Dynamic properties of superconductors: Anderson-Bogoliubov mode and Berry phase in the BCS and BEC regimes

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We analyze the evolution of the dynamics of a neutral s-wave superconductor between the BCS and BEC regimes. We consider 2d case when a BCS-BEC crossover occurs already at weak coupling as a function of the ratio of the two scales—the Fermi energy E_F and the bound state energy for two fermions in a vacuum, E_0. The BCS and BEC limits correspond to $E_F \gg E_0$ and $E_F \ll E_0$, respectively. The chemical potential $\mu = E_F - E_0$ changes the sign between the two regimes. We use the effective action approach, derive the leading terms in the expansion of the effective action in the spatial and time derivative of the slowly varying superconducting order parameter $\Delta(r,\tau)$, and express the action in terms of a derivatives of the phase $\phi(r,\tau)$ of $\Delta(r,\tau) = \Delta e^{i\phi(r,\tau)}$. The action contains $(\nabla\phi)^2$ and $\dot{\phi}^2$ terms, which determine the dispersion of collective phase fluctuations, and $i\pi A\dot{\phi}$ term. For continuous $\phi(r,\tau)$, the latter reduces to the contribution from the boundary and does not affect the dynamics. We show that this long-wavelength action does not change through the BCS-BEC crossover. We apply our approach to a moving vortex, for which ϕ is singular at the center of the vortex core, and $i\pi A_{\rm vort}\dot{\phi}$ term affects vortex dynamics. We find that this term has two contributions. One comes from the states away from the vortex core and has $A_{\rm vort,1} = n/2$, where n is the fermion density. The other comes from electronic states inside the vortex core and has $A_{\rm vort,2} = -n_0/2$, where n_0 is the fermion density at the vortex core. This last term comes from the continuous part of the electronic spectrum and has no contribution from discrete levels inside the core; it also does not change if we add impurities. We interpret this term as the contribution to vortex dynamics in the continuum limit, when the spacing between energy levels ω is set to zero, while fermionic lifetime τ can be arbitrary. The total $A_{\rm vort} = (n - n_0)/2$ determines the transversal force acting on the vortex core, $\pi A_{\rm vort}\dot{\mathbf{R}} \times \hat{z}$, where $\dot{\mathbf{R}}$ is the velocity of the vortex core and $\hat{z}$ a unit vector perpendicular to the 2d sample. The difference $(n - n_0)/2$ changes through the BEC-BCS crossover as n_0 nearly compensates n in the BCS regime, but vanishes in the BEC regime.

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I. INTRODUCTION

The evolution of the static properties of a superconductor between the BCS regime, when bound pairs of fermions condense immediately once they form, and Bose-Einstein condensation (BEC) regime, when bound pairs of fermions form at a higher $T_{\rm ins}$ and condense at a smaller T_c, has been extensively discussed in the condensed matter context [1–19] and also for optical lattices of ultracold atoms [20,21]. Experimental evidence for preformed pairs has been reported for high-T_c cuprates [22] and, more recently, for Fe-based superconductor $FeSe_{1-x}Te_x$ (Ref. [23]).

In this paper, we discuss the evolution of the dynamic properties of a neutral superconductor between the BCS and BEC regimes. We consider a 2d superconductor and for definiteness focus on s-wave gap symmetry and assume Galilean invariance, i.e., assume rotational symmetry and $k^2/(2m)$ fermionic dispersion. Extensions to non-s-wave pairing and lattice systems are straightforward. We consider 2d case because in 2d, the BCS-BEC crossover can be analyzed already within weak coupling, when calculations are under control. Indeed, in 2d systems, two fermions form a bound state already at arbitrary small attraction g. (In 3d systems, the bound state of two fermions in a vacuum emerges only once the interaction exceeds a certain cutoff, generally of the order of fermionic bandwidth [6].) Such a bound state has energy $2E_0 = 2\Lambda e^{-2/(N_0 g)}$, where $N_0 = m/(2\pi)$ is the free particle density of states per spin in 2d and Λ is the upper cutoff for the attraction [1,3,24].

The crossover between the BCS and BEC regimes occurs as a function of E_0/E_F. For $E_F \gg E_0$ the system is in the BCS regime, and bound pairs condense almost instantly after they form at $T_{\rm ins} \sim (E_F E_0)^{1/2}$. For $E_0 \gg E_F$, bounds pairs form at $T_{\rm ins} \sim E_0/\ln E_0/E_F$ and condense at a much smaller $T_c \sim E_F$, leaving a wide intermediate region of preformed pair behavior. The chemical potential μ at $T = 0$ changes sign between the two regimes: $\mu = E_F - E_0$.

We use the effective action approach, and expand the action in terms of time derivatives of the slowly varying order parameter $\Delta(r,\tau)$. We obtain the generic expressions for the terms

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up to second order in spatial and time derivatives, in terms of the eigenfunctions of the Bogolubov-De Gennes equation, and then apply the results to the case when the variations of the order parameter predominantly occur via variations of its phase, i.e., $\Delta(r,\tau) \approx \Delta e^{i\phi(r,\tau)}$. We obtain the action in terms of spatial and time derivatives of ϕ. The action contains the conventional terms $\nabla^2\phi$ and $\partial^2\phi/\partial\tau^2$, which fully describe the dynamics when ϕ is a continuous function of a coordinate and time

$$\mathcal{S}_{\rm cont} \propto \sum_{q,\Omega} |\phi_{q,\Omega}|^2 \left(\Omega^2 - q^2 \frac{v_F^2}{2}\right). \tag{1}$$

The form of $\mathcal{S}_{\rm cont}$ does not change between the BCS and BEC limits, and the velocity of phase fluctuations remains $v_F/\sqrt{2}$ through the crossover.

Using our approach we also study the dynamics of vortices and, in particular, the transverse force acting on a slowly moving vortex. Such force is typically attributed to the terms in the effective action that are linear in time derivatives of the phase, i.e., proportional to $\int dr d\tau \dot{\phi}$. This term is often referred to in the literature as the Berry phase term [25–32]. It reduces to the contribution from a boundary and does not contribute to the dynamics if ϕ is well defined at any r and τ. However, for vortices, as well as for other topological defects, such as phase slips [33], such term does contribute to the dynamics due to non-analytic behavior of ϕ at the center of the vortex core, and gives rise to an effective transverse force acting on a vortex [25,27,28,30,31,34–38]. The action associated with this transverse force can be written as

$$S_{\rm Berry}^{\rm vort} = i\pi A_{\rm vort} \int dt (X(t)\dot{Y}(t) - Y(t)\dot{X}(t)), \tag{2}$$

with X and Y being the coordinates of the vortex core.

We show that the prefactor $A_{\rm vort}$ has two contributions, $A_{\rm vort} = A_{{\rm vort},1} + A_{{\rm vort},2}$. The first one is the hydrodynamic contribution, associated with superfluid motion of fermions at the peripheral region of the vortex (this term is often termed as the Magnus force). We find $A_{{\rm vort},1} = n/2$, where $n = 2N_0E_F$ is the actual fermionic density. Another contribution is a reaction force from normal fermions at the vortex core. For this term, we find $A_{{\rm vort},2} = -n_0/2$, where $n_0 = 2N_0\mu\theta(\mu)$ is the density of free fermions with the same chemical potential $\mu = E_F - E_0$ (same as the density of fermions inside the vortex core). In the BCS regime $E_F > E_0$, the difference $n - n_0 = 2N_0E_0 \ll n$, i.e., these two forces nearly cancel each other. The resulting $A_{\rm vort} = N_0E_0$. In the BEC regime, $\mu < 0$, i.e., all states of free fermions are above the chemical potential and therefore are empty. Then $n_0 = 0$, and only the Magnus force contributes to $A_{\rm vort} = A_{{\rm vort},1} = n/2 = N_0E_F$. The vanishing of n_0 once μ becomes negative is consistent with the generic reasoning in Ref. [37] that free-fermion contribution to $A_{\rm vort}$ vanishes once the system undergoes a (fictitious) Lifshitz transition, in which the (fictitious) Fermi surface of free fermions with renormalized μ disappears. In our case, this happens once E_0 becomes larger than E_F.

The Berry phase term in the effective action has been analyzed earlier [25–31,34,37,39,40]. Several authors [26,34,37,39,40] argued on general grounds that the value of $A_{\rm vort}$ depends on the interplay between the fermionic lifetime τ and the separation between discrete levels in the vortex core $\omega_0 \sim \Delta^2/E_F$ (Refs. [34,41,42]). They argued that the result $A_{\rm vort} = (n - n_0)/2$, which we report, holds in the hydrodynamic limit $\omega_0\tau \ll 1$, when the quantization of fermionic states inside the vortex core can be neglected. In the opposite limit $\omega_0\tau \gg 1$ (which includes the most studied case of $\tau = \infty$), these and other researchers argued [26,29,31,34,37,39] that there should be a contribution to $A_{\rm vort}$ from discrete levels in the vortex core, which cancels out $A_{{\rm vort},2}$, such that the total $A_{\rm vort} = A_{{\rm vort},1} = n/2$.

In our approach, $A_{{\rm vort},1} = n/2$ comes from the states far away from the vortex core, and by this reason is unsensitive to whether $\omega_0\tau$ is small or large, i.e., it exists at $\omega_0\tau \ll 1$ and $\omega_0\tau \gg 1$. The contribution $A_{{\rm vort},2} = n_0/2$ is more tricky. On one hand, it does come from the states within the vortex core, and in our calculations we kept fermionic lifetime to be infinite. On the other, this contribution to the Berry phase comes solely from the term in the action at vanishing $\Delta \to 0$, when the distance between energy levels in the core vanishes, and the fermionic spectrum in the vortex core becomes continuous. This implies that in our analysis $\omega_0 = 0$ and $\tau = \infty$, so $\omega_0\tau$ is ill defined. Still, we emphasize that our $A_{{\rm vort},2}$ comes from continuous (i.e., nonquantized) states, and we will also argue that it remains the same in the presence of impurities, i.e., at a finite τ. By these two reasons, we believe that it is meaningful to compare our $A_{\rm vort}$ with the one obtained in earlier works at $\omega_0\tau \ll 1$. Then our and earlier results fully agree.

There is, however, one aspect in which our result seems to differ from earlier works. Namely, these works speculated [26,34,37,39,40] that at a finite $\omega_0\tau$ there should be a contribution to $A_{\rm vort}$ from discrete levels in the vortex core. We did not find such contribution in our microscopic approach (in our case, this would be the contribution to $A_{\rm vort}$ from discrete levels in the vortex core in the term in the action in which Δ is finite). We explicitly show this using a finite system size D for the regularization of the integrals and taking the limit $D \to \infty$ only at the end of calculations. It remains to be seen whether such contribution emerges if one goes beyond the approximations we made in the derivation of the action for phase variable.

There is one additional element in our calculation, which was not emphasized in earlier works [25,27,28,38] and forced us to derive the action in step-by-step calculations, starting from Hubbard-Stratonovich transformation and keep terms at a finite Δ and at $\Delta \to 0$. Namely, in some earlier calculations, $A_{{\rm vort},1}$ was argued to come from the action at a finite Δ. The full contribution to the Berry phase from this term, however, contains an additional, formally infinite piece, $\sum_k(1)$, where the sum is over all momenta. The authors of earlier works have argued that this "parasitic" term should not be present in $A_{{\rm vorx},1}$ by physical reasons, but did not show explicitly how this term get canceled. In our calculation below, we show explicitly that the parasitic divergent term from a finite Δ piece in the action get canceled by the counter-term from the part of the action at $\Delta \to 0$ (the original divergent contribution and the counter-term both come from the states far away from a vortex core).

The paper is organized as follows. In the next section, we introduce the effective action of a superconductor in terms of its fluctuating order parameter $\Delta(\mathbf{r},\tau)$. In Sec. III, we develop a systematic expansion of the action in terms of (imaginary)

time derivatives of the order parameter. We evaluate the zeroth order term and obtain the condensation energy of a superconductor. We then obtain terms with one and two time derivatives, and express them in terms of eigenfunctions of the Bogoliubov-de Gennes equation. We next focus on the small wavelength limit and express the action in terms of spatial and time derivatives of the phase of a superconducting order parameter, including the term, linear in time derivative. This last term becomes a meaningful Berry phase term when the phase of the superconducting order parameter is not defined globally, which is the case of a vortex. In Sec. IV, we compute the effective action for a moving vortex in a neutral *s*-wave superconductor in 2d. Section V is the summary of our results.

II. GENERAL FORMULATION

The effective action for an order parameter of an *s*-wave superconductor can be obtained by departing from a microscopic model with the local four-fermion attractive interaction $-g$ ($g>0$) and introducing the pairing field $\Delta(r,\tau)$ to decoupling the four-fermion interaction via the Hubbard-Stratonovich transformation [43]. This procedure is well documented (see, e.g., Ref. [44]), and we just quote the results.

The partition function Z is expressed via the integral over the Grassmann fields as

$$Z=\int d\psi d\bar{\psi} e^{-S[\psi,\bar{\psi}]}, \tag{3}$$

where $\psi=\psi_\alpha(\mathbf{r},\tau)$ and $\bar{\psi}=\bar{\psi}_\alpha(\mathbf{r},\tau)$ are the spin-full coordinate and time dependent Grassmann fields, and

$$S[\psi,\bar{\psi}]=\int d\mathbf{r}d\tau(\bar{\psi}_\alpha(\mathbf{r},\tau)\partial_\tau\psi_\alpha(\mathbf{r},\tau)+H[\psi,\bar{\psi}]). \tag{4}$$

Here, τ is the imaginary (Matsubara) time $\tau=it$ and

$$H[\psi,\bar{\psi}]=\left[\bar{\psi}_\sigma(\mathbf{r},\tau)\left(-\frac{\nabla^2}{2m}-\mu\right)\psi_\sigma(\mathbf{r},\tau)\right] -g\bar{\psi}_\uparrow(\mathbf{r},\tau)\bar{\psi}_\downarrow(\mathbf{r},\tau)\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau). \tag{5}$$

The four-fermion interaction is decoupled by the Hubbard-Stratonovich transformation

$$e^{\frac{ax^2}{2}}=\frac{1}{\sqrt{2\pi a}}\int dy\, e^{(-\frac{y^2}{2a}+yx)}. \tag{6}$$

In our case, we introduce two Hubbard-Stratonovich fields $\Delta(\mathbf{r},\tau)$ and $\Delta^*(\mathbf{r},\tau)$ and rewrite the partition function as

$$Z=\int d\psi d\bar{\psi} d\Delta d\Delta^* e^{-S[\psi,\bar{\psi},\Delta,\Delta^*]}, \tag{7}$$

where now

$$S[\psi,\bar{\psi},\Delta,\Delta^*]=\int d\mathbf{r}d\tau\left(\bar{\psi}_\alpha(\mathbf{r},\tau)\partial_\tau\psi_\alpha(\mathbf{r},\tau) +\frac{|\Delta(\mathbf{r},\tau)|^2}{g}+H[\psi,\bar{\psi},\Delta,\Delta^*]\right) \tag{8}$$

and

$$H[\psi,\bar{\psi},\Delta,\Delta^*]=\left[\bar{\psi}_\sigma(\mathbf{r},\tau)\left(-\frac{\nabla^2}{2m}-\mu\right)\psi_\sigma(\mathbf{r},\tau) +\Delta(\mathbf{r},\tau)\bar{\psi}_\uparrow(\mathbf{r},\tau)\bar{\psi}_\downarrow(\mathbf{r},\tau) +\Delta^*(\mathbf{r},\tau)\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)\right]. \tag{9}$$

The action $S[\psi,\bar{\psi},\Delta,\Delta^*]$ can be reexpressed in a more compact form by introducing a Gorkov-Nambu spinor $\psi=[\psi_\uparrow,\bar{\psi}_\downarrow]^T$. Then

$$S[\psi,\bar{\psi},\Delta,\Delta^*]=\int d\mathbf{r}d\tau\frac{|\Delta(\mathbf{r},\tau)|^2}{g} -\int d\mathbf{r}d\tau\,\bar{\psi}(\mathbf{r},\tau)\hat{G}^{-1}\psi(\mathbf{r},\tau), \tag{10}$$

where the $\hat{G}^{-1}$ is an operator

$$\hat{G}^{-1}=-\partial_\tau-\hat{K}(\mathbf{r})-\hat{\Delta}(\mathbf{r},\tau), \tag{11}$$

with

$$\hat{K}(\mathbf{r})=\begin{bmatrix}-(1/2m)\nabla^2-\mu & 0\\ 0 & (1/2m)\nabla^2+\mu\end{bmatrix}$$

and

$$\hat{\Delta}(\mathbf{r},\tau)=\begin{bmatrix}0 & \Delta(\mathbf{r},\tau)\\ \Delta^*(\mathbf{r},\tau) & 0\end{bmatrix}.$$

