \varphi^\dagger \gamma^0$. In this language, the anisotropic term will act as a momentum-dependent mass $\mu(p) = s p_3^2$ whose sign is given by s. Time reversal flips the sign of the fermion mass in 2 + 1 dimensions, which in our case amounts to $s \to -s$. Thus, a time-reversal-invariant system has at least two copies of the Lagrangian (4) with opposite choices for s. The minimal model is one of a single Majorana fermion χ obeying $C\bar{\chi}^T = \chi$.

A generalization is to take the Lifshitz scaling exponent arbitrary $(\omega, p_\perp, p_3) \to (\lambda\omega, \lambda p_\perp, \lambda^z p_3)$. In this case, the momentum-dependent mass term takes the form $\mu(p) = s|p_3|^{1/z}$. As emphasized before, our Lifshitz scaling differs from the one usually employed in the literature in that the anisotropic direction is a space direction and not time, as is the case for example in Galilean physics. We also note that in the limit $z \to 0$, the momentum in the third direction P_3 becomes a central element of the Lifshitz algebra since $[D, P_3] = -zP_3$ and P_3 commutes with the other generators.

Our theory in the limit $z \to 1$ does not reduce to the case of Weyl fermions with linear isotropic dispersion. Formally, this is because the relevant term in the Lagrangian $\mu(p) = s|p_3|^{1/z}$ is defined by taking the absolute value of the momentum. The theory at $z = 1$ is still essentially anisotropic (i.e., it cannot be deformed to the isotropic case by stretching the axis or similar). We also note that our theory at $z = 1$ still breaks time reversal whereas the usual theory of Weyl fermions breaks parity.

We will compute the Hall viscosity tensor for this class of models in the linear response regime, showing that indeed it is nonzero at finite temperature. In order to do this, we will couple the system to a curved space-time with nonvanishing torsion that will allow us to properly define the stress generators and the Kubo formulas in the Lifshitz case.[4] The paper is organized as follows. In Sec. II we present the nonrelativistic Newton-Cartan–type geometry. In Sec. III we discuss the derivation of the relevant Kubo formulas and in Sec. IV we give the main steps in the Kubo formulas computation, summarize the results on Hall viscosities, and comment on the simplifications happening in the $z \to 0$ limit of (4) from the point of view of effective field theory. In Sec. V we conclude with a few remarks and open questions for further discussion. The (many) technical details are relegated to the Appendices. Throughout we use greek letters $\mu, \nu, \rho \ldots$ for space-time indices, lower-case latin letters $a, b, c \ldots$ for SO(1, 3) tangent space indices, and upper case latin letters $A, B, C \ldots$ for the unbroken SO(1, 2) tangent space indices.

Let us also briefly remark on the presence of additional gapped bands as is generically the case in a crystal. We are not aware if it has been established that such bands can give rise to Hall viscosity in the way it happens in 2D. Even if this happens, the contribution of a gapped band is necessarily independent of the temperature (as long as the temperature is much smaller than the gap). In contrast, our work will focus on the unique temperature-dependent contribution that arises in the anisotropic ungapped theory. This temperature dependence should also be a unique signature to be tested in an experiment.

II. COUPLING TO CURVED SPACE-TIME

A first step in determining the properties of a system is to examine its symmetries. In particular, we will be interested in the way the symmetry currents of our Lifshitz system couple to (external) gauge fields. This allows us to derive the most general form for the conserved currents and to compute their responses to external perturbation through the Kubo formalism.

While standard relativistic systems with the full Lorentz symmetry couple to a pseudo-Riemannian geometry, this is in

[4]A similar approach in the 2D case was developed in [18,19].

general not possible for their nonrelativistic analogs. In this section we review the geometric structure to which a Lifshitz theory should couple and explain how it can be recovered as a limit of Newton-Cartan geometry. As a by-product, we will see that a curved space-time version of theory in (4) emerges as the lowest-order derivative action which breaks T symmetry with Lifshitz scaling.

What we want to implement is a diffeomorphism covariant geometry with a preferred and covariantly constant one-form field l_μ, which reduces to δ^3_μ in the flat limit. Once this one form is specified, there are various ways to approach the problem. One is to follow the standard treatment of Newton-Cartan geometry [7,8] and then restrict the set of geometric data to be compatible with the Lifshitz scaling symmetry. We will follow an ultimately equivalent prescription, commenting in the end about the connection with Newton-Cartan geometry.

A. Geometry

The starting point for us will be a space-time metric $g_{\mu\nu}$ and a one-form field l_μ which is normalized to one $l^\mu l_\mu = 1$, with $l^\mu = g^{\mu\nu} l_\nu$. This defines a splitting of the metric

$$g_{\mu\nu} = l_\mu l_\nu + h_{\mu\nu}, \tag{5}$$

where $h_{\mu\nu} l^\mu = 0$. To specify the geometry, we further need to define the parallel transport of tensors. This requires the introduction of a connection Γ to build a covariant derivative ∇. It acts as

$$\nabla_\mu V^\alpha_\beta = \partial_\mu V^\alpha_\beta + \Gamma^\alpha_{\gamma\mu} V^\gamma_\beta - \Gamma^\gamma_{\beta\mu} V^\alpha_\gamma. \tag{6}$$

We will require the metric to be covariantly constant $\nabla_\mu g_{\alpha\beta} = 0$. This fixes the connection to the Levi-Civita form plus the undetermined contorsion tensor [20] $K^\lambda{}_{\mu\nu} = \frac{1}{2}(T^\lambda{}_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu)$, and $T^\lambda{}_{\mu\nu}$ being the torsion

$$\Gamma^\rho{}_{\mu\nu} = \tfrac{1}{2} g^{\rho\tau}(-\partial_\tau g_{\mu\nu} + \partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau}) + K^\rho{}_{\mu\nu}. \tag{7}$$

We will suppose that the torsion is purely of the form $T^\lambda{}_{\mu\nu} = l^\lambda T_{\mu\nu}$. Then, demanding l_μ to be covariantly constant fixes the torsion

$$T_{\mu\nu} = -(\partial_\mu l_\nu - \partial_\nu l_\mu). \tag{8}$$

Covariant constancy of the metric and l_μ also implies $\nabla_\mu h_{\alpha\beta} = 0$, which in turn gives

$$\mathcal{L}_l h_{\mu\nu} = 0, \tag{9}$$

with $\mathcal{L}$ denoting the Lie derivative. Even so, l^μ will not be a Killing vector for the metric in the presence of torsion

$$\mathcal{L}_l l_\mu = T_{\alpha\mu} l^\alpha \equiv G_\mu. \tag{10}$$

After some algebra, one can write the connection as

$$\begin{aligned}\Gamma^\rho{}_{\mu\nu} &= l^\rho \partial_\nu l_\mu + \tfrac{1}{2} h^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) \\ &= l^\rho \partial_\nu l_\mu + \hat{\Gamma}^\rho_{\mu\nu}[h].\end{aligned} \tag{11}$$

For a bosonic system, this is enough to determine the coupling to geometry completely. Since we are dealing with fermions, we will also need vielbein fields e^A_μ coupling to the internal spin degrees of freedom. These are defined through the splitting $h_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB}$. They also satisfy $e^A_\mu l^\mu = 0$. We will also introduce inverse vielbein fields E^μ_A defined through the orthonormality conditions

$$e^A_\mu E^\mu_B = \delta^A_B\,, \quad l_\mu E^\mu_A = 0. \tag{12}$$

As customary, we also introduce a spin connection $\omega_\mu{}^A{}_B$ which acts on fermionic fields and on the vielbein. Given the connection Γ this is uniquely determined as a function of the geometric data by demanding the vielbein to be covariantly constant

$$\nabla_\mu e^A_\nu = \partial_\mu e^A_\nu - \Gamma^\gamma_{\nu\mu} e^A_\gamma + \omega_\mu{}^A{}_B e^B_\nu = 0 \tag{13}$$

as

$$\omega_\mu{}^{AB} = -E^{\nu B}\big(\partial_\mu e^A_\nu - \hat{\Gamma}^\rho_{\nu\mu}[h] e^A_\rho\big). \tag{14}$$

Notice that in this case the spin connection is torsionless, in form language $de^A + \omega^A{}_B \wedge e^B \equiv T^A = 0$.

Let us compare this construction to the one in the Newton-Cartan formalism. We start by defining a Newton-Cartan structure through a one-form l_μ and a symmetric twice contravariant tensor $h^{\mu\nu}$ whose kernel is spanned by l_μ, namely, $h^{\mu\nu} l_\nu = 0$. One can further define the vector l^μ and the symmetric twice covariant tensor $h_{\mu\nu}$ through the algebraic relations

$$l^\mu l_\mu = 1\,, \quad l^\mu h_{\mu\nu} = 0\,, \quad h^{\mu\alpha} h_{\alpha\nu} = \delta^\mu_\nu - l^\mu l_\nu, \tag{15}$$

such that $h^{\mu\alpha} h_{\alpha\nu} = P^\mu_\nu$ is a projector orthogonal to both l^μ and l_μ. The ambient metric is then defined as

$$g_{\mu\nu} = l_\mu l_\nu + h_{\mu\nu}. \tag{16}$$

To define our geometric setup, we further need to specify a connection to parallel transport tensors. The standard way of doing this is by demanding the original data to be covariantly constant

$$\nabla_\mu l_\nu = \nabla_\mu h^{\alpha\beta} = 0, \tag{17}$$

with the further restriction that the torsion tensor $T^\lambda_{\alpha\beta}$ satisfies

$$h_{\tau\lambda} T^\lambda_{\alpha\beta} = 0. \tag{18}$$

Solving these equations fixes the connection to the same form we have found apart from an undetermined two-form $F_{\mu\nu}$:

$$\Gamma^\mu_{\nu\rho} = l^\mu \partial_\rho l_\nu + \hat{\Gamma}^\mu_{\nu\rho}[h] + h^{\mu\sigma} l_{(\nu} F_{\rho)\sigma}. \tag{19}$$

Our data are not completely specified, indeed the Milne boosts

$$l'^\mu = l^\mu + h^{\mu\nu}\Psi_\nu, \tag{20}$$

$$h'_{\alpha\beta} = h_{\alpha\beta} - 2 l_{(\alpha} P^\nu_{\beta)} \Psi_\nu + l_\alpha l_\beta h^{\mu\nu} \Psi_\mu \Psi_\nu \tag{21}$$

leave the orthonormality relations invariant. These two pieces of data are problematic for our Lifshitz effective theory. They are responsible for the U(1) particle number (for which $F_{\mu\nu}$ is interpreted as a field strength) and Galilean boost symmetries of nonrelativistic theories. We will be interested in theories that are invariant under charge conjugation, so that a real representation of the relevant degrees of freedom should exist. This suggests that we should set $F_{\mu\nu} = 0$. In parallel, Lifshitz theories with $z \neq \frac{1}{2}$ do not seem to be compatible with the Milne redefinition above [21]. We should thus fix the Milne frame by some physical consideration. A useful way to fix Ψ_ν

is to notice that, with our choice for the connection, neither l^μ nor $h_{\alpha\beta}$ are covariantly constant. A quick calculation setting $F_{\mu\nu} = 0$ gives [9]

$$\nabla_\mu l^\nu = \tfrac{1}{2} h^{\alpha\nu} \mathcal{L}_l h_{\alpha\mu}, \tag{22}$$

$$\nabla_\mu h_{\alpha\beta} = l_{(\alpha} \mathcal{L}_l h_{\beta)\mu}. \tag{23}$$

These equations are not independent, but one implies the other once the orthogonality condition $l^\mu h_{\mu\nu} = 0$ is imposed. One then sees that our geometry corresponds to a Newton-Cartan setting in which no boost symmetry is allowed and no U(1) symmetry is present either.

