Reciprocity in diffusive spin-current circuits

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Similarly to their purely electric counterparts, spintronic circuits may be presented as networks of lumped elements. Due to the interplay between spin and charge currents, each element is described by a matrix conductance. We establish reciprocity relations between the entries of the conductance matrix of a multiterminal linear device, comprising normal metallic and strong-ferromagnetic elements with vanishing spin-orbit interactions and spin-inactive interfaces. In particular, reciprocity equates the spin transmissions through a two-terminal element in opposite directions. When applied to "geometric spin ratchets," reciprocity shows that certain effects, announced for such devices, are, in fact, impossible. We describe the relation between our work and the spintronic circuit theory formalism and contrast our results with the requirements following from the Onsager symmetry of kinetic coefficients. Consequences of finite spin-orbit interactions are also discussed.

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I. INTRODUCTION

Spin currents have been actively discussed in the context of spintronics, a field where memory and logic devices employ electron spin on par with its charge. A number of theoretical concepts have been developed to describe operation of such devices. As the field matures, one needs to build and work with ever larger networks of connected spintronic elements—akin to how electric circuits are composed of elementary resistors, capacitors, etc. To this end, a spin circuit theory was proposed in the pioneering paper Ref. [1]. The principles of the latter approach were then used to formulate circuit descriptions that may be more convenient for applications [2,3].

In dc electric circuits a textbook resistor is characterized by a single parameter, the resistance R that encapsulates the element's material properties, shape, size, and contact positions. In spintronics, where spin and charge currents are interconnected, even the simplest element is characterized not by a single number but by a conductance matrix [2,3]. In this paper we show—within the assumptions detailed below—that the entries of the spintronic conductance matrix obey certain general relations that are independent of shape, size, and material constants of the actual physical elements and are similar to classic reciprocity relations for electric circuits [4–6]. Generally, these relations are different from the Onsager reciprocity relations [7] for spintronic devices.

The ultimate goal of a circuit theory is to describe spintronic circuits using generalized Kirchhoff rules. Realizing this program, one has to keep in mind, however, that certain differences between spin and electric currents invalidate much of the intuition accumulated in electric circuits. First, unlike electric current, spin current is not conserved, and in a two-terminal element the incoming and outgoing spin currents are generally different: an element cannot be characterized by a single value of spin current. Second, spin currents behave differently from electric currents when potentials applied to the two terminals of an element are interchanged. For electric current, Ohm's law $I = G(V_1 - V_2)$, expressed through the conductivity $G = 1/R$, states that by interchanging V_1 and V_2, one flips the sign of the current but preserves its magnitude. In this sense, the resistor is a directionless element. As detailed below, this does not apply to two-terminal spin elements, where interchanging the terminal potentials generally changes the magnitudes of both incoming and outgoing spin currents. However, the relations between the entries of the conductance matrix, obtained in this paper, show that a two-terminal spin element behaves in a familiar way with respect to interchange of potentials in a special case where a driving spin potential is applied to one terminal, and the resulting spin current is measured at the other, grounded terminal. This means that the transmission of spin current through a spin-dissipating element is directionless.

II. RECIPROCITY IN THE DIFFUSION REGIME

We consider metallic devices in the diffusion regime, with the mean-free paths of charge carriers being much shorter than any other length scale in the problem. In this approximation, the electron state is completely described by the distributions of electric potential $\mu^e(r)$ and spin potential $\mu^s(r)$ [8–11]. Instead of electric current density j_i, we will work with particle current density $j_i^e = j_i/e$. Here the index $i = \{x, y, z\}$ denotes direction in real space. The spin-current density $j_i^{s\alpha}$ has two indices, with $\alpha = \{x, y, z\}$ denoting direction in spin space. Spin current is also defined in terms of the number of particles: passage of one spin-up electron per second through a mathematical plane contributes one, not $1/2$, to j^s flowing through it. In the diffusion regime, currents are driven by the

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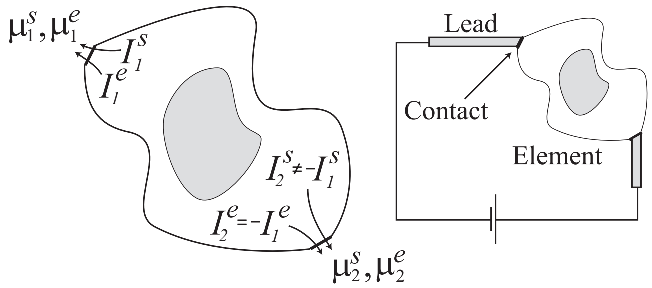

FIG. 1. Left: A two-terminal element, an island with two contacts, where spin potentials $\mu^s_{1,2}$ and electric potentials $\mu^e_{1,2}$ are applied. Conserved electric current ($I^e_1 = -I^2_2$) and nonconserved spin current ($I^s_1 \neq -I^s_2$) are shown schematically. White denotes a normal metal, gray denotes a ferromagnet. Right: The same connected to ferromagnetic leads (gray) in a spin circuit.

gradients of electric potential $\mu^e(r)$ and spin potential $\mu^{s\alpha}(r)$ [8–11].

A sample device is shown in Fig. 1. The element has arbitrary shape and may contain magnetic and nonmagnetic metal parts. Two contacts connect it to the outside world. They are assumed to be small enough for the electric and spin potentials to be considered constant across each of them. In order to apply spin potentials to the element, the external circuit must involve magnetic elements, producing the required spin imbalance.

A. A two-terminal normal-metal element

We start with a conceptually simpler case of a normal-metal element. The currents are related to the potentials as per

$$j^e_i = -\frac{\sigma}{e^2}\nabla_i \mu^e \,, \tag{1}$$

$$j^{s\alpha}_i = -\frac{\sigma}{2e^2}\nabla_i \mu^{s\alpha}, \tag{2}$$

where σ is the (possibly nonuniform) electric conductivity of the metal. We will study steady-state solutions, where the continuity equation $\partial_t\rho + \nabla\cdot j = 0$ yields

$$\nabla_i j^e_i = 0 \tag{3}$$

for the electric current and

$$\nabla_i j^{s\alpha}_i = \frac{\nu}{\tau_s}\mu^{s\alpha} \tag{4}$$

for the spin current, with ν being the density of states of the normal metal and τ_s the spin relaxation time.

Equations (1) and (3) for electric potential, and (2) and (4) for spin potential are decoupled. Once the system (2), (4) is solved, the spin-current density $j^{s\alpha}_i$ can be found everywhere, and the total spin current flowing through each contact is given by

$$I^{s\alpha}_t = \int_{S_t} j^{s\alpha}_i dA_i,$$

where the integration goes over the contact surface S_t, and $t = 1, 2$ labels the two contacts. It is, of course, assumed that the spin current does not leak in or out anywhere else at the sample boundary. By definition, the current is considered positive if it flows out of the element, i.e., surface element $d\mathbf{A}$ points along the outward normal. Due to the linearity of Eq. (2), the total spin currents must be linearly related to the spin potentials of the terminals

$$I^{s\alpha}_t = G^s_{tt'}\mu^{s\alpha}_{t'} \tag{5}$$

via the matrix spin conductance $G^s_{tt'}$, which is determined by the solution of system (2), (4). Since both equations in the system are diagonal in the spin index α, the conductance is diagonal in it as well. We will thus suppress the spin index in the equations for normal-metal elements.

Note that a purely electric two-terminal device can be described by a matrix conductance similar to Eq. (5). However, Ohm's law $I_2 = -I_1 = G(V_1 - V_2)$ constrains the electric conductance matrix to a form

$$\hat{G}^e = \begin{vmatrix} -G & G \\ G & -G \end{vmatrix},$$

with a single independent entry. The spin conductance matrix

$$\hat{G}^s = \begin{vmatrix} G^s_{11} & G^s_{12} \\ G^s_{21} & G^s_{22} \end{vmatrix}$$

has four entries, and one may ask whether there are any relations between them that hold regardless of the shape and material of the spintronic element.

We now prove that the answer to the question above is affirmative and the off-diagonal elements of $\hat{G}^s$ are always equal. The proof is based on the so-called reciprocity property [5] of the solutions of Eq. (4), summarized in Appendix A. Imagine solving this equation for mixed boundary conditions, specified by constant spin potentials $\mu^s(r) = \mu^s_t$ at the contacts and $j^s_i n_i = 0$ (no current penetrating the boundary) outside the contact areas, where n_i is the local normal to the surface of the element. Consider two solutions, each for a separate pair of potentials μ^s_t applied at the contacts $t = 1, 2$. These solutions will be denoted $\mu^s(r, c)$, with a "case label" $c = 1, 2$. Knowing $\mu^s(r, c)$, one can find the currents $j^s_i(r, c) = -(\sigma/2e^2)\nabla_i\,\mu^s(r, c)$. Now, let us use the functions $\mu^s(r, c)$ and $j^s_i(r, c)$ to calculate the integral:

$$Q = \int \big[\mu^s(r, 1)\nabla_i j^s_i(r, 2) - \mu^s(r, 2)\nabla_i j^s_i(r, 1)\big]dV \,.$$

On the one hand, Eq. (4) tells us that $Q = 0$. On the other hand, the identity (A2) of Appendix A transforms Q into the surface integral

$$Q = \oint [\mu^s(r, 1)(-\sigma\nabla_i\mu^s(r, 2)) - \mu^s(r, 2)(-\sigma\nabla_i\mu^s(r, 1))]dA_i.$$

Since spin potentials are constant across each contact, the contacts do not overlap, and j^s_i crosses the surface only at the contacts, we obtain

$$Q = \mu^s_t(1)I^s_t(2) - \mu^s_t(2)I^s_t(1) = 0$$

with summation over repeating indices t. Expressing currents through potentials via (5), we find

$$\mu^s_t(1)G^s_{tt'}\mu^s_{t'}(2) - \mu^s_t(2)G^s_{tt'}\mu^s_{t'}(1) = 0$$

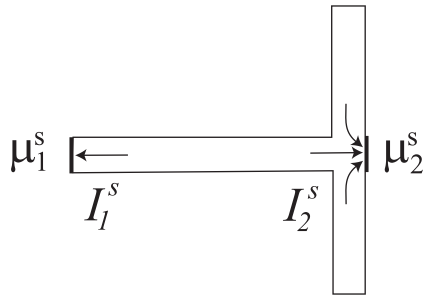

FIG. 2. Asymmetric normal element.

or

$$\mu_t^s(1)\mu_{t'}^s(2)\big(G_{tt'}^s - G_{t't}^s\big) = 0.$$

As the potentials $\mu_t^s(1)$ and $\mu_t^s(2)$ can be chosen arbitrarily, the above means $G_{tt'}^s = G_{t't}^s$. Thus, a 2×2 matrix $G_{tt'}^s$ obeys the constraint

$$G_{12}^s = G_{21}^s. \tag{6}$$

The physical meaning of Eq. (6) manifests itself in experiments with a single driving terminal, i.e., $\mu_t^s(c) = \mu^{s0}\delta_{tc}$. In the first case ($c = 1$), spin potential $\mu^{s0} > 0$ is applied to the first (driving) terminal, while the second terminal is kept at zero spin potential (ground terminal). In the second case ($c = 2$), the driving terminal and the ground terminal are interchanged. It is physically clear (and can be mathematically proven) that, in the first case, a current $(-I_1^s) > 0$ will enter the element at the driving terminal, and a current $I_2^s > 0$ will leave it at the ground terminal. As already discussed, the transmitted current will be smaller due to spin dissipation: $(-I_1^s) > I_2^s$, i.e., $G_{12} \leqslant G_1\,,\ G_2$. Equation (5) now reads $I_2^s = G_{21}^s\mu^{s0}$, i.e., the matrix conductance element G_{21}^s parameterizes the transmission of spin current through the device from the driving terminal to the ground terminal. In the second case, spin current is driven by the same spin potential, applied to terminal two. The current, transmitted from the driving terminal to the ground terminal, is now given by $I_1^s = G_{12}^s\mu^{s0}$. The reciprocity property $G_{12}^s = G_{21}^s$ means that, for equal potentials applied to the driving terminal, the spin current transmitted to the ground terminal is independent of which of the two terminals is driven. In other words, spin transmission from the driving terminal to the ground terminal is directionless.

