Impact of electron-electron interactions on the superfluid density of dirty superconductors

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Landau's theory of the Fermi liquid is adapted to analyze the impact of electron-electron (interactions on the deficit of the superfluid density $\rho_{s0} = \rho_s(T = 0)$ in dirty superconducting electron systems in which the damping γ of single-particle excitations exceeds the zero-temperature BCS gap Δ_0. In the dirty strong-coupling limit $\gamma/\Delta_0 \gg 1$, $m^*/m_e \gg 1$, the formula derived for ρ_{s0} is shown to coincide with the well-known empirical Uemura relation provided pair-breaking contributions are nonexistent. The roles of the crystal lattice and magnetic pair-breaking effects in the observed decline of the zero-temperature superfluid density ρ_{s0} in overdoped $La_{1-x}Sr_xCuO_4$ compounds are also discussed, and our procedure is applied to elucidation of results from the pioneering experimental studies performed recently by Bozovic and collaborators in these compounds.

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I. INTRODUCTION

The phenomenon of high-temperature superconductivity, discovered in two-dimensional (2D) electron systems of copper oxides in 1986 [1], is still a subject of hot debate, defying consistent explanation. Serious new challenges are presented by recent experimental studies in overdoped $La_{1-x}Sr_xCuO_4$ (LSCO) compounds [2–4] that reveal an unexpected deficit of the superfluid density ρ_{s0}.

Related discussions [5–7] of the implications of this anomalous behavior have focused on the penetration depth

$$\lambda_0^2(x) = [4\pi e^2 \rho_{s0}(x)/m_e]^{-1} \tag{1}$$

associated with the Meissner effect, which is responsible for the exponential decay of the external magnetic field at the interior surface of these compounds at doping values x lower than the critical value $x_c \simeq 0.3$ at which LSCO superconductivity terminates.

In a major portion of the phase diagram of the family $La_{1-x}Sr_xCuO_4$, but excluding the heavily overdoped region where $x_c - x \ll x_c$, one is dealing with a type-II superconductor, since the ratio of $\lambda_0(x)$ to the zero-temperature coherence length $\xi_0(x) = v_F/\Delta_0(x)$ markedly exceeds unity. In this case, the relation between an electric current $\mathbf{j}$ and the applied vector potential $\mathbf{A}$ generating the current turns out to be local [8–11], i.e.,

$$j(\mathbf{r}) = -\frac{e^2\rho_{s0}}{m_e}A(\mathbf{r}). \tag{2}$$

It is a fundamental result of the weak-coupling BCS theory of type-II superconductors that in the clean limit where $\gamma \ll \Delta_0$ the superfluid density ρ_{s0} coincides with the total electron density:

$$\rho_{s0} = n. \tag{3}$$

Importantly, more sophisticated scrutiny by Larkin and Migdal [12] affirms that the relation (3) remains unchanged when all interactions between particles in the normal state are taken into account within the framework of Landau theory [13]. The same conclusion was reached later in a different analysis by Leggett [14].

Conversely, the ratio ρ_{s0}/n in strongly correlated superconducting electron systems is suppressed, as established in multiple studies beginning with the well-known paper by Uemura *et al.* [15]. Among such studies, the high-quality measurements of Ref. [2], performed on thousands of films of LSCO compounds, are especially valuable, since the loss of $\rho_{s0}(x)$ has been traced with unprecedented accuracy, warranting the unambiguous conclusion that the standard BCS approach fails to explain the new experiments. It is intriguing that the substantial reduction of ρ_{s0} persists even at optimal doping $x_o \simeq 0.17$, where $\rho_{s0}(x_o) \simeq 0.15\,n$ [2], while upon approach to the critical value $x_c \simeq 0.3$ the superfluid density declines to zero in harmony with the critical temperature T_c—quite as if one is dealing with Bose-Einstein condensation (BEC) of bound electron pairs [16–20].

However, the observed loss of ρ_{s0} does not require the BEC phenomenon to be invoked for its explanation. Rather, in dirty superconductors where $\gamma > \Delta_0$, such behavior of ρ_{s0} is well documented [21–23]. It would appear reasonable that $\gamma(x)$, which grows linearly with doping x due to its proportionality to the impurity content, may eventually reach values comparable with the gap Δ_0, especially in the overdoped region where $T_c(x) \propto \Delta_0(x)$ is known to fall off rapidly with $x \to x_c$. It is just such a characteristic dome shape of both $T_c(x)$ and $\rho_{s0}(x)$ that was uncovered recently in Nb-doped $SrTiO_3$ [24,25].

A common objection to applicability of such a dirty-limit scenario to the LSCO compounds is based on the fact that angle-resolved photoemission spectroscopy (ARPES) data [26] and numerous observations of Lifshitz-Kosevich oscillations support a large and well-defined Fermi surface.

However, it follows from arguments first advanced by Landau that the presence of a well-defined Fermi surface and applicability of the dirty limit are not mutually exclusive [10,27]. In essence, by virtue of the elasticity of impurity scattering, the problem to be solved reduces to a quantum-mechanical one of electron motion in an external potential field, just as in the theory of finite Fermi systems [28] developed within the framework of Fermi-liquid (FL) methods. Observing that the momentum $\mathbf{p}$ remains a good quantum number in crystals, we thus infer that the FL formalism is applicable to dirty superconductors as well, provided the impact of damping effects on the structure of the pole part of the electron Green's function is properly taken into account.