Of course, it would be interesting to understand if generalizations are possible in order to still accommodate fermionic Lifshitz systems with $z \neq 2$, but for this work we will not need them. Torsion appears in this geometry as a necessary tool to make the one-form l_μ covariantly constant. It is not independent data.[5]

B. Ward identities

Now we have defined the geometric background. This allows us to derive the Ward identities obeyed by the currents which couple to the set of external fields $\{e^A_\mu, l_\mu, \omega_\mu{}^A{}_B, T_{\mu\nu}\}$. Since the spin connection and the torsion are functions of e^A_μ and l_μ, they automatically lead to improved conserved currents. In the following, we will stay faithful to the quantum field-theory literature, in which the improved currents are used as generators for the symmetries. This is the natural choice if the spin connection is torsionless.

We begin by writing a general variation of the effective action

$$\delta S = -\int \sqrt{g}\big(t^\mu_A \delta e^A_\mu + p^\mu \delta l_\mu + S^\mu{}_{AB} \delta\omega_\mu{}^{AB} + \Lambda^{\mu\nu}\delta T_{\mu\nu}\big). \tag{24}$$

Here, $t^\mu{}_A$ is the unimproved energy-momentum tensor, $S^\mu{}_{AB}$ the spin current, and p^μ the anisotropic momentum current. The inverse vielbein and the vector l^μ are treated as dependent objects, whose variations are

$$\delta E^\nu_B = -E^\mu_B E^\nu_A \delta e^A_\mu - l^\nu E^\mu_B \delta l_\mu, \tag{25}$$

$$\delta l^\nu = -l^\mu E^\nu_A \delta e^A_\mu - l^\nu l^\mu \delta l_\mu. \tag{26}$$

The Ward identities follow from the local invariance of the action under diffeomorphism and tangent space rotations on the independent fields

$$\delta_\xi l_\mu = \nabla_\mu(\xi^\nu l_\nu) - T_{\nu\mu}\xi^\nu, \tag{27}$$

$$\delta_\xi e^A_\mu = \nabla_\mu\big(\xi^\nu e^A_\nu\big) - \xi^\lambda \omega_\lambda{}^A{}_B e^B_\mu, \tag{28}$$

for the diffeomorphism generated by ξ^μ and

$$\delta_\Omega l_\mu = 0, \tag{29}$$

$$\delta_\Omega e^A_\mu = \Omega^A{}_B e^B_\mu \tag{30}$$

for tangent space rotations generated by $\Omega_{AB} = -\Omega_{BA}$. The last term in (28) is not covariant under tangent space transformations, as is the case for connections. However, we may combine it together with a Lorentz variation with $\Omega^{AB}_\xi = \xi^\lambda \omega_\lambda{}^{AB}$ to cancel it. We will use such "covariantized" variation in what follows.

In view of the application of the Kubo formalism, we will find it useful to saturate the space-time indices of the objects by contracting either with the vielbein or the vector l^μ in order to better distinguish Lorentz-invariant objects. Thus, we will often use splittings of the form $V^\mu = l^\mu v + E^\mu_A v^A$. Doing this for the diffeomorphism generator $\xi^\mu = \theta l^\mu + E^\mu_A \xi^A$ gives for the covariant diffeomorphism variation

$$\delta_\theta l_\mu = \partial_\mu\theta - \theta G_\mu, \tag{31}$$

$$\delta_\theta e^A_\mu = 0, \tag{32}$$

$$\delta_\xi l_\mu = -T_{A\mu}\xi^A, \tag{33}$$

$$\delta_\xi e^A_\mu = \nabla_\mu \xi^A. \tag{34}$$

The variation of the spin connection is recovered by using the identity

$$\delta\omega_\mu{}^{AB} = -\tfrac{1}{2}\big(E^{\nu A}\nabla_\mu \delta e^B_\nu + E^{\nu B}\nabla_\nu \delta e^A_\mu - E^{\nu A}E^{\mu B} e_{\mu C}\nabla_\nu \delta e^C_\rho\big) - (A \leftrightarrow B). \tag{35}$$

This, together with the explicit dependence of $T_{\mu\nu}$ on l_μ, defines the improved currents

$$\tau^\mu{}_A = t^\mu{}_A + \tfrac{1}{2} l^\mu(\nabla^B - G^B)\sigma_{BA} + \tfrac{1}{2}[E^{\mu B}(\nabla^C - G^C)(s_{CBA} + s_{BAC} - s_{ABC}) + \nabla_l \sigma_{BA}] \tag{36}$$

and

$$\pi^\mu = p^\mu - (\nabla_\nu - G_\nu)\Lambda^{\nu\mu}, \tag{37}$$

where $\nabla_l \equiv l^\mu \nabla_\mu$, whereas s_{ABC} and σ_{AB} are defined through the splitting of the spin connection by

$$S^\mu{}_{AB} = E^{\mu C} s_{CAB} + l^\mu \sigma_{AB}. \tag{38}$$

To derive the conservation laws, one needs the identity

$$\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} = \Gamma^\nu_{\mu\nu} = \Gamma^\nu_{\nu\mu} + G_\mu, \tag{39}$$

to integrate by parts in our torsionful geometry. Plugging in the variations of the independent fields we get the diffeomorphism and Lorentz Ward identities

$$(\nabla_\mu - G_\mu)\tau^\mu{}_A = T_{A\mu}\pi^\mu, \tag{40}$$

$$(\nabla_\mu - 2G_\mu)\pi^\mu = 0, \tag{41}$$

$$e_{\mu[A}\tau^\mu{}_{B]} = 0, \tag{42}$$

[5]This differs from the setup in [18,19] in which torsion plays the role of an independent external field.

which can be recast by further saturating the contracted space-time indices through

$$\tau^{\mu}{}_{A} = E^{\mu B}\tau_{BA} + l^{\mu}\Sigma_{A}, \tag{43}$$

$$\pi^{\mu} = E^{\mu A}\pi_{A} + l^{\mu}\pi \tag{44}$$

into the form

$$(\nabla_A - G_A)\tau^{AB} + \nabla_l \Sigma^B = T^B{}_A\pi^A + G^B\pi, \tag{45}$$

$$(\nabla_A - 2G_A)\pi^A + \nabla_l\pi = 0, \tag{46}$$

$$\tau_{[AB]} = 0. \tag{47}$$

Since the theory is also Lifshitz invariant, one may introduce the following transformation rule under Weyl rescalings:

$$\delta_\sigma l_\mu = -z\sigma l_\mu, \tag{48}$$

$$\delta_\sigma e^A_\mu = -\sigma e^A_\mu, \tag{49}$$

which give rise to the Lifshitz Ward identity

$$\tau^A{}_A + z\pi = 0. \tag{50}$$

The improved stress tensor τ_{AB}, anisotropic momentum π_A, and anisotropic strain Σ_A will be the quantities used in the linear response formulation.

III. LIFSHITZ HYDRODYNAMICS AND KUBO FORMULAS FOR ANISOTROPIC HALL VISCOSITY

Now, we develop a linear response formalism to changes in the external vielbein e^A_μ and l_μ. This will give us a clear definition of the relevant Kubo formulas, together with the necessary contact (seagull) terms that may arise during the computation. In doing this, we also make contact with the hydrodynamic expansion for a fluid in a Lifshitz space-time, in which case the viscosity tensor is defined through the response of the stress tensor to a velocity gradient. Since the systems we are going to study have a spacelike, rather than timelike, vector field dictating the anisotropic direction, we will end up with a system quite different from previous studies [11,12] and from Galilean hydrodynamics. The reason is that we cannot identify our vector field l_μ with the velocity field of the long-distance hydrodynamic description as it is customarily done. Rather, the two have to be introduced separately and with a reduced tangent space bundle in order to consistently couple a Lifshitz space-time. In the end we will see that the link between viscosity (that is response to velocity gradients), from the perspective of an external relativistic observer, and time variation of the vielbein is not accurate for gradients of the l_μ components of the velocity field. Instead, the geometric response to such gradients is encoded in the torsion. We provide physical intuition behind this picture at the end of the section.

As always, one should start the hydrodynamic formulation by introducing a velocity vector field. In a fully relativistic theory this may be thought of as a tangent vector u^a normalized to $u^a u_a = -1$. This is related by a local Lorentz boost to the rest-frame field $u^a = (1, \vec{0})$. The hydrodynamic equations then follow by substituting in the Ward identities for the conserved current the most general expansion for their one-point functions in terms of gradients of the velocity vector and (possibly) external gauge fields.

In our case, however, the boost symmetry is restricted, so that, if we define a velocity field

$$u^\mu = \theta l^\mu + v^A E^\mu_A, \tag{51}$$

only the latter part of the above expression may be brought in a canonical form $v^A = (\mathrm{v}, \tilde{0})$ via local Lorentz boosts. Thus, in our case the anisotropic velocity θ should be viewed as an intrinsic property of the flow and it will be instructive to divide such flows in two parts, depending on whether or not $\theta = 0$.

Another, probably more intuitive, interpretation is as follows. In a flat geometry with $e^A_\mu = \delta^A_\mu$ and $l_\mu = \delta^3_\mu$ we have the conservation equation $\partial_A\pi^A + \partial_3\pi = 0$. This shows that $\int d^3x\,\pi^0$ is a conserved charge and we can define a grand canonical ensemble with chemical potential conjugate to this charge. In fact, this charge is nothing but the momentum in the 3 direction. This chemical potential should be identified with the parameter θ in the same way as fluid velocity v^A is the chemical potential for the other momentum components. In this interpretation, then we define the rest frame as $v^A = (1, 0, 0)$ and $\theta = 0$.

Let us start by considering the case $\theta = 0$. We will work in a derivative expansion around the rest frame $v^A = (1, 0, 0)$ and in metric perturbations around the "flat" geometry $e^A_\mu = \delta^A_\mu$, $l_\mu = \delta^3_\mu$. We are interested here only in the viscosity tensor that appears at first order in derivatives and will not discuss lower-order terms.