We now show that the diagonal elements of $\hat{G}^s$ may differ from each other. Consider a geometrically asymmetric element such as the one in Fig. 2. Apply potentials $\mu_t^s(c) = \mu^{s0}\delta_{tc}$ as discussed above and measure the current flowing through the driving terminal. On the one hand, it equals $I_1^s(1) = \mu^{s0}G_{11}^s$ in the first case and $I_2^s(2) = \mu^{s0}G_{22}^s$ in the second. On the other hand, it is physically clear that for such an element these currents are different, since spins injected into the $t = 1$ terminal can diffuse in only one direction, whereas spins injected into the $t = 2$ terminal can also diffuse into the vertical bar (cf. Ref. [12], Sec. III C), thus increasing the total spin current entering the element. Due to our definition of current signs, we have $I_2^s(2) < I_1^s(1) < 0$. As a result, $G_{22}^s < G_{11}^s < 0$. Put more generally, in an asymmetric element of a size exceeding the spin-diffusion length, the diagonal elements G_{11}^s and G_{22}^s are primarily defined by the geometry and material properties of the device within a few diffusion lengths from the corresponding contact.

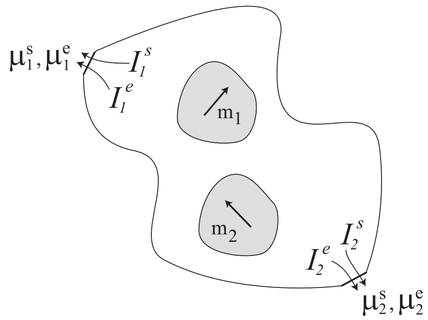

FIG. 3. Two-terminal element with normal (white) and ferromagnetic (shaded) parts.

The reciprocity equation (6) presents the main result of our work in the simplest setting of a two-terminal diffusive normal-metal element: transmission of spin current between the terminals is direction independent. We now proceed to describe the reciprocity relations that emerge in more general settings.

B. Composite elements incorporating normal metals and strong ferromagnets

In this section we consider a two-terminal element comprising normal ferromagnetic (F) as well as nonmagnetic (N) regions (Fig. 3). In each ferromagnetic region, magnetization is assumed to be uniform; magnetizations of different F regions are not expected to be collinear. We restrict our analysis to strong ferromagnets, where itinerant electron spins are polarized along the direction of local magnetization. It is further assumed that different ferromagnetic parts do not border each other directly but are always separated by a nonmagnetic region. The boundaries between the normal and ferromagnetic regions are assumed to be Ohmic (no tunnel barriers).

Inside a strong ferromagnet, the spin potential may be presented as $\mu^{s\alpha}(r) = \mu^s(r)m^\alpha$, where m^α is a unit vector along the magnetization. The currents are given by the expressions [8–11]

$$j_i^e = -\frac{\sigma}{e^2}\left(\nabla_i\mu^e + \frac{1}{2}p\nabla_i\mu^s\right), \tag{7}$$

$$j_i^{s\alpha} = -\frac{m^\alpha\sigma}{e^2}\left(\frac{1}{2}\nabla_i\mu^s + p\nabla_i\mu^e\right), \tag{8}$$

where p is the spin polarization parameter, characterizing the material of a given ferromagnetic part. Since $j_i^{s\alpha} \propto m^\alpha$, it follows that $(\delta^{\alpha\beta} - m^\alpha m^\beta)j^{s\beta} = 0$.

To streamline the formulas, we combine the electric and spin potentials into a four-component rescaled potential $\tilde{\mu}^a(r) = \{\mu^e, \mu^{sx}/2, \mu^{sy}/2, \mu^{sz}/2\}$, where $a = \{e, sx, sy, sz\}$. Likewise, the currents are combined into $j_i^a = \{j_i^e, j_i^{sx}, j_i^{sy}, j_i^{sz}\}$. Then, Eqs. (7) and (8) take the form

$$j_i^a = -\Sigma^{ab}\nabla_i\tilde{\mu}^b, \tag{9}$$

where Σ is the generalized conductivity matrix. The use of $\tilde{\mu}^a$ renders Σ^{ab} symmetric (Appendix B), which allows us to apply the identity (A3)

$$Q \equiv \int \big(\tilde{\mu}^a(1)\nabla_i j_i^a(2) - \tilde{\mu}^a(2)\nabla_i j_i^a(1)\big)dV$$
$$= \oint \big(\tilde{\mu}^a(1) j_i^a(2) - \tilde{\mu}^a(2) j_i^a(1)\big)dA_i \,. \qquad (10)$$

The divergences $\nabla_i j_i^a$ in the volume integral are nonzero for the spin part only. In the bulk, be it normal or ferromagnetic, one has $\nabla_i j_i^a = \{0, \nu(r)\mu^\alpha/\tau(r)\}$, with material-specific effective densities of states and relaxation times [10,11]. A direct check shows that bulk relaxation gives zero contribution to Q.

However, in a composite device spin relaxation is not limited to the bulk but acquires an additional contribution from the F/N interfaces. Here we will assume Ohmic, spin-inactive interfaces. At the interface S, the potentials are continuous,

$$\mu^a(N)|_S = \mu^a(F)|_S = \{\mu^e, \mu^s m_x, \mu^s m_y, \mu^s m_z\}, \qquad (11)$$

but the currents j^a are not [13,14]. Spin current may have arbitrary direction in spin space on the normal-metal side of the interface, but it has to be parallel to m^α on its ferromagnetic side. The spin-current component perpendicular to m^α is absorbed in a thin boundary layer near the interface, while the current component parallel to m^α is continuous. In the strong-ferromagnet approximation the absorption layer thickness is infinitesimally small, so the boundary conditions for currents read

$$j_i^e(N)n_i\big|_S = j_i^e(F)n_i\big|_S, \qquad (12)$$

$$m^\alpha j_i^{s\alpha}(N)n_i\big|_S = m^\alpha j_i^{s\alpha}(F)n_i\big|_S, \qquad (13)$$

with n_i being the normal to the interface. The discontinuity of the perpendicular spin current gives rise to a surface absorption term

$$\nabla_i j_i^{s\alpha} = R^\alpha(r) = (\delta^{\alpha\beta} - m^\alpha m^\beta) j_i^{s\beta}(N)\, n_i \delta_S(r) \qquad (14)$$

proportional to the surface delta function δ_S at the F/N interface. In the expression (14) the spin current is evaluated on the normal-metal side of the interface.

We now show that Q also vanishes in the presence of surface absorption (14). Indeed, since spin potential is continuous at the N/F interface, $\mu^{s\alpha} = m^\alpha \mu^s$ on both sides of the surface, and

$$\tilde{\mu}^{s\alpha}(c)\nabla_i j_i^{s\alpha}(c') = \tfrac{1}{2} m^\alpha \mu^s(c) R^\alpha = 0,$$

where we used $m^\alpha(\delta^{\alpha\beta} - m^\alpha m^\beta) = 0$. We therefore have

$$\oint \big(\tilde{\mu}^a(1) j_i^a(2) - \tilde{\mu}^a(2) j_i^a(1)\big)dA_i = 0,$$

and hence after integration,

$$\tilde{\mu}_t^a(1) I_t^a(2) - \tilde{\mu}_t^a(2) I_t^a(1) = 0. \qquad (15)$$

In a composite two-terminal element Eq. (5) is generalized to

$$I_t^a = G_{tt'}^{ab} \tilde{\mu}^b, \qquad (16)$$

and thus (15) means

$$\tilde{\mu}_t^a(1)\, G_{tt'}^{ab}\, \tilde{\mu}_{t'}^b(2) - \tilde{\mu}_t^a(2)\, G_{tt'}^{ab}\, \tilde{\mu}_{t'}^b(1) = 0$$

or

$$\tilde{\mu}_t^a(1)\tilde{\mu}_{t'}^b(2)\big(G_{tt'}^{ab} - G_{t't}^{ba}\big) = 0.$$

Since we are free to choose the potentials $\tilde{\mu}_t^a(1)$ and $\tilde{\mu}_t^a(2)$ arbitrarily, the above equality means

$$G_{tt'}^{ab} = G_{t't}^{ba}. \qquad (17)$$

This equation generalizes our result (6) to a composite two-terminal diffusive element.

The symmetry requirement (17) applied to an $n \times n$ matrix produces $n(n-1)/2$ relations between its entries. For an 8×8 matrix $G_{tt'}^{ab}$ this yields 28 relations between 64 entries. Note that relations between the elements with $t \neq t'$ and $a = b$ have a meaning similar to that of (6): transmission from one contact to another is directionless. In particular, for $a = b = e$ one recovers the direction independence of the charge transport, already well known from elementary physics. For $a = e$ and $b = sx, sy, sz$ we find additional relations between the spin currents generated by the electric potential and vice versa.

C. Multiterminal elements

An element with N terminals is described by the conductance $G_{tt'}^{ab}$ with $t, t' = 1 \ldots N$: conductance is a $4N \times 4N$ matrix. Applying the procedure of the previous section, we can prove the relation

$$\mu_t^a(1)\mu_{t'}^b(2)\big(G_{tt'}^{ab} - G_{t't}^{ba}\big) = 0$$

for any choice of $4N$-dimensional vectors $\mu_t^a(1)$ and $\mu_t^a(2)$. Thus Eq. (17) holds for $G_{tt'}^{ab}$, and the $4N \times 4N$ conductance matrix $G_{tt'}^{ab}$ is symmetric as well.

III. CONSEQUENCES OF ELECTRIC CURRENT CONSERVATION

A. Two-terminal elements

Let us return to the two-terminal case. The tensor $G_{tt'}^{ab}$ may be represented as an 8×8 matrix in two ways: first as

$$\begin{pmatrix} I_1^a \\ I_2^a \end{pmatrix} = \left|\begin{array}{c|c} G_{11}^{ab} & G_{12}^{ab} \\ \hline G_{21}^{ab} & G_{22}^{ab} \end{array}\right| \begin{pmatrix} \tilde{\mu}_1^b \\ \tilde{\mu}_2^b \end{pmatrix}, \qquad (18)$$

with 4×4 matrix entries in every block, and second as

$$\begin{pmatrix} I_t^e \\ I_t^{sx} \\ I_t^{sy} \\ I_t^{sz} \end{pmatrix} = \left|\begin{array}{c|c|c|c} G_{tt'}^{ee} & G_{tt'}^{e,sx} & G_{tt'}^{e,sy} & G_{tt'}^{e,sz} \\ \hline G_{tt'}^{sx,e} & G_{tt'}^{sx,sx} & G_{tt'}^{sx,sy} & G_{tt'}^{sx,sz} \\ \hline G_{tt'}^{sy,e} & G_{tt'}^{sy,sx} & G_{tt'}^{sy,sy} & G_{tt'}^{sy,sz} \\ \hline G_{tt'}^{sz,e} & G_{tt'}^{sz,sx} & G_{tt'}^{sz,sy} & G_{tt'}^{sz,sz} \end{array}\right| \begin{pmatrix} \mu_{t'}^e \\ \tilde{\mu}_{t'}^{sx} \\ \tilde{\mu}_{t'}^{sy} \\ \tilde{\mu}_{t'}^{sz} \end{pmatrix}, \qquad (19)$$

with 2×2 matrices in every block. The order of the tensor's elements in two cases is different, but in both representations the resulting 8×8 matrix is symmetric.