Given these conclusions, we now analyze the impact of electron-electron (*e-e*) interactions as they relate to perplexing behavior exhibited by strongly correlated electron systems, assuming the onset of superconductivity to be caused by Cooper pairing with total momentum $\mathbf{P} = 0$. Accordingly, in calculation of the superfluid density ρ_s, we adopt the BCS formalism without recourse to any alternative propositions. We concentrate on the dirty-limit situation $\epsilon_F^0 \gg \gamma > \Delta_0$ as described by the Abrikosov-Gor'kov (AG) theory of superconducting alloys [9,10,21]. As opposed to the clean-limit result (3), the superfluid density is predicted to behave as

$$\rho_{s0}(x) \propto n\frac{\Delta_0(x)}{\gamma}. \tag{4}$$

This implies a corresponding penetration depth of the form

$$\lambda_0^2 = (4\pi e^2 n_s/m_e)^{-1}, \tag{5}$$

with the AG effective density $n_s \propto n\Delta_0/\gamma$ of superfluid electrons appearing in place of the electron density n in the famous London formula.

We shall demonstrate that incorporation of the *e-e* interactions leads to further loss of superfluid density and growth of the penetration depth. This effect has its origin in the presence of a velocity-dependent component in the amplitude of the effective interaction between quasiparticles, which is responsible for the enhancement of the effective mass m^* in strongly correlated electron systems. The result so obtained is shown to be in agreement with the empirical Uemura relation [15]:

$$\lambda_0^2 = (4\pi e^2 n_s/m^*)^{-1}. \tag{6}$$

We shall also discuss the pros and cons of the AG pair-breaking scenario in attempting to explain the observed change from linear $\rho_{s0} \propto \Delta_0$ (4) to bilinear $\rho_{s0} \propto \Delta_0^2$ behavior of the superfluid density upon approach to the critical doping x_c at which superconductivity terminates.

II. GENERIC FORMULAS FOR CONVENTIONAL FERMI LIQUIDS

We begin by recalling that in BCS theory the electric current $\mathbf{j}(\mathbf{k})$ is connected with the weak vector potential $\mathbf{A}$ by

$$j_i(\mathbf{k}) = -\frac{ne^2}{m_e}Q_{ij}(\mathbf{k})A_j(\mathbf{k}), \tag{7}$$

where $Q_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j/k^2)Q(k)$. Henceforth we adopt the transverse gauge satisfying the condition $k_j A_j = 0$. Thereupon the analysis is simplified considerably, in that a part of the tensor $Q_{ij}(\mathbf{k})$ emergent from the change of the gap Δ in the external magnetic field turns out to be proportional to the factor $k_i k_j/k^2$ [12]. This contribution to the current $\mathbf{j}$ then vanishes identically, and we are left with $Q_{ij}(\mathbf{k}) = Q(k)\delta_{ij}$.

The existing weak-coupling BCS-AG theory of superconductivity properly describes the experimental situation in conventional metals. However, this theory fails in strongly correlated electron systems of high-temperature superconductors. In dealing with the superfluid density, its failure is evident from comparison of Eqs. (5) and (27) and will clearly be due to the neglect of so-called Fermi-liquid effects that arise from the fact that the single-particle energy $\epsilon(\mathbf{p})$ is itself a functional of the quasiparticle momentum distribution $n(\mathbf{p})$.

The magnitude of these FL effects is determined by the variational derivative $f(\mathbf{p}, \mathbf{p}_1) = \delta\epsilon[\mathbf{p}, n(\mathbf{p}_1)]/\delta n(\mathbf{p}_1)$, known in FL theory as the Landau interaction function. In homogeneous matter, this phenomenological quantity is identified by a set of parameters, namely, dimensionless harmonics of its Legendre polynomial expansion. In strongly correlated Fermi systems, their magnitudes are of order 1; hence their inclusion is imperative. This can be accomplished in different ways (see, for example, Ref. [29]); however, as we shall see, application of FL methods to evaluation of the tensor Q_{ij} is advantageous in allowing us to obtain final results in analytical and persuasive form.

In conventional three-dimensional Fermi liquids where the damping of single-particle excitations is immaterial, the original FL formula for the tensor Q_{ij} reads (for details, see Ref. [12])

$$Q_{ij}(\mathbf{k}) = \delta_{ij} + \frac{2}{nm_e}\int p_i L(\mathbf{p}, \mathbf{k})\mathcal{T}(p_j; \mathbf{k})\frac{d\mathbf{p}}{(2\pi)^3}, \tag{8}$$

the particle-hole propagator L being given by the integral

$$L(\mathbf{p}, \mathbf{k}) = \int [G_s(\mathbf{p}+\mathbf{k}, \varepsilon)G_s(\mathbf{p}, \varepsilon) + F(\mathbf{p}+\mathbf{k}, \varepsilon)F(\mathbf{p}, \varepsilon)]\frac{d\varepsilon}{2\pi i}, \tag{9}$$

where G_s and F are the Gor'kov quasiparticle propagators

$$G_s(\mathbf{p}, \varepsilon) = \frac{\varepsilon + \epsilon(\mathbf{p})}{\varepsilon^2 - \epsilon^2(\mathbf{p}) - \Delta^2(\mathbf{p})},$$
$$F(\mathbf{p}, \varepsilon) = -\frac{\Delta(\mathbf{p})}{\varepsilon^2 - \epsilon^2(\mathbf{p}) - \Delta^2(\mathbf{p})}. \tag{10}$$