The Green's function for the fermions $\hat{G}(\mathbf{r},\tau;\mathbf{r}',\tau')$ satisfies the following operator identity:

$$(-\partial_\tau-\hat{K}(\mathbf{r})-\hat{\Delta}(\mathbf{r},\tau,\lambda))\hat{G}(\mathbf{r},\tau;\mathbf{r}',\tau') =\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \tag{12}$$

Integrating over ψ and $\bar{\psi}$ we then obtain

$$Z=\int d\Delta d\Delta^* e^{-S[\Delta,\Delta^*]} \tag{13}$$

and

$$S[\Delta,\Delta^*]=\int d\mathbf{r}d\tau\frac{|\Delta(\mathbf{r},\tau)|^2}{g}-\mathrm{Tr}\ln\hat{G}^{-1}. \tag{14}$$

The logarithm in the effective action can be eliminated by introducing an auxiliary variable λ and making Δ, and hence G, λ-dependent, subject to $\Delta(\mathbf{r},\tau,1)=\Delta(\mathbf{r},\tau)$ and $\Delta(\mathbf{r},\tau,0)=0$. Indeed, because $\hat{G}^{-1}(r,\tau)=\hat{G}_0^{-1}(r,\tau)-\hat{\Delta}(r,\tau)$, we have $\ln\hat{G}^{-1}=\ln\hat{G}_0^{-1}-\sum_{n=1}(\hat{G}_0\hat{\Delta})^n/n$. The product $\hat{G}\hat{\Delta}=\sum_{n=1}(\hat{G}_0\hat{\Delta})^n$ is represented by the same expansion, but without $1/n$. The $1/n$ can be reintroduced by using the identity

$$\int_0^1 d\lambda\,\mathrm{Tr}\left[\frac{\partial\hat{\Delta}(\lambda)}{\partial\lambda}\hat{G}_0(\hat{\Delta}(\lambda)\hat{G}_0)^{n-1}\right]=\frac{1}{n}\mathrm{Tr}[(\hat{\Delta}(1)\hat{G}_0)^n].$$

Using this trick, we can replace $\mathcal{S}$ in (14) by

$$\mathcal{S}=\int_{-\infty}^{\infty}d\tau\int d\mathbf{r}\int_0^1 d\lambda\,\mathrm{Tr}\left[\hat{\mathbf{G}}_\lambda(\mathbf{r},\tau;\mathbf{r},\tau)\frac{\partial}{\partial\lambda}\hat{\Delta}(\mathbf{r},\tau,\lambda)\right] +\int d\tau\int d\mathbf{r}\frac{|\Delta(\mathbf{r},\tau,1)|^2}{g}+\mathcal{S}_{\mathrm{norm}}, \tag{15}$$

where

$$\mathcal{S}_{\rm norm} = -{\rm Tr}\ln \hat{G}^{-1}_{\Delta\to 0}. \tag{16}$$

Note that in Eq. (15) the trace is over the Gorkov-Nambu 2×2 matrices only, while in Eq. (16) the trace is assumed to be over the infinite-dimensional matrix $\hat{G}^{-1}$ as well as over the Gorkov-Nambu 2×2 structure.

A similar trick has been used in Ref. [45], where an auxiliary variation of the coupling constant was introduced instead of λ dependence. Writing the action in the form of Eq. (15) helps with the computations and will also allow us to establish a connection with the Wess-Zumino formalism, which has been used in earlier works [31,37,40] to evaluate the Berry phase and the Magnus force for topological defects, such as vortices.

III. ADIABATIC EXPANSION

We set $\hat{\Delta}(\mathbf{r},\tau,\lambda)$ to be a slowly varying function of τ and expand it around a particular τ' as

$$\hat{\Delta}(\mathbf{r},\tau,\lambda) = \hat{\Delta}(\mathbf{r},\tau',\lambda) + \partial_{\tau'}\hat{\Delta}(\mathbf{r},\tau',\lambda)(\tau-\tau') + (1/2)\partial^2_{\tau'}\hat{\Delta}(\mathbf{r},\tau',\lambda)(\tau-\tau')^2 + \dots \tag{17}$$

Consequently, we seek for the solution of Eq. (12) in the form

$$\hat{\mathbf{G}}_\lambda(\mathbf{r},\tau;\mathbf{r}',\tau') = \hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\tau-\tau';\mathbf{r}',\tau') + \hat{\mathbf{G}}^{(1)}_\lambda(\mathbf{r},\tau-\tau';\mathbf{r}',\tau') + \hat{\mathbf{G}}^{(2)}_\lambda(\mathbf{r},\tau-\tau';\mathbf{r}',\tau') + \dots, \tag{18}$$

with $\hat{\mathbf{G}}^{(0)}_\lambda$ being of the order $(\partial_\tau\Delta)^0$, $\hat{\mathbf{G}}^{(1)}_\lambda$ being of the order $(\partial_\tau\Delta)^1$, etc. The functional in Eq. (15) can then be written as a series

$$\begin{aligned}\mathcal{S} &= \mathcal{S}_{\rm norm} + \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \dots \\ &= \mathcal{S}_{\rm norm} + \int d\tau\left[L^{(0)}(\tau) + \int d\mathbf{r}\frac{|\Delta(\mathbf{r},\tau,1)|^2}{g}\right] \\ &\quad + \int d\tau L^{(1)}(\tau) + \int d\tau L^{(2)}(\tau) + \dots\end{aligned} \tag{19}$$

with

$$L^{(k)}(\tau) = \int d\mathbf{r}\int_0^1 d\lambda\,{\rm Tr}\left[\hat{\mathbf{G}}^{(k)}_\lambda(\mathbf{r},\tau;\mathbf{r},\tau)\frac{\partial}{\partial\lambda}\hat{\Delta}(\mathbf{r},\tau,\lambda)\right]. \tag{20}$$

Again we emphasize that in Eq. (20) the trace is taken over the 2×2 matrices only. In what follows, we derive the first three terms in the expansion in Eqs. (19) and (20) and obtain $S = S_0 + S_1 + S_2 + S_{\rm norm}$.

A. The expansion of the action for a generic $\Delta(\tau,\mathbf{r})$

We start with Eq. (19). Substituting Eqs. (17) and (18) into Eq. (12), we find that the zeroth-order Green's function $\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\tau-\tau';\mathbf{r}',\tau')$ satisfies the operator identity

$$[-\partial_\tau - \mathcal{H}(\mathbf{r},\tau',\lambda)]\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\tau;\mathbf{r}',\tau') = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'), \tag{21}$$

where

$$\mathcal{H}(\mathbf{r},\tau',\lambda) = \hat{K}(\mathbf{r}) + \hat{\Delta}(\mathbf{r},\tau',\lambda). \tag{22}$$

The solution of this equation can be written as

$$\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\tau-\tau';\mathbf{r}',\tau') = \int\frac{d\omega}{2\pi}\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\omega;\mathbf{r}',\tau')\,e^{-i\omega(\tau-\tau')}, \tag{23}$$

with

$$\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\omega;\mathbf{r}',\tau') = \sum_n\frac{|\chi_{n,\lambda}(\mathbf{r},\tau')\rangle\langle\chi_{n,\lambda}(\mathbf{r}',\tau')|}{i\omega - E_{n,\lambda}}, \tag{24}$$

where $|\chi_{n,\lambda}(\mathbf{r},\tau')\rangle$ are the eigenfunctions of the corresponding Bogolubov-De Gennes equation:

$$\mathcal{H}(\mathbf{r},\tau',\lambda)|\chi_{n,\lambda}(\mathbf{r},\tau')\rangle = E_{n,\lambda}(\tau')|\chi_{n,\lambda}(\mathbf{r},\tau')\rangle, \tag{25}$$

which satisfy the completeness relation

$$\sum_n|\chi_{n,\lambda}(\mathbf{r})\rangle\langle\chi_{n,\lambda}(\mathbf{r}')| = \delta(\mathbf{r}-\mathbf{r}'). \tag{26}$$

The eigenfunctions $|\chi_{n,\lambda}(\mathbf{r},\tau')\rangle$ depend parametrically on λ and τ'. Continuing with the expansion, we find higher-order contributions in Eq. (18) to be

$$\hat{\mathbf{G}}^{(1)}_\lambda(\mathbf{r},\omega;\mathbf{r}',\tau') = i\int d\mathbf{r}_1\,\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\omega;\mathbf{r}_1,\tau')[\partial_{\tau'}\hat{\Delta}(\mathbf{r}_1,\tau',\lambda)]\frac{\partial}{\partial\omega}\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r}_1,\omega;\mathbf{r}',\tau'), \tag{27}$$

$$\begin{aligned}\hat{\mathbf{G}}^{(2)}_\lambda(\mathbf{r},\omega;\mathbf{r}',\tau') &= -\frac{1}{2}\int d\mathbf{r}_1\,\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\omega;\mathbf{r}_1,\tau')\left[\partial^2_{\tau'}\hat{\Delta}(\mathbf{r}_1,\tau',\lambda)\right]\frac{\partial^2}{\partial\omega^2}\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r}_1,\omega;\mathbf{r}',\tau') - \lambda^2\int d\mathbf{r}_1 d\mathbf{r}_2\,\frac{\partial}{\partial\omega}\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r},\omega;\mathbf{r}_1,\tau') \\ &\quad\times[\partial_{\tau'}\hat{\Delta}(\mathbf{r}_1,\tau',\lambda)]\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r}_1,\omega;\mathbf{r}_2,\tau')[\partial_{\tau'}\hat{\Delta}(\mathbf{r}_2,\tau',\lambda)]\frac{\partial}{\partial\omega}\hat{\mathbf{G}}^{(0)}_\lambda(\mathbf{r}_2,\omega;\mathbf{r}',\tau').\end{aligned} \tag{28}$$

We now substitute Eqs. (23), (24), (27), and (28) into Eqn. (20). The zeroth-order term gives

$$L^{(0)}(\tau) = \int_0^1 d\lambda\int\frac{d\omega}{2\pi}e^{i\omega\epsilon^+}\sum_n\frac{\langle\chi_{n,\lambda}|\partial_\lambda\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle}{i\omega - E_{n,\lambda}} = \sum_n\int_0^1 d\lambda\,\langle\chi_{n,\lambda}|\partial_\lambda\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle\theta(-E_{n,\lambda}), \tag{29}$$

where $\theta(x) = 1$ for $x>0$ and $\theta(x)=0$ for $x<0$. Here and below, we use the notation

$$\langle\chi_{n,\lambda}|\dots|\chi_{n,\lambda}\rangle = \int d\mathbf{r}\langle\chi_{n,\lambda}(\mathbf{r})|\dots|\chi_{n,\lambda}(\mathbf{r})\rangle. \tag{30}$$

The integral over λ in (29) can be evaluated if we note that $\partial_\lambda \hat{\Delta}(\tau) = \partial_\lambda \hat{H}$. Then

$$\langle \chi_{n,\lambda}|\partial_\lambda \hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle = \langle \chi_{n,\lambda}|\partial_\lambda \hat{H}|\chi_{n,\lambda}\rangle = \partial_\lambda \langle \chi_{n,\lambda}|\hat{H}|\chi_{n,\lambda}\rangle = \partial_\lambda E_{n,\lambda}. \tag{31}$$

Substituting this into (29), integrating over λ, and substituting the result into (19), we obtain the zeroth-order (adiabatic) term in the expansion of $\mathcal{S}$:

$$\mathcal{S}_0 = \sum_n \int_{-\infty}^{\infty} d\tau\, [E_n(\tau) - E_n^{(|\Delta|\to 0)}(\tau)]\theta[-E_n(\tau)] + \int_{-\infty}^{\infty} d\tau \int d\mathbf{r} \frac{|\Delta(\mathbf{r},\tau)|^2}{g}. \tag{32}$$

$E_n(\tau)$ in this expression are the eigenenergies $E_n(\tau)$ of the Bogolubov-De Gennes equation (25) with $\lambda = 1$. [Note that $\sum_n E_n(\tau)$ is proportional to the area S of a 2d system, so both terms in (32) scale as S.] The counter-term with $E_n^{(|\Delta|\to 0)}$ comes from the lower limit of the integration over λ.

To derive the term in the action with the first derivative over time, $\mathcal{S}_1 = \int d\tau L^{(1)}(\tau)$, we substitute Eq. (27) into Eq. (20). Evaluating the trace with the use of (26), we obtain

$$\begin{aligned}\mathcal{S}_1 &= i\int d\tau \int_0^1 d\lambda \int \frac{d\omega}{2\pi} \sum_{n,m} \frac{\langle \chi_{n,\lambda}|\partial_\tau \hat{\Delta}|\chi_{m,\lambda}\rangle}{i\omega - E_{n,\lambda}} \frac{\partial}{\partial\omega} \frac{\langle \chi_{m,\lambda}|\partial_\lambda \hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle}{i\omega - E_{m,\lambda}} \\ &= \int d\tau \int_0^1 d\lambda \sum_{n,m} \frac{\langle \chi_{n,\lambda}|\partial_\tau \hat{\Delta}|\chi_{m,\lambda}\rangle\langle \chi_{m,\lambda}|\partial_\lambda \hat{\Delta}|\chi_{n,\lambda}\rangle(\theta_{n,\lambda} - \theta_{m,\lambda})}{(E_{n,\lambda} - E_{m,\lambda})^2}.\end{aligned} \tag{33}$$

We used Eq. (30) and a shorthand notation $\theta_{n,\lambda} \equiv \theta(E_{n,\lambda})$. To proceed further, we use the identities

$$\langle \chi_{n,\lambda}|\partial_\lambda \hat{\Delta}(\tau)|\chi_{m,\lambda}\rangle = \langle \chi_{n,\lambda}|\partial_\lambda \hat{H}|\chi_{m,\lambda}\rangle = \langle \partial_\lambda \chi_{n,\lambda}|\chi_{m,\lambda}\rangle (E_{n,\lambda} - E_{m,\lambda})\ (m \neq n), \tag{34}$$

$$\langle \chi_{n,\lambda}|\partial_\tau \hat{\Delta}(\tau)|\chi_{m,\lambda}\rangle = \langle \chi_{n,\lambda}|\partial_\tau \hat{H}|\chi_{m,\lambda}\rangle = \langle \partial_\tau \chi_{n,\lambda}|\chi_{m,\lambda}\rangle (E_{n,\lambda} - E_{m,\lambda})\ (m \neq n). \tag{35}$$

Substituting them into (33), we get rid of the denominator. Integrating the rest by parts, we obtain

$$\mathcal{S}_1 = \int d\tau \int_0^1 d\lambda \sum_n [\langle \partial_\lambda \chi_{n,\lambda}|\partial_\tau \chi_{n,\lambda}\rangle - \langle \partial_\tau \chi_{n,\lambda}|\partial_\lambda \chi_{n,\lambda}\rangle]\theta_{n,\lambda}. \tag{36}$$

One can make sure [46] that the term inside the brackets is

$$\frac{d}{d\lambda}\left[\langle \chi_n(\tau)|\partial_\tau \chi_n(\tau)\rangle - \left\langle \chi_n^{(|\Delta|\to 0)}(\tau)\middle|\partial_\tau \chi_n^{(|\Delta|\to 0)}(\tau)\right\rangle\right]. \tag{37}$$

Then

$$\begin{aligned}\mathcal{S}_1 = \sum_n \int d\tau \Big[&\langle \chi_n(\tau)|\partial_\tau \chi_n(\tau)\rangle \\ &- \left\langle \chi_n^{(|\Delta|\to 0)}(\tau)\middle|\partial_\tau \chi_n^{(|\Delta|\to 0)}(\tau)\right\rangle\Big]\theta_{n,\lambda}.\end{aligned} \tag{38}$$

The derivation of the second order term in the action $S_2 = \int d\tau L^{(2)}(\tau)$ is more cumbersome. We present the details in Appendix. The result is

$$\begin{aligned}\mathcal{S}_2 = -\frac{1}{2}\sum_{m\neq n}\int d\tau \Bigg[&\frac{|\langle \chi_n(\tau)|\partial_\tau \chi_m(\tau)\rangle|^2(\theta_n - \theta_m)}{E_n(\tau) - E_m(\tau)} \\ &- \frac{\left|\left\langle \chi_n^{(|\Delta|\to 0)}(\tau)\middle|\partial_\tau \chi_m^{(|\Delta|\to 0)}(\tau)\right\rangle\right|^2(\theta_n - \theta_m)}{E_n^{(|\Delta|\to 0)}(\tau) - E_m^{(|\Delta|\to 0)}(\tau)}\Bigg].\end{aligned} \tag{39}$$

We emphasize that in Eqs. (38) and (39), the wave functions $|\partial_\tau \chi_{n,\lambda}(\tau)\rangle$ and energies $E_n(\tau)$ satisfy Bogoluibov-De Gennes equations, in which the order parameter $\Delta(\mathbf{r},\tau)$ depends on coordinate $\mathbf{r}$ and on τ.