To first order in derivatives, we of course need to take into account the covariant derivative of the velocity field $\nabla_\mu v^A$. However, at the same order in derivatives we should also keep track of other independent data in our chosen geometry. This is the background torsion

$$T_{\mu\nu} = -(\partial_\mu l_\nu - \partial_\nu l_\mu). \tag{52}$$

These data are now to be projected such that they are orthogonal to the velocity field v^A, through the projector

$$P^A_B = \delta^A_B + v^A v_B, \tag{53}$$

which we often leave implicit to avoid cluttering of notation. Furthermore, space-time indices, when present, will be saturated using the geometric data e^A_μ, l^μ and then projected. This gives the following set of data:

$$\nabla_\mu v_A = l_\mu\nabla_l v_A + e^B_\mu(\hat{\sigma}_{AB} + \eta_{AB}\Theta + \epsilon_{AB}\omega), \tag{54}$$

having defined the shear $\hat{\sigma}_{AB} = \nabla_{(A}v_{B)} - \frac{1}{2}\eta_{AB}\nabla_C v^C$, the expansion $\Theta = \nabla_C v^C$, and the vorticity $\omega = \epsilon^{ABC}v_A\nabla_B v_C$, with $\epsilon_{AB} = \epsilon_{ABC}v^C$. In much the same way, the torsion tensor also has an electric-magnetic decomposition through

$$T_{\mu\nu} = 2\big[l_{[\mu}e^A_{\nu]}G_A + e^A_{[\mu}e^B_{\nu]}(\zeta_{[B}v_{A]} + \epsilon_{AB}m)\big], \tag{55}$$

where ζ_A and m are the analogs of electric and magnetic field for three-dimensional electrodynamics, with torsion playing the role of field strength. From an ambient metric point of view, the magnetic component is somewhat analogous to a gravitomagnetic field. Note that contrary to the usual case here this "gravitomagnetic" field is a covariant tensor and can appear independently in the response. In this sense it seems related to a response pattern that is familiar from the chiral vortical effect [22]. For now we defer study of anomalous transport patters analogous to chiral vortical (and chiral magnetic) effects in the Lifshitz model to future investigation, although we present some partial results in the discussion section.

At this point, we would be ready to develop the most general hydrodynamic response formalism for our Lifshitz theories. However, for this work we focus on the nondissipative, time-dependent responses in the strain tensor and the anisotropic momentum current.

First, let us briefly recall the basic definitions of the viscosity tensor. In isotropic theories it is defined as the response of the strain to gradients of the velocity fields

$$\langle \tau^{\mu\nu} \rangle = \eta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma + O(\nabla^2), \tag{56}$$

due to the symmetry of the strain tensor it satisfies $\eta^{\mu\nu\rho\sigma} = \eta^{\nu\mu\rho\sigma} = \eta^{\mu\nu\sigma\rho}$ where the last equality follows from the fact that the viscosity may be computed as a two-point function of strain tensors. The viscosity tensor, furthermore, may be divided in a dissipative and nondissipative (Hall) part according to the symmetry of the two couples of indices

$$\eta_D^{\mu\nu\rho\sigma} = \eta_D^{\rho\sigma\mu\nu}, \quad \eta_H^{\mu\nu\rho\sigma} = -\eta_H^{\rho\sigma\mu\nu}. \tag{57}$$

The dissipative part of the viscosity may be further decomposed in symmetric traceless (shear) and trace-part (bulk) viscosities, while the Hall viscosity requires the introduction of a dimension-dependent tensor. In 2D this is given by the projector

$$P^H_{\mu\nu\rho\sigma} = \tfrac{1}{4}(h_{\mu\rho}\epsilon_{\nu\sigma} + h_{\nu\rho}\epsilon_{\mu\sigma} + h_{\mu\sigma}\epsilon_{\nu\rho} + h_{\nu\sigma}\epsilon_{\mu\rho}), \tag{58}$$

with $\epsilon_{\mu\nu} = \epsilon_{\mu\nu\rho}u^\rho$ and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. It is clear that, in $3+1$ dimensions, one then needs also the presence of an additional vector field, say b_μ, orthogonal to the velocity field to mimic this construction, now using $\tilde{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} b_\rho u_\sigma$ to construct the projector

$$\tilde{P}^H_{\mu\nu\rho\sigma} = \tfrac{1}{4}(h_{\mu\rho}\tilde{\epsilon}_{\nu\sigma} + h_{\nu\rho}\tilde{\epsilon}_{\mu\sigma} + h_{\mu\sigma}\tilde{\epsilon}_{\nu\rho} + h_{\nu\sigma}\tilde{\epsilon}_{\mu\rho}). \tag{59}$$

This is not, however, the only tensor structure with the required properties, in fact,

$$\Pi^{(1)}_{\mu\nu\rho\sigma} = b_\mu b_\rho \tilde{\epsilon}_{\nu\sigma}, \quad \Pi^{(2)}_{\mu\nu\rho\sigma} = \Pi^{(1)}_{\nu\mu\sigma\rho},$$
$$\Pi^{(3)}_{\mu\nu\rho\sigma} = \Pi^{(1)}_{\mu\nu\sigma\rho} + \Pi^{(1)}_{\nu\mu\rho\sigma} \tag{60}$$

also satisfy the required conditions. Thus, one expects four independent Hall viscosity components to be present. In our formulation, the fixed vector b_μ is substituted by l_μ and the corresponding indices are automatically saturated, thus, we remain with only two projectors $P_{ABCD} = \epsilon^{(A(C}\eta^{B)D)}$ and $\epsilon_{AB} = \epsilon_{ABC}v^C$, while we have to explicitly distinguish the operators τ_{AB}, Σ_A, π_A due to the lack of Lorentz invariance.

The most general expansion for the Hall coefficients then reads as

$$\langle \tau^{AB} \rangle = \eta_\tau^{ABCD} \hat{\sigma}_{CD}, \tag{61}$$

$$\langle \pi^A \rangle = \eta^\pi \epsilon^{AB} \zeta_B + \eta^{\pi\Sigma} \epsilon^{AB} \nabla_l v_B, \tag{62}$$

$$\langle \Sigma^A \rangle = \eta^\Sigma \epsilon^{AB} \nabla_l v_B + \eta^{\pi\Sigma} \epsilon^{AB} \zeta_B, \tag{63}$$

where $\eta_\tau^{ABCD} = \eta_\tau P^{ABCD}$. Notice that at this stage we have not included gradients of θ since these terms are captured in the response to torsion ζ_A as we will argue in the following.

To derive Kubo formulas for the above coefficients, we expand to first order in the external geometric data, setting $v^A = (1, \vec{0})$ to its rest-frame value. By using

$$\nabla_\mu v^A \sim \partial_t e^A_\mu, \quad \zeta_A \sim E^\mu_A \partial_t l_\mu, \tag{64}$$

and with $\partial_t = v^A \partial_A$ denoting the time derivative, this gives

$$\langle \tau^{AB} \rangle = \eta_\tau^{ABCD} E^\mu_C \partial_t e_{\mu D}, \tag{65}$$

$$\langle \pi^A \rangle = \eta^\pi \epsilon^{AB} E^\mu_B \partial_t l_\mu + \eta^{\pi\Sigma} \epsilon^{AB} l^\mu \partial_t e_{\mu B}, \tag{66}$$

$$\langle \Sigma^A \rangle = \eta^\Sigma \epsilon^{AB} l^\mu \partial_t e_{\mu B} + \eta^{\pi\Sigma} \epsilon^{AB} E^\mu_b \partial_t l_\mu. \tag{67}$$

Upon functional differentiation with respect to e^A_μ, l_μ, we find the Kubo formulas:

$$\eta^\tau = \lim_{\omega\to 0} \frac{-i}{\omega} P_H^{ABCD} \big[G^{\tau\tau}_{ABCD}(\omega, 0) + C_{ABCD}(\omega, 0) \big], \tag{68}$$

$$\eta^\pi = \lim_{\omega\to 0} \frac{-i}{\omega} \epsilon^{AB} G^{\pi\pi}_{AB}(\omega, 0), \tag{69}$$

$$\eta^\Sigma = \lim_{\omega\to 0} \frac{-i}{\omega} \epsilon^{AB} \big[G^{\Sigma\Sigma}_{AB}(\omega, 0) + C_{AB}(\omega, 0) \big], \tag{70}$$

$$\eta^{\pi\Sigma} = \lim_{\omega\to 0} \frac{-i}{\omega} \epsilon^{AB} G^{\Sigma\pi}_{AB}(\omega, 0), \tag{71}$$

where we have defined the retarded Green's function

$$G^{UV}(\omega, \vec{k}) = \int d^4x\, e^{i(\omega t - \vec{k}\cdot\vec{x})} \mathrm{tr}(\rho_\beta [U(\vec{x}, t), V(0, 0)]) \theta(t)\,, \tag{72}$$

and C_{ABCD}, C_{AB} stand for contact terms which arise due to the explicit connection dependence of the strain generators. For our specific model, they are computed in Appendix B.

We will use such formulas in the next section to compute the odd viscosities. Let us stop for a moment to examine what we have found so far. Contrary to the isotropic case, there are four independent Hall viscosities

$$\eta^\tau\,, \quad \eta^\pi\,, \quad \eta^\Sigma\,, \quad \eta^{\pi\Sigma}. \tag{73}$$

In 3D such coefficients can arise only because we have broken the full rotational symmetry. Under time reversal, all of the above coefficients have to be odd in order to be nonvanishing. Thus, the microscopic theory supporting them should break this discrete symmetry.

In our formulation of hydrodynamics we have not gauged the Lifshitz scaling symmetry. Imposing it on the viscosities (73) through the Weyl scalings $l_\mu \to e^{-z\Omega} l_\mu$, $e^A_\mu \to e^{-\Omega} e^A_\mu$. This gives the following Lifshitz scaling dimensions for the viscosities:

$$[\eta^\tau]_L = 2 + z, \quad [\eta^\pi]_L = 3z, \quad [\eta^\Sigma]_L = 4 - z,$$
$$[\eta^{\pi\Sigma}]_L = 2 + z. \tag{74}$$

Thus, nonvanishing Hall viscosities need the state in which our theory is in to break the scaling symmetry. In our case, the breaking will be due to finite temperature.

It may seem that from the point of view of the Lifshitz theory, only η^τ and η^Σ can be interpreted as viscosities since they are explicitly related to gradients of the velocity field v^A. The coefficients η^π and, in part $\eta^{\pi\Sigma}$, instead, are related to the torsional response, which is more akin to an electric conductivity. However, a moment of thought shows that the response to electric torsion needs to go together with the gradients of θ in the hydrodynamic expansion. There are two ways to justify this dual description. First, one uses the embedding in the full relativistic UV system to perform an SO(1, 3) frame redefinition. In particular, at the linearized level

$$l_\mu \to l'_\mu = l_\mu + \xi_A e^A_\mu. \tag{75}$$

Suppose we start with a geometry with vanishing θ but nonvanishing torsion. Then, the transformation above sends us to a geometry with torsion

$$T'_{\mu\nu} = T_{\mu\nu} + 2e^A_{[\mu}\partial_{\nu]}\xi_A + O(e^3) \tag{76}$$

and θ component

$$\theta = u_A\xi^A, \tag{77}$$

we may thus choose ξ^A to make the torsion vanish, ending up with a nontrivial velocity gradient and vice versa.