For convenience, the factor z identifying the quasiparticle weight in single-particle states is absorbed into the definition of the quasiparticle propagators G_s and F, and likewise for the vertex part $\mathcal{T}(\mathbf{p}, \mathbf{k})$, which incorporates FL effects in satisfying the equation [9,10]

$$\mathcal{T}(\mathbf{p}, \mathbf{k}) = \mathbf{p} + 2\int f(\mathbf{n}, \mathbf{n}')L(\mathbf{p}', \mathbf{k})\mathcal{T}(\mathbf{p}'; \mathbf{k})\frac{d\mathbf{p}'}{(2\pi)^3}, \tag{11}$$

where $\mathbf{n} = \mathbf{p}/p_F$. For the homogeneous electron liquid, only the first harmonic f_1 of the Landau interaction function f enters this equation.

Throughout the whole $T - x$ phase diagram, except for the heavily overdoped region $|x - x_c| \ll x_c$, the London case $k = 0$ applies. Accordingly, straightforward calculation of the

integral (9) establishes that the function $L(\epsilon) = L(\mathbf{p}; k = 0)$ vanishes identically at *any momentum* $\mathbf{p}$ and for *any form of the single-particle spectrum* $\epsilon(\mathbf{p})$ [28]. Such a conclusion remains valid for the tight-binding model spectrum $\epsilon(\mathbf{p})$ employed in the calculations of ρ_{s0} by Lee-Hone *et al.* [6], to guarantee the property

$$Q(0) = 1. \tag{12}$$

This result immediately triggers *recovery* of the BCS-FL relation (3) in superconducting electron systems of solids, provided damping of single-particle excitations is nonexistent (more details are provided below).

III. FERMI-LIQUID EFFECTS IN DIRTY SUPERCONDUCTORS

A. Incorporation of *e-e* interactions

From the forgoing developments we infer that in the London limit the underlying cause of the reduction of ρ_{s0} is the damping of single-particle excitations in a correlated electron system. To facilitate analysis of the impact of FL effects on this reduction, we first address the case of ordinary impurities where the Anderson theorem [30] holds, i.e., the critical temperature T_c is not changed by the presence of impurities. The textbook dirty-limit formulas, written for homogeneous matter in the Matsubara representation, then take the forms [9,10]

$$G_s(\epsilon, \zeta) = -\frac{i\zeta\eta(\zeta) + \epsilon}{\left(\zeta^2 + \Delta_0^2\right)\eta^2(\zeta) + \epsilon^2},$$
$$F(\epsilon, \zeta) = \frac{\Delta_0\eta(\zeta)}{\left(\zeta^2 + \Delta_0^2\right)\eta^2(\zeta) + \epsilon^2}, \tag{13}$$

where

$$\eta(\zeta) = 1 + \frac{\gamma}{2\left(\zeta^2 + \Delta_0^2\right)^{1/2}}. \tag{14}$$

It is easily verified that in dirty superconductors the function $L(\epsilon, k = 0)$, given by Eq. (9) with dirty-limit propagators (13), *no longer vanishes*, thus destroying the coincidence between ρ_{s0} and n that occurs in the London limit at $\gamma = 0$.

In a dirty homogeneous system of interacting electrons, the solution of Eq. (8), rewritten in the form

$$Q(\gamma) = 1 + \frac{p_F}{3m_e n}\mathcal{T}_1(p_F, 0)L(\gamma), \tag{15}$$

is expressed in terms of two quantities: the particle-hole propagator L of Eq. (9) and the first harmonic $\mathcal{T}_1$ of vertex part $\mathcal{T}$, determined by Eq. (11). Their explicit forms are as follows:

$$L(\gamma) = p_F^2 \int [G_s(\epsilon, \zeta)G_s(\epsilon, \zeta) + F(\epsilon, \zeta)F(\epsilon, \zeta)]\frac{d\zeta d\epsilon d\Omega}{(2\pi)^4 v(\epsilon)}, \tag{16}$$

with the group velocity $v(\epsilon) = d\epsilon(p)/dp$ expressed in terms of the energy ϵ itself, and

$$\mathcal{T}_1(p_F, 0) = p_F[1 - f_1 L(\gamma)/3]^{-1}, \tag{17}$$

where f_1 is the first harmonic of the Landau interaction function.

An inherent problem associated with calculation of $L(\gamma)$ by Eq. (16) is the poor convergence of the integral, an obstacle usually overcome by subtracting the corresponding result for normal metals, where $Q(k)$ vanishes [9,29]. However, this procedure works flawlessly only in the weak-coupling limit where the vertex part $\mathcal{T}$ remains the same in both superconducting and normal states. Otherwise, an additional contribution proportional to the corresponding difference of the vertex parts comes into play, introducing complications.