Finally, consider $\mathcal{S}_{\rm norm} = -{\rm Tr}\ln \hat{G}^{-1}_{\Delta\to 0}$, Eq. (16). We argue that it also contains the term linear in time derivative. The most straightforward way to see this is to keep Δ small but finite and apply a gauge transformation under the logarithm to get rid of the ϕ dependence of $\Delta(\phi) = \Delta e^{i\phi}$, i.e., replace $\hat{G}^{-1}_{\Delta\to 0}$ by $\hat{U}^\dagger \hat{G}^{-1}_{\Delta\to 0}\hat{U}$, where $\hat{U}$ is chosen such that in $\hat{U}^\dagger \hat{G}^{-1}_{\Delta\to 0}\hat{U}$, Δ appears without $e^{i\phi}$ factor (Ref. [25,47]). A simple experimentation shows that one should choose $\hat{U}$ in the form

$$\hat{U}(\mathbf{r},\tau) = \begin{bmatrix} e^{i\phi(\mathbf{r},\tau)/2} & 0 \\ 0 & e^{-i\phi(\mathbf{r},\tau)/2}\end{bmatrix}. \tag{40}$$

Once $\Delta(\phi)$ is stripped of the phase, its magnitude can be safely set to zero. However, because $\hat{G}^{-1}$ contains time and spatial derivatives, $\hat{U}^\dagger \hat{G}^{-1}_{\Delta\to 0}\hat{U}$ acquires the terms with $\dot{\phi}$ and $\nabla\phi$. These terms are additional to the ones in S_1 because to obtain the latter we used the expansion in powers of Δ, while here we treat Δ as infinitesimally small and do not expand in it.

Using (12) and (40), we obtain, keeping only time derivative of ϕ,

$$\mathcal{S}_{\rm norm} = -{\rm Tr}\ln\left[\hat{G}_0^{-1} - \frac{i}{2}\sigma_z\dot{\phi}(\mathbf{r},\tau)\right], \tag{41}$$

where

$$\hat{G}_0^{-1} = \begin{bmatrix} -\partial_\tau + (1/2m)(\nabla + (i/2)\nabla\phi)^2 + \mu & 0 \\ 0 & -\partial_\tau - (1/2m)(\nabla - (i/2)\nabla\phi)^2 - \mu\end{bmatrix}. \tag{42}$$

$\mathcal{S}_{\rm norm}$ in Eqs. (41) and (42), as well as $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_2$, can be expanded in terms of space and time derivatives of ϕ. This will be carried out in the next section, where we will analyze the long wavelength and low-frequency limit of the effective action derived in this section and obtain the Anderson-Bogolubov-Goldstone (ABG) mode of gapless phase fluctuations. A special attention is required when the phase ϕ contains a vortex, in which case an expansion in $\nabla\phi$ fails in the vicinity of the vortex core. Instead we expand of $\mathcal{S}_{\rm norm}$ in terms of small displacements of the vortex core and show that there is a transverse reaction force associated with the readjustments of the normal component to the vortex displacement. The corresponding analysis will be carried out in Sec. IV.

B. The long-wavelength limit

The expansion of the action in the previous section holds for any function $\Delta(\tau, \mathbf{r})$. In this section, we consider slowly varying the order parameter and derive an effective action in terms of its spatial and time derivatives. We compute each term in $S = S_0 + S_1 + S_2 + S_{\rm norm}$ separately

1. The action $\mathcal{S}_0$: the condensation energy and the $(\nabla\phi)^2$ term

The term $\mathcal{S}_0$ is given by Eq. (32). To express it in terms of spatial derivatives of Δ, we need to find the solutions to the Bogolubov-de Gennes equation:

$$\begin{bmatrix} -(1/2m)\nabla^2 - \mu & |\Delta(\mathbf{r},\tau)|e^{i\phi(\mathbf{r},\tau)} \\ |\Delta(\mathbf{r},\tau)|e^{-i\phi(\mathbf{r},\tau)} & (1/2m)\nabla^2 + \mu \end{bmatrix} \begin{bmatrix} u_n(\mathbf{r},\tau) \\ v_n(\mathbf{r},\tau) \end{bmatrix} = E_n(\tau) \begin{bmatrix} u_n(\mathbf{r},\tau) \\ v_n(\mathbf{r},\tau) \end{bmatrix}. \tag{43}$$

To get rid of the complex phase of $\Delta(\mathbf{r},\tau)$, we redefine the wave function in Eq. (43) as $|\chi_n(\mathbf{r},\tau)\rangle = e^{i\phi(\mathbf{r},\tau)\sigma_z/2}|\tilde{\chi}_n(\mathbf{r},\tau)\rangle$. The eigenfunction $|\tilde{\chi}_n(\mathbf{r},\tau)\rangle$ satisfies the equation

$$\left\{\hat{H}_0 - \frac{i}{4m}[\nabla(\nabla\phi) + (\nabla\phi)\nabla] + \frac{\hat{\sigma}_z}{8m}(\nabla\phi)^2\right\}|\tilde{\chi}_n(\mathbf{r},\tau)\rangle = E_n(\tau)|\tilde{\chi}_n(\mathbf{r},\tau)\rangle, \tag{44}$$

where

$$\hat{H}_0 = \begin{bmatrix} -(1/2m)\nabla^2 - \mu & |\Delta(\mathbf{r},\tau)| \\ |\Delta(\mathbf{r},\tau)| & (1/2m)\nabla^2 + \mu \end{bmatrix}. \tag{45}$$

Due to slow variation of ϕ on $\mathbf{r}$, the last two terms in Eq. (44) can be treated as perturbations. We define $\hat{V}_1$ and $\hat{V}_2$ as follows:

$$\hat{V}_1 = -\frac{i}{4m}[\nabla(\nabla\phi) + (\nabla\phi)\nabla], \quad \hat{V}_2 = \frac{\hat{\sigma}_z}{8m}(\nabla\phi)^2. \tag{46}$$

In $\hat{V}_1$, the free-standing gradient in the first term acts on the bra state on the left, and in the second term, it acts on the ket state on the right.

In the following, we restrict our analysis to terms quadratic in spatial derivatives. It is easy to verify that to this order, one can neglect the spatial fluctuations of $|\Delta(\mathbf{r},\tau)|$ as the spatial dependence of $|\Delta|$ only gives rise to third-order terms like $(\nabla\phi)^2(\nabla\Delta)$, etc.

The eigenstates of (44) at $\hat{V}_1 = \hat{V}_2 = 0$ are the conventional Bogolubov solutions, for which n is a continuous 2d variable, which we label as $\mathbf{k}$. For the particle branch, we have

$$|\tilde{\chi}^{(+)}_{\mathbf{k}}(\mathbf{r})\rangle \equiv \begin{bmatrix} \tilde{u}_{\mathbf{k}} \\ \tilde{v}_{\mathbf{k}} \end{bmatrix} e^{i\mathbf{kr}} = \begin{bmatrix} \sqrt{\frac{1}{2} + \frac{\xi_{\mathbf{k}}}{2|E_{\mathbf{k}}|}} \\ \sqrt{\frac{1}{2} - \frac{\xi_{\mathbf{k}}}{2|E_{\mathbf{k}}|}} \end{bmatrix} \frac{e^{i\mathbf{kr}}}{\sqrt{S}}, \tag{47}$$

where $E_n(\tau) = E^{(+)}_{\mathbf{k}}(\tau) = +\sqrt{\xi^2_{\mathbf{k}} + |\Delta(\tau)|^2}$ and we remind that S is the area of the sample. For the hole branch, we have

$$|\tilde{\chi}^{(-)}_{\mathbf{k}}(\mathbf{r})\rangle \equiv \begin{bmatrix} \tilde{v}_{\mathbf{k}} \\ -\tilde{u}_{\mathbf{k}} \end{bmatrix} e^{i\mathbf{kr}} = \begin{bmatrix} \sqrt{\frac{1}{2} - \frac{\xi_{\mathbf{k}}}{2|E_{\mathbf{k}}|}} \\ -\sqrt{\frac{1}{2} + \frac{\xi_{\mathbf{k}}}{2|E_{\mathbf{k}}|}} \end{bmatrix} \frac{e^{i\mathbf{kr}}}{\sqrt{S}} \tag{48}$$

where $E_n(\tau) = E^{(-)}_{\mathbf{k}}(\tau) = -\sqrt{\xi^2_{\mathbf{k}} + |\Delta(\tau)|^2}$.

(a) The condensation energy. We label by $\mathcal{S}_{0,a}$ the term in $\mathcal{S}_0$, which does not contain gradients. It is given by

$$\mathcal{S}_{0,a}(\Delta) = \int d\tau \int d\mathbf{r}\left[-\int \frac{d^2\mathbf{k}}{(2\pi)^2}\left\{\sqrt{\xi^2_{\mathbf{k}} + |\Delta(\tau)|^2} - |\xi_{\mathbf{k}}|\right\} + \frac{|\Delta(\tau)|^2}{g}\right]. \tag{49}$$

In equilibrium, $\Delta(\tau) = \Delta_0 + \delta\Delta(\tau)$, where $\Delta_0 \gg \delta\Delta(\tau)$. Substituting this $\Delta(\tau)$ into (49) and using $\partial\mathcal{S}_{0,a}(\Delta)/\partial(\delta\Delta) = 0$, we obtain a conventional gap equation

$$S\frac{\Delta_0}{g} = \Delta_0 \sum_{\omega,\mathbf{k}} \frac{\Delta_0}{\omega^2 + \Delta_0^2 + \xi^2_{\mathbf{k}}}, \tag{50}$$

which after the integration over Matsubara frequency becomes

$$\frac{1}{g} = \frac{N_0}{2}\int_{-\mu}^{\Lambda} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}}. \tag{51}$$

Integrating further over ξ in (51) and re-expressing the result in terms of the bound state energy $2E_0 = 2\Lambda e^{-2/(N_0 g)}$, we obtain the relation [1,3,24]

$$\sqrt{\mu^2 + \Delta_0^2} - \mu = 2E_0\,. \tag{52}$$

The self-consistency equation for μ in turn follows from the condition that the total number of fermions, including bound pairs, is conserved [3]. This gives another relation

$$\sqrt{\mu^2 + \Delta_0^2} + \mu = 2E_F\,. \tag{53}$$

Solving Eqs. (52) and (53), we obtain

$$\mu = E_F - E_0\,, \quad \Delta_0 = 2\sqrt{E_F E_0}. \tag{54}$$

We will use these formulas below when we evaluate the prefactors for $\dot{\phi}$, $(\dot{\phi})^2$, and $(\nabla\phi)^2$ terms in the crossover region between the BCS and BEC behavior. We recall that BCS behavior holds when the bound state energy E_0 is much smaller than E_F (and $\Delta_0 \ll \mu$) and BEC behavior holds when $E_0 \gg E_F$. A negative μ at $E_F < E_0$ implies that the Fermi momentum k_F, defined as position of the minimum of the fermionic dispersion $E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta_0^2}$, is zero [7].

Equations (53) and (54) allow one to obtain the condensation energy of a superconductor in the whole crossover range

between the BCS and BEC regimes. We have

$$E_{\rm cond} - NS(\mu-\mu_0) = \mathcal{S}_{0,a} + \delta\mathcal{S}_{\rm free}, \tag{55}$$

where $\delta\mathcal{S}_{\rm free}$ is the difference between $2\sum_k \xi_k n_k$ in the normal state at chemical potentials μ and μ_0. Using $N = 2N_0E_F$, $\mu - \mu_0 = -E_0$ and evaluating $\delta\mathcal{S}_{\rm free} = SN_0(\mu_0^2 - \mu^2)$ for $\mu > 0$ and $\delta\mathcal{S}_{\rm free} = SN_0\mu_0^2$ for $\mu < 0$, we obtain

$$\begin{aligned} E_{\rm cond} &= -SN_0E_0^2 + \mathcal{S}_{0,a}, \quad \mu > 0; \\ &= -SN_0\left(\frac{\Delta^2}{2} - E_F^2\right) + \mathcal{S}_{0,a}, \quad \mu < 0. \end{aligned} \tag{56}$$

Using (51) and introducing $x = \mu/\Delta_0$ and $y = \xi/\Delta_0$, we re-express $S_{0,a}$ in (49) as

$$\mathcal{S}_{0,a} = SN_0\frac{\Delta_0^2}{2}I(x), \tag{57}$$

where

$$I(x) = \int_{-x}^{\infty} dy\left(\frac{1}{\sqrt{y^2+1}} - 2\sqrt{y^2+1} + |y|\right). \tag{58}$$

This integration yields

$$\begin{aligned} I(x) &= -\tfrac{1}{2} + x^2 - x\sqrt{1+x^2}, \quad x > 0; \\ &= -\tfrac{1}{2} - x^2 + |x|\sqrt{1+x^2}, \quad x < 0. \end{aligned} \tag{59}$$

Substituting the expressions for μ and Δ, we obtain

$$\begin{aligned} \mathcal{S}_{0,a} &= SN_0\left(-\frac{\Delta_0^2}{2} + E_0^2\right), \quad \mu > 0; \\ &= -SN_0E_F^2, \quad \mu < 0. \end{aligned} \tag{60}$$

The combination of (56) and (60) yields

$$E_{\rm cond} = -SN_0\frac{\Delta_0^2}{2} \tag{61}$$

independent on the ratio E_0/E_F. The same result (the independence of $E_{\rm cond}$ on E_0/E_F) has been also obtained [24,32] by directly evaluating the kinetic and the potential energy of a superconductor (see also Ref. [48]).

(b) The $\nabla\phi$ term. The leading term in $\nabla\phi$ in $\mathcal{S}_0$ can be calculated by treating $\nabla\phi$ terms in the Hamiltonian in Eq. (44) as perturbations. To the first order, this contribution is given by $\hat{V}_2$. We label the corresponding term in $\mathcal{S}_0$ as $\mathcal{S}_{0,b}$. We have

$$\begin{aligned} \mathcal{S}_{0,b} &= -\int d\tau \sum_{\mathbf{k}} [\langle\tilde{\chi}_{\mathbf{k}}^{(-)}|\hat{V}_2|\tilde{\chi}_{\mathbf{k}}^{(-)}\rangle_\Delta \\ &\quad - \langle\tilde{\chi}_{\mathbf{k}}^{(-)}|\hat{V}_2|\tilde{\chi}_{\mathbf{k}}^{(-)}\rangle_{\Delta\to 0}]\theta(-E_k) \\ &= -\left[\frac{1}{S}\sum_{\mathbf{k}}\left(\frac{\xi_{\mathbf{k}}}{|E_{\mathbf{k}}|} - \frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|}\right)\right]\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m}. \end{aligned} \tag{62}$$

The $\mathbf{k}$ integral is ultraviolet convergent. Note that due to the presence of $\theta(-E_n)$ in Eq. (32), the summation in Eq. (62) involves only the hole states $|\tilde{\chi}_{\mathbf{k}}^{(-)}\rangle$. Evaluating the integral, we find

$$\begin{aligned} -\frac{1}{S}\sum_{\mathbf{k}}\left(\frac{\xi_{\mathbf{k}}}{|E_{\mathbf{k}}|} - \frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|}\right) &= -\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\frac{\xi_{\mathbf{k}}}{|E_{\mathbf{k}}|} - \frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|}\right) \\ &= N_0(\sqrt{\mu^2 + |\Delta(\tau)|^2} - |\mu|). \end{aligned} \tag{63}$$

This term can be equivalently re-expressed as

$$\begin{aligned} &\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(1 - \frac{\xi_{\mathbf{k}}}{|E_{\mathbf{k}}|}\right) - \int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(1 - \frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|}\right) \\ &\quad = 2\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\left(\tilde{v}_k^2\right)_\Delta - \tilde{v}_k^2(\Delta = 0)\right) = n - n_0, \end{aligned} \tag{64}$$

where

$$n = 2\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\tilde{v}_k^2\right)_\Delta = (\sqrt{\mu^2 + |\Delta(\tau)|^2} + |\mu|) = 2N_0E_F \tag{65}$$

is the density of fermions and

$$n_0 = 2\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\tilde{v}_k^2\right)_{\Delta=0} = 2N_0\mu\theta(\mu) \tag{66}$$

is the density of free electrons in the normal state at the actual chemical potential μ. Using these notations, we find

$$\mathcal{S}_{0,b} = (n - n_0)\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m}. \tag{67}$$