The second representation is more convenient for taking into account the electric current conservation: for any set of applied potentials $I_1^e = -I_2^e$ (the minus on the left-hand side appears due to our definition of current signs). This gives

$G^{ea}_{1t} = -G^{ea}_{2t}$ for every t and a, where, in the notation of Eqs. (9), (18), and (19), a takes the values e, sx, sy, and sz. This amounts to eight more constraints on the entries of the conductance matrix, which further reduces the number of its independent entries to 28 = 64–28 (reciprocity)–8 (electric current conservation).

Note that, together with the reciprocity condition, the electric current conservation yields $G^{ae}_{t1} = -G^{ae}_{t2}$. That is, every current I^a_t depends only on the difference $\mu^e_1 - \mu^e_2$, as required by gauge invariance. This is simply a manifestation of the intimate relation between gauge invariance and charge conservation [15]. Equivalently, we could impose gauge invariance via $G^{ae}_{t1} = -G^{ae}_{t2}$, which would then imply eight constraints and, of course, yield electric current conservation. Needless to say, the conductance matrix ends up with the same 28 independent entries.

With current conservation taken into account,

$$G^{ee}_{tt'} = \begin{vmatrix} -G & G \\ G & -G \end{vmatrix}, \quad G^{s\alpha,e}_{tt'} = \begin{vmatrix} C^\alpha_1 & C^\alpha_2 \\ -C^\alpha_1 & -C^\alpha_2 \end{vmatrix}.$$

Now we can write down the 8 × 8 matrix $G^{ab}_{tt'}$ that obeys all the constraints and has 28 independent entries:

$$\begin{pmatrix} I^e_1 \\ I^e_2 \\ I^{sx}_1 \\ I^{sx}_2 \\ I^{sy}_1 \\ I^{sy}_2 \\ I^{sz}_1 \\ I^{sz}_2 \end{pmatrix} = \begin{vmatrix} -G & G & C^x_1 & C^x_2 & C^y_1 & C^y_2 & C^z_1 & C^z_2 \\ G & -G & -C^x_1 & -C^x_2 & -C^y_1 & -C^y_2 & -C^z_1 & -C^z_2 \\ C^x_1 & -C^x_1 & S^x_1 & S^x_c & S^{xy}_{11} & S^{xy}_{12} & S^{xz}_{11} & S^{xz}_{12} \\ C^x_2 & -C^x_2 & S^x_c & S^x_2 & S^{xy}_{21} & S^{xy}_{22} & S^{xz}_{21} & S^{xz}_{22} \\ C^y_1 & -C^y_1 & S^{xy}_{11} & S^{xy}_{21} & S^y_1 & S^y_c & S^{yz}_{11} & S^{yz}_{12} \\ C^y_2 & C^y_2 & S^{xy}_{12} & S^{xy}_{22} & S^y_c & S^y_2 & S^{yz}_{21} & S^{yz}_{22} \\ C^z_1 & -C^z_1 & S^{xz}_{11} & S^{xz}_{21} & S^{yz}_{11} & S^{yz}_{21} & S^z_1 & S^z_c \\ C^z_2 & -C^z_2 & S^{xz}_{12} & S^{xz}_{22} & S^{yz}_{12} & S^{yz}_{22} & S^z_c & S^z_2 \end{vmatrix} \begin{pmatrix} \mu^e_1 \\ \mu^e_2 \\ \tilde{\mu}^{sx}_1 \\ \tilde{\mu}^{sx}_2 \\ \tilde{\mu}^{sy}_1 \\ \tilde{\mu}^{sy}_2 \\ \tilde{\mu}^{sz}_1 \\ \tilde{\mu}^{sz}_2 \end{pmatrix}. \quad (20)$$

For electric current we obtain the expression

$$I^e_t = (-1)^t \left[G\left(\mu^e_1 - \mu^e_2\right) + C^\alpha_{t'} \tilde{\mu}^{s\alpha}_{t'} \right]. \quad (21)$$

For spin current we find

$$I^{s\alpha}_t = C^\alpha_t \left(\mu^e_1 - \mu^e_2\right) + S^{\alpha\beta}_{tt'} \tilde{\mu}^{s\beta}_{t'}, \quad (22)$$

with a symmetric spin conductance matrix $S^{\alpha\beta}_{tt'} = S^{\beta\alpha}_{t't}$. The matrix elements of $S^{\alpha\beta}_{tt'}$ with $\alpha \neq \beta, t \neq t'$ describe the precession of spin injected at one terminal while it is transmitted to the other terminal. For instance, S^{xy}_{12} describes spin precession from y to x that may occur due to, e.g., the presence of magnetic parts in the element. Note that the reciprocity relations do not connect S^{xy}_{12} and S^{xy}_{21}, that is, transmission in the opposite spatial directions with the same spin precession. Instead, the equation $S^{xy}_{12} = S^{yx}_{21}$ connects the processes that are opposite in both the spatial direction and the sense of spin precession.

An interesting special case is found when both terminals of an element are strong ferromagnets (Fig. 4). The magnetization directions of the terminals, m^α_1 and m^α_2, may be noncollinear. Spin potentials and spin currents at the terminals are restricted to the form $\mu^{s\alpha}_1 = m^\alpha_1 \mu^s_1$ and $\mu^{s\alpha}_2 = m^\alpha_2 \mu^s_2$, $I^{s\alpha}_1 = m^\alpha_1 I^s_1$ and $I^{s\alpha}_2 = m^\alpha_2 I^s_2$. Equation (20) then reduces to a simpler one involving a 4 × 4 conductance matrix as per

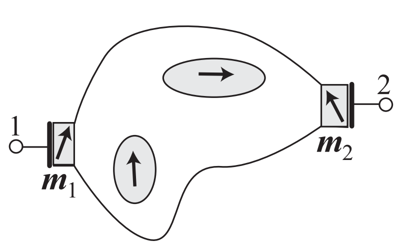

FIG. 4. Composite element with ferromagnetic contacts.

$$\begin{pmatrix} I^e_1 \\ I^e_2 \\ I^s_1 \\ I^s_2 \end{pmatrix} = \begin{vmatrix} -G & G & C_1 & C_2 \\ G & -G & -C_1 & -C_2 \\ C_1 & -C_1 & S_1 & S_c \\ C_2 & -C_2 & S_c & S_2 \end{vmatrix} \begin{pmatrix} \mu^e_1 \\ \mu^e_2 \\ \tilde{\mu}^s_1 \\ \tilde{\mu}^s_2 \end{pmatrix}, \quad (23)$$

with $C_1 = C^\alpha_1 m^\alpha_1$, $C_2 = C^\alpha_2 m^\alpha_2$, $S_1 = m^\alpha_1 S^{\alpha\beta}_{11} m^\beta_1$, $S_2 = m^\alpha_2 S^{\alpha\beta}_{22} m^\beta_2$, and $S_c = m^\alpha_1 S^{\alpha\beta}_{12} m^\beta_2$. We see that the conductance of such an element is defined by six independent parameters.

A related special case, admitting an equally simple description, is found when the magnetizations of all ferromagnetic parts of a composite element are collinear. The terminals may be ferromagnetic or normal, but it is required that the applied spin potentials are collinear with the magnetization direction. The situation then reduces to the one described by Eq. (23) with $m^\alpha_1 = m^\alpha_2$.

Returning to the general expressions (21) and (22) for the currents, we stress that, unlike the electric potentials μ^e_t, spin potentials do not have to appear only in the form of a difference $\mu^{s\alpha}_1 - \mu^{s\alpha}_2$. In other words, the coefficients C^α_1 and C^α_2 are not necessarily equal in absolute value and opposite in sign. The same is true for $S^{\alpha\beta}_{t1}$ and $S^{\alpha\beta}_{t2}$. The absence of such a requirement becomes transparent in a collinear setup, where $\mu^{s\alpha}$ differs from zero for a single direction α in spin space. Here it is evident that $\mu^s_t = \mu^\uparrow_t - \mu^\downarrow_t$ is already gauge invariant for each t as the difference of spin-up and spin-down potentials. Thus electric and spin currents may depend separately on μ^s_1 and μ^s_2 without violating gauge invariance.

We illustrate this point by an explicit example of a composite element consisting of ferromagnetic and normal parts, shown in Fig. 5. The parts have lengths $L_{F,N}$, much larger than the spin-diffusion lengths $\lambda_{F,N}$ in either material. We assume

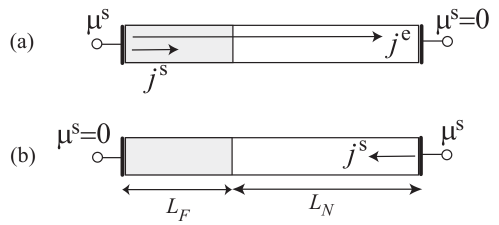

FIG. 5. Composite element consisting of a ferromagnetic (shaded) and normal-metal (white) parts. Electric current is generated by spin potential applied to one terminal (a) but not the other (b).

that the ferromagnet is magnetized along the x direction. No electric potentials are applied to the element.

In the first experiment, a spin potential is applied along the magnetization, to the left ($t = 1$) contact only, $\mu_t^{s\alpha} = \mu^{s0}\delta^{\alpha x}\delta_{t1}$. Thus spin current j^{sx} is injected and propagates along the ferromagnetic part of the element over a distance λ_F before dissipating. The presence of nonzero j^{sx} in a ferromagnet, in turn, generates electric current j^e according to the Johnson-Silsbee physics [16,17]. Electric current, once generated, reaches the right terminal of the element. The total electric current is given by $I_1^e = -I_2^e = C_1^x\mu^{s0} \neq 0$. Thus $C_1^x \neq 0$.

In the second experiment, spin potential is applied only to the right ($t = 2$) contact $\mu_t^{s\alpha} = \mu^{s0}\delta^{\alpha x}\delta_{t2}$, also injecting spin current. As in the previous case, the spin current completely dissipates before reaching the boundary between the parts of the element. However, in a normal metal, pure spin current generates no electric current and thus $I_1^e = -I_2^e = C_2^x\mu^{s0} = 0$. Therefore, $C_2^x = 0 \neq C_1^x$.

B. Multiterminal elements

An easy way to find the additional constraints arising from electric current conservation in a multiterminal element is to work with the generalizations of Eqs. (21) and (22):

$$I_t^e = G_{tt'}^e\mu_{t'}^e + C_{tt'}^\alpha\tilde{\mu}_{t'}^{s\alpha}, \tag{24}$$

$$I_t^{s\alpha} = C_{t't}^\alpha\mu_{t'}^e + S_{tt'}^{\alpha\beta}\tilde{\mu}_{t'}^{s\beta}. \tag{25}$$

The reciprocity requirements translate into $G_{tt'}^e = G_{t't}^e$, $S_{tt'}^{\alpha\beta} = S_{t't}^{\beta\alpha}$, and the indices of C in the second equation being transposed compared with the first.

Conservation of electric current requires $\sum_{t=1}^N I_t^e = 0$, where the summation is performed over all terminals. Two conditions emerge from it:

$$\sum_{t=1}^N G_{tt'}^e = 0, \quad \sum_{t=1}^N C_{tt'}^\alpha = 0. \tag{26}$$

The first of them is the standard requirement satisfied in any multiterminal electric element.