This obstacle can be surmounted in a different way, aided by the relation

$$\frac{\partial G_s(\epsilon, \zeta)}{\partial \epsilon} = G_s(\epsilon, \zeta)G_s(\epsilon, \zeta) - F(\epsilon, \zeta)F(\epsilon, \zeta). \tag{18}$$

Indeed, upon inserting Eq. (18) into Eq. (16) and performing some manipulations, we are led to

$$L(\gamma) = \frac{2p_F^2}{(2\pi)^3}\left(\int \frac{\partial n(\epsilon)}{\partial \epsilon}\frac{d\epsilon d\Omega}{v(\epsilon)} + 2\iint F^2(\epsilon, \zeta)\frac{d\zeta d\epsilon d\Omega}{2\pi v(\epsilon)}\right). \tag{19}$$

Both of the integrals involved converge rapidly in the energy interval of order Δ_0 adjacent to the Fermi surface. Therefore the group velocity $v(\epsilon)$ can be freely replaced by the Fermi velocity v_F, to arrive finally at

$$L(\gamma, \Delta_0) = -\frac{p_F m^*}{\pi^2}[1 - I(\gamma, \Delta_0)], \tag{20}$$

where

$$I(\gamma, \Delta_0) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^2(\epsilon, \zeta)\frac{d\zeta d\epsilon}{2\pi} = \Delta_0^2 \int_0^{\infty} \frac{d\zeta}{\left[\zeta^2 + \Delta_0^2\right]\left[\left(\zeta^2 + \Delta_0^2\right)^{1/2} + \gamma/2\right]}. \tag{21}$$

At small Δ_0, the integrand diverges as $1/\Delta_0^2$, which implies that the integral $I(\Delta_0)$ varies *linearly* with the gap value as $\Delta_0 \to 0$.

It is straightforward to show that accounting for impurity-induced effects in these equations (as well as those that follow) reduces to the replacement of the total damping γ by its transport version $\gamma_{\rm tr}$.

With Eqs. (20) and (21) in hand, Eq. (15) takes the form

$$Q(\gamma, \alpha) = 1 - \frac{\alpha[1 - I(\gamma)]}{1 + \alpha F_1^0[1 - I(\gamma)]/3}, \tag{22}$$

where $\alpha = m^*/m_e$ and $F_1^0 = f_1 p_F m_e/\pi^2$. Invoking the FL relation [9,10]

$$m_e/m^* = 1 - F_1^0/3, \tag{23}$$

the constant F_1^0 may be eliminated from Eq. (22) to yield

$$\frac{\rho_{s0}(z, \alpha)}{n} \equiv Q(z, \alpha) = \frac{I(z)}{1 + (\alpha - 1)[1 - I(z)]}, \tag{24}$$

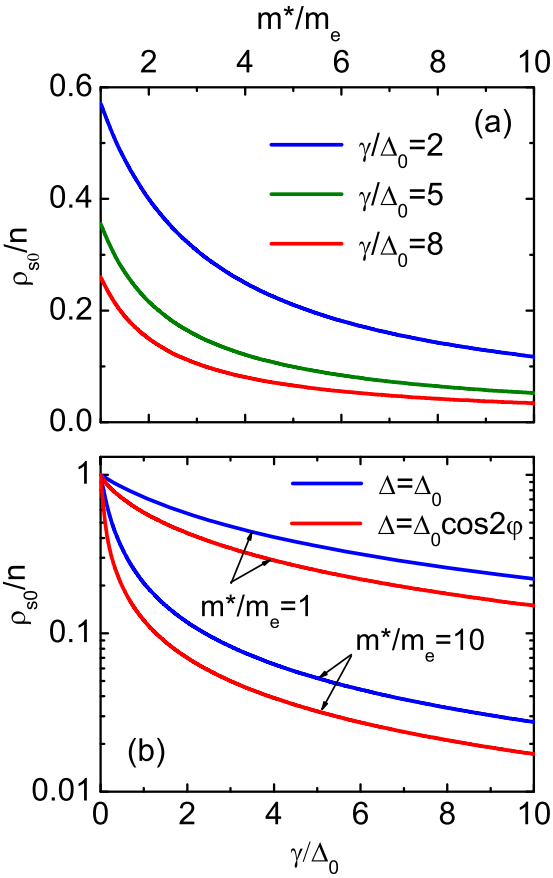

FIG. 1. (a) Ratio ρ_{s0}/n vs m^*/m_e evaluated from Eq. (24) at $\gamma = 2\Delta_0$ (blue line), $\gamma = 5\Delta_0$ (green line), and $\gamma = 8\Delta_0$ (red line). (b) Ratio ρ_{s0}/n vs γ/Δ_0 evaluated from Eq. (24) at $m^*/m_e = 1$ and 10 for the S-wave gap $\Delta = \Delta_0$ (blue lines) and D-wave gap $\Delta = \Delta_0 \cos 2\varphi$ (red lines).