The last contribution to $\mathcal{S}_0$ comes from $\hat{V}_1$. The first-order contribution from $\hat{V}_1$ is zero. The contribution to order $(\hat{V}_1)^2$ is given by

$$\mathcal{S}_{0,c} = (\mathcal{S}_{0,c})_\Delta - (\mathcal{S}_{0,c})_{\Delta=0}, \tag{68}$$

where

$$\begin{aligned} (\mathcal{S}_{0,c})_\Delta &= \sum_{\mathbf{k},\mathbf{k}',i=\pm}\frac{\langle\tilde{\chi}_{\mathbf{k}}^{(-)}|\hat{V}_1|\tilde{\chi}_{\mathbf{k}'}^{(i)}\rangle\langle\tilde{\chi}_{\mathbf{k}'}^{(i)}|\hat{V}_1|\tilde{\chi}_{\mathbf{k}}^{(-)}\rangle}{E_{\mathbf{k}}^{(-)} - E_{\mathbf{k}'}^{(i)}} \\ &= (\mathcal{S}_{0,c}^{-+})_\Delta + (\mathcal{S}_{0,c}^{--})_\Delta. \end{aligned} \tag{69}$$

Consider $(\mathcal{S}_{0,c}^{-+})_\Delta$ and $(\mathcal{S}_{0,c}^{--})_\Delta$ separately. For $i = +$, we use Eq. (47) for $|\tilde{\chi}_{\mathbf{k}}^{(+)}(\mathbf{r})\rangle$ and Eq. (48) for $|\tilde{\chi}_{\mathbf{k}}^{(-)}(\mathbf{r})\rangle$ and obtain

$$\begin{aligned} &\langle\tilde{\chi}_{\mathbf{k}}^{(-)}|\hat{V}_1|\tilde{\chi}_{\mathbf{k}'}^{(+)}\rangle\langle\tilde{\chi}_{\mathbf{k}'}^{(+)}|\hat{V}_1|\tilde{\chi}_{\mathbf{k}}^{(-)}\rangle \\ &\quad = (\tilde{u}_{\mathbf{k}}\tilde{v}_{\mathbf{k}'} - \tilde{v}_{\mathbf{k}}\tilde{u}_{\mathbf{k}'})^2\frac{(k_j + k_j')^2}{2m} \\ &\qquad \times\int d\mathbf{r}d\mathbf{r}'\frac{(\nabla_j\phi)(\nabla_j'\phi)}{8m}e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}, \end{aligned} \tag{70}$$

where $j = x, y$. Using the forms of $\tilde{u}_k$ and $\tilde{v}_k$, we then obtain

$$\begin{aligned} (\mathcal{S}_{0,c}^{-+})_\Delta &= -\frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\frac{E_{\mathbf{k}}E_{\mathbf{k}'} - \xi_{\mathbf{k}}\xi_{\mathbf{k}'} - \Delta_0^2}{2E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}} + E_{\mathbf{k}'})}\frac{(k_j + k_j')^2}{2m} \\ &\quad\times\int d\mathbf{r}d\mathbf{r}'\frac{(\nabla_j\phi)(\nabla_j'\phi)}{8m}e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}, \end{aligned} \tag{71}$$

where in the prefactor we can use the zero-order expression $E_{\mathbf{k}} = \sqrt{\xi_k^2 + \Delta_0^2}$. For $(\mathcal{S}_{0,c}^{--})_\Delta$ the computation along the same lines yields

$$\begin{aligned} (\mathcal{S}_{0,c}^{--})_\Delta &= -\frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\frac{E_{\mathbf{k}}E_{\mathbf{k}'} + \xi_{\mathbf{k}}\xi_{\mathbf{k}'} + \Delta_0^2}{2E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}} - E_{\mathbf{k}'})}\frac{(k_j + k_j')^2}{2m} \\ &\quad\times\int d\mathbf{r}d\mathbf{r}'\frac{(\nabla_j\phi)(\nabla_j'\phi)}{8m}e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}. \end{aligned} \tag{72}$$

The total contribution $(\mathcal{S}_{0,c})_\Delta = (\mathcal{S}^{-+}_{0,c})_\Delta + (\mathcal{S}^{--}_{0,c})_\Delta$ is, after symmetrization over $\mathbf{k}$ and $\mathbf{k}'$

$$(\mathcal{S}_{0,c})_\Delta = -\frac{1}{2S^2}\sum_{\mathbf{k},\mathbf{k}'} \frac{E_{\mathbf{k}}E_{\mathbf{k}'} - (\xi_{\mathbf{k}}\xi_{\mathbf{k}'} + \Delta_0^2)}{E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}} + E_{\mathbf{k}'})} \frac{(k_j + k'_j)^2}{2m} \times \int d\mathbf{r}d\mathbf{r}' \frac{(\nabla_j\phi)(\nabla'_j\phi)}{8m} e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}. \quad (73)$$

To proceed, assume that $(\nabla_j\phi(\mathbf{r}))(\nabla'_j\phi(\mathbf{r}'))$, viewed as a function of $\delta\mathbf{r} = \mathbf{r} - \mathbf{r}'$, drops at some characteristic scale D_0, which is much smaller than the system size $2D$, but much larger than interatomic spacing a_0. The corresponding characteristic $\delta_{\mathbf{k}} = |\mathbf{k} - \mathbf{k}'|$ are of order $1/D_0$, which satisfies $1/D \ll 1/D_0 \ll 1/a_0$. Such δk are, on one hand, much smaller than k_F, and, on the other hand, are large enough such that the discreteness of momentum $\delta k_m = \pi m/D$ is irrelevant. As the consequence, the expression for $(\mathcal{S}_{0,c})_\Delta$ can be re-expressed, to leading order in the derivatives, as

$$(\mathcal{S}_{0,c})_\Delta = -\lim_{\delta\mathbf{k}\to 0}\chi_j(\delta k)\int d\mathbf{r}\frac{(\nabla_j\phi)^2}{8m}, \quad (74)$$

where we introduced

$$\chi_j(\delta\mathbf{k}) = \frac{1}{S}\sum_{\mathbf{k}} \frac{E_{\mathbf{k}_-}E_{\mathbf{k}_+} - \left(\xi_{\mathbf{k}_+}\xi_{\mathbf{k}_-} + \Delta_0^2\right)}{E_{\mathbf{k}_+}E_{\mathbf{k}_-}(E_{\mathbf{k}_+} + E_{\mathbf{k}_-})} \frac{k_j^2}{m} \quad (75)$$

with $\mathbf{k}_\pm = \mathbf{k} \pm \delta\mathbf{k}/2$, and used

$$\int \frac{d(\delta k_j)}{2\pi}\int_{-D_0}^{D_0} d(\delta r_j)e^{i\delta k_j\delta r_j} = \frac{2}{\pi}\int_0^\infty \frac{\sin x}{x}dx = 1. \quad (76)$$

The quantity $\chi(\delta\mathbf{k})$ is, up an overall factor, a particle-hole bubble made out of superconducting Green's functions. At a finite Δ, it vanishes at $\delta\mathbf{k} \to 0$ because the term in the numerator in (75) tends to zero in this limit. Accordingly, $(\mathcal{S}_{0,c})_\Delta = 0$. However, for $(\mathcal{S}_{0,c})_{\Delta\to 0}$, the corresponding $\chi(\delta\mathbf{k})$ is a free-fermion static susceptibility in the normal state, and it tends to a finite value when $\delta\mathbf{k}$ is small but finite. We now use the fact that at small $\delta\mathbf{k}$ the integration over $\mathbf{k}$ in (75) is confined to $\mathbf{k} = \mathbf{k}_F$ and pull $k_j^2/m \approx (k_F^2)_j/m$ from the sum. Performing the remaining integration with $E_{\mathbf{k}} = |\xi_{\mathbf{k}}|$ and using the symmetry between $j = x$ and $j = y$ and the fact that $N_0k_F^2/m = 2N_0\mu\theta(\mu) = n_0$, we obtain

$$(\mathcal{S}_{0,c})_{\Delta\to 0} = -N_0\frac{k_F^2}{m}\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m} = -n_0\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m}. \quad (77)$$

Substituting this into (68), we obtain

$$\mathcal{S}_{0,c} = n_0\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m}. \quad (78)$$

Combining (67) and (78), we obtain the total term with $(\nabla\phi)$ and no time derivative in the form

$$\mathcal{S}_{0,b} + \mathcal{S}_{0,c} = n\int d\tau\int d\mathbf{r}\frac{(\nabla\phi)^2}{8m}. \quad (79)$$

We see that the prefactor for the $(\nabla\phi)^2$ term in the action is the full density. The consideration can be readily extended to the case when impurity scattering is present. The result is that n is replaced by the superfluid density n_s. In our consideration, we do not distinguish between n and n_s.

2. The actions $\mathcal{S}_1$ and $\mathcal{S}_{\rm norm}$—the linear term in $\dot{\phi}$

The calculation of the first-order term in the derivative over τ is quite straightforward. We start with S_1 term. From Eqs. (38) and (48), we obtain

$$\sum_n \langle\chi_n(\tau)|\partial_\tau\chi_n(\tau)\rangle - \left\langle\chi_n^{(|\Delta|\to 0)}(\tau)\middle|\partial_\tau\chi_n^{(|\Delta|\to 0)}(\tau)\right\rangle\theta_{n,\lambda} \quad (80)$$

$$= i\int d\mathbf{r}\,\frac{\dot{\phi}(\mathbf{r},\tau)}{2}\frac{1}{S}\sum_{\mathbf{k}}\left[\left(\tilde{v}_{\mathbf{k}}^2 - \tilde{u}_{\mathbf{k}}^2\right)_{\Delta\neq 0} - \left(\tilde{v}_{\mathbf{k}}^2 - \tilde{u}_{\mathbf{k}}^2\right)_{\Delta=0}\right]$$

$$= i\int d\mathbf{r}\,\frac{\dot{\phi}(\mathbf{r},\tau)}{2}\frac{1}{S}\sum_{\mathbf{k}}\left(\frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|} - \frac{\xi_{\mathbf{k}}}{|E_{\mathbf{k}}|}\right). \quad (81)$$

Substituting this into Eq. (38) and using Eq. (63), we obtain

$$\mathcal{S}_1 = \frac{iN_0}{2}\int d\tau\int d\mathbf{r}\,(\sqrt{\mu^2 + |\Delta(\tau)|^2} - |\mu|)\,\dot{\phi}(\mathbf{r},\tau) = i\int d\tau\int d\mathbf{r}\frac{n(\tau) - n_0}{2}\dot{\phi}(\mathbf{r},\tau). \quad (82)$$

Note that this expression again contains fluctuating $\Delta(\tau)$ rather than equilibrium Δ_0.

Equation (82) can be cast in the form of the Wess-Zumino action for a superconductor [31,37,40]. To see this, let us recall the derivation of $\mathcal{S}_1$, e.g., Eq. (38), and write it as a slightly modified version of Eq. (33),

$$\mathcal{S}_1 = \int_{-\infty}^{\infty} d\tau\int_0^1 d\lambda\sum_{n,m}\frac{\langle\chi_{n,\lambda}|\partial_\lambda\hat{\Delta}|\chi_{m,\lambda}\rangle\langle\chi_{m,\lambda}|\partial_\tau\hat{\Delta}|\chi_{n,\lambda}\rangle(\theta_{n,\lambda} - \theta_{m,\lambda})}{(E_{n,\lambda} - E_{m,\lambda})^2}. \quad (83)$$

Since Eq. (83) already contains double gradients (over τ and λ), we can treat states $|\chi_{n,\lambda}\rangle$ and energies $E_{n,\lambda}$ in this equation adiabatically, i.e., use Eq. (47) and $E^{(+)}_{\mathbf{k},\lambda}(\tau) = +\sqrt{\xi_{\mathbf{k}}^2 + |\Delta(\tau,\lambda)|^2}$ for the particle branch and use Eq. (48) and $E^{(-)}_{\mathbf{k},\lambda}(\tau) = -E^{(+)}_{\mathbf{k},\lambda}(\tau)$ for the hole branch.

The integrand in Eq. (83) can then be written as

$$\frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\frac{1}{(E_{\mathbf{k},\lambda}+E_{\mathbf{k}',\lambda})^2}\int d\mathbf{r}d\mathbf{r}' e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}\{[\partial_\lambda\Delta(\mathbf{r})v_{\mathbf{k},\lambda}v_{\mathbf{k}',\lambda}-\partial_\lambda\Delta^*(\mathbf{r})u_{\mathbf{k},\lambda}u_{\mathbf{k}',\lambda}][\partial_\tau\Delta^*(\mathbf{r}')v_{\mathbf{k},\lambda}v_{\mathbf{k}',\lambda}-\partial_\tau\Delta(\mathbf{r}')u_{\mathbf{k},\lambda}u_{\mathbf{k}',\lambda}]$$
$$-(\text{same with }\Delta\to\Delta^*)\}\,. \tag{84}$$

In the long-wavelength limit, we can replace $(1/S^2)\sum_{\mathbf{k},\mathbf{k}'}e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}v^2_{\mathbf{k},\lambda}v^2_{\mathbf{k}',\lambda}/(E_{\mathbf{k},\lambda}+E_{\mathbf{k}',\lambda})^2$ by $C_v\,\delta(\mathbf{r}-\mathbf{r}')$ with

$$C_v=\frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\frac{v^2_{\mathbf{k},\lambda}v^2_{\mathbf{k}',\lambda}}{(E_{\mathbf{k},\lambda}+E_{\mathbf{k}',\lambda})^2}(2\pi)^2\delta(\mathbf{k}-\mathbf{k}')=\frac{1}{S}\sum_{\mathbf{k}}\frac{v^4_{\mathbf{k},\lambda}}{4E^2_{\mathbf{k},\lambda}}\,,$$

etc. Then Eq. (84) reduces to a single integral over $\mathbf{r}$, which can be expressed as

$$\frac{1}{S}\sum_{\mathbf{k}}\frac{u^2_{\mathbf{k},\lambda}-v^2_{\mathbf{k},\lambda}}{4E^2_{\mathbf{k},\lambda}}\int d\mathbf{r}\,(\partial_\lambda\Delta\partial_\tau\Delta^*-\partial_\tau\Delta\partial_\lambda\Delta^*)\,, \tag{85}$$

where we have used that $v^2_{\mathbf{k},\lambda}+u^2_{\mathbf{k},\lambda}=1$. Finally, using

$$\frac{1}{S}\sum_{\mathbf{k}}\frac{u^2_{\mathbf{k},\lambda}-v^2_{\mathbf{k},\lambda}}{4E^2_{\mathbf{k},\lambda}}=\frac{1}{2}\frac{\partial n}{\partial(|\Delta|^2)}\,,$$

where n is particle density, we express the action S_1 as

$$\mathcal{S}_1=\frac{1}{2}\int d\mathbf{r}\int_{-\infty}^{\infty}d\tau\int_0^1 d\lambda\frac{\partial n}{\partial(|\Delta|^2)}(\partial_\lambda\Delta\partial_\tau\Delta^*-\partial_\tau\Delta\partial_\lambda\Delta^*). \tag{86}$$

This action has the same form as Wess-Zumino action for s-wave superconductor [31,37]. Note, however, that Eq. (86) is only valid in the long wave length limit, e.g., it does not account for the bound states that may arise in a vortex core [41], whereas Eq. (38) is more general because it includes all types of states.