IV. NO GEOMETRIC SPIN RATCHETS

While spin electronics may promise various advantages, spin dissipation hinders spin transmission and is clearly an obstacle. This naturally raises the issue of finding systems with longer spin propagation lengths.

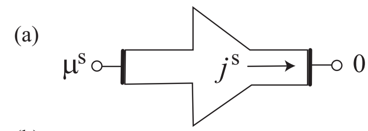

FIG. 6. Normal-metal elements (a) with directional arrow shape and (b) reference strip.

One interesting proposal [18,19] involves optimizing the geometric shape of a conductor—and a claim that, in an arrow-shaped normal wire [Fig. 6(a)], spin transmission is enhanced as compared with a rectangular wire [Fig. 6(b)]. Indeed, depending on the precise shape of the arrow, the spin conductance G_{12}^s of the wire with an arrow may or may not be enhanced compared with a rectangular strip. But this does not yet mean that the reason for the enhancement is the orientation of the arrow. The presence or absence of propagation boost due to the geometric asymmetry of the wire should be inferred from a comparison between spin propagation along the arrow direction and opposite to it. And this is precisely where the reciprocity relation (6) applies. It tells us that spin propagation through the arrow-shaped element is the same in both directions. The conductances G_{11}^s and G_{22}^s may differ, and thus the current drawn from the injector may depend on the side where μ^s is applied. But, at a given μ^s, the transmitted spin current remains the same regardless of the arrow orientation. We must conclude that an arrow pointing against the spin-current flow "amplifies" it as much as the one pointing along the flow. This conclusion holds for any passive spintronic element of a kind described above, to which the reciprocity relations apply.

V. COMPARISON WITH THE CIRCUIT THEORY FORMALISM

The "circuit theory" (CT) of Ref. [1] is a finite-element (lumped element) theory, operating with two types of elementary units: normal or ferromagnetic "nodes," each characterized by a spatially uniform electron distribution function, and "contacts" that define the conductance between the nodes. Spin relaxation may take place in the nodes but not in the contacts. A special type of node, the "reservoirs," sets the voltages and spin potentials applied to the device.

Here we illustrate the correspondence between the matrix conductance $G_{tt'}^{ab}$ of the diffusion-equation description of the preceding sections and the CT matrix conductance. We focus on a simple two-terminal F/N element in Fig. 5. To begin with, the terminology of the two approaches is different: In the diffusion-equation approach, an "element" connects two "contacts", each characterized by its electric potential μ^e and spin potential $\mu^{s\alpha}$. In the CT approach, a "contact" connects two "nodes", each characterized by its μ^e and $\mu^{s\alpha}$. Thus an "element" of the diffusion-equation description should be compared with a CT "contact", while a diffusion-equation "contact" corresponds to a CT "node".

In a CT F/N contact, the electric and spin currents are determined by spin-resolved real conductances $G_\uparrow$, $G_\downarrow$ and a complex mixing conductance $G_{\uparrow\downarrow}$, which in total makes four real parameters [1]. At the same time, the conductance matrix $G^{ab}_{tt'}$ in Eq. (20) involves 28 independent parameters. The relation between the parameter sets of the CT and the diffusion-equation description is discussed below.

The settings studied in Ref. [1] and in our work are generally different, and comparison is meaningful only where the validity domains of the two approaches overlap. First, Ref. [1] assumed no spin dissipation in the contact. Second, it considered an F terminal with the spin potential set to zero, $\mu^{s\alpha}_F = 0$. Without loss of generality, we can also choose $\mu^e_F = 0$, since currents depend only on the difference, $\mu^e_N - \mu^e_F$. The spin potential $\mu^{s\alpha}_N$ of the N contact is allowed to have an arbitrary direction, not necessarily collinear with the magnetization direction m^α of the F electrode. Third, Ref. [1] studied the currents in the N terminal. Therefore, we shall compare the matrix conductance of a CT contact with that of a diffusive F/N element with spin relaxation lengths $\lambda_{F,N} \to \infty$.

The CT operates with a 2×2 matrix current $\hat{I}$, related to the electric and spin currents as per $\hat{I} = (I^e\hat{E} + I^{s\alpha}\hat{\sigma}_\alpha)/2$, where $\hat{\sigma}_\alpha$ are the Pauli matrices. Likewise, the matrix potential is given by $\hat{\mu} = \mu^e\hat{E} + \tilde{\mu}^{s\alpha}\hat{\sigma}_\alpha$. If the z axis is chosen along m^α, CT provides the following formula for the current in the N contact:

$$\hat{I}_N = -\begin{pmatrix} G_\uparrow\mu_{\uparrow\uparrow}(N) & G_{\uparrow\downarrow}\mu_{\uparrow\downarrow}(N) \\ G^*_{\uparrow\downarrow}\mu_{\downarrow\uparrow}(N) & G_\downarrow\mu_{\downarrow\downarrow}(N) \end{pmatrix} = -\begin{pmatrix} G_\uparrow\left(\mu^e_N + \tilde{\mu}^{sz}_N\right) & G_{\uparrow\downarrow}\left(\tilde{\mu}^{sx}_N - i\tilde{\mu}^{sy}_N\right) \\ G^*_{\uparrow\downarrow}\left(\tilde{\mu}^{sx}_N + i\tilde{\mu}^{sy}_N\right) & G_\downarrow\left(\mu^e_N - \tilde{\mu}^{sz}_N\right) \end{pmatrix}.$$

By recasting this formula in the form $I^a_N = \mathcal{G}^{ab}\tilde{\mu}^a_N$, one gets the conductance,

$$\hat{\mathcal{G}} = -\begin{vmatrix} G_\uparrow + G_\downarrow & 0 & 0 & G_\uparrow - G_\downarrow \\ 0 & 2\,\mathrm{Re}[G_{\uparrow\downarrow}] & 2\,\mathrm{Im}[G_{\uparrow\downarrow}] & 0 \\ 0 & -2\,\mathrm{Im}[G_{\uparrow\downarrow}] & 2\,\mathrm{Re}[G_{\uparrow\downarrow}] & 0 \\ G_\uparrow - G_\downarrow & 0 & 0 & G_\uparrow + G_\downarrow \end{vmatrix}. \quad (27)$$

The matrix $\mathcal{G}^{ab}$ should be compared with the 4×4 sector G^{ab}_{NN} of the 8×8 matrix $G^{ab}_{tt'}$ (18),

$$G^{ab}_{NN} = \begin{vmatrix} -G & C^x_N & C^y_N & C^z_N \\ C^x_N & S^x_N & S^{xy}_N & S^{xz}_N \\ C^y_N & S^{xy}_N & S^y_N & S^{yz}_N \\ C^z_N & S^{xz}_N & S^{yz}_N & S^z_N \end{vmatrix}. \quad (28)$$

The sectors G^{ab}_{NF} and G^{ab}_{FF} are not related to $\mathcal{G}^{ab}$.

The G^{ab}_{NN} of Eq. (28) assumes the form of $\mathcal{G}^{ab}$ in Eq. (27) if its entries satisfy a number of conditions:

First, zero entries of $\hat{\mathcal{G}}$ should be matched by $C^x_N = C^y_N = S^{xz}_N = S^{yz}_N = 0$. Appendix C shows that this is the case for collinear devices, of which ours is a particular example. Appendix D illustrates this for the device in Fig. 5 by direct calculation along the lines of Refs. [1,13,14,20].

Second, the equality of the first and the last diagonal entries of $\hat{\mathcal{G}}$ requires $S^z_N = -G$. Appendix C shows that this property relies both on the collinear character of the device and on the absence of spin relaxation. Under these conditions, the equality $S^z_N = -G$ is protected by a peculiar symmetry of the diffusion equations and boundary conditions. Appendix D illustrates this by direct calculation and shows that, in the presence of spin relaxation, S^z_N and $-G$ are different.

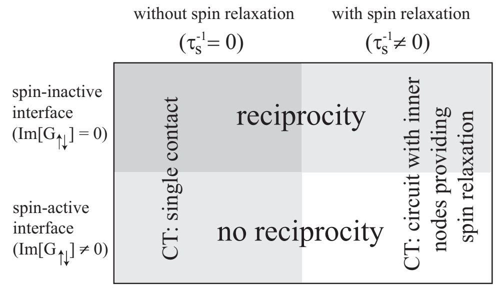

FIG. 7. Regimes of a diffusive F/N element: Reciprocity properties discussed in our work break down in the presence of a spin-active interface. In terms of circuit theory, an element can be described as a single CT contact if $\tau_s^{-1} = 0$ but has to be modeled by a CT circuit with spin-relaxing inner nodes when $\tau_s^{-1} \neq 0$.

Third, since the entry Im[$G_{\uparrow\downarrow}$] appears in $\hat{\mathcal{G}}$ antisymmetrically, while the symmetry of $\hat{G}_{NN}$ is our main statement, $\hat{\mathcal{G}}$ and $\hat{G}_{NN}$ can be equal only if Im[$G_{\uparrow\downarrow}$] = 0. In terms of Eq. (28), this means $S^{xy}_N = 0$. In the diffusion-equation description, this can be traced back to the F/N interface being spin-inactive (Appendix D).

To conclude, in the absence of spin relaxation, a diffusive F/N element with a spin-inactive interface can indeed be modeled as a single CT contact with symmetric (reciprocal) $\hat{\mathcal{G}}$. By contrast, in the presence of spin relaxation, $S^z_N \neq -G$, and thus the F/N element cannot be modeled by a single CT contact of the form (27). Instead, the model shall involve a CT circuit with at least one inner node, accounting for spin relaxation. All of this is schematically summarized in Fig. 7.

Finally, we wish to note that diffusive elements can also be described by equations for the spatially nonuniform 2×2 spin distribution function [21]. As the node size reduces below the diffusion length, this description explicitly crosses over to the CT formalism in the form of Ref. [1]. Yet another way of introducing spin relaxation was developed in Ref. [22].

VI. RECIPROCITY AND THE ONSAGER RELATIONS

There exist universal relations between the entries of the spin conductance matrix [7] that follow from Onsager's principle [23] of the symmetry of the kinetic coefficients. In this section we show that such relations are generally different from (and more general than) the reciprocity relations of Sec. II. To demonstrate this difference, we consider a two-terminal device with all magnetizations and applied spin potentials being collinear. Onsager relations have a different meaning for systems with and without the microscopic reversibility of equations of motion. Below we consider these cases one by one.

For a device described in the preceding paragraph, microreversibility holds only in the absence of ferromagnetic elements. A derivation presented in Appendix E shows that Onsager's principle imposes the following form of the conductance matrix, given by Eq. (E10):

$$\begin{pmatrix} I_1^e \\ I_2^e \\ I_1^s \\ I_2^s \end{pmatrix} = \begin{pmatrix} -G & G & B_1 & B_2 \\ G & -G & -B_1 & -B_2 \\ -B_1 & B_1 & S_{11} & S_{\rm mix} \\ -B_2 & B_2 & S_{\rm mix} & S_{22} \end{pmatrix} \begin{pmatrix} \mu_1^e \\ \mu_2^e \\ \tilde{\mu}_1^s \\ \tilde{\mu}_2^s \end{pmatrix}.$$

Here all spin currents and spin potentials point along the same axis and thus require no extra indices to define their direction in spin space.

The reciprocity requirements for the very same collinear setup are given by Eq. (23). This equation and Eq. (E10) impose the same symmetry constraints on conductance coefficients for potentials of the same parity under time reversal: those relating spin currents to spin potentials, and those relating charge current to electric potentials. By contrast, the constraints on the entries relating spin currents to electric potentials and vice versa are different: Eq. (23) imposes symmetry, while Eq. (E10) demands antisymmetry—precisely due to the *opposite* parity of spin and electric potentials with respect to time reversal.