where $z = \gamma/\Delta_0$. The function $I(z)$ can in fact be evaluated explicitly, with the results [31]

$$I(z) = \frac{\pi}{z}\left(1 + \frac{8\arctan\frac{z-2}{\sqrt{4-z^2}}}{\pi\sqrt{4-z^2}}\right), \quad z \leqslant 2,$$
$$I(z) = \frac{\pi}{z}\left(1 + \frac{8\,\mathrm{arctanh}\frac{2-z}{\sqrt{z^2-4}}}{\pi\sqrt{z^2-4}}\right), \quad z > 2. \tag{25}$$

Importantly, Eq. (24) simplifies in the dirty strong-coupling limit $\gamma_{\rm tr}/\Delta_0 \gg 1, \alpha \gg 1$, becoming

$$\frac{\rho_{s0}}{n} = \frac{I(z \gg 1)}{\alpha} \simeq \frac{\pi}{z\alpha} = \pi\frac{\Delta_0}{\gamma_{\rm tr}}\left(\frac{m_e}{m^*}\right). \tag{26}$$

We conclude that the linear relation between the superfluid density $\rho_{s0}(x)$ and the gap value $\Delta_0(x)$, emergent in the dirty limit, comes from the presence of the damping γ in the denominator of the integrand of Eq. (21), while incorporation of e-e interactions leads to a further decline of superfluid density $\propto m_e/m^*$ relative to the AG result [21], as documented in Fig. 1.

It is significant that in the strong-coupling dirty limit defined by $z \gg 1$ and $m^*/m_e \gg 1$, the FL penetration depth, determined by Eq. (1), can be rewritten in the Uemura form [15,32]:

$$\lambda_0^2 = (4\pi e^2 n_s/m^*)^{-1}. \tag{27}$$

Here n_s stands for the AG superfluid density evaluated with the aid of Eq. (25) in the dirty limit $z \gg 1$, while the additional dependence of λ_0^2 on the effective mass m^* comes from the e-e interactions. Thus, in the strong-coupling dirty limit, the penetration depth λ_0 *diverges* at the quantum critical point, in tandem with the effective mass.

Let us now turn to the issue of Cooper pairing in copper oxides, where the gap $\Delta(\phi)$ has D-wave structure. First of all, we observe that upon keeping the transverse gauge of the vector potential $\mathbf{A}$ and restricting attention to the first harmonic f_1 of the Landau interaction, the structure of the vertex part $\mathcal{T}$ itself remains unchanged, i.e., $\mathcal{T}(n_i) = \mathcal{T}_1 n_i$. In this situation, the Larkin-Migdal analysis [12] informs us that the correction to $\mathcal{T}_1$ coming from variation of the gap in the external magnetic field is necessarily proportional to the product $k_i(\mathbf{kn})/k^2$ (other contributions of this variation yielding zero upon multiplication with the vector $\mathbf{n}$ and angular integration). However, the contribution of this correction to the electric current vanishes identically with the gauge chosen for the vector potential $\mathbf{A}$. In the case of the D-wave gap, evaluation of Eq. (21) with the aid of a mean-value theorem for integrals yields

$$I_D(\gamma_{\rm av}) = \frac{1}{\pi}\int_0^\infty\int_0^\pi \frac{\Delta_D^2(\phi)d\zeta\,d\phi}{\left[\zeta^2+\Delta_D^2(\phi)\right]\left\{\left[\zeta^2+\Delta_D^2(\phi)\right]^{1/2}+\gamma_{\rm av}/2\right\}}, \tag{28}$$

where $\gamma_{\rm av}$ stands for an averaged damping value in the integration interval.

B. Impact of paramagnetic impurities on the relation between $\rho_{s0}(x)$ and $\Delta_0(x)$

A comprehensive analysis of this problem, including discussion of non-Born-limit corrections to the damping γ_s (see Refs. [33,34]) and the interplay between Kondo screening and Cooper pairing (see Ref. [35] and works cited therein), is beyond the scope of the present paper, as its primary aim has been to demonstrate the importance of e-e interactions in quenching the superfluid density in copper oxides. In what follows, we focus on the role of magnetic effects in the profound change of the dirty-limit linear relation (24) between the superfluid density $\rho_{s0}(x)$ and the gap value $\Delta_0(x)$, which prevails over a substantial portion of the LSCO phase diagram, but yields to bilinear behavior, $\rho_{so} \propto \Delta_0^2$, near the critical doping x_c at which superconductivity terminates.

A key point underlying this rearrangement is that in overdoped $La_{1-x}Sr_xCuO_4$ compounds the Anderson theorem [30], which establishes that the gap value is insensitive to the presence of impurities, and has been substantially involved in the preceding calculations, no longer holds. This breakdown is caused by the appearance of a magnetic component γ_s in the damping of single-particle excitations due to electron scattering on localized Sr magnetic moments [36,37]. Such an effect cannot be absorbed into the chemical potential, as is done in treating ordinary impurities associated with the formulas (13). As a result, each of the denominators of the two Gor'kov propagators F involved in the integrand of Eq. (21) acquires

an additive term proportional to the damping γ_s, which renders them *finite* along with the resulting integral. Consequently, instead of the linear relation (21), a new behavior applies, namely,

$$I(\gamma, \Delta_0) \propto \frac{\Delta_0^2}{\gamma_s \gamma_{\rm tr}}. \tag{29}$$

Accordingly, inclusion of the effects of *e-e* interactions modifies the result for $I(\gamma, \Delta_0)$ in the same way as it does for ordinary impurities [see Eq. (6)], implying that

$$\frac{\rho_{s0}}{n} \propto \frac{\Delta_0^2}{\gamma_s \gamma_{\rm tr} \alpha}, \tag{30}$$

also exploiting the fact that $I \ll 1$ is valid in the relevant region of the LSCO phase diagram.