Another, perhaps, clearer way to see this is to remember that in a boosted frame the hydrodynamic ensemble is constructed by coupling the conserved momentum charges P^μ to the fluid velocity u^μ. In particular, we may define

$$\tau^{\mu\nu} = \tau^{AB}E^\mu_A E^\nu_B + \pi^\mu l^\nu + l^\mu\Sigma^\nu + l^\mu l^\nu\pi, \tag{78}$$

putting all of the conserved currents in the same multiplet. The conserved momentum is obtained by integrating the conserved charge $\tau_{\mu\nu}u^\nu$ on a spatial surface of the foliation generated by u^μ:

$$u^\mu P_\mu = v^A\int\tau_{At} + \theta\int\pi_t, \tag{79}$$

so that θ behaves as a chemical potential for the momentum in the anisotropic direction. Recall that, in the electromagnetic case, the chemical potential appears in conjunction to the electric field so that together they form the Lie derivative of the vector potential along the velocity flow (provided we choose a gauge such that $A_t = \mu$). Then, at fixed temperature $\mathcal{L}_u A = \nabla\mu - E$ as expected. In our case, the role of the connection is played by l_μ (more precisely by $\delta l_\mu = l_\mu - \delta^3_\mu$), its Lie derivative reads as

$$\mathcal{L}_u l_\mu = \nabla_\mu\theta + \theta G_\mu - \zeta_A. \tag{80}$$

Vanishing of the right-hand side of this equation should be seen as the expression of "chemical equilibrium" for anisotropic translations. Notice that, in this way, the viscosity coefficients that enter though the response to the gradient of θ will also be expressible as conductivities for the electric part of the torsion.

With this understanding we can now give the physical interpretation of the different Hall viscosities in a flat background $e^A_\mu = \delta^A_\mu$, $l_\mu = e^z_\mu$, $A \in \{t, x, y\}$, and $\mu \in \{0, 1, 2, 3\}$. To this end, we first note that τ^{AB} contains the energy density and the pressures in the diagonal. Its off-diagonal entries can be interpreted either as the x, y components of the energy current or the momentum densities in the x, y direction. Entries with two spatial indices are components of the strain tensor. The momentum density in the z direction is π^t, π is the zz component of the pressure. The current Σ^t is the energy current in the z direction, etc.

The first viscosity component η^τ is the analog of the well known two-dimensional Hall viscosity; it is activated if the flow and gradients are all orthogonal to l_μ. If the flow is in the x direction but has a gradient in the z direction, $\eta^{\pi\Sigma}$ and η^Σ describe the generation of the strain components π^y and Σ^y. If the flow is in the z direction and has a gradient in the x direction, $\eta^{\pi\Sigma}$ and η^π describe the generation of π^y and Σ^y. We note that these last viscosities are chiral in the sense that they involve all three directions x, y, z and have a definite handedness that is determined by the parameter s in the microscopic Lagrangian.

IV. HALL VISCOSITY OF LIFSHITZ FERMIONS

Let us now come to the question of determining whether or not the viscosities (73), even though allowed by the symmetries of the problem, are nonzero for a quantum critical theory such as Lifshitz fermions. Also, if they are nonvanishing, it is interesting to ask if such coefficients contain universal information about the nature of the critical point.

We will explicitly compute the value of the coefficients (73) below, while at the end of the section we give a partial answer to the question of universality, at least in a particular limit. To this end, we consider the following effective description of the Lifshitz system, which is the minimal model compatible with anisotropic Lifshitz scaling and breaking time reversal while preserving charge conjugation and parity.

A. Microscopic model

The action in the curved geometry of Sec. II reads as

$$S_z = \int_{\mathcal{M}}\sqrt{-g}\left(\bar\chi i\gamma^A E^\mu_A\overleftrightarrow{\nabla}_\mu\chi + s\chi^T M(\nabla_l)^{1/2z}C^{-1}\chi\right), \tag{81}$$

with γ^A denoting a Majorana representation of the three-dimensional Clifford algebra $Cl(1,2)$, $M(\nabla_l) = \overleftarrow{\nabla}_l\overrightarrow{\nabla}_l$ and $s = \pm$ the T-odd parameter. The covariant derivative acts on fermions through $\nabla_\mu\chi = \partial_\mu\chi + \omega_\mu{}^{AB}\gamma_{AB}\chi$ where $\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B]$ are the Lorentz generators.

Notice that, strictly speaking, the Lagrangian is local only when $z = 1/2n$. In particular, $z = \frac{1}{2}$ represents the critical point of the Weyl-semimetal–insulator transition and $z = 1/2n$ can be adiabatically reached from this by tuning infrared irrelevant couplings (see Appendix A). To work in a unified way with Majoranas is expedient to introduce the matrices $\beta^A = C^{-1}\gamma^A$ which may be represented as $\beta_0 = -1$, $\beta_1 = -\sigma_x$, $\beta_2 = \sigma_z$. For A a spatial index these fulfill $\{\beta_A, C^{-1}\} = 0$, $[\beta_1, \beta_2] = 2C^{-1}$.

Notice also that $M(\nabla_l)$ is a positive operator, which can be seen as a mass term for 2D Majorana fermions. From this perspective, the sign of s is the sign of the mass of the fermionic excitations. We will be interested in defining a strain tensor and an anisotropic momentum current for the theory in question. Using (25) one can first compute the unimproved currents t^μ_A, p^μ:

$$t^\mu_A = iE^\mu_B \chi^T \beta^B \overleftrightarrow{\nabla}_A \chi + \frac{s}{2z} l^\mu \chi^T [\overleftarrow{\nabla}_l M(\nabla_l)^{1/2z-1} \overrightarrow{\nabla}_A + \overleftarrow{\nabla}_A M^{1/2z-1}(\nabla_l) \overrightarrow{\nabla}_l] C^{-1} \chi, \tag{82}$$

$$p^\mu = iE^\mu_A \chi^T \beta^A \overleftrightarrow{\nabla}_l \chi + \frac{s}{z} l^\mu \chi^T M(\nabla_l)^{1/2z} C^{-1} \chi. \tag{83}$$

The spin current is given by ${S_\mu}^{AB} = e_{\mu C} s^{CAB} + l_\mu \sigma^{AB}$ with

$$s_{CAB} = i\bar{\chi}(\gamma_C \gamma_{AB} + \gamma_{AB}\gamma_C)\chi, \tag{84}$$

$$\sigma_{AB} = -\frac{s}{2z} \chi^T M(\nabla_l)^{1/2z-1} (\overleftarrow{\nabla}_l \gamma_{AB} C^{-1} - \overrightarrow{\nabla}_l C^{-1} \gamma_{AB})\chi, \tag{85}$$

while the torsion current $\Lambda^{\mu\nu}$ vanishes identically.

The improvement procedure has been explained in Sec. II and can be carried out in a straightforward way. To this end, notice that the spin current entering in τ_{AB} has the same structure as the isotropic free-fermion one, plus contributions from σ_{AB}. Since these are by nature antisymmetric but $\tau_{[AB]} = 0$ by the Lorentz Ward identity they will cancel on shell against contributions coming from the covariant derivative acting on s_{CAB}. The final result will be equal to the one obtained for the isotropic fermion. On the other hand, the momentum current receives no further contribution. We thus have

$$\tau_{AB} = i\chi^T \beta_{(A} \overleftrightarrow{\nabla}_{B)} \chi, \quad \pi_A = i\chi^T \beta_A \overleftrightarrow{\nabla}_l \chi, \tag{86}$$

$$\Sigma_A = \frac{s}{2z} \chi^T [\overleftarrow{\nabla}_l M(\nabla_l)^{1/2z-1} \overrightarrow{\nabla}_A + \overleftarrow{\nabla}_A M^{1/2z-1}(\nabla_l) \overrightarrow{\nabla}_l] C^{-1} \chi + \frac{1}{2} \nabla^B \sigma_{BA}. \tag{87}$$

Notice that in the above the order of the covariant derivatives matters since, in our geometry,

$$[\nabla_\mu, \nabla_\nu] = 2\partial_{[\mu} l_{\nu]} \nabla_l + {R_{\mu\nu}}^{AB} \gamma_{AB}, \tag{88}$$

when acting on fermions. Even though these expressions look complicated, they simplify considerably in momentum space and we will be able to analytically extract the viscosities. In order to compute them, we follow the standard technique for computing retarded Green's functions from analytic continuation of Euclidean ones. For this we analytically continue the Majorana fermions to Euclidean signature [23].

The Euclidean correlators are given by the following (imaginary time) Feynman diagrams:

$$G^{\pi\pi}_{AB}(\omega) = \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\big[S(k, \omega_n)\beta_A S(k, \omega + \omega_n)\beta_B k_3^2\big], \tag{89}$$

$$G^{\tau\tau}_{ABCD}(\omega) = \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}[S(k, \omega_n)\beta_{(A} S(k, \omega + \omega_n)\beta_{(C} k_{B)} k_{D)}], \tag{90}$$

$$C_{ABCD}(\omega) = -\frac{\delta_{AC}}{16} \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}[\beta_{[B}\beta_{D]} \omega \beta_0 S(k, \omega_n)] + A \leftrightarrow B\,, C \leftrightarrow D, \tag{91}$$

$$\begin{aligned} G^{\Sigma\pi}_{AB}(\omega) = {}& \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\left[S(k, \omega_n) C^{-1} k_A \frac{s}{2z} |k_3|^{1/z} k_3^{-1} S(k, \omega + \omega_n) \beta_B\right] \\ & + \frac{\omega}{4} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\left[S(k, \omega_n) \frac{s}{2z} |k_3|^{1/z} k_3^{-1} \beta_B S(k, \omega_n) C^{-1} \beta_A S(k, \omega_n)\right], \end{aligned} \tag{92}$$

$$\begin{aligned} G^{\Sigma\Sigma}_{AB}(\omega) = {}& \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\left[S(k, \omega_n) C^{-1} k_A \frac{s}{2z} |k_3|^{1/z} S(k, \omega + \omega_n) C^{-1} k_B \frac{s}{2z} |k_3|^{1/z}\right] \\ & + \omega \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\left[S(k, \omega_n) C^{-1} k_B \frac{s}{2z} |k_3|^{1/z} k_3^{-1} S(k, \omega + \omega_n) |k_3|^{1/z} k_3^{-1} \frac{s}{z} \beta_A C^{-1}\right], \end{aligned} \tag{93}$$

$$C_{AB}(\omega) = \frac{1}{\beta} \sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3} \mathrm{tr}\left[|k_3|^{1/z-2} \frac{1}{4z^2} (\omega_n C^{-1} \beta_A \beta_B + k_A C^{-1} \beta_B) S(k, \omega_n)\right], \tag{94}$$

where we have introduced the Majorana propagator $S(p) = [\beta^A p_A + sM(p)^{1/2z} C^{-1}]^{-1}$. The form of the contact terms, which require quite a lengthy computation, is justified in Appendix B. At this point, the external $\omega = 2\pi nT$ is a bosonic Matsubara frequency, and the Lorentzian continuation is defined by the substitution $\omega \to i(\omega_L + i\epsilon)$ after the sum over internal frequencies has been performed. The retarded and Euclidean Green's functions are related by

$$G(\omega,\vec{k}) = -iG_E(\omega+i\epsilon,\vec{k}). \tag{95}$$

The Matsubara sum over fermionic frequencies is evaluated using the integral representation of the fermionic sums

$$\frac{1}{\beta}\sum_n f(\omega_n) = \frac{1}{2}\int_C \frac{dz}{2\pi i}\tanh(\beta z/2) f(z), \tag{96}$$

where C is a contour encircling the poles of the hyperbolic tangent. By contour deformation the sum is expressed as a sum over the residues of the poles of the function $f(z)$. Notice that in the case of Majorana fermions no antiparticles are present, so that the sum over frequencies gives half of the result of that for a Dirac fermion. Following the analysis of the previous section, we expect the viscosities to scale as

$$\eta^\tau \sim T^{2+z},\quad \eta^\pi \sim T^{3z},\quad \eta^\Sigma \sim T^{4-z},\quad \eta^{\pi\Sigma} \sim T^{2+z}. \tag{97}$$

Thus, we may safely drop all of the vacuum contributions to the thermal sums since they have no intrinsic parameter which scales under the Lifshitz symmetry.