We now turn to the contribution from $\mathcal{S}_{\rm norm}$, Eqs. (41) and (42). Expanding then to first order in $\dot\phi$, we obtain

$$\mathcal{S}_{\rm norm}=\mathcal{S}_0+\frac{i}{2}\int d\tau\int d\mathbf{r}\dot\phi(\mathbf{r},\tau){\rm Tr}[G_0(\mathbf{r},\tau;\mathbf{r},\tau)\sigma_z], \tag{87}$$

where $\mathcal{S}_0$ does not depend on ϕ. Introducing Fourier transformation for relative time and relative coordinate, replacing the integral over momentum by $N_0\int d\xi$, and keeping $e^{\pm i\omega\delta}$ factors (with infinitesimally small $\delta>0$) for particle and hole components of the Nambu Green's function in the normal state, we obtain for the second term in Eq. (87)

$$\frac{i}{2}\int d\tau\int d\mathbf{r}\dot\phi(\mathbf{r},\tau){\rm Tr}[G_0(\mathbf{r},\tau;\mathbf{r},\tau)\sigma_z]=\frac{i}{2}\int d\tau\int d\mathbf{r}\,\dot\phi(\mathbf{r},\tau)N_0\int_{-\mu}^{\infty}d\xi\int\frac{d\omega}{2\pi}\left(\frac{e^{i\omega\delta}}{i\omega-\xi}-\frac{e^{-i\omega\delta}}{i\omega+\xi}\right)$$
$$=i\int d\tau\int d\mathbf{r}\,\dot\phi(\mathbf{r},\tau)N_0\int_{-\mu}^{\infty}d\xi\theta(-\xi)=i\frac{n_0}{2}\int d\tau\int d\mathbf{r}\,\dot\phi(\mathbf{r},\tau)\,. \tag{88}$$

Then

$$\mathcal{S}_{\rm norm}=\mathcal{S}_0+i\frac{n_0}{2}\int d\tau\int d\mathbf{r}\,\dot\phi(\mathbf{r},\tau)\,. \tag{89}$$

Combining $\mathcal{S}_1$ from (82) and $\mathcal{S}_{\rm norm}$, we obtain

$$\mathcal{S}_1+\mathcal{S}_{\rm norm}=\mathcal{S}_0+\frac{i}{2}\int d\tau n(\tau)\int d\mathbf{r}\,\dot\phi(\mathbf{r},\tau)\,. \tag{90}$$

3. *The action $\mathcal{S}_2$: the $\dot\phi^2$ term*

(a) Contribution from Eq. (39). To obtain the $\dot\phi^2$ term from Eq. (39), we we need the matrix elements $\langle\chi_{n,\lambda}|\partial_\tau\chi_{m,\lambda}\rangle$ between particle and hole states. Using Eqs. (48) and (47), we obtain after straightforward algebra

$$\mathcal{S}_2=\int d\tau\int d\mathbf{r}d\mathbf{r}'\,\dot\phi(\mathbf{r},\tau)\dot\phi(\mathbf{r}',\tau)B(\mathbf{r}-\mathbf{r}')\,, \tag{91}$$

where

$$B(\mathbf{r}-\mathbf{r}')=\frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\left[\frac{E_{\mathbf{k}}E_{\mathbf{k}'}-\xi_{\mathbf{k}}\xi_{\mathbf{k}'}+\Delta_0^2}{E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}}+E_{\mathbf{k}'})}-\frac{|\xi_{\mathbf{k}}||\xi_{\mathbf{k}'}|-\xi_{\mathbf{k}}\xi_{\mathbf{k}'}}{|\xi_{\mathbf{k}}||\xi_{\mathbf{k}'}|(|\xi_{\mathbf{k}}|+|\xi_{\mathbf{k}'}|)}\right]e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{r}-\mathbf{r}')}\,, \tag{92}$$

where the last term due to the $\Delta \to 0$ term in Eq. (39). In the long-wavelength limit, $B(\mathbf{r}-\mathbf{r}')$ can be approximated as $B_0\delta(\mathbf{r}-\mathbf{r}')$, where

$$B_0 = \frac{1}{S^2}\sum_{\mathbf{k},\mathbf{k}'}\left[\frac{E_{\mathbf{k}}E_{\mathbf{k}'}-\xi_{\mathbf{k}}\xi_{\mathbf{k}'}+\Delta_0^2}{E_{\mathbf{k}}E_{\mathbf{k}'}(E_{\mathbf{k}}+E_{\mathbf{k}'})}-\frac{\mathrm{sign}(-\xi_{\mathbf{k}})-\mathrm{sign}(-\xi_{\mathbf{k}'})}{\xi_{\mathbf{k}}-\xi_{\mathbf{k}'}}\right](2\pi)^2\delta(\mathbf{k}-\mathbf{k}')\,. \tag{93}$$

In Eq. (93), we have rewritten the last term in the brackets of Eq. (92), which corresponds to the familiar susceptibility of a normal (free) electron gas. This contribution is, however, canceled out by the second-order contribution from $\mathcal{S}_{\rm norm}$ in Eq. (41),

$$\mathcal{S}^{(2)}_{\rm norm} = \mathcal{S}^{(1)}_{\rm norm} + (1/8)\mathrm{Tr}[\hat{G}_0\sigma_z\dot{\phi}\hat{G}_0\sigma_z\dot{\phi}]\,, \tag{94}$$

where we have expanded the logarithm up to the second order in $\dot{\sigma}$. The Green's function $\hat{G}_0$ in Eq. (94) can be written in (Fourier representation) as

$$\hat{G}_0(\omega,\mathbf{k}) = \begin{bmatrix}(i\omega-\xi_{\mathbf{k}})^{-1} & 0\\ 0 & (i\omega+\xi_{\mathbf{k}})^{-1}\end{bmatrix}, \tag{95}$$

where we have dropped $\nabla\phi$-dependent terms as they lead to higher- (than the second-) order contributions. Then we obtain that

$$\mathrm{Tr}[\hat{G}_0\sigma_z\dot{\phi}\hat{G}_0\sigma_z\dot{\phi}] = \int\frac{d\mathbf{q}}{(2\pi)^2}\int\frac{d\omega}{2\pi}|\omega\phi(\omega,\mathbf{q})|^2\int\frac{d\Omega}{2\pi} \times\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{1}{i\Omega-\xi_{\mathbf{k}}}\frac{1}{i(\Omega+\omega)-\xi_{\mathbf{k}+\mathbf{q}}} + \frac{1}{i\Omega+\xi_{\mathbf{k}}}\frac{1}{i(\Omega+\omega)+\xi_{\mathbf{k}+\mathbf{q}}}\right]. \tag{96}$$

Integrating over Ω and taking low-frequency limit, i.e., setting $\omega = 0$ in the resulting expression, one obtains

$$\begin{aligned}\mathrm{Tr}[\hat{G}_0\sigma_z\dot{\phi}\hat{G}_0\sigma_z\dot{\phi}] &= \int\frac{d\mathbf{q}}{(2\pi)^2}\int\frac{d\omega}{2\pi}|\omega\phi(\omega,\mathbf{q})|^2\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{\mathrm{sign}(-\xi_{\mathbf{k}})-\mathrm{sign}(-\xi_{\mathbf{k}+\mathbf{q}})}{\xi_{\mathbf{k}}-\xi_{\mathbf{k}+\mathbf{q}}}\right]\\ &= \int d\tau|\dot{\phi}(\mathbf{q})|^2\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{\mathrm{sign}(-\xi_{\mathbf{k}})-\mathrm{sign}(-\xi_{\mathbf{k}+\mathbf{q}})}{\xi_{\mathbf{k}}-\xi_{\mathbf{k}+\mathbf{q}}}\right].\end{aligned} \tag{97}$$

It is clear now that the second order term in the right-hand side (RHS) of Eq. (94) is exactly the negative of the contribution produced by the last term in the brackets in Eqs. (92) and (93) and therefore only the first term in the RHS of Eq. (93) contributes.

Performing integration over the momenta, we obtain

$$\mathcal{S}_2 = \int d\tau N_0\left(1+\frac{\mu}{\sqrt{\mu^2+\Delta^2(\tau)}}\right)\int d\mathbf{r}\,\frac{\dot{\phi}^2}{8}\,. \tag{98}$$

(b) Another contribution to the prefactor for the $\dot{\phi}^2$ term. We now show that fluctuations of $|\Delta(\tau)|$ give rise to an additional term in the action, $S_{\rm extra}$, with the same structure as in Eq. (98). For this we note that the components of actions $\mathcal{S}_0$, $\mathcal{S}_1$, etc., are expressed in terms of fluctuating $|\Delta(\tau)|$ rather than in terms of constant Δ_0. That is, the magnitude of Δ fluctuates around its equilibrium value Δ_0: $|\Delta(\tau)| = \Delta_0 + \delta\Delta(\tau)$, and these longitudinal fluctuations are present in the action $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$. They are small at weak coupling and are not important for the $(\nabla\phi)^2$ in Eq. (79) and for the $\dot{\phi}^2$ term in Eq. (98), but they give rise to $(\delta\Delta)^2$ term in $\mathcal{S}_0$, coming from $\mathcal{S}_{0,a}(\Delta)$ in Eq. (49) and to $\dot{\phi}\delta\Delta$ term in $\mathcal{S}_1$, coming from expanding the prefactor for $\dot{\phi}$ term in (82) in $\delta\Delta(\tau)$. The combination of these two pieces gives rise to the additional $\dot{\phi}^2$ term in the action, which we now compute.

Within our approximation, $\Delta(\tau)$ is independent of $\mathbf{r}$, hence one can simply expand Eqn. (49) to the second order in $\delta\Delta$. The linear term is zero because Δ_0 corresponds to the minimum in the free energy, but the second-order term is finite. Using Eqs. (50), (52), and (53), we obtain after some straightforward algebra that

$$\mathcal{S}_{0,a}(\Delta) = \mathcal{S}_{0,a}(\Delta_0) + SN_0\int d\tau\frac{\Delta_0^2}{4E_0(E_0+E_F)}(\delta\Delta)^2, \tag{99}$$

where, we recall, S is the area of a 2d sample.

Similarly, we expand in Eq. (82) to linear order in $\delta\Delta(\tau)$, use Eqs. (52) and (53), and obtain $\mathcal{S}_1$ in terms of Δ_0 with the extra term with the product of first derivatives:

$$\mathcal{S}_1 = -(iN_0/2)\int d\tau\int d\mathbf{r}\left[(\sqrt{\mu^2+\Delta_0^2}-|\mu|)\dot{\phi} + \frac{\Delta_0}{E_0+E_F}\dot{\phi}\delta\Delta\right]. \tag{100}$$

Combining the last terms in (99) and (100) together, we obtain the extra piece in the action, $\delta\mathcal{S}$, associated with longitudinal gap fluctuations:

$$\delta\mathcal{S} = \frac{N_0}{2}\int d\tau\int d\mathbf{r}\left[\frac{\Delta_0^2}{2E_0(E_0+E_F)}(\delta\Delta)^2 - i\frac{\Delta_0}{E_0+E_F}\dot{\phi}\delta\Delta\right]. \tag{101}$$

Averaging over the Gaussian fluctuations of $\delta\Delta$ [which is the same as completing the square in (101)], we obtain an

additional contribution to the action, $\mathcal{S}_{\rm extra}$, in the form

$$\mathcal{S}_{\rm extra} = \int d\tau N_0 \left(1 - \frac{\mu}{\sqrt{\mu^2 + |\Delta_0|^2}}\right) \int d\mathbf{r} \frac{\dot{\phi}^2}{8}. \quad (102)$$

Combining this with $\dot{\phi}^2$ term in $\mathcal{S}_2$ in (98), we obtain

$$\mathcal{S}_2 + \mathcal{S}_{\rm extra} = N_0 \int d\tau \int d\mathbf{r} \frac{\dot{\phi}^2}{4}. \quad (103)$$

C. The full long-wavelength action

Combining the $(\nabla\phi)^2$ and $\dot{\phi}^2$ terms, Eqs. (79) and (103), we obtain the regular part of the action in the form

$$\mathcal{S}_{\rm reg} = N_0 \int d\tau \int d\mathbf{r} \left[(\sqrt{\mu^2 + |\Delta(\tau)|^2} + \mu) \frac{(\nabla\phi)^2}{8m} + \frac{\dot{\phi}^2}{4} \right]. \quad (104)$$

To our accuracy, the prefactor for $(\nabla\phi)^2$ term can be evaluated at $|\Delta(\tau)|^2 = \Delta_0^2$. We then obtain

$$\mathcal{S}_{\rm reg} = N_0 \int d\tau \int d\mathbf{r} \left[\frac{E_F}{4m} (\nabla\phi)^2 + \frac{\dot{\phi}^2}{4} \right]. \quad (105)$$

The coefficient in front of $(\nabla\phi)^2$ in Eq. (105) can be rewritten as more familiar $n/8m$ (Refs. [10,49]). Transforming to Fourier components (momentum q and real frequency Ω), we obtain from (105) $\mathcal{S}_{\rm cont} \propto \sum_{q,\Omega} |\phi_{q,\Omega}|^2 (\Omega^2 - q^2 v_F^2/2)$. The prefactor for $|\phi_{q,\Omega}|^2$ is the inverse susceptibility of phase fluctuations. We see that it has a pole at $\Omega = \pm(v_F/\sqrt{2})q$. The pole position corresponds to the frequency of a gapless phase fluctuation mode, whose velocity is $v_F/\sqrt{2}$, independent on the ratio of E_F/E_0.

The full term linear in $\dot{\phi}$ (the Berry phase term) is obtained by setting $\Delta(\tau) = \Delta_0$ in Eq. (90):

$$\mathcal{S}_{\rm Berry} = iA \int d\tau \int d\mathbf{r} \dot{\phi}, \quad (106)$$

where

$$A = \frac{n}{2} = \frac{N_0}{2} \left(\sqrt{\mu^2 + \Delta_0^2} + \mu \right) \quad (107)$$

and n is the actual electron density.

The result for A agrees with Refs. [25,27,28,34,38]. Note, however, that there is one element in our calculation, which has not been emphasized in earlier works. Namely, the absence of n_0 in (107) could be interpreted as if there is no contribution from $\Delta \to 0$. We argue that this is not entirely true. In our calculation, there are two contributions from $\Delta \to 0$: the term $\mathcal{S}_1$ in the limit $\lambda = 0$ and the term $\mathcal{S}_{\rm norm}$. The $n_0/2$ pieces from these two terms do cancel out, however, the full contribution to A from $\Delta \to 0$ does not vanish and gives $(1/2)\sum_k [\xi_k/|\xi_k| + (1 - \xi_k/|\xi_k|)] = (1/2)\sum_k (1)$. This formally divergent piece cancels out the divergence in the contribution to the prefactor from $\mathcal{S}_1$ at $\lambda = 1$ (i.e., at nonzero Δ), which is $(-1/2)\sum_k (\xi_k/E_k) = n/2 - (1/2)\sum_k (1)$. Without this cancellation, the coefficient for $\dot{\phi}$ term would contain a parasitic, formally infinite term. The same holds for the coefficient for the $(\nabla\phi)^2$ term in the action: if we were to neglect the contributions from $\Delta \to 0$, the prefactor would be $(-1/8m)\sum_k (\xi_k/E_k) = (1/8m)(n - \sum_k (1))$. The parasitic $\sum_k (1)$ term is canceled out by the sum of the two contributions from $\Delta \to 0$, as we showed above.

If ϕ is well defined for all $\mathbf{r}$ and τ, the Berry phase term reduces to the contribution from the boundary and does not affect the dynamics. The situation changes when ϕ is singular, as in the case of a moving vortex with coordinates $X(\tau)$ and $Y(\tau)$. Then the Berry phase term in the action becomes proportional to $\int d\tau X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)$, which cannot be expressed as a total derivative and contributes to the vortex dynamics. We show that the action (106) describes the contribution to the Berry term from fermions far away from the vortex core. We show that there is another contribution, which comes from the states right at the center of the vortex core. This last term originates from $\nabla\phi$ terms in $S_{\rm norm}$.

IV. THE BERRY PHASE TERM IN THE ACTION FOR A MOVING VORTEX

The order parameter for a moving vortex in 2d can be written as

$$\Delta(\tau, \mathbf{r}) = \Delta[\mathbf{r} - \mathbf{R}(\tau)] = |\Delta[\mathbf{r} - \mathbf{R}(\tau)]| e^{i\phi[\mathbf{r} - \mathbf{R}(\tau)]}, \quad (108)$$

where $\mathbf{R}(\tau)$ is vortex center,

$$\phi(\tau, \mathbf{r}) = \tan^{-1} \left[\frac{y - Y(\tau)}{x - X(\tau)} \right], \quad (109)$$

and $|\Delta(\mathbf{r})| \to \Delta_0$ for $r \gg \lambda$, where λ is the penetration depth.

The spectrum of the Bogolubov-DeGennes equation [Eq. (43)] near a vortex has been extensively studied [27–29,38,41,42,47,50] and is known to posses both continuous and discrete branches corresponding to delocalized and localized eigenstates, respectively. The localized eigenstates are known as Caroli, de Gennes, Matricon states [41]. The continuous part of the spectrum covers the range $|E_n^c| > \Delta_0$, while discrete states have energies $|E_n^d| < \Delta_0$.

The contributions to the vortex motion come from the occupied states with negative energies. A generic eigenstate, corresponding to $E_n < 0$, can be expressed as

$$|\chi_\nu^-(\mathbf{r})\rangle = e^{i(\hat{\sigma}_z/2)\phi(\mathbf{r}-\mathbf{R})} |\tilde{\chi}_\nu^-(\mathbf{r})\rangle; \; |\tilde{\chi}_\nu^-(\mathbf{r})\rangle = e^{i\nu\phi(\mathbf{r}-\mathbf{R})} \begin{bmatrix} \tilde{v}_\nu(|\mathbf{r} - \mathbf{R}|) \\ -\tilde{u}_\nu(|\mathbf{r} - \mathbf{R}|) \end{bmatrix}, \quad (110)$$

where $\nu = n + 1/2$ and n is an integer, $\phi(\mathbf{r} - \mathbf{R})$ is given by Eq. (109), and the radial functions $u_\nu^b(r)$ and $v_\nu^b(r)$ *and their derivatives* with respect to r are continuous for all r.