IV. MODIFICATION OF THE FL FORMALISM IN CRYSTALS

We next consider how the modified FL approach, adapted above for the description of damping effects in strongly correlated homogeneous superconducting electron systems, must be extended to accommodate lattice-induced phenomena.

A. Preservation of $\rho_{s0} = n$ equality in the absence of damping

It instructive to begin the analysis of crystal-lattice effects with a validation of the BCS-FL result (3) in crystals for the conventional FL situation in which the damping γ vanishes. In this case, the bare Green's function associated with propagation of electrons in the external field of the crystal lattice has the common form

$$G(\mathbf{r}_1, \mathbf{r}_2, \varepsilon) = \sum \frac{\psi_{\mathbf{p}}(\mathbf{r}_1)\psi_{\mathbf{p}}(\mathbf{r}_2)}{\varepsilon - \epsilon(\mathbf{p}) + i\delta\,\mathrm{sgn}(\varepsilon)} \tag{31}$$

in terms of the corresponding Bloch wave functions $\psi_{\mathbf{p}}(\mathbf{r})$. Analogous expressions apply to the Gor'kov propagators G_s and F, with matrix elements $G_s(\mathbf{p}, \varepsilon)$ and $F(\mathbf{p}, \varepsilon)$ given by Eq. (10). Calculations in which integration is performed in coordinate space are greatly simplified in the London limit, where only matrix elements of propagators G_s and F evaluated at the same momentum $\mathbf{p}$ produce a nonzero result, which in fact coincides with Eq. (10) by virtue of the orthogonality of different Bloch wave functions. Moreover, as before, integration over ε leads to nullification of the propagator L, thus again yielding $Q_{ij}(0) = \delta_{ij}$. Accordingly, the BCS-FL result $\rho_{s0} = n$ is recovered in the standard FL case $\gamma = 0$.

B. FL prescription for the crystal-lattice case

Explicit treatment of the effects of damping within the crystal-lattice system may proceed as follows, focusing on 2D electron systems and employing the formula $d^2p = dp_t dp_n = p_F d\epsilon d\phi / v(\phi, \epsilon)$, with $v = |\nabla\epsilon|$. Since the propagators G_s and F depend just on ζ and ϵ, integrations over ϵ and ϕ may be separated, facilitating calculation. In particular, consider evaluation of the first contribution

$$L^{(1)}(z) = \frac{1}{2}\int\!\!\int \frac{\partial n(\epsilon)}{\partial \epsilon} \frac{p_F d\phi\, d\epsilon}{v(\phi, \epsilon)} \tag{32}$$

to the 2D propagator $L(z)$, which has a form analogous to Eqs. (20) and (21). Since the derivative $\partial n(\epsilon)/\partial\epsilon$ is peaked at the Fermi surface, the group velocity $v(\phi, \epsilon)$ can be replaced by the Fermi velocity $v_F(\phi) = v(\phi, \epsilon = 0)$ to yield

$$L^{(1)}(z) = -N(0)/2, \tag{33}$$

where

$$N(0) = p_F \int \frac{d\phi}{v_F(\phi)} \tag{34}$$

is the real 2D density of states. Similarly, one finds

$$L(z) = -N(0)[1 - I(z)]/2, \tag{35}$$

the function $I(z)$ being given by Eq. (21).

Further, observing that in the major share of the $T - x$ phase diagram of the LSCO compounds the Fermi line has approximately circular shape [26], the relation (17) remains unchanged, leading after the requisite manipulations to the following result:

$$\frac{\rho_{s0}(z, \alpha_c, x)}{n} = \frac{I(z)}{1 + (\alpha_c(x) - 1)[1 - I(z)]}, \tag{36}$$

in which $\alpha_c(x) = N(0, x)/N^0_{\rm FL}(0)$ with $N^0_{\rm FL}(0) = m_e/\pi$.