After this has been done, one can divide by the external frequency and take the limit of $\omega_L \to 0$. The remaining momentum-space integrals are evaluated using the representations

$$\eta_D(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} n_F(t) \tag{98}$$

for the Dirichlet eta function and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = 2\int_0^{\pi/2} d\phi\, \sin(\phi)^{2b-1}\cos(\phi)^{2a-1} \tag{99}$$

for the Euler beta function. We give details of the various computations in Appendix C.

The final results are

$$\eta^\pi = \frac{s}{4\pi^2}\frac{z}{3z+1}T^{3z}\Gamma(3z)\eta_D(3z), \tag{100}$$

$$\eta^\tau = \frac{s}{4\pi^2}T^{2+z}\frac{z(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2), \tag{101}$$

$$\eta^{\pi\Sigma} = \frac{s}{4\pi^2}T^{2+z}\frac{(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2), \tag{102}$$

$$\eta^\Sigma = \frac{s}{4z\pi^2}T^{4-z}\frac{(6-z)}{(5-z)(3-z)}\Gamma(4-z)\eta_D(4-z). \tag{103}$$

Notice that it holds $\eta^\tau = z\eta^{\pi\Sigma}$. Furthermore, by rescaling $\Sigma \to z\Sigma$ the last three viscosities obey the compact relation

$$\eta^{\rm Hall}(\xi) = z\frac{s}{4\pi^2}T^\xi\frac{(\xi+2)}{(\xi+1)(\xi-1)}\Gamma(\xi)\eta_D(\xi), \tag{104}$$

being ξ their Lifshitz scaling dimension. Notice that all of the coefficients are proportional to the time-reversal-breaking parameter s, as it should be.

B. A Chern-Simons interpretation as $z \to 0$

The values of the viscosities do not seem to bear any universality since they explicitly depend on the Dirichlet eta function which regulates the thermal sums. However, at least for the anisotropic momentum current, a suggestive interpretation of the result may be given when the scaling exponent z approaches zero. In this case, the temperature dependence of η^π vanishes and one may hope to derive an effective action for the result within the framework of effective field theory. Also, if one uses Eq. (104) for the remaining three viscosities, they all vanish in this limit. On the other hand,

$$\lim_{z\to 0}\eta^\pi = \frac{s}{24\pi^2}. \tag{105}$$

The intuition behind the $z \to 0$ limit is that the system actually undergoes a dimensional reduction. This can be seen, for example, by computing the density of states with energy for the single-particle excitations $\rho(\epsilon) \sim \epsilon^{1+z}$.

In this case, the effective action is described by a $(2+1)$-dimensional field theory on a manifold which is obtained by integrating over the (possibly noncompact) anisotropic direction. Indeed, (105) is consistent with a Chern-Simons type of action

$$S_{\rm CS} = \kappa\int l\wedge dl,\quad \kappa = \frac{s}{48\pi^2} \tag{106}$$

as can be checked by functional differentiation. In this case, the Chern-Simons level needs not to be quantized since the symmetry it is associated with is noncompact.

There are various way to interpret this phenomenon, and we give two complementary explanations. They are both based on the idea that our model can be seen as an (infinite) tower of massive Majorana fermions. Notice that as $z \to 0$ the mass $\mu(k_3) = |k_3|^{1/z}$ is either arbitrarily small or big depending on whether $k_3 < 1$ or $k_3 > 1$. However, such masses are all mapped between each other by the Lifshitz symmetry, and should give the same contribution to the effective action.

This is analog to the fact that, once a massive Dirac fermion is integrated out in $2+1$ dimensions, it generates a Chern-Simons theory with half-quantized level $-\mathrm{sgn}(m)/2$. The half-quantization in our case is still present since the fact that our fermions are Majoranas balances the double occurrences of positive masses (for $k_3 > 0$ and $k_3 < 0$). As usual, one should extract the value for the Hall coefficients by comparing the result with the one with inverse sign for the mass ($s \to -s$ in our case) and subtract them.

The coefficient $\frac{1}{48}$ is explained as follows. Imagine that the anisotropic direction is compact, in the limit $z \to 0$ its radius is a number which we may set to one. Fermionic modes on this circle have quantized momenta in half-integer units $k_3 = 2n-1, n = 1, 2, \ldots$. These numbers can be interpreted as the charge of the particle under the translation current π_A. They enter the Chern-Simons action through their value squared as in the case for e^2 in the quantum Hall effect. This gives an infinite sum

$$\kappa = -\frac{s}{4\pi}\frac{1}{2\pi}\sum_{n=1}^\infty (2n-1)^2, \tag{107}$$

where the further factor $1/2\pi$ comes from the dk_3 integral converted into a sum. The sum over charges may be regulated by zeta function regularization to give $\sum_{n=1}^\infty (2n-1)^2 =$

$4[\zeta(-2)-\zeta(-1)]+\zeta(0)=\frac{1}{3}-\frac{1}{2}=-\frac{1}{6}$ so that

$$\kappa=\frac{s}{48\pi^2}, \tag{108}$$

as anticipated.

Another way to find the same result is to use the quadratic form for the Chern-Simons coefficient at finite temperature [24]:

$$\kappa(\mu(q))=-\frac{1}{8\pi}s\tanh(\beta\mu(q)/2)q^2, \tag{109}$$

where we take $\mu(q)=q^z$ and take the limit $z\to 0$ afterward; in this way, the temperature acts as a UV regulator. Integrating over the modes gives

$$\kappa=-\int_0^\infty \frac{dq}{2\pi}\frac{1}{8\pi}s\tanh(\beta\mu(q)/2)q^2, \tag{110}$$

and regulating to zero the vacuum contribution this reduces to

$$\kappa=T^{3z}\frac{s}{8\pi^2}z\int_0^\infty dt\, t^{3z-1}n_F(t)=\frac{s}{24\pi^2}\eta(3z)\Gamma(3z+1)T^{3z}, \tag{111}$$

which tends to the previous value as the limit of small z is taken.

In the case of a complex Lifshitz fermion, one could go one step further and take the quantized Hall conductivity of the single fermion $e^2\sigma_H^{2\mathrm{D}}$. Summing over the momentum in the anisotropic direction, one is led to conjecture the relationship

$$\eta^\pi=2\kappa=-\frac{1}{6\pi}\sigma_H^{2\mathrm{D}}+O(z). \tag{112}$$

We were not able to find such a nice interpretation for the remaining viscosities.

Why is the Chern-Simons interpretation valid in such limit? A partial explanation comes from the symmetry algebra of our Lifshitz system. In fact, apart from the ISO(1, 2) commutation relations, the anisotropic momentum $P_3=\int\pi_t$ appears only through the nontrivial commutator with the Lifshitz generator D:

$$[D,P_3]=-zP_3\,, \tag{113}$$

and is otherwise a central element. One thus clearly sees that, as $z\to 0$, this commutator vanishes and P_3 behaves effectively as an Abelian charge, which is dimensionless. Since in this limit the system dimensionally reduces to 2D, it is possible that a nonzero Hall conductivity may develop for such Abelian charge in the presence of massive fermions.

Incidentally, the fact that such a Chern-Simons interpretation may be given tells us that a magnetic torsion m will induce a momentum density $\pi^A v_A\equiv\pi_t$ in the anisotropic direction given by

$$\pi_t=2\kappa\, m. \tag{114}$$

Integrating the above equation relates the total anisotropic momentum with the line integral of the Burgers vector in the anisotropic direction.

This is reminiscent of the chiral vortical conductivity for Weyl fermions once the torsion is rewritten in terms of the ambient metric $g_{\mu\nu}$. For $z\neq 0$, κ does not coincide with (twice) the viscosity η^π, as the finite-temperature summation is not the same if the limits of zero frequency and momentum are interchanged. It can, however, be easily computed to be

$$\kappa=\frac{s}{8\pi^2}T^{3z}z\Gamma(3z)\eta(3z)=2(3z+1)\eta^\pi. \tag{115}$$

V. DISCUSSION

We have shown that the anisotropic fermionic Lifshitz theory possesses a nonvanishing Hall viscosity. Due to its dissipationless nature it is possible compute these particular transport coefficients at weak coupling. It should therefore give rise to measurable effects even when an essentially noninteracting quasiparticle description applies. Signatures of two-dimensional Hall viscosity in graphene in a magnetic field have recently been reported in [25]. It will be interesting to see if the Hall viscosities reported here can be measured in three-dimensional materials along similar lines.

While the Hall viscosities found here do not seem to bear any universality in general, one of its components may be given a Chern-Simons interpretation in the limit $z\to 0$. In this case, we have related it to the intrinsic 2D Hall conductivity of the dimensionally reduced system.

Also, the kind of torsional response we have uncovered is extremely reminiscent of the (much debated) torsional contribution to the mixed anomaly in 3D by the Nieh-Yan term [26,27]

$$\mathrm{NY}[e]=T^a\wedge T_a-R(\omega)_{ab}\wedge e^a\wedge e^b, \tag{116}$$

which in our case should reduce to

$$\mathrm{NY}[l]=T\wedge T. \tag{117}$$

Such contribution has been studied in the context of quantum Hall systems and Weyl semimetals in various occasions [18,19]. Because of dimensionality reasons it always comes together with an unspecified UV scale which makes its interpretation very difficult.

The Lifshitz theory may provide a more natural setup to relate torsion to the underlying anomalies of the quantum field theory. In particular, as the exponent z approaches zero, it makes sense (on dimensional grounds) to write an equation like

$$(\nabla_\mu-2G_\mu)\pi^\mu=c_\pi\epsilon^{\mu\nu\rho\sigma}T_{\mu\nu}T_{\rho\sigma} \tag{118}$$

since in this limit l does not scale under the Lifshitz symmetry, as an Abelian connection should. Consistency of this Ward identity demands $c_\pi=\eta^\pi/8$.

We have studied a broad class of Lifshitz critical points, with arbitrary scaling exponent $z\leqslant 1$. We can obtain such models as relevant (in the UV) deformations of the Weyl-semimetal model, although subject to an increasing number of fine-tuning conditions (see Appendix A for further discussion). It would be interesting to see whether a lattice realization of such low-energy theories may also be given.

Finally, we note that it should be interesting to work out the full Lifshitz hydrodynamics including all the dissipative and possible additional nondissipative transport coefficients.