The resolution of this apparent contradiction hides in the observation that Eq. (E10) relies only on the general thermodynamic properties of the global quantities N_t and N_{st} in Appendix E, whereas the diffusion reciprocity relations appeal to rather specific equations, Eqs. (1) and (2), for charge and spin currents. In a normal-metal device $B_{1,2} = 0$ (see Sec. II A), and both the Onsager reciprocity and diffusion reciprocity are satisfied.

The vanishing of B coefficients results from the decoupling of charge and spin currents in Eqs. (1) and (2). The latter decoupling stems from our implicit disregard of spin-orbit interaction (SOI), as is commonly done for lighter metals such as Cu. In a normal metal with noticeable SOI, such as Pt, charge current generates a transverse spin current—and vice versa [24]. In a four-terminal device this effect induces transverse spin current in response to longitudinal electric bias and thus gives rise to nonzero conductance coefficients of the $B_{1,2}$ kind [25]. Hence, as shown in Appendix F, the diffusion reciprocity breaks down, while the Onsager reciprocity holds indeed. While the condition of vanishing SOI is quantitative, one may turn it around and state that precision of the diffusion reciprocity relations is limited by the ratio of the spin Hall conductivity to the electric conductivity—for instance, as done in a different context for platinum [26].

For devices with ferromagnetic elements, the microscopic equations of motion are not time-reversal invariant. In this case, Onsager's principle [23] relates the entries of the conductance matrix of a device to those of a *different* device that has all the magnetization vectors reversed. By contrast, the diffusion reciprocity connects the entries of the conductance matrix of the *very same* device. In diffusive devices with ferromagnetic elements, both the Onsager's reciprocity and the diffusion reciprocity are valid, but their content is different, and they do not follow from each other.

Onsager reciprocity hinges on the time-reversal symmetry of microscopic equations of motion, a very general symmetry. Diffusive reciprocity is valid *in addition to* that when extra constraints are present. A somewhat similar situation is described in Ref. [7] for materials with sublattice symmetry: their conductance matrix is symmetric with respect to a simultaneous interchange of contact and spin indices—also *in addition* to the Onsager symmetry. Similarly, diffusive reciprocity arises as a result of certain special properties, such as absorption of transverse spin current due to strong ferromagnetism, as encapsulated in Eqs. (12) and (13). These conditions were essential for the arguments of Sec. II B.

VII. DISCUSSION

In this work, we established reciprocity relations for a class of devices, with or without spin relaxation, where (a) both spin and charge propagate diffusively, (b) the carrier spin aligns itself with magnetization of a ferromagnetic element over a vanishingly short distance, (c) the F/N interfaces are Ohmic and spin inactive, and (d) spin-orbit interaction is negligible. Together with charge conservation, the reciprocity relations constrain the form of the conductance matrix—for example, the 64 entries of an 8×8 conductance matrix of a two-terminal element are reduced to only 28 independent values.

We showed that diffusive reciprocity is different from the more general Onsager reciprocity, and emerges as an additional relation in a narrower class of systems. For normal-metal elements, reciprocity relations prove the impossibility of "geometric spin ratchets" [18,19] that would amplify spin current or even transmit it differently in two directions. In fact, this conclusion stems from the symmetry of the spin-sector matrix $S_{tt'}^{\alpha\beta}$, protected by the Onsager principle even when diffusive reciprocity fails.

ACKNOWLEDGMENTS

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APPENDIX A: IDENTITIES

For two functions $u(\mathbf{r})$ and $v(\mathbf{r})$, the Gauss theorem gives

$$\int_D (u\Delta v - v\Delta u)dV = \oint_S (u\nabla v - v\nabla u)d\mathbf{A}, \tag{A1}$$

with D being the integration volume with surface S. Furthermore, for an arbitrary $a(\mathbf{r})$,

$$\int [u\nabla(a\nabla v) - v\nabla(a\nabla u)]dV = \oint [ua\nabla v - va\nabla u]d\mathbf{A}. \tag{A2}$$

This may be generalized for tensors. For $u_\alpha(\mathbf{r})$, $v_\beta(\mathbf{r})$, and symmetric $A_{\alpha\beta}(\mathbf{r}) = A_{\beta\alpha}(\mathbf{r})$ one has

$$\int [u_\alpha \nabla(A_{\alpha\beta}\nabla v_\beta) - v_\alpha \nabla(A_{\alpha\beta}\nabla u_\beta)]dV = \oint [u_\alpha A_{\alpha\beta}\nabla v_\beta - v_\alpha A_{\alpha\beta}\nabla u_\beta]d\mathbf{A}, \tag{A3}$$

with repeated index summation assumed.

APPENDIX B: MATRIX CONDUCTIVITY OF STRONG FERROMAGNETS

In a strong ferromagnet with constant magnetization direction m^α, the vector spin potential satisfies $\mu^\alpha(r) = m^\alpha \mu^s(r)$, and the spin current satisfies $j_i^\alpha(r) = m^\alpha j_i^s(r)$. Currents j_i^e, j_i^s and potentials μ^e, μ^s are related by the equations

$$j_i^e = -\sigma(r)\nabla_i\mu^e - p(r)\sigma(r)\nabla_i(\mu^s/2),$$

$$j_i^s = -p(r)\sigma(r)\nabla\mu^e - \sigma(r)\nabla_i(\mu^s/2).$$

Therefore

$$\begin{aligned} j_i^e &= -\sigma(r)\nabla_i\mu^e - p(r)\sigma(r)m^\alpha\nabla_i(\mu^{s\alpha}/2),\\ j_i^{s\alpha} &= -p(r)\sigma(r)m^\alpha\nabla\mu^e - \sigma(r)\nabla_i(\mu^{s\alpha}/2). \end{aligned} \tag{B1}$$

These equations can be combined into

$$j_i^a = -\Sigma^{ab}\nabla_i\tilde{\mu}^b, \tag{B2}$$

with $a = \{e, x, y, z\} = \{0, 1, 2, 3\}$, rescaled potentials

$$\tilde{\mu}^b = \left\{\mu^e, \frac{\mu^{sx}}{2}, \frac{\mu^{sy}}{2}, \frac{\mu^{sz}}{2}\right\},$$

and a 4×4 matrix of generalized conductivity

$$\Sigma^{ab} = \begin{pmatrix} \sigma & p\sigma m^x & p\sigma m^y & p\sigma m^z \\ p\sigma m^x & \sigma & 0 & 0 \\ p\sigma m^y & 0 & \sigma & 0 \\ p\sigma m^z & 0 & 0 & \sigma \end{pmatrix}. \tag{B3}$$

When defined in terms of $\tilde{\mu}^a$, the generalized conductivity tensor is symmetric.

APPENDIX C: SYMMETRY CONSTRAINTS ON CONDUCTANCE MATRICES OF COLLINEAR DEVICES

All of the constrains obtained in this Appendix rely on the device being magnetically collinear. That is, magnetization in all the ferromagnetic parts points along or opposite one and the same direction, denoted as m^α.

1. In a collinear device, $C_N^x = C_N^y = S_N^{xz} = S_N^{yz} = 0$

In such a device, all the equations for the electric current density j_i^e and for the component $j_i^{sz} = m^\alpha j_i^{s\alpha}$ of the spin current are invariant with respect to uniform spin rotation $(\mu^{sx}, \mu^{sy}, \mu^{sz}) \Rightarrow (\mu^{sx}\cos\varphi - \mu^{sy}\sin\varphi, \mu^{sy}\cos\varphi + \mu^{sx}\sin\varphi, \mu^{sz})$ by an arbitrary angle φ around m^α. This includes the continuity equations (3) and (4), expressions (7) and (8) for the currents, and the boundary conditions (12), (13), and (14) at the F/N interface. Therefore, both the electric current I_N^e and the I_N^{sz} component of the spin current are invariant under such a rotation, accompanied by the corresponding rotation $(\mu_N^{sx}, \mu_N^{sy}, \mu_N^{sz}) \Rightarrow (\mu_N^{sx}\cos\varphi - \mu_N^{sy}\sin\varphi, \mu_N^{sy}\cos\varphi + \mu_N^{sx}\sin\varphi, \mu_N^{sz})$ of the boundary conditions. For the electric current I_N^e, Eq. (28) yields

$$I_N^e = -G\mu_N^e + C_N^x\mu_N^{sx} + C_N^y\mu_N^{sy} + C_N^z\mu_N^{sz}.$$

This expression is invariant with respect to the rotation above only if $C_N^x = C_N^y = 0$. The same argument for the component I_N^{sz} of the spin current

$$I_N^{sz} = C_N^z\mu_N^e + S_N^{xz}\mu_N^{sx} + S_N^{yz}\mu_N^{sy} + S_N^z\mu_N^{sz}$$

leads us to conclude that $S_N^{xz} = S_N^{yz} = 0$. In a device with noncollinear ferromagnetic parts, C_N^x, C_N^y, S_N^{xz}, and S_N^{yz} are generally nonzero.

2. Collinearity and no spin relaxation lead to $G = -S_N^z$

In a collinear device with no spin relaxation, an extra symmetry guarantees that $G = -S_N^z$. To see this, we apply potentials μ_N^e, μ_N^{sz} to the N terminal, while the remaining components of spin potential are set to zero ($\mu_N^{sx} = \mu_N^{sy} = 0$). Using the notation $\tilde{\mu}^s \equiv \mu^s/2$ [cf. Eq. (9)], Eqs. (7) and (8) yield

$$j_i^e = -\frac{\sigma}{e^2}(\nabla_i\mu^e + p\nabla_i\tilde{\mu}^{sz}), \tag{C1}$$

$$j_i^{sz} = -\frac{\sigma}{e^2}(\nabla_i\tilde{\mu}^{sz} + p\nabla_i\mu^e). \tag{C2}$$

In the normal part, Eqs. (1) and (2) yield

$$j_i^e = -\frac{\sigma}{e^2}\nabla_i\mu^e, \tag{C3}$$

$$j_i^{sz} = -\frac{\sigma}{e^2}\nabla_i\tilde{\mu}^{sz}. \tag{C4}$$

We observe that the system of Eqs. (C1)–(C4) is invariant under the transmutation $\tilde{\mu}^e \leftrightarrow \tilde{\mu}^{sz}$, $j_i^e \leftrightarrow j_i^{sz}$. Spin relaxation breaks this invariance, since the continuity equations (3) and (4) for spin and charge are different. However, as $\tau_s \to \infty$ in Eq. (4), the symmetry is restored, and the full problem becomes symmetric under the replacement $\tilde{\mu}^e \leftrightarrow \tilde{\mu}^{sz}$, $j_i^e \leftrightarrow j_i^{sz}$, completed by interchanging the driving potentials μ_N^e and μ_N^{sz} at the N terminal. All the equations describing the element, including the boundary conditions (12) and (13) at the F/N interface, are invariant under this transformation.

For the total currents I_N^e and I_N^{sz}, Eq. (28) implies

$$I_N^e = -G\mu_N^e + C_N^z\mu_N^{sz},$$

$$I_N^{sz} = C_N^z\mu_N^e + S_N^z\mu_N^{sz}.$$

Upon the symmetry transformation above ($\mu_N^e \leftrightarrow \mu_N^{sz}$, $I_N^e \leftrightarrow I_N^s$), these two equations turn into

$$I_N^e = S_N^z\mu_N^e + C_N^z\mu_N^{sz},$$

$$I_N^{sz} = C_N^z\mu_N^e - G\mu_N^{sz}\ .$$

At the same time, the two equations must remain intact for any μ_N^e and μ_N^{sz}. This is the case only if $S_N^z = -G$.