The integrand in Eq. (34) determining the total density of states $N(0)$ is calculated on the basis of a modified Pitaevskii equation [9,38]

$$\mathbf{v}(\mathbf{n}) = \mathbf{v}_0(\mathbf{n}) + 2\int f(\mathbf{n}, \mathbf{n}_1) \frac{\partial n(\mathbf{p}_1)}{\partial \mathbf{p}_1} \frac{d^2\mathbf{p}_1}{(2\pi)^2} \tag{37}$$

for the group velocity $\mathbf{v}(\mathbf{p}) = \nabla\epsilon(\mathbf{p})$, adapted to the 2D case. The free term $\mathbf{v}_0(\mathbf{n})$ is the sum of the gradient of a lattice-induced electric field and the so-called ω limit of the vertex part, determined as $\mathcal{T}(\mathbf{p}, \omega \to 0, k \to 0, kv_F/\omega \to 0)$. In the homogeneous liquid, where the momentum $\mathbf{p}$ commutes with the total Hamiltonian of the problem, this term coincides with the corresponding Landau result $\mathbf{p}/m_e$ [28,38]. In LSCO compounds, $\mathbf{v}_0(\mathbf{n})$ is replaced by $\nabla\epsilon_{\rm ARPES}(\mathbf{p})$ [6], the parameters of which are extracted from available ARPES data [26]. Evidently, by virtue of the presence of the crystal lattice, the momentum $\mathbf{p}$ ceases to commute with the total Hamiltonian, which, strictly speaking, leads to the occurrence of gradients of the external potential in the right side of Eq. (37), and hence to some renormalization of the term p/m_e appearing in the corresponding Landau equation. To avoid further complications, such contributions are hereafter neglected.

The second term of Eq. (37), the integrand of which contains the Landau interaction function f, accounts for the functional dependence of the single-particle spectrum $\epsilon(\mathbf{p})$ on the quasiparticle momentum distribution $n(\mathbf{p})$. Comparison of Eqs. (26) and (36) demonstrates that in solids the structure of the Uemura relation remains unchanged, with the real density of states $N(0)$ absorbing both the lattice-induced and interaction-induced effects.

There is a widespread belief that elucidation of the electronic properties of crystals within FL theory is impossible, since its basic equation relating the single-particle spectrum and the quasiparticle momentum distribution was derived by Landau under the assumption of Galilean invariance, which breaks down for electrons inhabiting crystals. We observe,

however, that Eq. (37) is almost identical to the original Landau equation [9]. The crucial distinction is that this equation has instead been derived on the basis of gauge invariance, which is known to hold in crystals as well as homogeneous systems. Accordingly, it is the Pitaevskii equation that should be employed in calculations of the real density of states $N(0)$, which is responsible for renormalization of the AG results in interacting electron systems of solids.

V. DISCUSSION AND SUMMARY

Let us first consider the situation that prevails in the London limit for conventional Fermi liquids in which the damping γ of single-particle excitations is negligible. In this case, it follows from our analysis that the superfluid density ρ_{s0} must coincide with the total electron density n, irrespective of the form of the single-particle spectrum, which is naturally quite complicated due to lattice effects and the character of the interactions between particles, sometimes giving rise to the occurrence of flat portions in $\epsilon(\mathbf{p})$.

In the case of dirty homogeneous superconductors with the conventional FL ground state, introduction of e-e interactions has been shown to alter the elementary AG behavior, in which the loss of superfluid density ρ_{s0} depends solely on the AG parameter $z = \gamma/\Delta_0$ [21,31]. As illustrated in panel (a) of Fig. 1, an additional decline of ρ_{s0} is found to be triggered by the presence of effective velocity-dependent interactions between quasiparticles that produce an enhancement of the density of states $N(0)$ associated with the ratio m^*/m_e. We have demonstrated that the Uemura relation $\lambda_0^2 = (4\pi e^2 n_s/m^*)^{-1}$ does in fact apply in the strong-coupling dirty limit $z \gg 1$, $m^*/m_e \gg 1$, with (i) the effective mass m^* characterizing the interaction-induced contribution to ρ_{s0} and (ii) the Uemura parameter n_s representing the AG superfluid density associated with the function $I(z)$ given by Eq. (25).

In connection with these results, it is worth noting that in their first paper [6] devoted to evaluation of the superfluid density of overdoped LSCO compounds the authors have sought to explain the unorthodox behavior of the superfluid density of LSCO compounds uncovered by Bozovic *et al.* [2] within the so-called semiclassical scheme [39,40]. In this procedure, the magnetic field is incorporated by making the replacement $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ solely in the assumed *single-particle spectrum*, the parameters of which are determined from the available ARPES data. In so doing, FL effects associated with the variation of the single-particle energy $\epsilon(p)$ due to the change of the quasiparticle momentum distribution $n(p)$—which are naturally incorporated for the homogeneous electron liquid in Secs. II and III above—are completely ignored within the framework of the semiclassical scheme. Without taking proper account of these effects, the magnitude of which increases with growth of the ratio m^*/m_e, elucidation of the Uemura relation (6) becomes impossible, because the decisive factor m^*/m_e is lost.

Clearly, this deficiency of the semiclassical approach persists in dealing with strongly correlated electron systems of high-T_c superconductors moving in the external field of their crystal lattice. [As seen from Eq. (37), it persists irrespective of whether proper treatment has been given to the logarithmiclike divergence of the tight-binding density of LSCO states [26,41], which is exhibited in the doping region where the Fermi line touches the zone boundary.] Furthermore, as seen from Eq. (37), the effects of the lattice and interaction, working in tandem, change the group velocity profoundly, and hence the density of states $N(0)$ itself. Consequently, the only way to proceed without extensive and problematic numerical calculations based on Eq. (37) is phenomenological. Within the modified FL theory presented here, these effects are naturally absorbed into a single density-of-states parameter α_c, and the same is true for the Sommerfeld coefficient in the specific heat $C(T)$. We refer to the available experimental information on the LSCO compounds [42–44] to extract the parameter α_c.