ACKNOWLEDGMENTS

This work has been supported by Grants No. FPA2015-65480-P(MINECO/FEDER), No. PIC2016FR6/PICS07480, and Severo Ochoa Excellence Program Grant No. SEV-2016-0597. The work of C.C. is funded by Fundación La Caixa under "La Caixa-Severo Ochoa" international predoctoral grant. We thank B. Bradlyn, M. Chernodub, A. Cortijo, and M. A. H. Vozmediano for discussions.

APPENDIX A: DETAILS OF THE FOUR-BAND MODEL

In this Appendix we review some details about the four-band model for the Weyl-semimetal–insulator transition, together with some of the salient features of the critical theory. We start with the four-band Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m + \gamma_\mu\gamma_5 b^\mu)\psi, \tag{A1}$$

which can be interpreted as a massive Dirac fermion in an axial background $\langle A^5_\mu\rangle = b_\mu$. Standard computations lead to the spectrum of the theory

$$\epsilon(k)^2_\pm = k^2 + m^2 + b^2 \pm 2|b|\sqrt{m^2 + (\hat{b}\cdot k)^2}. \tag{A2}$$

The bands responsible for the low-energy behavior are those for which the minus sign is chosen above. The low-energy phase is determined by the respective magnitude of b, m. For $|b| > |m|$ the lowest bands touch at $\vec{k}_\pm = \pm\alpha\vec{b}$, being $\alpha = \sqrt{1 - m^2/b^2}$ the screening factor for the chiral charge. In the opposite case, the system is gapped, with the gap given by $\Delta_{gap} = 2\sqrt{m^2 - b^2}$. Since we will mostly give an effective description of the two lowest bands, we should notice that the gap between these and the upper one is given by

$$\Delta_{EFT} = \min_k \epsilon_+(k) - \epsilon_-(k) = 2\max(|m|, |b|), \tag{A3}$$

thus we should always think of our results as valid below these scales. This means, for example, that in the thermal case the temperature should always be much smaller than Δ_{EFT}. Of particular interest for us will be the point $|m| = |b|$ at which the lowest bands have the approximate dispersion relation (near $k = 0$)

$$\epsilon^2/m^2 = \frac{k_\perp^2}{m^2} + \frac{(k\cdot\hat{b})^4}{4m^4} + O((k\cdot b/m)^6), \tag{A4}$$

which exhibits $z = \frac{1}{2}$ Lifshitz scaling as long as we lie below Δ_{EFT}. As one can clearly see, the parameter m has no dimensions from the point of view of the Lifshitz scaling and was thus omitted in the main text. However, in order to connect the results presented with the complete field theoretical answer one needs to reintroduce it explicitly. This simply amounts to have all the quantities scale in the right way according to the UV counting, in which m has dimensions of energy. In particular, the matrix $M^{1/2z}$ in the fermionic Lagrangian gets replaced by $\frac{1}{m^{1/z-1}}M^{1/2z}$. The viscosities also scale with m. In this case, the trick is to substitute T with the UV dimensionless quantity $\tau = T/m$ and remember that viscosity has dimensions of energy cubed, then,

$$\eta^\tau \sim m^{1-z}T^{2+z}, \quad \eta^\pi \sim m^{3-3z}T^{3z}, \tag{A5}$$

$$\eta^\Sigma \sim m^{z-1}T^{4-z}, \quad \eta^{\pi\Sigma} \sim m^{1-z}T^{2+z}. \tag{A6}$$

While for our realization of the $z = \frac{1}{2}$ theory we have a definite interpretation for the parameter m, for different values of the anisotropic scaling exponent m will in general depend on the particular UV completion one will choose. The appearance of an ultraviolet scale should not be surprising as this is often the case when dealing with torsionful theories. However, we stress that from the perspective of the critical point alone, such scale does not play any physical role.

We may furthermore generalize the model (A1) to support critical points with critical exponent $z = 1/2n$, $n \geqslant 1$. The idea is to add couplings to the higher-spin counterparts of the chiral current j_5^μ:

$$j_5^{\mu_1\ldots\mu_s} = \mathrm{Str}\big[\bar{\psi}\gamma_5\gamma^{\mu_1}\overset{\leftrightarrow\mu_2}{\partial}\ldots\overset{\leftrightarrow\mu_s}{\partial}\psi\big], \tag{A7}$$

where Str refers to the symmetric traceless projection of the tensor. As for the chiral current, such higher-spin counterparts are not conserved in the presence of a nonvanishing mass and will in general be irrelevant deformation of the infrared physics. However, let us examine

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi + \sum_{s=1} b_{\mu_1\ldots\mu_s} j_5^{\mu_1\ldots\mu_s}; \tag{A8}$$

the requirement of maintaining at least SO(1, 2) symmetry forces $b_{\mu_1\ldots\mu_s} = b_s b_{\mu_1}\ldots b_{\mu_s}$. The energy dispersion relation then becomes

$$\epsilon(k)^2_\pm = k^2 + m^2 + b(k)^2 \pm 2|b(k)|\sqrt{m^2 + (\hat{b}\cdot k)^2}, \tag{A9}$$

where we have defined $b(x) = \sum_{s=1} b_s(x\cdot\hat{b})^{s-1}$. We would like to choose the b function such that a critical point of Lifshitz scaling $z = 1/2n$ is reached for small momenta. Furthermore, we would like to have to tune only a finite number of current couplings b_s to achieve such result. We thus put the momentum in the orthogonal directions to zero and solve the scaling equation ($k_3 = k\cdot\hat{b}$)

$$k_3^2 + m^2 + b(k_3)^2 - 2|b(k_3)|\sqrt{m^2 + k_3^2} = k_3^{2n}f^2(k_3), \tag{A10}$$

for $b(k_3)$, subject to the requirement that $f^2(k_3)$ is finite at $k_3 = 0$. This gives, supposing $b, f > 0$,

$$b(k_3) = \sqrt{m^2 + k_3^2} - k_3^n f(k_3). \tag{A11}$$

At this point, we may series expand $b(k_3)$ and $f(k_3) = \sum_s f_s k_3^s$ and fix the first $2n$ coefficients to match the expression on the right-hand side. Furthermore, without loss of generality, we may set $b_s = 0$ for $s > 2n$ and thus fix the function f. The final result is a low-energy Lifshitz fixed point with $z = 1/n$, obtained by tuning $2n$ parameters through

$$b_{2s} = m^2\frac{(1/2)_s}{s!}\left(\frac{k_3}{m}\right)^{2s}, \quad s \leqslant n \tag{A12}$$

$$f_{2s} = m^2\frac{(1/2)_{s+n}}{(s+n)!}\left(\frac{k_3}{m}\right)^{2s}. \tag{A13}$$

Of course, such a critical point still has a huge amount of fine tuning. Also, it can be continuously reached by deforming the $z = \frac{1}{2}$ critical point without breaking any further symmetries. In this sense, we expect the physics at different z to belong to the same universality class.

It could be interesting to see whether less fine-tuned versions of such critical points exist, and if so what is their lattice realization.

APPENDIX B: SEAGULL TERMS

Before moving to the calculation itself, it is, however, important to verify whether any contact (seagull) term may arise from the dependence of the strain tensor on connection and torsion. Seagull terms typically arise in quantum field theory due to the explicit dependence of the curved space-time stress tensor and currents on the spin or Christoffel connection. This causes functional differentiation to give rise to terms proportional to

$$\delta^\rho_\mu \delta^A_D \frac{\delta}{\delta e^D_\rho(x)} \omega_\nu{}^{BC}(y) \equiv Z^{ABC}_{\mu\nu\alpha} \partial^\alpha \delta(x-y), \tag{B1}$$

where, in the flat space-time limit,

$$Z^{ABC}_{\mu\nu\alpha} = \tfrac{1}{2}\left(\eta_{\nu\alpha}\delta^B_\mu \delta^C_A + \delta^C_\alpha \delta^B_A \delta^\mu_\nu - \delta^B_\alpha \delta^C_\mu \delta^A_\nu\right) - (B \leftrightarrow C). \tag{B2}$$

These contribute to the linear response theory with finite terms, that are computed from a one-loop diagram with no external momenta present. In particular, the external momentum is carried by the derivative of the delta function, so that in order to compute viscosities we set $\alpha = 0$. Apart from these, other contact terms may arise by functional differentiation of the vielbein itself.[6] These terms do not carry any derivative of the vielbein and so do not contribute to the viscosity tensor. We will disregard such contributions.

Let us start from the correlators of two τ. In this case, one has to compute the classic contact term of a free-fermionic stress tensor. This is a well-known computation (see, for example, [28]) and the final result gives

$$\begin{aligned} C_{ABCD}(x,y) &= -\frac{i}{16}\delta_{AB}\chi^T(x)\left(\left\{\frac{1}{4}[\beta_B,\beta_D],\beta_0\right\}\right)\chi(x)\partial_0\delta(x-y) \\ &\quad + A \leftrightarrow B,\, C \leftrightarrow D \end{aligned} \tag{B3}$$

which in momentum space gives the contact term integral we will compute in the next section. There are three further cases to be examined. The first is the correlator of two anisotropic momentum currents π_A, π_B. Seagull terms in this case arise from the dependence of the anisotropic current on torsion. Since we work with the SO(1, 2) connection only, no such dependence arises in the covariant derivative and the contact term vanishes.

[6]They arise because we consider the basic object to be the energy-momentum current that is obtained by variation with respect to the vielbein, i.e., τ^μ_A but we use τ^A_B to calculate the viscosity and this introduces a trivial dependence on the vielbein since $\tau^A_B = e^A_\mu \tau^\mu_B$. This dependence on the vielbein therefore is not related to the viscosity tensor.

A second contact term may contribute to the $\Sigma_A \Sigma_B$ correlator due to the vielbein dependence of Σ. To start, recall that in position space this reads as

$$C^{AB} = \frac{\partial \Sigma_A}{\partial \omega_\nu{}^{CD}} Z^{BCD}_{\mu\nu\alpha} \partial^\alpha \delta(x-y) l^\mu, \tag{B4}$$

where the last l^μ projects on the right component of the vielbein variation. We will be interested of the part of said contact term which is proportional to ϵ_{AB}, thus encoding the nondissipative viscosity. First, one may notice, using the expression above for Z, that

$$Z^{BCD}_{\mu\nu\alpha} l^\mu = \tfrac{1}{2} l_\nu \delta^D_\alpha \delta^{BC} - (B \leftrightarrow C), \tag{B5}$$

thus the only contributions to the contact term will come from derivatives ∇_l in Σ. The contributions may be divided in two parts, the first stemming from the unimproved strain $\hat{\Sigma}$ and the latter from the improvement term coming from the spin current.