The eigenfunctions for the localized states are proportional to $J_{|\nu \pm 1/2|}(k_F r)$ at small $r \ll \lambda$ (upper sign for $\tilde{u}_\nu$, lower for $\tilde{v}_\nu$). At large distances, when $r \gg \lambda$, both $|\tilde{v}_\nu|^2$ and $|\tilde{u}_\nu|^2$ decay exponentially (Ref. [50]). The eigenfunctions for continuous states are expressed via $J_{|\nu \pm 1/2|}(k r)$ at small $r \ll \lambda$ (where k is generally a function of ν), while for $r \gg \lambda$, they are parameterized by ν and momentum k, which becomes an independent variable (Refs. [41,42,47]):

$$\tilde{u}_\nu(r) = \tilde{u}_{\nu,k}(r) = u_k J_{|\nu|}(kr), \; \tilde{v}_\nu(r) = \tilde{v}_{\nu,k}(r) = v_k J_{|\nu|}(kr), \quad (111)$$

where

$$u_k = \left(\frac{1}{2} + \frac{\xi_k}{2|E_k^{(-)}|}\right)^{1/2}, \; v_k = \left(\frac{1}{2} - \frac{\xi_k}{2|E_k^{(-)}|}\right)^{1/2}, \quad (112)$$

and $E_k^{(-)} = -\sqrt{\xi_k^2 + |\Delta_0|^2}$. The full solution of the Bogolubov-DeGennes equation for a vortex is expressed via Hankel functions, which are linear combinations of the Bessel and Neumann functions. The Neumamm functions $Y_{|\nu\pm 1/2|}(x)$ and $Y_{|\nu|}(x)$, however, grow when the index ν becomes larger than the argument x, and the sums over ν in (115) do not converge. The Bessel functions $J_{|\nu|}(x)$, on the contrary, decay exponentially when ν gets larger than x. By this reason, we only consider the solutions expressed via the Bessel functions.

Because both localized and extended states are present, specified by a discrete parameter ν, it is not *a priori* guaranteed that we can use the results from the previous section, which were obtained using the eigenfunctions far away from the vortex core, when ν can be treated as a continuous variable.

In this section, we re-evaluate the prefactor for $\dot{\phi}$ term using the exact eigenfunctions $|\chi_\nu^-(\mathbf{r})\rangle$. We first re-evaluate the terms $\mathcal{S}_1$ and $\mathcal{S}_{\rm norm}$ and show that they are determined by fermions far away from the vortex core and have the same forms as we found in the previous section. Then we take a closer look at seemingly innocent part of $S_{\rm norm}$, which does not contain $\dot{\phi}$, but does depend on $\nabla\phi$. We argue that it also contributes to the Berry phase term for a moving vortex, and this contribution comes from fermions inside the vortex core.

A. The $\mathcal{S}_1$ term for the vortex motion

The $\mathcal{S}_1$ term in the action is given by Eq. (38), which is valid for arbitrary $|\chi_\nu^-(\mathbf{r})\rangle$. To obtain $\mathcal{S}_1$ for a vortex we need to evaluate

$$\sum_\nu \int d\mathbf{r} \langle \chi_\nu^-(\tau,\mathbf{r})|\partial_\tau \chi_\nu^-(\tau,\mathbf{r})\rangle \,. \quad (113)$$

with $|\chi_\nu^-(\mathbf{r})\rangle$ from (110). Substituting these $|\chi_\nu^-(\mathbf{r})\rangle$ into (113), we obtain

$$\sum_\nu \int d\mathbf{r} \langle \chi_\nu^-(\tau,\mathbf{r})|\partial_\tau \chi_\nu^-(\tau,\mathbf{r})\rangle = i\int d\mathbf{r}\, \Phi(\mathbf{r}-\mathbf{R}(\tau))\partial_\tau\phi[\mathbf{r}-\mathbf{R}(\tau)]\,, \quad (114)$$

where

$$\Phi(|\mathbf{r}|) = \sum_\nu \left[|\tilde{u}_\nu(\mathbf{r})|^2\left(\nu - \frac{1}{2}\right) + |\tilde{v}_\nu(\mathbf{r})|^2\left(\nu + \frac{1}{2}\right)\right]. \quad (115)$$

Using (109), one can re-express $\partial_\tau\phi[\mathbf{r}-\mathbf{R}(\tau)]$ as

$$\partial_\tau\phi[\mathbf{r}-\mathbf{R}(\tau)] = \dot{X}(\tau)\left[\frac{y-Y(\tau)}{(x-X(\tau))^2+(y-Y(\tau))^2}\right] - \dot{Y}(\tau)\left[\frac{x-X(\tau)}{(x-X(\tau))^2+(y-Y(\tau))^2}\right], \quad (116)$$

such that

$$\sum_\nu \int d\mathbf{r} \langle \chi_\nu^-(\tau,\mathbf{r})|\partial_\tau \chi_\nu^-(\tau,\mathbf{r})\rangle = i[\dot{X}(\tau)Q_x - \dot{Y}(\tau)Q_y], \quad (117)$$

where

$$Q_x = \int d\mathbf{r}\left[\frac{y-Y(\tau)}{(x-X(\tau))^2+(y-Y(\tau))^2}\right]\Phi(\mathbf{r}-\mathbf{R}(\tau))\,,$$
$$Q_y = \int d\mathbf{r}\left[\frac{x-X(\tau)}{(x-X(\tau))^2+(y-Y(\tau))^2}\right]\Phi(\mathbf{r}-\mathbf{R}(\tau)). \quad (118)$$

We show below that $Q_x \propto Y(\tau)$ and $Q_y \propto X(\tau)$. It is then tempting to compute the prefactors by evaluating the derivatives dQ_y/dX and dQ_x/dY. This has to be done with extra care as the integrals for dQ_y/dX and dQ_x/dY are infrared singular and have to be properly regularized (see below). We use a different computational procedure and evaluate the integrals in (118) directly assuming that our system has a finite size $2D$ in both x and y directions. We show that Q_x and Q_y remain finite if we set D to infinity at the end of the calculation. We verified that the result does not depend on the We checked that the result does not depend on the geometry of the integration range, as long as the symmetry between x and y is preserved, i.e., Q_x and Q_y remains the same if we assume that the boundary of our system is, e.g., a circle instead of a square.

Let us evaluate Q_x first. Shifting the variables of integration from x and y to $\tilde{x} = x - X(\tau)$ and $\tilde{y} = y - Y(\tau)$, we obtain from (118)

$$Q_x = \int_{-D-X(\tau)}^{D-X(\tau)} d\tilde{x} \int_{-D-Y(\tau)}^{D-Y(\tau)} d\tilde{y}\left[\frac{\tilde{y}}{\tilde{x}^2+\tilde{y}^2}\right]\Phi(\tilde{r})\,, \quad (119)$$

where $\tilde{r} = (\tilde{x}^2+\tilde{y}^2)^{1/2}$. Using that the integrand is odd in $\tilde{y}$, we rewrite (119) as

$$Q_x = -\int_{-D-X(\tau)}^{D-X(\tau)} d\tilde{x} \int_{D-Y(\tau)}^{D+Y(\tau)} d\tilde{y}\left[\frac{\tilde{y}}{\tilde{x}^2+\tilde{y}^2}\right]\Phi(\tilde{r})\,. \quad (120)$$

This shows that the result comes from a tiny range of $\tilde{y}$ around $\tilde{y} = D$. A simple experimentation then shows that typical $\tilde{x}$ are also of order D. Assuming that $\Phi(\tilde{r})$ tends to the value at $\tilde{r} \sim D$, independent on the ratio $\tilde{x}/\tilde{y}$, we pull $\Phi(D)$ from the RHS of (120). The integration over $\tilde{x}$ is then elementary, and the result is

$$Q_x = -\pi Y(\tau)\Phi(D)\,. \quad (121)$$

Evaluating Q_y the same way, we find $Q_y = -\pi X(\tau)\Phi(D)$. Substituting Q_x and Q_y into (117), we obtain

$$\sum_\nu \int d\mathbf{r} \langle \chi_\nu^-(\tau,\mathbf{r})|\partial_\tau \chi_\nu^-(\tau,\mathbf{r})\rangle = i\pi\Phi(D)[X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)]. \quad (122)$$

We emphasize that Q_x and Q_y are determined by distances of order D, i.e., the contribution comes from fermions far away from the vortex core. There is no contribution from $r = 0$, contrary to what has been reported in some earlier

papers (see, e.g., Ref. [27]). In these earlier works the authors computed dQ_y/dX and dQ_x/dY by differentiating only in the term in the brackets (118) [i.e., not differentiation $\Phi(\mathbf{r}-\mathbf{R}(\tau))$], and set $X=Y=0$ in the integrand before evaluating the integral over $d\mathbf{r}$. Then the result comes from the smallest $\mathbf{r}=0$, as we will see below. However, the full dQ_x/dY contains also the derivative of $\Phi(\mathbf{r}-\mathbf{R}(\tau))$, i.e., if we differentiate under the integral in (118) and set $X=Y=0$, we obtain

$$\frac{dQ_x}{dY}=\int_{-D}^{D}dxdy\left\{\left[-\frac{1}{x^2+y^2}+2\frac{y^2}{(x^2+y^2)^2}\right]\Phi(x,y)-\frac{y}{x^2+y^2}\frac{\partial\Phi(x,y)}{\partial y}\right\}. \tag{123}$$

The first term is formally zero (it contains y^2-x^2 as the overall factor), but it also diverges at $x=y=0$. To regularize this term, we introduce an infinitesimally small "mass" term in the denominator, i.e., replace x^2+y^2 by $x^2+y^2+\epsilon^2$, evaluate the integral with a finite ϵ, and then set it to zero. Transforming to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ and using $\partial\Phi(x,y)/\partial y=(y/r)d\Phi/dr=\sin\theta d\Phi/dr$, we then obtain from (123)

$$\frac{dQ_x}{dY}=-2\pi\epsilon^2\int\frac{rdr}{(r^2+\epsilon^2)^2}\Phi(r)-\pi\int_0^D dr\frac{d\Phi(r)}{dr}. \tag{124}$$

The evaluation of the integral is now elementary. In the first term, the integral comes from $r\sim\epsilon$ and cancels ϵ^2 in the numerator ($r\epsilon^2/(r^2+\epsilon^2)^2$ acts as $\delta(r)$). The term then yields $-\pi\Phi(0)$. This is what has been obtained in Ref. [27]) and earlier papers cited in that work. The full result, however, also contains the contribution from the second term. It obviously gives $-\pi(\Phi(D)-\Phi(0))$. The sum of the two terms is $-\pi\Phi(D)$, with no contribution from $\mathbf{r}=0$. This agrees with (121).

Substituting Eq. (122) into Eq. (38), we find

$$\mathcal{S}_1=i\pi A_1\int d\tau(X(\tau)\dot{Y}(\tau)-Y(\tau)\dot{X}(\tau)), \tag{125}$$

where $A_1=\Phi^{\Delta}(D)-\Phi^{\Delta\to 0}(D)$. We emphasize that $X(\tau)\dot{Y}(\tau)-Y(\tau)\dot{X}(\tau)$ is *not* a full derivative, hence $\mathcal{S}_1$ term does contribute to the equation of motion for a vortex.

To obtain $\Phi^{\Delta}(D)\equiv\Phi(r=D)$ we need to know the forms of $\tilde{u}_\nu$ and $\tilde{v}_\nu$ at large r. The eigenfunctions for localized eigenstates decay exponentially when $r\sim D$ and hence are irrelevant for our purpose. The eigenfunctions for continuous states with a negative energy at distances larger than the penetration depth are given by Eq. (48).

Using these forms, we obtain

$$\Phi(x)=\sum_{\nu,k}\left[u_k^2(\nu-1/2)J_{|\nu|}^2(x)+v_k^2(\nu+1/2)J_{|\nu|}^2(x)\right]=\sum_k\left\{\left[\left(u_k^2+v_k^2\right)\sum_\nu(\nu-1/2)J_\nu^2(x)\right]+v_k^2\sum_\nu J_{|\nu|}^2(x)\right\}. \tag{126}$$

To evaluate the $\sum_\nu J_{|\nu|}^2(x)$, we note that at large x and $\nu<x$, the Bessel function can be approximated as

$$J_\nu(x)\approx\sqrt{\frac{2}{\pi}}\frac{1}{(x^2-\nu^2)^{1/4}}\sin\left(x+\frac{\nu^2}{2x}-\frac{\pi}{4}(2\nu+1)\right). \tag{127}$$

This formula is valid up to $\nu=x-O(x^{1/3})$. The sum over ν is determined by large $\nu=O(x)$, for which the summation over ν can be replaced by integration. We assume and then verify that the integral is determined by $\nu=O(x)$, but $x-\nu\gg x^{1/3}$. The contribution from this range is

$$2\int_0^{x-O(x^{1/3})}J_\nu^2(x)=\frac{2}{\pi}\int_0^{x-O(x^{1/3})}\frac{dx}{\sqrt{x^2-\nu^2}}=1-O(x^{-1/6}). \tag{128}$$

One can easily verify that the contribution from $|\nu-x|\leqslant x^{1/3}$ scales as $x^{-1/6}$, and the contribution from larger $\nu>x+O(x^{1/3})$ is even smaller because at such ν, $J_{|\nu|}(x)$ decays exponentially. Then $\sum_\nu J_{|\nu|}^2(x)=1$ up to corrections, which vanish at $x\to\infty$

Further, $\sum_\nu(\nu J_{|\nu|}^2(x))$ vanishes because of cancellation between terms with positive and negative ν. As the consequence,

$$\left[\left(u_k^2+v_k^2\right)\sum_\nu(\nu-1/2)J_\nu^2(x)\right]+v_k^2\sum_\nu J_{|\nu|}^2(x)=-\frac{1}{2}+v_k^2 \tag{129}$$

and, hence,

$$\Phi^{\Delta}(D)=-\frac{1}{2}\sum_k(1)+\sum_k\left(v_k^2\right)^{\Delta}. \tag{130}$$

This formula could also be obtained if we assumed from the beginning that at large $\mathbf{r}$, the eigenfunctions for the continuous spectrum approach those for a superconductor with a constant gap Δ, i.e., radial quantum number ν becomes momentum k, and $\tilde{\chi}_\nu^-(\tau,\mathbf{r})\rangle$ becomes

$$|\tilde{\chi}_{\mathbf{k}}^{(-)}(\mathbf{r})\rangle=\begin{bmatrix}v_k\\-u_k\end{bmatrix}\frac{e^{i\mathbf{kr}}}{\sqrt{S}}. \tag{131}$$

The second term in (130) can be easily evaluated

$$\sum_k\left(v_{\mathbf{k}}^2\right)^{\Delta}=\frac{N_0}{2}\int_{-\mu}^{\infty}d\xi\left(1-\frac{\xi}{\sqrt{\xi^2+\Delta_0^2}}\right)=\frac{N_0}{2}\left(\sqrt{\mu^2+\Delta_0^2}+\mu\right)=\frac{n}{2}, \tag{132}$$

where, we remind, $n=2N_0E_F$ is the actual density of fermions. However, the first term in (130) is the sum over all momenta, and is formally infinite. We now recall that the prefactor in the S_1 term in the action, Eq. (125), contains the difference $\Phi^{\Delta}(D)-\Phi^{\Delta\to 0}(D)$. We assume that the distances $r\sim D$ are outside the vortex core even when $\Delta\to 0$. Then $\Phi^{\Delta\to 0}(D)$ is determined by the same Eq. (130) as $\Phi^{\Delta}(D)$, the

only difference is that now

$$u_k = \left(\frac{1}{2} + \frac{\xi_k}{2|\xi_k|}\right)^{1/2}, \quad v_k = \left(\frac{1}{2} - \frac{\xi_k}{2|\xi_k|}\right)^{1/2}, \tag{133}$$

and

$$\sum_k \left(v_{\mathbf{k}}^2\right)^{\Delta\to 0} = \frac{N_0}{2}\int_{-\mu}^{\infty} d\xi \left(1 - \frac{\xi}{|\xi|}\right) = \frac{N_0}{2}(|\mu| + \mu) = \frac{n_0}{2}, \tag{134}$$

where n_0 is the density of free fermions at the same chemical potential μ. Accordingly,

$$\Phi^{\Delta\to 0}(D) = -\frac{1}{2}\sum_k (1) + \sum_k \left(v_k^2\right)^{\Delta=0}. \tag{135}$$

Combining (130) and (135), we obtain

$$A_1 = \Phi^{\Delta}(D) - \Phi^{\Delta\to 0}(D) = \sum_k \left(v_k^2\right)^{\Delta} - \sum_k \left(v_k^2\right)^{\Delta=0} = \frac{n - n_0}{2}. \tag{136}$$

We emphasize that this result is free from ultraviolet divergencies—the term $\sum_k(1)$ cancels out between (130) and (135). To obtain this cancellation one has to keep the contributions to A_1 from the action at $\Delta \to 0$.