APPENDIX D: CONDUCTANCE OF A DIFFUSIVE F/N ELEMENT

1. Equations and boundary conditions

Here we consider an F/N element, comprising a thin ferromagnetic wire of length L_F in series with a normal wire of length L_N, as shown in Fig. 5. All quantities thus depend only on the coordinate x along the wire, with the origin at the F/N boundary. In the absence of spin-orbit coupling, the spin and coordinate spaces are decoupled. Thus, without loss of generality, we will assume that the magnetization of the F wire points along the z axis.

Electric and spin potentials are assumed to take zero values at the ferromagnetic terminal:

$$\mu^e(-L_F) = 0\,,$$
$$\mu^{s\alpha}(-L_F) = 0\,.$$

At the normal terminal,

$$\mu^e(L_N) = \mu^e_N\,,$$
$$\mu^{s\alpha}(L_N) = \mu^{s\alpha}_N.$$

Our goal is to find the currents $j^e_x(L_N) \equiv j^e_N$ and $j^{s\alpha}_x(L_N) \equiv j^{s\alpha}_N$ at the normal terminal. To do this, one has to find potentials $\mu^e(x)$ and $\mu^{s\alpha}(x)$ on the interval $[-L_F, L_N]$. The spin potential obeys equations [10,11]

$$\lambda^2_{N,F}\frac{d^2\mu^{s\alpha}(x)}{dx^2} = \mu^{s\alpha}(x), \tag{D1}$$

with $\lambda_{N,F}$ being the spin-diffusion lengths in the N and F wires. General solutions of these equations can be written in the form

$$\mu^{s\alpha}(x) = X^\alpha_F e^{x/\lambda_F} + Y^\alpha_F e^{-x/\lambda_F} \quad (-L_F < x \leqslant 0),$$
$$\mu^{s\alpha}(x) = X^\alpha_N e^{x/\lambda_N} + Y^\alpha_N e^{-x/\lambda_N} \quad (0 \leqslant x < L_N),$$

with coefficients to be determined from the continuity of μ^e, μ^s, j^e, and j^{sz} at $x = 0$ [13,14] and from the values of potentials at the terminals.

2. General solutions

Since the F wire is assumed to be a strong ferromagnet, in addition to (D1), the spin potential satisfies $\mu^{s\alpha}(x) = m^\alpha\mu^s(x)$ in the ferromagnetic part of the element, with m^α being the unit vector along the magnetization. Thus we seek the spin potential in the F wire in the form

$$\begin{aligned}\mu^{sx}(x) &= 0,\\ \mu^{sy}(x) &= 0, \qquad\qquad (-L_F < x \leqslant 0)\\ \mu^{sz}(x) &= A\sinh\frac{x+L_F}{\lambda_F}.\end{aligned}$$

The last expression is written so that it automatically satisfies the boundary condition at $x = -L_F$. The spin potential in the N wire can be sought in the form

$$\begin{aligned}\mu^{sx}(x) &= a_x\sinh\frac{x}{\lambda_N},\\ \mu^{sy}(x) &= a_y\sinh\frac{x}{\lambda_N}, \qquad\qquad (0 \leqslant x < L_N)\\ \mu^{sz}(x) &= a_z\sinh\frac{x}{\lambda_N} + b_z\cosh\frac{x}{\lambda_N},\end{aligned}$$

with unknown $a_{x,y,z}$ and b_z. The first two equations ensure the continuity of μ^{sx} and μ^{sy} at $x = 0$. Matching the spin potentials at the normal terminal we get

$$a_x\sinh\frac{L_N}{\lambda_N} = \mu^{sx}_N, \tag{D2}$$

$$a_y\sinh\frac{L_N}{\lambda_N} = \mu^{sy}_N, \tag{D3}$$

$$a_z\sinh\frac{L_N}{\lambda_N} + b_z\cosh\frac{L_N}{\lambda_N} = \mu^{sz}_N\,. \tag{D4}$$

From the continuity of spin potential on the F/N boundary,

$$A\sinh\frac{L_F}{\lambda_F} = b_z\,. \tag{D5}$$

To shorten the expressions in the remainder of this section, we introduce the following notation:

$$sh = \sinh(L/\lambda),$$
$$ch = \cosh(L/\lambda),$$
$$th = \tanh(L/\lambda).$$

The coefficients a_x and a_y are then expressed as

$$a_x = \frac{\mu^{sx}_N}{sh_N}, \quad a_y = \frac{\mu^{sy}_N}{sh_N}.$$

To find A, a_z, and b_z, the conditions of continuity for μ^e, j^e, and j^{sz} have to be invoked.

3. Electric current continuity

Electric potential in the N wire obeys the equation

$$\frac{d^2\mu^e(x)}{dx^2} = 0 \quad (0 \leqslant x < L_N)\,.$$

Its solutions are linear functions, so

$$\mu^e(x) = \mu^e_N\frac{x}{L_N} + \mu^e(0)\left(1 - \frac{x}{L_N}\right) \quad (0 \leqslant x < L_N)$$

with yet unknown $\mu^e(0)$.

The electric potential equation in the F wire is more complicated and couples electric and spin potentials. However, its use can be avoided because in the present 1D case the conservation of electric current means $j^e_x = \text{const} = j^e_N$. Equation (7) then gives a relation for potentials in the F wire:

$$-\frac{e^2 j^e_N}{\sigma_F} = \frac{d\mu^e}{dx} + \frac{p}{2}\frac{d\mu^{sz}}{dx} \quad (-L_F < x \leqslant 0).$$

Integrating it from $x = -L_F$ to $x = 0$, and using $\mu^e(-L_F) = 0$, gives

$$\mu^e(0) = -\frac{p}{2}\mu^s(0) - \frac{e^2L_F j^e_N}{\sigma_F} = -\frac{p\,b_z}{2} - \frac{e^2L_F j^e_N}{\sigma_F}\,. \tag{D6}$$

Electric current flowing through the element can be alternatively expressed by applying formula (1) to the N wire:

$$j^e_N = -\frac{\sigma_N}{e^2}\,\frac{\mu^e_N - \mu^e(0)}{L_N}. \tag{D7}$$

Following Ref. [20], we introduce the notation

$$\frac{1}{R} = \frac{\sigma}{e^2L}\,.$$

Combining Eqs. (D6) and (D7), we find

$$\begin{aligned}\mu^e(0) &= \frac{R_F}{R_N+R_F}\mu^e_N - \frac{R_N}{R_N+R_F}\frac{p}{2}b_z\,,\\ j^e_N &= -\frac{1}{R_N+R_F}\left(\mu^e_N + \frac{p}{2}b_z\right).\end{aligned} \tag{D8}$$

4. Spin current continuity

Finally, we use the continuity of j^{sz} at the F/N boundary. Combining Eqs. (7) and (8), we find in the ferromagnet

$$j^{sz}(x) = pj^e - \frac{\sigma_F(1-p^2)}{2e^2}\frac{d\mu^{sz}}{dx} \quad (-L_F < x \leqslant 0).$$

In the normal metal, Eq. (2) gives

$$j^{sz}(x) = -\frac{\sigma_N}{2e^2}\frac{d\mu^{sz}}{dx} \quad (0 \leqslant x \leqslant L_N).$$

Expressing the derivatives of μ^{sz} if F and N wires at $x=0$ in terms of the unknown coefficients, we get the continuity condition

$$-\frac{\sigma_N}{e^2\lambda_N}\frac{a_z}{2} = pj_N^e - \frac{\sigma_F}{e^2\lambda_F}(1-p^2)\frac{ch_F}{2}A\,.$$

Substituting the electric current from (D8), we recast the preceding equation in the final form:

$$\frac{L_N}{\lambda_N R_N}\frac{a_z}{2} = \frac{p\big(\mu_N^e + pb_z/2\big)}{R_N+R_F} + \frac{L_F(1-p^2)}{\lambda_F R_F}\frac{ch_F}{2}A. \quad \text{(D9)}$$

5. Solving for unknown coefficients

Equations (D4), (D5), and (D9) can now be solved to give the unknown coefficients. The results can be presented in a more compact way using the notation

$$t = \frac{\lambda}{L}\tanh\frac{L}{\lambda}, \quad s = \frac{\lambda}{L}\sinh\frac{L}{\lambda}\,,$$
$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_N t_N} + \frac{p^2}{R_N+R_F} + \frac{1-p^2}{R_F t_F}\,.$$

This gives

$$\begin{aligned} a_z &= \frac{2pR_{\text{eff}}}{R_N+R_F}\frac{\mu_N^e}{th_N} + \left(1-\frac{R_{\text{eff}}}{t_N R_N}\right)\frac{\mu_N^{sz}}{sh_N}\,,\\ b_z &= -\frac{2pR_{\text{eff}}}{R_N+R_F}\mu_N^e + \frac{R_{\text{eff}}}{s_N R_N}\mu_N^{sz}. \end{aligned} \quad \text{(D10)}$$

6. Currents at the normal terminal

Spin currents at the normal terminal ($x = L_N$) are given by

$$j_N^{sx} = -\frac{L_N}{2\lambda_N R_N}a_x ch_N,$$
$$j_N^{sy} = -\frac{L_N}{2\lambda_N R_N}a_y ch_N,$$
$$j_N^{sz} = -\frac{L_N}{2\lambda_N R_N}(a_z ch_N + b_z sh_N).$$

Substituting the expressions for $a_{x,y,z}$ and b_z into the equations above, we get

$$\begin{aligned} j_N^{sx} &= -\frac{1}{R_N t_N}\frac{\mu_N^{sx}}{2},\\ j_N^{sy} &= -\frac{1}{R_N t_N}\frac{\mu_N^{sy}}{2},\\ j_N^{sz} &= -\frac{1}{R_N}\left(\frac{1}{t_N} - \frac{R_{\text{eff}}}{s_N^2 R_N}\right)\frac{\mu_N^{sz}}{2} - \frac{pR_{\text{eff}}}{s_N R_N(R_N+R_F)}\mu_N^e\,. \end{aligned} \quad \text{(D11)}$$

Electric current is obtained by substituting expression (D10) for b_z into Eq. (D8):

$$j_N^e = -\frac{1}{R_N+R_F}\left(1-\frac{p^2R_{\text{eff}}}{R_N+R_F}\right)\mu_N^e - \frac{pR_{\text{eff}}}{s_N R_N(R_N+R_F)}\frac{\mu_N^{sz}}{2}\,. \quad \text{(D12)}$$

For completeness, we also give an expression for

$$\mu^e(0) = \left(\frac{R_F}{R_N+R_F} + \frac{p^2R_N R_{\text{eff}}}{(R_N+R_F)^2}\right)\mu_N^e - \frac{pR_{\text{eff}}}{s_N(R_N+R_F)}\frac{\mu_N^s}{2}\,.$$

7. Conductance matrix

Using (D11) and (D12), we can write down the entries of the 4×4 sector G_{NN}^{ab}, Eq. (18). Recall here that they are defined so that currents are positive when they flow out of the element. The nonzero entries are

$$G = \frac{1}{R_N+R_F}\left(1-\frac{p^2R_{\text{eff}}}{R_N+R_F}\right), \quad \text{(D13)}$$
$$C_N^z = -\frac{pR_{\text{eff}}}{s_N R_N(R_N+R_F)}, \quad \text{(D14)}$$
$$S_N^x = S_N^y = -\frac{1}{t_N R_N}, \quad \text{(D15)}$$
$$S_N^z = -\frac{1}{R_N}\left(\frac{1}{t_N} - \frac{R_{\text{eff}}}{s_N^2 R_N}\right). \quad \text{(D16)}$$

The remaining ones are equal to zero, in accordance with the explanations of Appendix C.