For the first term we have, using (82),

$$\begin{aligned} \frac{\partial \hat{\Sigma}_A}{\partial \omega^{CD}_\nu} &= \frac{s}{z}\left(\frac{1}{2z} - 1\right) l^\nu \chi^T [\overleftarrow{\partial}_l \overleftrightarrow{\partial}_A C^{-1}\beta_C C^{-1}\beta_D C^{-1} \\ &\quad + \overrightarrow{\partial}_l \overleftrightarrow{\partial}_A \beta_C C^{-1}\beta_D] M^{1/2z-2}\chi \\ &\quad + \frac{s}{z}\chi^T [\overleftarrow{\partial}_A C^{-1}\beta_C C^{-1}\beta_D C^{-1} \\ &\quad + \overrightarrow{\partial}_A \beta_C C^{-1}\beta_D] M^{1/2z-1}\chi \end{aligned} \tag{B6}$$

up to terms orthogonal to l^ν. Going to momentum space and remembering that one of the two β matrices is the identity because of (B5), one is left with an anticommutator $\beta_D C^{-1} + C^{-1}\beta_D = 0$ if D is spatial. So, the whole contribution vanishes. Thus, the possible contact terms may come from the improvement only.

The second term gives

$$\begin{aligned} \frac{\partial \Sigma^A_{\rm imp}}{\partial \omega^{CD}_\nu} &= \frac{s}{z} l^\nu \partial_B \chi^T \left[\left(\frac{1}{2z} - 1\right) M^{1/2z-2} \overleftrightarrow{\partial}_l (\overleftarrow{\partial}_l \gamma^{BA} C^{-1}\gamma^{CD} \right. \\ &\quad \left. - \overrightarrow{\partial}_l \gamma^{CD}\gamma^{BA} C^{-1})\right]\chi \\ &\quad + \frac{s}{z} l^\nu \partial_B \chi^T [M^{1/2z-1}(\gamma^{BA}C^{-1}\gamma^{CD} + \gamma^{CD}\gamma^{BA}C^{-1})]\chi. \end{aligned} \tag{B7}$$

This simplifies in a considerable way in momentum space, where the two contributions above sum if the external frequency is set to zero. The result is

$$\frac{\partial \Sigma^A_{\rm imp}}{\partial \omega^{CD}_\nu}(q) = l^\nu \frac{s}{2z^2} q^B \chi^T |ql|^{1/z-2} X^{CD}_{AB}\chi, \tag{B8}$$

with

$$X^{CD}_{AB} = \gamma^{BA} C^{-1}\gamma^{CD} + \gamma^{CD}\gamma^{BA}C^{-1}; \tag{B9}$$

the expression for X can be recasted as either a commutator or an anticommutator depending on whether $CD = 0i$ or $CD = ij$. In our case, the relevant part will be

$$X^{CD}_{AB} = [\gamma^{CD}, \gamma_{AB}]C^{-1}\left(\delta^C_0 - \delta^D_0\right), \tag{B10}$$

one may now use the Lorentz algebra

$$[\gamma_{CD},\gamma_{AB}]=\eta_{CA}\gamma_{DB}+(\text{cyclic}) \tag{B11}$$

to simplify the expression further. The final result taking into account the fact the either C or D are in the time direction reads as

$$\frac{\partial\Sigma^A_{\text{imp}}}{\partial\omega^{CD}_\nu}(q)=l^\nu\frac{s}{4z^2}\chi^T|ql|^{1/z-2}(q_0C^{-1}\beta^A\beta_D+q_DC^{-1}\beta^A)\delta^C_0\chi. \tag{B12}$$

We will use this term in the next section for the computation of the linear response.

One last contact term may come from the $\Sigma_A\,\pi_B$ correlator, and can be seen either through the torsion dependence of Σ_A or through the spin connection dependence of π_A. The first of the two is simpler to compute, in this case, in fact, the only dependence on torsion comes from the $G^B\sigma_{BA}$ term in the definition of Σ, recalling the definition of G_μ one has

$$\delta G_\mu=-l^\nu(\partial_\nu\delta l_\mu-\partial_\mu\delta l_\nu)-\delta l^\nu(dl)_{\nu\mu}; \tag{B13}$$

this gives a seagull contribution to $\eta^{\pi\Sigma}$ only if the derivative is in the time direction and δl_μ is in a spatial direction. This is, however, not possible since the only time derivative comes with the anisotropic component of l_μ which does not contribute to the correlator we are interested in.

APPENDIX C: RELEVANT FEYNMAN GRAPHS AND MATSUBARA SUMS

In this Appendix we review the detailed calculations of the 3D Hall viscosity. The main steps of the procedure have already been outlined in the main text in Sec. IV. Here, we reproduce the essential details of the computations.

1. Computation of η^π

We start with the computation of the $\pi_A\pi_B$ correlator. Since we are interested only in the contributions to the Hall viscosity tensor, we will always implicitly extract the part of the correlators that goes like the appropriate projector. The $\pi_A\pi_B$ correlator is computed by the Lorentzian continuation of the following Euclidean diagram:

$$\begin{aligned}&\langle\pi_A(-\omega,0)\pi_B(\omega,0)\rangle\\&\quad=\frac{1}{\beta}\sum_n\int\frac{d^2k\,dk_3}{(2\pi)^3}\text{tr}\big[S(k,\omega_n)\beta_AS(k,\omega+\omega_n)\beta_Bk_3^2\big],\end{aligned} \tag{C1}$$

where $\omega=2\pi mT$ is a bosonic Matsubara frequency, while the discrete sum runs over fermionic frequencies $\omega_n=(2n+1)\pi T$. In Majorana notation, the fermionic propagator is

$$S(p)=(\beta^Ap_A+sM(p)^{1/2z}C^{-1})^{-1},\ \ \beta^A=C^{-1}\gamma^A. \tag{C2}$$

Due to the Majorana nature of the computation and thus the absence of antiparticles, the Matsubara sums will only give half of the expected result, as it can be explicitly checked that the poles for particles and antiparticles give the same contributions to the odd viscosity.

To begin, we have to evaluate the trace over the Dirac indices to extract the odd projector. We will often encounter such traces in the various computations, in this case the key result is that

$$\text{tr}[\beta_A\beta_BC^{-1}]=2\epsilon_{AB}, \tag{C3}$$

where $\epsilon_{AB}=\epsilon_{ABC}u^C$ and u^C represents the time direction. This can be readily checked via the representation $\beta_0=-1$, $\beta_1=-\sigma_x$, $\beta_2=\sigma_z$, $C=-i\sigma_y$, which we will use in practical computations.

In (C1) one readily sees that the trace can be saturated only in the case in which we have an $M(k)$ contribution from the first propagator and a $\beta_C\omega^C\equiv-\omega$ one from the second. The contribution from the internal Matsubara frequency cancels because of the ordering of the matrices.

The Hall contribution then reads as

$$\begin{aligned}&\langle\pi_A(-\omega,0)\pi_B(\omega,0)\rangle_H\\&\quad=\epsilon_{AB}\omega\frac{4s}{4\pi^2}\int_0^\infty dk_3k_3^{1/z+2}\int_0^\infty dk\,kg(\epsilon,\omega),\end{aligned} \tag{C4}$$

where

$$\begin{aligned}g(\epsilon,\omega)&=\sum_n\frac{1}{\omega_n^2+\epsilon^2(k,k_3)}\frac{1}{(\omega+\omega_n)^2+\epsilon^2(k,k_3)},\\ \epsilon^2(k,k_3)&=k^2+k_3^{2/z}\end{aligned} \tag{C5}$$

is the Matsubara sum. Its evaluation of the Matsubara sum is straightforward and gives

$$g(\epsilon,\omega)=-\frac{\tanh(\beta\epsilon/2)}{8\epsilon(\epsilon^2+\omega^2/4)}, \tag{C6}$$

where we stress that ω must be kept as a bosonic Matsubara frequency.

We will be eventually interested in continuing the result to the Lorentzian sector to extract the retarded propagator. This is done as customary by the replacement $\omega=2\pi mT\to i(\omega_L+i0)$, followed by the $\omega_L\to0$ limit. However, in this case, since the transport we are interested in is nondissipative, we expect the density of states $\rho^{\pi\pi}_{AB}(\omega)=\text{Im}G^{\pi\pi}_{AB}(\omega)$ to vanish as the frequency is set to zero. This can be explicitly checked by computing the residue of the integrand of $G^{\pi\pi}$, which scales as ω^{3z+1}, so that both its value and its derivatives vanish in the zero-frequency limit. A similar reasoning holds for the other integrals. We may then take the naive $\omega\to0$ limit inside the integral after performing the Matsubara sums.

At this point, we divide vacuum from thermal contributions through the identity

$$\tanh(x/2)=1-2n_F(x), \tag{C7}$$

where $n_F(x)=1/(1+e^x)$ is the Fermi-Dirac distribution. Since the vacuum has no intrinsic Lifshitz scaling parameter, its contribution vanishes in any sensible regulation scheme. On the other hand, the thermal part gives the Hall conductivity to be

$$\eta^\pi=\frac{s}{4\pi^2}\int_0^\infty dk_3k_3^{1/z}\int_0^\infty dk\,k\frac{n_F(\beta\epsilon(k,k_3))}{\epsilon(k,k_3)^3}. \tag{C8}$$

We now change variables to $u=\beta k_3^{1/z}$, $v=\beta k$ to get

$$\eta^\pi=\frac{s}{4\pi^2}T^{3z}I_{3z}, \tag{C9}$$

where

$$I_{3z} = z\int_0^\infty du\, u^{3z}\int_0^\infty dv\, v\frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{3/2}} = z\int_0^\infty d\rho\, \rho^{3z-1}n_F(\rho)\int_0^{\pi/2} d\phi\, \sin(\phi)\cos(\phi)^{3z} = \frac{z}{3z+1}\Gamma(3z)\eta_D(3z), \tag{C10}$$

by going to polar coordinates $u = \rho\cos(\phi)$, $v = \rho\sin(\phi)$. Finally,

$$\eta^\pi = \frac{s}{4\pi^2}T^{3z}\frac{z}{3z+1}\Gamma(3z)\eta_D(3z). \tag{C11}$$

Most of the other computations go along the same lines, in particular, we will make the same series of changes of variables, as well as computing largely the same Matsubara sums. We will thus focus on the technical differences to speed up the presentation.