In comparing our results with experimental data on the LSCO superfluid density $\rho_{s0}(x)$, we focus on the overdoped region $0.20 < x < 0.25$, which is free of pseudogap influence [26] and where experimental values of the key input parameter $\gamma(x)$ are available [4]. As is known, the experimental curve $T_c(\rho_{s0})$ consists of a dominant linear portion and, in a relatively small region adjacent to the origin, T_c behaves as $\sqrt{\rho_{s0}}$ [2,3]. The linear segment of the curve is associated with the Uemura-like portion given by the theory as expressed in Eq. (26) and characterized by its slope $dT_c/d\rho_{s0} = (2.5 \pm 0.1) \times 10^2$ K [2,3]. In accord with Eq. (26), the slope depends on the product of the damping γ and the density-of-states factor α. Since the doping region involved is quite narrow, one might expect that the x variations of the two input parameters involved can be neglected. If so, at the midpoint $x = 0.22$ of the doping interval implicated, where $\gamma = 75$ K [4] is known from experimental data, simple numerical calculations based on Eq. (28) derived for D pairing, as appropriate for the LSCO compounds, yield a theoretical slope of 2.2×10^2 K. This is close to the experimental value $(2.5 \pm 0.1) \times 10^2$ K, provided the BCS relation $2\Delta_0 = 4.28T_c$ is adopted and the effective mass value is chosen to be $m^* = 12m_e$, in accordance with the relevant experimental data [43].

However, the issues raised by the hypothetical assumption of permanent input parameters as functions of doping are more involved. In the doping range under consideration, the damping $\gamma(x)$ is doubled, increasing linearly toward x_c [4]. Moreover, the change of $\alpha_c(x)$ associated with the aforementioned logarithmic divergence of the tight-binding density of states on the left edge on the doping interval, occurring at a critical value $x_t \simeq 0.2$ [26], is even more profound. Fortunately, the variations of $\gamma(x)$ and $\alpha_c(x)$ swing in opposite directions, thereby suppressing the net change of $\rho_{s0}(x)$ and allowing it to be neglected in a first approximation. The next step toward improving the reliability of the results obtained within the extended FL approach would involve numerical solution of Eq. (37) to obtain a realistic quasiparticle group velocity $v(\mathbf{p})$ for insertion into the integral (21). The energy dependence of the damping γ, known from experiment [2], should be properly taken into account as well.

In explanation of the second segment of $\rho_{s0}(x)$, characterized by its bilinear dependence on Δ_0 and situated adjacent to the critical doping x_c, an idea advanced many years ago by Abrikosov and Gor'kov [36] has been invoked to attribute the rearrangement of the linear regime to the presence of a *magnetic* part γ_s of the damping of single-particle excitations. In their original model, the authors of Ref. [36] considered a pair-breaking mechanism associated with electron scattering

by impurity magnetic moments. Within this model, a particular behavior $T_c^2(x) \propto x_c - x$ observed experimentally is reproduced.

However, the application of this idea to elucidation of the available LSCO experimental data [2–4] encounters some difficulties. For example, these experiments have shown no trace of gapless superconductivity, which is an integral feature of the AG pair-breaking mechanism. Moreover, the BCS approach fails to explain basic features of high-temperature superconductivity, including the enhancement of the critical temperature T_c itself. In this situation, results from application of the BCS gap equation to the problem appear to be inconsistent. Given these considerations, the version of the AG paramagnetic scenario adopted in Ref. [6] becomes questionable.

A potential source of the observed discrepancy of predictions of extended FL theory applied here from the experimentally established behavior of the superfluid density $\rho_{s0} \propto T_c^2$ upon approach to critical doping, as well as the challenging temperature dependence of the superfluid density $\rho_s(T)$ [2,3], may be related to a rearrangement of normal states of strongly correlated electron systems associated with violation of their topological stability [45]. Such a phenomenon is now actively discussed following publication of a series of papers devoted to the occurrence of *flat bands* in magic-angle twisted bilayer graphene [46–49]. In the same vein, we may point to the recent observation [50] of a magnetic-field dependent electronic gap in the point-contact spectrum of dirty graphite. This observation is indicative of local superconductivity having an estimated critical temperature $T_c \approx 14\,\mathrm{K}$, with possible implication of a flat-band mechanism [51,52]. Remarkably, it is in exactly the present case of *overdoped LSCO compounds* considered here that arguments favoring the emergence and agency of flat bands in strongly correlated electron systems of cuprates have recently been reiterated in Ref. [53]. In future work, we plan to investigate the role that flat bands may have in quenching the superfluid density $\rho_s(T)$ and in its unexpected temperature dependence.

To summarize, we have demonstrated that the basic regime of behavior of the LSCO superfluid density $\rho_{s0}(T_c)$, where it changes linearly with T_c, is properly reproduced within the AG-FL theory, the calculated slope being in agreement with experiment. As for the second regime, operative near critical doping x_c where $\rho_{s0}(T_c) \propto T_c^2$, effort toward its quantitative explanation remains inconclusive.

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