8. Limit of zero spin dissipation

The limit of zero spin dissipation corresponds to infinite spin-diffusion lengths $\lambda_{N,F} \to \infty$. In this limit, $s_{N,F} \to 1$, $t_{N,F} \to 1$, and thus

$$\frac{1}{R_{\text{eff}}} \to \frac{1}{R_N} + \frac{p^2}{R_N+R_F} + \frac{1-p^2}{R_F}$$

or

$$R_{\text{eff}} \to \frac{R_N R_F(R_N+R_F)}{(R_N+R_F)^2 - p^2R_N^2}\,.$$

Using these properties we find that conductances G and $-S_N^z$ indeed approach the same limit (see Fig. 8):

$$-S_N^z, G \to G_{\text{lim}} = \frac{(R_N+R_F) - p^2R_N}{(R_N+R_F)^2 - p^2R_N^2}\,. \quad \text{(D17)}$$

This expression for G_{lim} reproduces the results of Refs. [1] and [13], where spin-diffusion length was set to infinity from the outset and thus completes the correspondence between G_{NN}^{ab} and $\mathcal{G}^{ab}$, as discussed in Sec. V.

APPENDIX E: ONSAGER RELATIONS FOR A TWO-TERMINAL ELEMENT

Consider a two-terminal element of arbitrary shape with internal magnetic regions. This Appendix applies the Onsager argument to derive relations between the spin currents flowing through its contacts. Two restrictive assumptions are

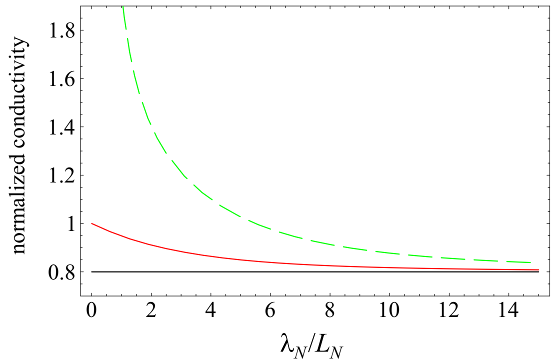

FIG. 8. Conductances G/G_0 of Eq. (D13) (red/gray), $-S_N^z/G_0$ of Eq. (D16) (green/gray-dashed), and their limiting value $G_{\lim}/G_0$ of Eq. (D17) (black) normalized to $G_0 \equiv G|_{p=0} = 1/(R_F + R_N)$ and plotted as a function of λ_N/L_N. For illustrative purposes, other parameters are set to $\lambda_F/L_F = 0.1(\lambda_N/L_N)$ and $R_F = 0.1R_N$.

made: first, that the applied spin potentials and the magnetization directions in the element are collinear, and second, that the regions of the element adjacent to the contacts are nonmagnetic.

The textbook derivation [23] of the Onsager relations starts by considering a closed system with partial equilibria defined by thermodynamic parameters x_i. At the global equilibrium all $x_i = 0$, and the entropy is maximized. Near that equilibrium the entropy is given by

$$S = S_0 - \tfrac{1}{2}A_{ij}x_i x_j. \tag{E1}$$

Thermodynamic forces are defined as $Y_i = A_{ij}x_j$. If the thermodynamic parameters satisfy the phenomenological equations

$$\dot{x}_i = -\gamma_{ij}Y_j , \tag{E2}$$

then the matrix γ_{ij} obeys the Onsager relations.

To recast spin currents in the framework above, imagine two large diffusive normal-metal reservoirs connected by the element in question. The system consisting of the element and the reservoirs is closed. The key feature of this thought experiment is that the reservoirs are assumed to have no spin relaxation. In the absence of the connecting element, a state with spin imbalance in either of the reservoirs will not relax to zero spin density. All spin relaxation is due to electrons flowing through the connecting element. Consequently, spin currents through the element can be related to the time derivatives $\dot{N}_\uparrow$ and $\dot{N}_\downarrow$ of the numbers $N_\uparrow$ and $N_\downarrow$ of spin-up- and spin-down electrons in the reservoirs. The $N_\uparrow$ and $N_\downarrow$ thus play the role of thermodynamic parameters x_i. Reservoirs without spin relaxation were previously considered [27] in the derivation of Onsager relations through the Landauer-Büttiker approach.

An important subtlety is that for this approach to work, there must be no spin relaxation at the contacts between the reservoirs and the element. If the reservoirs are connected to the normal-metal parts of the element, this requirement is automatically satisfied.

It will be further assumed that the spin-polarized state of each reservoir is spatially uniform. In the present thought experiment this can be guaranteed by assuming sufficiently fast diffusion in each of them. A spatially uniform, spin-polarized state of a reservoir is specified either by electric and spin potentials (μ^e, μ^s) or by the numbers of up and down electrons $N_\uparrow$ and $N_\downarrow$. In the latter case we can use a picture of two noninteracting Fermi gases. The state of each gas is a thermal equilibrium characterized by the Fermi-Dirac distribution function $n_\sigma(p) = n_F[(\epsilon(p) - \mu_\sigma)/T]$, where $\sigma = \uparrow, \downarrow = \pm 1$, $\mu_\sigma = \mu^e + \sigma\mu^s/2$, and $\epsilon(p)$ is the band energy of electrons. In our thought experiment we can neglect the electrostatic potential energy $e\phi$ because (a) the reservoirs are not connected to external electric batteries and (b) they are assumed to be so large that electron transfer from one to the other changes ϕ by a negligible amount (limit of infinite capacity).

From the properties of an ideal Fermi gas, we know that the parameters $\mu_{\uparrow,\downarrow}$ are the actual chemical potentials of gases in the reservoirs, i.e., the energy of each gas obeys $dE_{\sigma r} = -T\,dS_{\sigma r} + P_{\sigma r}dV_r + \mu_{\sigma r}dN_{\sigma r}$, where $r = 1, 2$ is the reservoir index. Since the system of two connected reservoirs is closed, the relation $\sum_{\sigma r} dE_{\sigma r} = 0$ is satisfied. Furthermore, the reservoir volumes stay constant, $dV_r = 0$. Heat exchange between the reservoirs is allowed so that their temperatures remain equal. Under these conditions, the total change of the entropy in reservoirs is given by the expression

$$dS \equiv \sum_{\sigma r} dS_{\sigma r} = -\frac{1}{T}\sum_{\sigma r} \mu_{\sigma r}dN_{\sigma r} \, . \tag{E3}$$

Neglecting the changes of entropy in the element, we can use (E3) to specify the coefficients A_{ij} in formula (E1) for the closed system considered in our thought experiment.

In the state of global equilibrium $N_{\uparrow,r} = N_{\downarrow,r} = N_{eq,r}$ $(r = 1, 2)$, and $\mu_{\sigma r} = \mu_0$ for all σ and r. Away from the global equilibrium the chemical potentials depend on the electron concentrations $n_{\sigma r}$ in each gas: $\mu_{\sigma r} = \mu_{Fr}(n_{\sigma r}, T) = \mu_{Fr}(N_{\sigma r}/V_r, T)$, where $\mu_{Fr}(n, T)$ is the thermodynamic function characterizing the ideal Fermi gas in it, taking into account all the properties of electron spectrum $\epsilon(p)$ in the reservoir material. Expanding $\mu_{Fr}(n_{\sigma r}, T)$ in Taylor series, one gets

$$\mu_{\sigma r} = \mu_0 + \left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{N_{\sigma r} - N_{eq,r}}{V_r} + \cdots \, . \tag{E4}$$

An expression for entropy S can be now obtained by integrating Eq. (E3). Recall that the deviations $\Delta N_{\sigma r} = N_{\sigma r} - N_{eq,r}$ are assumed to be small. We will see below that it is sufficient to use Eq. (E4) truncated to two terms to get the leading-order contribution to S. Integration gives

$$S = S_0 - \frac{1}{T}\sum_{\sigma r}\int_{N_{eq,r}}^{N_{\sigma r}} \left[\mu_0 + \left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{\Delta N_{\sigma r}}{V_r}\right] dN_{\sigma r}.$$

Using conservation of the total number of electrons $\sum_{\sigma r} dN_{\sigma r} = 0$, one gets

$$S = S_0 - \frac{1}{2T}\sum_{\sigma r}\left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{(\Delta N_{\sigma r})^2}{V_r}. \tag{E5}$$

With the identification $x_i \leftrightarrow \Delta N_{\sigma r}$, Eq. (E5) looks similar to the relation (E1) of the Onsager approach. However, it cannot yet be identified with (E1) for two reasons. First, only three of four variables $\Delta N_{\sigma r}$ are independent, since these numbers are still connected by the electron number conservation relation. Second, the validity of the Onsager relations requires each x_i to have definite parity with respect to time reversal $\hat{T}$. However, variables $\Delta N_{\sigma r}$ do not satisfy this property; time reversal changes the direction of spin, so $\hat{T}[\Delta N_\uparrow] = \Delta N_\downarrow$ and vice versa.

Both difficulties can be overcome by introducing new variables,

$$\Delta N_r = \Delta N_{\uparrow r} + \Delta N_{\downarrow r}\,,$$
$$\Delta N_{sr} = \Delta N_{\uparrow r} - \Delta N_{\downarrow r}\,,$$

which do have definite parities with respect to $\hat{T}$: ΔN_r is even, and ΔN_{sr} is odd. Furthermore, by rewriting Eq. (E5) in terms of ΔN_r and ΔN_{sr}, one gets

$$S = S_0 - \frac{1}{2T}\sum_r \left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{\Delta N_r^2 + \Delta N_{sr}^2}{2V_r}\,,$$

where the summation over σ is already performed. The particle conservation condition $\Delta N_1 = -\Delta N_2$ allows one to rewrite the above as

$$S = S_0 - \frac{1}{2T}\left\{\left[\sum_r \left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{1}{V_r}\right]\Delta N_1^2 + \left(\frac{\partial \mu_{F1}}{\partial n}\right)_{eq} \frac{\Delta N_{s1}^2}{2V_1} + \left(\frac{\partial \mu_{F2}}{\partial n}\right)_{eq} \frac{\Delta N_{s2}^2}{2V_2}\right\}. \quad \text{(E6)}$$

Now the three variables $x_1 = \Delta N_1$, $x_2 = \Delta N_{s1}$, and $x_3 = \Delta N_{s2}$ have all the required properties of Onsager thermodynamic coordinates, and Eq. (E6) can be identified with (E1). The corresponding thermodynamic forces are then calculated as

$$Y_1 = \frac{1}{T}\left[\sum_t \left(\frac{\partial \mu_{Fr}}{\partial n}\right)_{eq} \frac{1}{V_r}\right]\Delta N_1\,,$$
$$Y_2 = \frac{1}{T}\left(\frac{\partial \mu_{F1}}{\partial n}\right)_{eq} \frac{\Delta N_{s1}}{2V_1}\,,$$
$$Y_3 = \frac{1}{T}\left(\frac{\partial \mu_{F2}}{\partial n}\right)_{eq} \frac{\Delta N_{s2}}{2V_2}\,.$$

Using Eq. (E4) and $\Delta N_1 = -\Delta N_2$, these expressions can be transformed to

$$Y_1 = \frac{\mu_1^e - \mu_2^e}{T}\,,$$
$$Y_2 = \frac{\mu_{\uparrow 1} - \mu_{\downarrow 1}}{T} = \frac{\mu_1^s}{2T}\,, \quad \text{(E7)}$$
$$Y_3 = \frac{\mu_{\uparrow 2} - \mu_{\downarrow 2}}{T} = \frac{\mu_2^s}{2T}\,.$$

The Onsager theory predictions for γ_{ij} depend on whether (a) the microscopic evolution of the system is invariant under time reversal and (b) thermodynamic variables x_i and x_j have the same or opposite parities under time reversal $\hat{T}$.