2. Computation of $\eta^{\pi\Sigma}$

We proceed to compute the Hall conductivity stemming from the correlator between π and Σ. In this case, the contribution splits into two parts, the first one given by the unimproved Σ, $\hat{\Sigma}_A = \frac{s}{z}\chi^T M^{1/2z-1}(\overleftarrow{\partial}_\nu l^\nu \overrightarrow{\partial}_A + \overleftarrow{\partial}_A l^\nu \overrightarrow{\partial}_\nu)C^{-1}\chi$ and a second one coming from the improvement term $\partial^B\sigma_{BA}$. The first of the two is given by the graph

$$\langle\hat{\Sigma}_A(-\omega,0)\pi_B(\omega,0)\rangle = \frac{1}{\beta}\sum_n\int\frac{d^2k\,dk_3}{(2\pi)^3}\mathrm{tr}\Big[S(k,\omega_n)C^{-1}k_A\frac{s}{z}|k_3|^{1/z-2}k_3S(k,\omega+\omega_n)\beta_Bk_3\Big]. \tag{C12}$$

The trace is evaluated in a similar way as before, only that now we will need a $\beta_C\omega^C$ and a β_Dk^D contribution from the propagators. The trace will be proportional to $-2\epsilon_{BD}k^D\omega$. Performing the angular integral d^2k amounts to the substitution $k_Ak^D \to \delta_A^Dk^2$ and a factor of 2π, so

$$\langle\hat{\Sigma}_A(-\omega,0)\pi_B(\omega,0)\rangle = \frac{2s}{4\pi^2}\epsilon_{AB}\omega\int_0^\infty dk_3k_3^{1/z}\int_0^\infty dk\,k^3g(\epsilon,\omega), \tag{C13}$$

and using the previous change of variables this gives

$$\eta^{\pi\Sigma}(2pf) = \frac{2s}{4z\pi^2}T^{2+z}I_{z+2}, \tag{C14}$$

where

$$I_{2+z} = \frac{z}{4}\int_0^\infty du\,u^z\int_0^\infty dv\,v^3\frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{3/2}} = \frac{z}{4}\int_0^\infty d\rho\,\rho^{z+1}n_F\,\rho\int_0^{\pi/2}d\phi\,\sin(\phi)^3\cos(\phi)^z = \frac{1}{2(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2), \tag{C15}$$

so

$$\eta^{\pi\Sigma}(2pf) = \frac{s}{4\pi^2}\tau^{z+2}m^3\frac{1}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2). \tag{C16}$$

For the improvement term we instead get

$$\frac{1}{2}\langle\sigma_{0A}(-\omega)\pi_B(\omega)\rangle = \frac{1}{4}\sum_n\int\frac{d^2k\,dk_3}{(2\pi)^3}\frac{s|k_3|^{1/z}}{z}\mathrm{tr}[S(k,\omega_n)\beta_AC^{-1}S(k,\omega+\omega_n)\beta_B], \tag{C17}$$

as both A and B are spatial, the only way to get an ϵ tensor is that the matrices from the two propagators contract between each other. The trace thus gives

$$\mathrm{tr}[S(k,\omega_n)\beta_AC^{-1}S(k,\omega+\omega_n)\beta_B] = 2\epsilon_{AB}\frac{\omega_n(\omega_n+\omega)+\epsilon(k,k_3)^2}{[\omega_n^2+\epsilon(k,k_3)^2][(\omega_n+\omega)^2+\epsilon(k,k_3)^2]}, \tag{C18}$$

which may be simplified, writing $\omega_n(\omega_n+\omega) = 1/2[\omega_n^2+(\omega+\omega_n)^2-\omega^2]$ to

$$\epsilon_{AB}\left[\frac{1}{\omega_n^2+\epsilon(k,k_3)^2}+\frac{1}{(\omega_n+\omega)^2+\epsilon(k,k_3)^2}-\frac{\omega^2}{[\omega_n^2+\epsilon(k,k_3)^2][(\omega_n+\omega)^2+\epsilon(k,k_3)^2]}\right], \tag{C19}$$

the first two sums are easily computed $\frac{1}{\beta}\sum_n \frac{1}{(\omega_n+\omega)^2+\epsilon(k,k_3)^2} = -\frac{\tanh\beta\epsilon(k,k_3)/2}{4\epsilon(k,k_3)}$ to be equivalent, while the third vanishes in the $\omega \to 0$ limit. We then get

$$\begin{aligned}\epsilon^{AB}\frac{1}{2}\langle\sigma_{0A}(-0)\pi_B(0)\rangle &= -\int \frac{d^2k\, dk_3}{4(2\pi)^3}\frac{s|k_3|^{1/z}}{z}\frac{1}{\epsilon(k,k_3)}\tanh\left(\beta\epsilon(k,k_3)/2\right)\\ &= \frac{s}{4\pi^2}T^{2+z}\int_0^\infty du\, u^z \int_0^\infty dv\, v\frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{1/2}}\\ &= \frac{s}{4\pi^2}T^{2+z}\int_0^\infty d\rho\, \rho^{z+1}\int_0^{\pi/2} d\phi\, \sin(\phi)\cos(\phi)^z = \frac{s}{4\pi^2}T^{2+z}\frac{1}{(z+1)}\Gamma(z+2)\eta_D(z+2). \quad \text{(C20)}\end{aligned}$$

Summing the two contributions we finally get

$$\eta^{\pi\Sigma} = \frac{s}{4\pi^2}T^{2+z}\frac{(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2). \quad \text{(C21)}$$

3. Computation of η^τ

We next move move to the intrinsic (2 + 1)-dimensional thermal Hall viscosity, for which one should compute both the two-point function $\tau\tau$ and the seagull term C_{ABCD}. The first of these is given by the integral

$$\langle\tau_{AB}(-\omega,0)\tau_{CD}(\omega,0)\rangle = \frac{1}{\beta}\sum_n\int\frac{d^2k\, dk_3}{(2\pi)^3}\mathrm{tr}[S(k,\omega_n)\beta_{(A}S(k,\omega+\omega_n)\beta_{(C}k_{B)}k_{D)}]. \quad \text{(C22)}$$

We are interested in the contribution proportional to P_{ABCD} of this correlator. To get the right factors, it is sufficient to work with one combination of indices, the full structure of the projector is then automatically recovered through symmetrization. The trace is computed in the same way as for η^π and we find

$$\langle\tau_{AB}(-\omega,0)\tau_{CD}(\omega,0)\rangle_H = \omega P_{ABCD}\frac{2s}{4\pi^2}\int_0^\infty dk_3 k^{1/z}\int_0^\infty dk\, k^3 g(\epsilon,\omega) = zP_{ABCD}\eta^{\pi\Sigma}(2pf). \quad \text{(C23)}$$

Confronting this expression with the computation of $\eta^{\pi\Sigma}$ we deduce that the two-point function contribution to this component of the viscosity will be given by

$$\eta^\tau(2pf) = z\eta^{\pi\Sigma}(2pf) = \frac{s}{4\pi^2}T^{z+2}\frac{z}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2). \quad \text{(C24)}$$

To get the full result, we still have to evaluate the contact term C_{ABCD}. In momentum space the seagull term gives the following diagram:

$$C_{ABCD}(\omega) = \frac{\delta_{AC}}{16}\frac{1}{\beta}\sum_n\int\frac{d^2k\, dk_3}{(2\pi)^3}\mathrm{tr}[\beta_{[A}C^{-1}\beta_{B]}\omega S(k,\omega_n)] + A\leftrightarrow B\,,C\leftrightarrow D, \quad \text{(C25)}$$

the trace is computed as before, and the index structure organizes to give a projector, so

$$\begin{aligned}C_{ABCD}(\omega) &= \omega P_{ABCD}\frac{s}{4\pi^2}T^{2+z}z\int_0^\infty du\, u^z\int_0^\infty dv\, v\frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{1/2}}\\ &= \omega P_{ABCD}\frac{s}{4\pi^2}T^{2+z}\frac{1}{(z+1)}\Gamma(z+2)\eta(z+2). \quad \text{(C26)}\end{aligned}$$

Summing all up, we get the relation

$$\eta^\tau = z\eta^{\pi\Sigma} = \frac{s}{4\pi^2}T^{2+z}\frac{z(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2). \quad \text{(C27)}$$

4. Computation of η^Σ

Finally, we inspect the value of η^Σ; this is the longest computation but we may use most of the tricks learned before to speed it up. It can be divided in three parts: the first coming from the correlators of the unimproved strains $\hat{\Sigma}$, the second coming from the correlator of one of these with the improvement term, and the last one stemming from the contact terms. It is simple to convince oneself that the unimproved correlator vanishes. This is because the Feynman diagram contains a term $k_A k_B$ which should be antisymmetrized.

The improvement term, on the other hand, behaves in much the same may as we have seen in the $\pi\Sigma$ correlator and gives a contribution

$$\eta^{\Sigma}_{\rm imp} = \lim_{\omega\to 0}\langle \sigma_{0A}(-\omega)\hat{\Sigma}_B(\omega)\rangle \epsilon^{AB} \tag{C28}$$

which reads as in terms of Feynman diagrams

$$\eta^{\Sigma}_{\rm imp} = \lim_{\omega\to 0}\frac{1}{\beta}\sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3}{\rm tr}\left[|k_3|^{1/z}k_3^{-1}k_B C^{-1}S(k,\omega_n)|k_3|^{1/z}k_3^{-1}\frac{1}{2}\beta_A C^{-1}S(k,\omega_n+\omega)\right], \tag{C29}$$

and, as before, the odd part of the trace may be computed by bringing up one term with the anisotropic momentum and one β matrix. This gives

$$\begin{aligned}\eta^{\Sigma}_{\rm imp} &= \lim_{\omega\to 0}\frac{1}{z^2}\frac{s}{2\pi^2}\frac{1}{\beta}\sum_n\int_0^\infty dk_3\int_0^\infty dk\, k^3 k_3^{3/z-2} g(\epsilon,\omega)\\ &= \frac{s}{8\pi^2 z}T^{4-z}\int d\rho\, \rho^{3-z} n_F(\rho)\int_0^{\pi/2} d\phi\, \sin(\phi)^3\cos(\phi) = \frac{s}{4z\pi^2}T^{4-z}\frac{\Gamma(4-z)\eta_D(4-z)}{(5-z)(3-z)}.\end{aligned} \tag{C30}$$

Finally, one has to take care of the contact term, whose form we had computed in the previous section. In this case, one has the Feynman graph

$$\eta^{\Sigma}_{\rm ct} = \frac{1}{2z^2}\frac{1}{\beta}\sum_n\int\frac{d^2k\, dk_3}{(2\pi)^3}{\rm tr}\left[|k_3|^{1/z-2}\frac{1}{2}(\omega_n C^{-1}\beta_A\beta_B + k_A C^{-1}\beta_B)S(k,\omega_n)\right]\epsilon^{AB}, \tag{C31}$$

where the trace gives

$${\rm tr}\left[|k_3|^{1/z-2}\frac{1}{2}(\omega_n C^{-1}\beta_A\beta_B + k_B C^{-1}\beta_A)S(k,\omega_n)\right]\epsilon^{AB} = 1 - \frac{|k_3|^{2/z}}{\omega_n^2+\epsilon(k,k_3)^2}. \tag{C32}$$

The first term is a vacuum contribution which may be regulated away, while the second Matsubara sum can be easily computed. One gets

$$\eta^{\Sigma}_{\rm ct} = \frac{s}{4z\pi^2}\int_0^\infty d\rho\, \rho^{3-z}n_F(\rho)\int_0^{\pi/2}d\phi\, \sin(\phi)\cos(\phi)^{2-z} = \frac{s}{4z\pi^2}T^{4-z}\frac{\Gamma(4-z)\eta_D(4-z)}{(3-z)}. \tag{C33}$$

Putting everything together we finally find

$$\eta^{\Sigma} = \frac{s}{4z\pi^2}T^{4-z}\frac{(6-z)}{(5-z)(3-z)}\Gamma(4-z)\eta_D(4-z). \tag{C34}$$

It is nice to notice that the three viscosities η^τ, η^Σ, and $\eta^{\pi\Sigma}$ can be compactly reexpressed (provided we renormalize $\Sigma \to z\Sigma$) as functions of their scaling dimension ξ:

$$\eta(\xi)/z = \frac{s}{4\pi^2}T^{\xi}\frac{(\xi+2)}{(\xi+1)(\xi-1)}\Gamma(\xi)\eta_D(\xi). \tag{C35}$$

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