B. The $\mathcal{S}_{\text{norm}}$ term for the vortex motion

Now we need to add to this result the contribution from $\mathcal{S}_{\text{norm}}$ (the term in the action at $\Delta \to 0$, proportional to $\dot{\phi}$). This contribution is computed in the same way as the one from $\mathcal{S}_1$. Namely, the contribution to $\partial\tau\phi[\mathbf{r} - \mathbf{R}]$ comes from fermions far away from the vortex core, hence we can just use $\mathcal{S}_{\text{norm}}$ from Eq. (88): $\mathcal{S}_{\text{norm}} = (in_0/2)\int d\tau \int d\mathbf{r}\dot{\phi}(\mathbf{r},\tau)$ and substitute $\int d\mathbf{r}\dot{\phi}(\mathbf{r},\tau) = \pi((X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau))$. This gives

$$\mathcal{S}_{\text{norm}} = i\pi A_{\text{norm}} \int d\tau (X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)), \tag{137}$$

where $A_{\text{norm}} = n_0/2$. Adding A_{norm} to A_1 from (136), we obtain at this stage the Berry phase term in the action in the form

$$\mathcal{S}_{\text{Berry},1} = i\pi A_{\text{vort},1} \int d\tau (X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)), \tag{138}$$

where

$$A_{\text{vort},1} = \frac{n}{2}. \tag{139}$$

We see that the prefactor is the same as in the action in Eq. (106). This is not surprising because the contribution comes solely from the states well outside the vortex core, where the eigenfunctions can be approximated by the ones in the absence of a vortex.

C. Contribution from the vortex core

We now show that there is another contribution to the Berry phase term in the action for a moving vortex, which comes from the vortex core. This extra contribution is "hidden" in the zero-order term in $S_{\text{norm}} = \text{Tr}\ln(G_0^{-1})$—the one which does not contain $\dot{\phi}$. We label this term as S^0_{norm}. This term does, however, depend on $\nabla\phi$, because we remind that to eliminate Δ from the Green's function we had to apply the unitary transformation $\hat{U}$ under Tr ln, with $\hat{U}$ given by (40). Under this unitary transformation the kinetic energy operator $\hat{\xi}(\nabla) = -\nabla^2/(2m)$ changes to $\hat{\xi}(\nabla + (i/2)\sigma_z\nabla\phi)$, where $\phi = \phi(\mathbf{r} - \mathbf{R}(\tau))$. As the result, S^0_{norm} does actually depend on $\mathbf{R}(\tau)$ via $\nabla\phi$.

Let us assume that $\mathbf{R}(\tau)$ is small and expand $S^0_{\text{norm}}(\mathbf{R}) = -\text{Tr}\ln G_0^{-1}$ to second order in $\mathbf{R}(\tau)$. A generic expansion yields

$$S^0_{\text{norm}}(\mathbf{R}) = S^0_{\text{norm}}(\mathbf{R}=0) - \int d\tau \eta_\alpha(\tau) R_\alpha(\tau) - \int d\tau d\tau' \eta_{\alpha\beta}(\tau - \tau') R_\alpha(\tau) R_\beta(\tau') + \ldots, \tag{140}$$

where the summation over repeated indices is assumed.

The first order response function η_α is zero because of translational invariance. To see this, we note that the matrix G_0 is diagonal, and so $\text{Tr}\ln G_0^{-1} = \text{Tr}\ln(G_0^+)^{-1} + \text{Tr}\ln(G_0^-)^{-1}$, where

$$(G_0^{\pm})^{-1} = -\partial_\tau \pm \hat{\xi}(\nabla \pm (i/2)\nabla\phi). \tag{141}$$

Then

$$\eta_\alpha = \text{Tr}\left[G_0^+ \frac{\partial\hat{\xi}^+}{\partial R_\alpha}\right] + \text{Tr}\left[G_0^- \frac{\partial\hat{\xi}^-}{\partial R_\alpha}\right], \tag{142}$$

where $G_0 = G_0(r,\tau;r,\tau)$. In Fourier representation,

$$\eta_\alpha = \sum_n \int \frac{d\omega}{2\pi}\left[\frac{\langle\chi_n^+|\partial\hat{\xi}^+/\partial R_\alpha|\chi_n^+\rangle\, e^{i\omega\delta}}{i\omega - \xi_n^+} + \frac{\langle\chi_n^-|\partial\hat{\xi}^-/\partial R_\alpha|\chi_n^-\rangle\, e^{-i\omega\delta}}{i\omega + \xi_n^-}\right], \tag{143}$$

where $|\chi_n^{\pm}\rangle$ ($\xi_n^{\pm}$) are the eigenfunctions (eigenvalues) of $\hat{\xi}^{\pm} = \hat{\xi}(\nabla \pm (i/2)\nabla\phi)$ (we present explicit expressions below). The eigenvalues of $\xi_n^{\pm}$ do not dependent on R_α in a translationally invariant system, hence $\partial\xi_n^{\pm}/\partial R_\alpha = 0$. Accordingly, $\langle\chi_n^{\pm}|\partial\hat{\xi}^{\pm}/\partial R_\alpha|\chi_n^{\pm}\rangle = \partial\langle\chi_n^{\pm}|\hat{\xi}^{\pm}|\chi_n^{\pm}\rangle/\partial R_\alpha = \partial\xi_n^{\pm}/\partial R_\alpha$ also vanish. Hence, $\eta_\alpha = 0$.

The second-order term in Eq. (140) is nonzero, as we will see. In the Fourier representation,

$$\int d\tau d\tau' \eta_{\alpha\beta}(\tau - \tau') R_\alpha(\tau) R_\beta(\tau') = \int \frac{d\omega}{2\pi} R_\alpha(-\omega) R_\beta(\omega) \eta_{\alpha\beta}(\omega) \tag{144}$$

The form of the Berry phase term is reproduced if we set $\eta_{\alpha\beta}(\omega) = \omega\epsilon_{\alpha\beta}\bar{\eta}$, where $\epsilon_{\alpha\beta}$ is antisymmetric tensor ($\epsilon_{xy} = -\epsilon_{yx} = 1$). Then

$$S^0_{\text{norm}}(\mathbf{R}) = S^0_{\text{norm}}(\mathbf{R}=0) - i\frac{\bar{\eta}}{2}(X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)). \tag{145}$$

Our goal therefore is to extract linear in ω and antisymmetric in α, β contribution to $\eta_{\alpha\beta}$. In τ space, $\eta_{\alpha\beta} = \omega\epsilon_{\alpha\beta}\bar{\eta}$ corresponds to $\eta_{\alpha\beta}(t) = i\epsilon_{\alpha\beta}\bar{\eta}d\delta(t)/dt$ $(t = \tau - \tau')$.

There are two contributions to $\eta_{\alpha\beta}(\tau - \tau')$. One comes from the second order term in Taylor expansion of $\hat{\xi}(\mathbf{R})$ and has the form

$$\eta^{(1)}_{\alpha\beta}(\tau - \tau') = \mathrm{Tr}\left[G_0 \frac{\partial^2\hat{\xi}}{\partial R_\alpha \partial R_\beta}\right]\delta(\tau - \tau'), \tag{146}$$

where again $G_0 = G_0(r, \tau, r, \tau)$. In Fourier representation, $\eta^{(1)}_{\alpha\beta}(\omega)$ does not depend on ω $(\eta^{(1)}_{\alpha\beta}(\omega) = \eta^{(1)}_{\alpha\beta})$ and is given by

$$\eta^{(1)}_{\alpha\beta} = \mathrm{Tr}\left[G_0 \frac{\partial^2\hat{\xi}}{\partial R_\alpha \partial R_\beta}\right] = \sum_n \int \frac{d\omega_1}{2\pi}\left[\frac{\langle\chi^+_n|\partial^2\hat{\xi}^+/\partial R_\alpha\partial R_\beta|\chi^+_n\rangle}{i\omega_1 - \xi^+_n} + \sum_n \int \frac{d\omega_1}{2\pi}\frac{\langle\chi^-_n|\partial^2\hat{\xi}^-/\partial R_\alpha\partial R_\beta|\chi^-_n\rangle}{i\omega_1 + \xi^-_n}\right]. \tag{147}$$

Evaluating the integral over ω_1, we obtain

$$\eta^{(1)}_{\alpha\beta} = \sum_n \theta(-\xi^+_n)\langle\chi^+_n|\partial^2\hat{\xi}^+/\partial R_\alpha\partial R_\beta|\chi^+_n\rangle - \sum_n \theta(-\xi^-_n)\langle\chi^-_n|\partial^2\hat{\xi}^-/\partial R_\alpha\partial R_\beta|\chi^-_n\rangle. \tag{148}$$

The second contribution to $\eta_{\alpha\beta}(\tau - \tau')$ is (we keep only τ dependence in G_0 to shorten the notations):

$$\int d\tau d\tau' \eta^{(2)}_{\alpha\beta}(\tau - \tau') = \mathrm{Tr}\left[G^s_0(\tau, \tau')\frac{\partial\hat{\xi}^s}{\partial R_\alpha(\tau)}G^s_0(\tau', \tau)\frac{\partial\hat{\xi}}{\partial R_\beta(\tau')}\right], \tag{149}$$

where $s = \pm$. In the Fourier representation,

$$\eta^{(2)}_{\alpha\beta}(\omega) = \sum_{n,m,s}\int \frac{d\omega_1}{2\pi}\frac{\langle\chi^s_n|\partial\hat{\xi}^s/\partial R_\beta|\chi^s_m\rangle\langle\chi^s_m|\partial\hat{\xi}^s/\partial R_\alpha|\chi^s_n\rangle}{(i(\omega_1 + \omega) - s\xi^s_n)(i\omega_1 - s\xi^s_m)}. \tag{150}$$

Performing the integration over ω_1, we obtain

$$\eta^{(2)}_{\alpha\beta}(\omega) = \sum_{n\neq m,s}\frac{\langle\chi^s_n|\partial\hat{\xi}^s/\partial R_\beta|\chi^s_m\rangle\langle\chi^s_m|\partial\hat{\xi}^s/\partial R_\alpha|\chi^s_n\rangle}{i\omega - s\xi^s_n + s\xi^s_m} \times\left[\theta\left(-s\xi^s_m\right) - \theta\left(-s\xi^s_n\right)\right]. \tag{151}$$

Let us now expand Eq. (151) in powers of ω,

$$\eta^{(2)}_{\alpha\beta}(\omega) = \eta^{(20)}_{\alpha\beta} + i\omega\eta^{(21)}_{\alpha\beta} + \dots, \tag{152}$$

where

$$\eta^{(20)}_{\alpha\beta} = \sum_{n\neq m,s}\frac{\langle\chi^s_n|\partial\hat{\xi}^s/\partial R_\beta|\chi^s_m\rangle\langle\chi^s_m|\partial\hat{\xi}^s/\partial R_\alpha|\chi^s_n\rangle}{s\xi^s_m - s\xi^s_n} \times\left[\theta\left(-s\xi^s_m\right) - \theta\left(-s\xi^s_n\right)\right],$$

$$\eta^{(21)}_{\alpha\beta} = -\sum_{n\neq m,s}\frac{\langle\chi^s_n|\partial\hat{\xi}^s/\partial R_\beta|\chi^s_m\rangle\langle\chi^s_m|\partial\hat{\xi}^s/\partial R_\alpha|\chi^s_n\rangle}{\left(\xi^s_m - \xi^s_n\right)^2} \times\left[\theta\left(-s\xi^s_m\right) - \theta\left(-s\xi^s_n\right)\right]. \tag{153}$$

We will need the antisymmetric part of $\eta^{(21)}_{\alpha\beta}$ term. Before evaluating it, we pause for a moment and show that the frequency independent term $\eta^{(20)}_{\alpha\beta}$ cancels $\eta^{(1)}_{\alpha\beta}$ from Eq. (148).

This follows from the identity

$$\langle\chi^s_n|\partial\hat{\xi}^s/\partial R_\beta|\chi^s_m\rangle = \left(\xi^s_n - \xi^s_m\right)\langle\chi^s_n|\partial/\partial R_\beta|\chi^s_m\rangle, \tag{154}$$

which is obtained from the condition $\partial/\partial R_\beta\langle\chi^s_n|\hat{\xi}^s|\chi^s_m\rangle = \partial/\partial R_\beta(\xi^s_m\langle\chi^s_n|\chi^s_m\rangle) = 0$, when $n \neq m$, by differentiating each term in $\partial/\partial R_\beta\langle\chi^s_n|\hat{\xi}^s|\chi^s_m\rangle$ over R_β. Substituting this identity into (153) and using the completeness relation $\sum_n|\chi^s_n\rangle\langle\chi^s_n| = 1$, we obtain for $\eta^{(20)}_{\alpha\beta}$ the same expression as in (148), but with the opposite sign.

We now return to $\eta^{(21)}_{\alpha\beta}$. Using the same identity and the completeness relation, we re-write $\eta^{(21)}_{\alpha\beta}$ in Eq. (153) as

$$\eta^{(21)}_{\alpha\beta} = \sum_n \theta(-\xi^+_n)\langle\chi^+_n|\frac{\partial}{\partial R_\alpha}\frac{\partial}{\partial R_\beta} - \frac{\partial}{\partial R_\beta}\frac{\partial}{\partial R_\alpha}|\chi^+_n\rangle - \sum_n \theta(-\xi^-_n)\langle\chi^-_n|\frac{\partial}{\partial R_\alpha}\frac{\partial}{\partial R_\beta} - \frac{\partial}{\partial R_\beta}\frac{\partial}{\partial R_\alpha}|\chi^-_n\rangle. \tag{155}$$

To evaluate the RHS of Eq. (155), we use the explicit form of $|\chi^s_n\rangle$. These are the eigenstates of $\hat{\xi}^s = -(\nabla + (is/2)\nabla\phi)^2/(2m) - \mu$. In polar coordinates r and ϕ, $\hat{\xi}^s$ can be written as

$$\hat{\xi}^s = -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\phi} + \frac{is}{2}\right)^2\right] - \mu. \tag{156}$$

The complete set of the eigenstates of (156) is

$$\chi^\pm_n \equiv \chi^\pm_{\nu,k}(r, \phi) = \sqrt{\frac{k}{2D}}e^{i\nu\phi}J_{|\nu\pm1/2|}(kr), \tag{157}$$

where $\nu = \pm1/2, \pm3/2, \dots$, and k is a quasicontinuous radial wave number quantized as $\Delta k = \pi/D$, where, we remind, $2D$ is the system size (the quantization of k originates from the boundary condition $J_{|\nu\pm1/2|}(kD) = 0$ at $kD \gg 1$).

Substituting these eigenstates and eigenvalues into Eq. (155) and using the fact that for $\phi(r - R)$ given by (109),

$$\left(\frac{\partial}{\partial R_x}\frac{\partial}{\partial R_y} - \frac{\partial}{\partial R_y}\frac{\partial}{\partial R_x}\right)\phi = 2\pi\delta(\mathbf{r} - \mathbf{R}), \tag{158}$$

we obtain after straightforward algebra $\eta^{(21)}_{\alpha\beta} = \epsilon_{\alpha\beta}\bar{\eta}$, where

$$\bar{\eta} = -2\pi\sum_{k,\nu=n+1/2}\theta\left(\mu - \frac{k^2}{2m}\right)\left(\frac{k}{2D}\right) \times\nu\left(J^2_{|\nu+1/2|}(0) - J^2_{|\nu-1/2|}(0)\right). \tag{159}$$

Because $J_n(0) = 0$ for integer $n > 0$ and $J_0(0) = 1$, the sum over ν gives -1. The summation over k gives, at $D \to \infty$,

$$\frac{\pi}{D}\sum_k k\theta\left(\mu - \frac{k^2}{2m}\right) = \int_0^\infty dkk\theta\left(\mu - \frac{k^2}{2m}\right) = m\mu = \pi n_0 \tag{160}$$

such that

$$\bar{\eta} = \pi n_0 \,. \tag{161}$$

Substituting into (145), we obtain the additional contribution to the Berry phase action of a moving vortex

$$S_{\text{Berry},2} = S^0_{\text{norm}}(\mathbf{R}) - S^0_{\text{norm}}(\mathbf{R}=0) = i\pi A_{\text{vort},2}(X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)) \,, \tag{162}$$

where

$$A_{\text{vort},2} = -\frac{n_0}{2}. \tag{163}$$

We emphasize that this term comes from the states right at the vortex core [see (158)] and, in this respect, is very different from $A_{\text{vort},1}$, which comes from the states far away from the vortex core.