Condition (a) is satisfied if the connecting element is made only of normal metals. We discuss this case first—elements with magnetic regions will be discussed later. The kinetic coefficients in Eq. (E2) obey $\gamma_{ij} = \gamma_{ji}$ when the variables x_i and x_j have the same parity under time reversal. This is the case for ΔN_{s1} and ΔN_{s2}. When the two variables have different parity—as for ΔN_1 and $\Delta N_{s1,2}$—the Onsager relations demand $\gamma_{ij} = -\gamma_{ji}$. Writing out an explicit matrix equation [minus signs are introduced for easier comparison with Eq. (23)]

$$\begin{pmatrix} \Delta\dot{N}_1 \\ \Delta\dot{N}_{s1} \\ \Delta\dot{N}_{s2} \end{pmatrix} = -\begin{pmatrix} G & -C_1 & -C_2 \\ -B_1 & -S_1 & -S_{12} \\ -B_2 & -S_{21} & -S_2 \end{pmatrix}\begin{pmatrix} \mu_1^e - \mu_2^e \\ \mu_1^s/2 \\ \mu_2^s/2 \end{pmatrix}, \quad \text{(E8)}$$

we can state that $B_i = -C_i$ and $S_{12} = S_{21} \equiv S_c$. Here the constant factor of $1/T$ in thermodynamic forces (E7) was absorbed into the definition of matrix entries.

Due to the assumed absence of spin relaxation in the reservoirs, the currents through the contacts are related to time derivatives of the particle numbers as per $I_1^e = \Delta\dot{N}_1$, $I_r^s = \Delta\dot{N}_{sr}$, where we used the current sign definition of Sec. II A. We then find

$$\begin{pmatrix} I_1^e \\ I_1^s \\ I_2^s \end{pmatrix} = \begin{pmatrix} -G & C_1 & C_2 \\ -C_1 & S_1 & S_c \\ -C_2 & S_c & S_2 \end{pmatrix}\begin{pmatrix} \mu_1^e - \mu_2^e \\ \mu_1^s/2 \\ \mu_2^s/2 \end{pmatrix}. \quad \text{(E9)}$$

Equation (E9) can be recast in a form allowing direct comparison with Eq. (23) for collinear systems in Sec. III A. Introducing the notation $\mu_r^s/2 = \tilde{\mu}_r^s$ as in Sec. II B, we find

$$\begin{pmatrix} I_1^e \\ I_2^e \\ I_1^s \\ I_2^s \end{pmatrix} = \begin{pmatrix} -G & G & C_1 & C_2 \\ G & -G & -C_1 & -C_2 \\ -C_1 & C_1 & S_1 & S_c \\ -C_2 & C_2 & S_c & S_2 \end{pmatrix}\begin{pmatrix} \mu_1^e \\ \mu_2^e \\ \tilde{\mu}_1^s \\ \tilde{\mu}_2^s \end{pmatrix}. \quad \text{(E10)}$$

We see that the requirement for the S entries coincides with those of Eq. (23), while for the B entries one finds antisymmetry rather than symmetry.

For a device with ferromagnetic elements, the time-reversal property (a) is not satisfied. The standard arguments [23] show that in this case the Onsager reciprocity relates the coefficients γ_{ij} for two *different* systems. If $\{\mathbf{M}\}$ denotes magnetizations of all the regions, the Onsager relations for the matrix in Eq. (E8) read $B_i(\{-\mathbf{M}\}) = -C_i(\{\mathbf{M}\})$ and $S_{12}(\{-\mathbf{M}\}) = S_{21}(\{\mathbf{M}\})$.

APPENDIX F: CONDUCTANCE MATRIX OF A NORMAL ELEMENT WITH SPIN-ORBIT INTERACTION

Here we derive Onsager relations directly from the diffusion equations rather than from general principles. The derivation uses the approach analogous to that employed in Sec. II for diffusive reciprocity. It is instructive to see how the presence of SOI invalidates the derivation of Sec. II, but not that of this Appendix.

Diffusive currents in centrosymmetric metals with SOI were discussed in Ref. [24]. Equations (5) and (6) of this paper are valid for arbitrary spin accumulation, including that found in a nonlinear regime. In a metal, spin accumulation is small, and the equations can be linearized:

$$j_i^e = -\frac{\sigma}{e^2}(\nabla_i\tilde{\mu}^e + \gamma\epsilon_{ij\alpha}\nabla_j\tilde{\mu}^{s\alpha})\,,$$
$$j_i^{s\alpha} = -\frac{\sigma}{e^2}(\nabla_i\tilde{\mu}^{s\alpha} - \gamma\epsilon_{i\alpha j}\nabla_j\tilde{\mu}^e)\,, \quad \text{(F1)}$$

where γ is a dimensionless coefficient characterizing the relative strength of the spin-orbit interaction.

In a steady state in the absence of external magnetic field, the charge and spin continuity equations read [24]

$$\nabla_i j_i^e = 0\,, \quad \nabla_i j_i^{s\alpha} = -\frac{\nu \mu^{s\alpha}}{\tau}\,. \tag{F2}$$

Expressions (F1) can be written in a compact form:

$$j_i^a = -\Sigma_{ij}^{ab}\nabla_j\tilde{\mu}^b\,.$$

Compared with the zero-SOI case as in Eq. (B2) of Appendix B, now the conductivity matrix Σ_{ij}^{ab} acquired additional spatial indices (i, j), and, more importantly, became asymmetric.

In this Appendix we wish to derive relations between the elements of the conductance matrix $G_{tt'}^{ab}$ in the presence of SOI by appropriately modifying the arguments of Sec. II. Contrary to the zero-SOI case, for an asymmetric Σ_{ij}^{ab} expressions such as (A3) are no longer valid. However, by trial and error, one can arrive at an analog of the quantity Q in Eq. (10). The integral

$$Q_{12} = \int \big(\tilde{\mu}^e(1)\nabla_i j_i^e(2) - \tilde{\mu}^{s\alpha}(1)\nabla_i j_i^{s\alpha}(2)\big)dV$$

has the sought properties. On the one hand, from Eqs. (F2) one immediately finds

$$Q_{12} = -\int \frac{\tilde{\mu}^{s\alpha}(1)\tilde{\mu}^{s\alpha}(2)}{\tau}dV. \tag{F3}$$

On the other hand, integrating by parts, one finds

$$Q_{12} = \oint \big[\tilde{\mu}^e(1)j_i^e(2) - \tilde{\mu}^{s\alpha}(1)j_i^{s\alpha}(2)\big]dA_i - \int \big[\nabla_i\tilde{\mu}^e(1)j_i^e(2) - \nabla_i\mu^{s\alpha}(1)j^{s\alpha}(2)\big]dV.$$

On the assumptions of Sec. II, the surface integral is expressed as a sum $\sum_t [\tilde{\mu}_t^e(1)I_t^e(2) - \tilde{\mu}_t^{s\alpha}(1)I_t^{s\alpha}(2)]$. After a few transformations, recasting currents in the volume integral via (F1), we find

$$\begin{aligned} Q_{12} = &\sum_t \big[\tilde{\mu}_t^e(1)I_t^e(2) - \tilde{\mu}_t^{s\alpha}(1)I_t^{s\alpha}(2)\big] \\ &+ \frac{\sigma}{e^2}\int\{[\nabla_i\tilde{\mu}^e(1)\nabla_i\tilde{\mu}^e(2) - \nabla_i\mu^{s\alpha}(1)\nabla_i\mu^{s\alpha}(2)] \\ &+ \gamma\epsilon_{ij\alpha}[\nabla_i\tilde{\mu}^e(1)\nabla_j\tilde{\mu}^{s\alpha}(2) + \nabla_i\tilde{\mu}^e(2)\nabla_j\tilde{\mu}^{s\alpha}(1)]\}dV. \end{aligned} \tag{F4}$$

Upon defining a complementary quantity

$$Q_{21} = \int \big(\tilde{\mu}^e(2)\nabla_i j_i^e(1) - \tilde{\mu}^{s\alpha}(2)\nabla_i j_i^{s\alpha}(1)\big)dV\,,$$

from (F3) it follows that $Q_{12} - Q_{21} = 0$, while (F4) shows that the volume integrals in Q_{12} and Q_{21} are equal, so

$$\begin{aligned} 0 = Q_{12} - Q_{21} = &\sum_t \big[\tilde{\mu}_t^e(1)I_t^e(2) - \tilde{\mu}_t^{s\alpha}(1)I_t^{s\alpha}(2)\big] \\ &- \big[\tilde{\mu}_t^e(2)I_t^e(1) - \tilde{\mu}_t^{s\alpha}(2)I_t^{s\alpha}(1)\big]. \end{aligned} \tag{F5}$$

This equation is the sought analog of Eq. (15) in Sec. II, and we will now use it to relate the elements of conductance matrix $G_{tt'}^{ab}$ in the presence of SOI.

Electric and spin currents can be written in the form

$$I_t^e = G_{tt'}^{ee}\tilde{\mu}_{t'}^e + A_{tt'}^{e\alpha}\tilde{\mu}_{t'}^{s\alpha}\,, \tag{F6}$$

$$I_t^{s\alpha} = B_{tt'}^{\alpha e}\tilde{\mu}_{t'}^e + S_{tt'}^{\alpha\beta}\tilde{\mu}_{t'}^{s\beta}\,, \tag{F7}$$

where $G_{tt'}^{ee}$, $A_{tt'}^{e\alpha}$, $B_{tt'}^{\alpha e}$, and $S_{tt'}^{\alpha\beta}$ are the blocks of $G_{tt'}^{ab}$. Substituting this into (F5) yields

$$\begin{aligned} 0 = &\tilde{\mu}_t^e(1)\tilde{\mu}_{t'}^e(2)\big[G_{tt'}^{ee} - G_{t't}^{ee}\big] - \tilde{\mu}_t^{s\alpha}(1)\tilde{\mu}_{t'}^{s\beta}(2)\big[S_{tt'}^{\alpha\beta} - S_{t't}^{\beta\alpha}\big] \\ &+ \tilde{\mu}_t^e(1)\tilde{\mu}_{t'}^{s\alpha}(2)\big[A_{tt'}^{e\alpha} + B_{t't}^{\alpha e}\big] + \tilde{\mu}_t^{s\alpha}(1)\tilde{\mu}_{t'}^e(2)\big[B_{tt'}^{\alpha e} + A_{t't}^{e\alpha}\big]. \end{aligned}$$

This identity should hold for arbitrary sets $(\tilde{\mu}_t^e(1), \tilde{\mu}_t^{s\alpha}(1))$ and $(\tilde{\mu}_t^e(2), \tilde{\mu}_t^{s\alpha}(2))$, which is only possible if all combinations in square brackets vanish, i.e.,

$$G_{tt'}^{ee} = G_{t't}^{ee}\,, \quad S_{tt'}^{\alpha\beta} = S_{t't}^{\beta\alpha}\,, \quad A_{tt'}^{e\alpha} = -B_{t't}^{\alpha e}\,. \tag{F8}$$

Requirements (F8) are precisely the Onsager relations, manifest in the right-hand side of Eq. (E10).

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