Effects of impurities

We expect the result $A_{\text{vort}} = -(n_0)/2$ to hold when the impurity potential is included. Indeed, Eq. (155) is valid when the impurity potential (U_{imp}) is present and the eigenfunctions $\chi_n^{\pm}\rangle$ can still be expressed as

$$\chi_n^{\pm}(\mathbf{r}) = e^{\pm i\phi(\mathbf{r}-\mathbf{R})/2}a_n(\mathbf{r}) \,, \tag{164}$$

where $a(\mathbf{r})$ satisfies the Schrodinger equation

$$[-\nabla^2/(2m) + U_{\text{imp}}((\mathbf{r}))]a_n(\mathbf{r}) = \xi_n a_n(\mathbf{r}). \tag{165}$$

Substituting this $\chi_n^{\pm}(\mathbf{r})$ into Eq. (155), we obtain

$$\eta^{(21)}_{\alpha\beta} = \pi\sum_n \theta(-\xi_n)\,|a_n(\mathbf{R})|^2 + \pi\sum_n \theta(-\xi_n)\,|a_n(\mathbf{R})|^2 = \pi n_0(\mathbf{R}), \tag{166}$$

where $n_0(\mathbf{R})$ is the fermion density at the vortex core, i.e., the same result as in the absence of impurity potential.

We caution, however, that the full analysis of a vortex flow in the presence of impurities is rather nontrivial (see, e.g., Ref. [51]) and may not be fully captured by treating the impurities with the τ approximation.

D. The total Berry phase term for a moving vortex

Combining the two contributions to S_{Berry}, we obtain the total Berry phase term for a moving vortex

$$S^{\text{vort}}_{\text{Berry}} = S_{\text{Berry},1} + S_{\text{Berry},2} = i\pi A_{\text{vort}}(X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau)), \tag{167}$$

where

$$A_{\text{vort}} = \frac{n - n_0}{2}. \tag{168}$$

We remind that $n - n_0 = 2N_0E_0$ when $E_0 < E_F$, and $n - n_0 = 2N_0E_F$ when $E_0 > E_F$, where $2E_0$ is the bound state energy of two fermions in a vacuum. The first limit corresponds to BCS, the second one to the BEC. More specifically, $n_0 = 0$ when $E_0 > E_F$, hence in this situation the prefactor in the Berry phase term in the action becomes just $i\pi n/2$. The vanishing of n_0 once μ becomes negative is consistent with the generic reasoning in Ref. [37] that free-fermion contribution to A_{vort} vanishes once the system undergoes a (fictitious) Lifshitz transition, in which the (fictitious) Fermi surface of free fermions with renormalized μ disappears. In our case, this happens once E_0 becomes larger than E_F.

We also note that the two contributions to S_{Berry} from $\Delta \to 0$—one given by Eq. (162) and the other by Eq. (137), are equal in magnitude, but differ in sign. As a result, the combined total contribution from the action at $\Delta \to 0$ vanishes. As the consequence, and the total Berry phase term in the action of a moving vortex is the same as in Eq. (136), obtained by expanding in Δ.

We argued above that $A_{\text{vort},2} = -n_0/2$ is not affected by impurities, i.e., the reaction force remains the same in the presence of imputity potential. By the same reason, Eq. (38) and the subsequent consideration in Sec. IV for the Magnus force also remains valid when the impurity potential is present. This is consistent with argument made by Ao and Thouless [25] that impurity scattering should not modify the value of the Magnus force. As the consequence, we expect $A_{\text{vort}} = (n - n_0)/2$ to hold when impurity scattering is present.

The interpretation of Eq. (167)

As we said, there are two contributions to the prefactor A_{vort} for the Berry phase term in the effective action for a moving vortex: one, $A_{\text{vorx},1}$, comes from states far away from a vortex core, and the other, $A_{\text{vort},2}$, comes from the states at the vortex core. Looking back at our derivation of $A_{\text{vort},1}$, we see that this term has two contributions: one comes from the first term in the action in Eq. (15), another comes from $\dot{\phi}$ piece in last term, S_{norm}, which is the normal state contribution to the action (more accurately, the contribution from $\Delta \to 0$). On more careful look, we note that there are in fact two contributions from $\Delta \to 0$ in $A_{\text{vort},1}$: the one from S_{norm} and the one from the lower limit in the integral $\int_0^1 d\lambda$ in the first term in (15) (the lower limit $\lambda = 0$ corresponds to $\Delta \to 0$). We separated the two contributions for convenience of the derivation and to show explicitly how parasitic ultraviolet divergent term $\sum_k(1)$ cancels out between the contributions from $\lambda = 1$ and 0 [see Eqs. (130) and (135)]. If we were to combine from the start the $\dot{\phi}$ piece in S_{norm} and the contribution from $\lambda = 0$ in the first term in (15), we would obtain that the n_0 terms cancel each other (one is $-n_0/2$, another is $n_0/2$). The cancellation implies that there is no contribution from $\Delta \to 0$ to the Berry phase term besides the counter-term to cancel the ultraviolet divergence. The full $A_{\text{vort},1} = n/2$ comes exclusively from the limit $\lambda = 1$ in the first term in (15), which describes the action at $\Delta(\mathbf{r}, \tau, 1) = \Delta(\mathbf{r}, \tau)$, where $\Delta(\mathbf{r}, \tau)$

is the actual gap function at distance $\mathbf{r}$ from a vortex. We recall that this contribution comes from fermionic states far away from a vortex core, where the gap amplitude approaches equilibrium value Δ_0. Obviously then, $A_{\text{vort},1} = n/2$ is the same as the prefactor for $\dot{\phi}$ term in the effective action for the case when $\phi(\mathbf{r}, \tau)$ is a regular function of its arguments.

The second contribution $A_{\text{vort},2}$ is additional contribution from the vortex core, i.e., from $\mathbf{r} = 0$. In general, low-energy fermionic states near the core, the ones with energies below Δ_0, are discrete levels with separation $\omega_0 \sim \Delta_0^2/E_F$ (Refs. [34,41,42]). We found no contribution from discrete levels in the vortex core from the part of the action with the actual $\Delta(\mathbf{r}, \tau)$ [this would be a contribution from the upper limit $\lambda = 1$ in the first term in (15)]. Our $A_{\text{vort},2}$ comes from the $\nabla\phi$ piece in $S_{\text{norm}} = \text{Tr}\ln(G_0^{-1})$, which is the contribution to the action from $\Delta \to 0$. In this limit, the spacing between discrete levels $\omega_0 \sim \Delta_0^2/E_F$ vanishes, and electronic states in the vortex core are not quantized and are described by a continuous variable k.

The Berry phase term in the effective action has been analyzed earlier. Several authors [26,34,37,39,40] argued on general grounds that quantization of fermionic states inside the vortex core can be neglected in the hydrodynamic limit $\omega_0\tau \ll 1$, where τ is fermionic lifetime. In this limit, earlier works [26,34,37,39,40] found the same $A_{\text{vort}} = (n - n_0)/2$ as in Eq. (168). In our consideration, $\omega_0 = 0$ and $\tau = \infty$, so $\omega_0\tau$ is not well determined. Still, our $A_{\text{vort},2}$ comes from continuous (i.e., nonquantized) states, and it remains the same in the presence of impurities, i.e., at a finite τ. In this respect, we believe that the agreement between our A_{vort} and the one obtained in earlier works at $\omega_0\tau \ll 1$ is meaningful.

There is, however, one aspect in which our result seems to differ from earlier works. Namely, these works speculated [26,34,37,39,40] that at finite $\omega_0\tau$ there should be a contribution to A_{vort} from discrete levels in the vortex core. These and several other authors have argued [29,31,34,37,39,40] that in the limit $\omega\tau \gg 1$, the total contribution from the vortex core $A_{\text{vort},2}$ should vanish, i.e., the total A_{vort} should reduce to $A_{\text{vort},1} = n/2$. We did not find in our microscopic approach the contribution to A_{vort} from discrete levels in the vortex core in the term in the action with a finite $\Delta(\mathbf{r}, \tau)$. It remains to be seen whether such contribution can be obtained by going beyond the approximations we made in our derivation of the effective action.

E. External superflow and the equation for the balance of forces

In the presence of an external supercurrent, the phase of the order parameter in Eq. (108) acquires an additional term, $2m\mathbf{v}_s\mathbf{r}$. The effect of this term on the action can be analyzed perturbatively if $\mathbf{v}_s$ is small. Performing the same gauge transformation as we used to move from Eq. (43) to Eq. (44), but for nonzero $\mathbf{v}_s$, we obtain the additional $-i\mathbf{v}_s \cdot \nabla$ term in the lhs of Eq. (44) (Ref. [52]). Evaluating now the correction to the ground state energy within the first order perturbation theory, we obtain

$$\delta E(v_s) = -i\sum_n \mathbf{v}_s \int d^2\mathbf{r}\langle\chi_n(\tau,\mathbf{r})|\nabla\chi_n(\tau,\mathbf{r})\rangle\theta(-E_n), \tag{169}$$

where $|\chi_n(\tau,\mathbf{r})\rangle$ are the solutions to the BdG equations in the presence of a vortex, but without $\mathbf{v}_s$, see Eq. (110). From (169), we then obtain the extra term in the action of the vortex:

$$\mathcal{S}_{v_s} = -i\int_{-\infty}^{\infty} d\tau \sum_n \int d^2\mathbf{r}\,\mathbf{v}_s\big[\langle\chi_n(\tau,\mathbf{r})|\nabla\chi_n(\tau,\mathbf{r})\rangle - \big\langle\chi_n^{|\Delta|\to 0}(\tau,\mathbf{r})\big|\nabla\chi_n^{|\Delta|\to 0}(\tau,\mathbf{r})\big\rangle\big]\theta(-E_n). \tag{170}$$

The sum and the matrix elements in the RHS of Eq. (170) are identical to those in Eq. (38), hence

$$\mathcal{S}_{v_s} = -i\pi B_{\text{vort}} \int d\tau (X(\tau)v_{sy} - Y(\tau)v_{sx}), \tag{171}$$

where $B_{\text{vort}} = A$, with A given in Eq. (136), e.g., $A = (n - n_0)/2$ in the BCS regime, and $A = n/2$ in the BEC regime.

As we have seen in the previous subsection, there exist another contribution to the force acting on a vortex associated with the vortex core. Similarly, one might expect that there is another contribution to the action in Eq. (171), i.e., to the constant B_{vort} that comes from the core. As we show below, this is not the case, we find that the normal part of the action, S_{norm} does not contribute to B_{vort}.

To see this, let us look at S_{norm} in Eqs. (41), (42) in the presence of extra phase superfluid velocity. We need to replace in this equation $\nabla\phi$ by $\nabla\phi + 2m\mathbf{v}_s$. This gives an extra term under the Tr ln … (Ref. [53]),

$$\hat{V}_s = -i\mathbf{v}_s\left(\nabla + \frac{i}{2}\sigma_z\nabla\phi\right). \tag{172}$$

Treating this extra term perturbatively, we expand the $\text{Tr}\ln(\dots)$ to the first order in v_s. The term $\text{Tr}(G_0\hat{V}_s)$ does not contribute because the expectation values of ∇ and of $i\sigma_z\nabla\phi/2$ in Eq. (172) cancel each other. The next-order term $\text{Tr}[G_0\hat{V}_sG_0(\partial\hat{\xi}/\partial R_\beta)R_\beta]$ is apparently relevant as it contains the same combination $v_{s\alpha}R_\beta$ as in (171). Using the same manipulations as in the previous section, we can write

$$\text{Tr}[G_0\hat{V}_sG_0(\partial\hat{\xi}/\partial R_\alpha)R_\alpha] = -i\int d\tau\, B_{\text{core}}^{\alpha,\beta}\, v_{s\alpha}R_\beta(\tau), \tag{173}$$

where

$$B_{\text{core}}^{\alpha,\beta} = \sum_{n\neq m,s} \frac{\langle\chi_n^s|[\nabla_\alpha + (is/2)(\nabla_\alpha\phi)]|\chi_m^s\rangle\langle\chi_m^s|\partial\hat{\xi}^s/\partial R_\beta|\chi_n^s\rangle}{\xi_n^s - \xi_n^s} \times\big[\theta\big(-\xi_m^s\big) - \theta\big(-\xi_n^s\big)\big] = \sum_{n,s}\big\langle\chi_n^s\big|\left[\nabla_\alpha + \frac{is}{2}(\nabla_\alpha\phi),\ \frac{\partial}{\partial R_\beta}\right]\big|\chi_n^s\big\rangle\theta\big(-\xi_n^s\big). \tag{174}$$

The term $(is/2)(\nabla_\alpha\phi)$ in Eq. (174) doe not contribute: Since $[(\nabla_\alpha\phi), \partial/\partial R_\beta] = -\partial^2\phi/\partial_\alpha\partial_\beta$, it cancels a correction to $\text{Tr}[G_0 i\sigma_z\mathbf{v}_s\nabla\phi/2]$ that arises when we expand $\nabla\phi$ as $\nabla_\alpha\phi(0) + \nabla_\alpha\nabla_\beta\phi(0)R_\beta + \dots$, i.e., the $\text{Tr}[G_0 i\sigma_z v_{s\alpha}\nabla_\alpha\nabla_\beta\phi(0)R_\beta/2]$ term. This is similar to cancellation between Eqs. (148) and (153) in the previous section. For the remaining ∇_α piece in Eq. (174), we obtain

$$B_{\text{core}}^{\alpha,\beta} = \sum_n \theta(-\xi_n^+)\,\langle\chi_n^+|\frac{\partial}{\partial R_\alpha}\frac{\partial}{\partial R_\beta} - \frac{\partial}{\partial R_\beta}\frac{\partial}{\partial R_\alpha}|\chi_n^+\rangle + \sum_n \theta(-\xi_n^-)\,\langle\chi_n^-|\frac{\partial}{\partial R_\alpha}\frac{\partial}{\partial R_\beta} - \frac{\partial}{\partial R_\beta}\frac{\partial}{\partial R_\alpha}|\chi_n^-\rangle\,. \tag{175}$$

The expression for $B_{\rm core}$ in Eq. (175) is similar to the formula for $\eta^{(21)}_{\alpha\beta}$ in Eq. (155), but with one important distinction: the relative sign between the two terms in (175) is plus, while in (155) it is minus. As a result, performing the same calculations as in Eqs. (156)–(159), we obtain

$$B^{\alpha,\beta}_{\rm core} = 2\pi \sum_{k,\nu=n+1/2} \theta\left(\mu - \frac{k^2}{2m}\right)\left(\frac{k}{2D}\right) \times \nu\left(J^2_{|\nu+1/2|}(0) + J^2_{|\nu-1/2|}(0)\right) = 4\pi \sum_{k,n} \theta\left(\mu - \frac{k^2}{2m}\right)\left(\frac{k}{2D}\right) n\, J^2_{|n|}(0). \quad (176)$$

Because the product $nJ^2_{|n|}(0)$ is zero for any integer n, $B^{\alpha,\beta}_{\rm core} = 0$. Hence, $S_{\rm norm}$ does not contribute to $B_{\rm vort}$ in Eq. (171).

Equation (171), together with Eq. (125), determines the balance of forces acting on a vortex. Converting from Matsubata to real time, we obtain

$$A_{\rm vort}\dot{\mathbf{R}} \times \mathbf{z} - B_{\rm vort}\mathbf{v}_s \times \mathbf{z} = 0\,, \quad (177)$$

where $\mathbf{z}$ is a unit vector perpendicular to the 2d plane. We see from Eq. (177) that

$$v_{\rm vort} = \dot{\mathbf{R}} = \frac{B_{\rm vort}}{A_{\rm vort}}\mathbf{v}_s. \quad (178)$$

Because $A_{\rm vort} = B_{\rm vort} = (n - n_0)/2$, we have

$$\mathbf{v}_{\rm vort} = \mathbf{v}_s. \quad (179)$$

This agrees with the reasoning based on translational invariance [29,31,37].

V. SUMMARY

In this paper, we analyzed the evolution of the $T = 0$ low-frequency dynamics of collective excitations of an s-wave neutral superconductor between the BCS and BEC regimes. The two regimes correspond to small and large ratio of E_0/E_F, respectively, where E_F is the Fermi energy, and E_0 is the bound state energy for two particles. In $d = 2$, bound state develops already at weak coupling, what allows one to analyze the crossover within a controllable weak coupling expansion. We obtained the terms in the long-wavelength action, proportional to $(\nabla\phi)^2$ and $\dot{\phi}^2$, where $\phi(\mathbf{r}, t)$ is the phase of the superconducting order parameter $\Delta(\mathbf{r}, t) = \Delta e^{i\phi(\mathbf{r},t)}$. We found that the phase velocity of the collective excitations remains $v_F/\sqrt{2}$ through the BCS-BEC crossover. We also obtained the topological Berry phase term in the long-wavelength action $i\int d\tau A\dot{\phi}$. We found that the prefactor $A = n/2$, where n is the actual fermion density, and does not change through the BCS-BEC crossover.

The Berry phase term in the action is meaningful when the phase of the superconducting order parameter is not defined globally, which is the case when the pairing gap vanishes at some point is space, like in the vortex core. We computed the effective action for a moving vortex in a neutral s-wave superconductor in 2d. The Berry phase term for a moving vortex has the form $S_{\rm Berry} = i\pi A_{\rm vort}\int d\tau(X(\tau)\dot{Y}(\tau) - Y(\tau)\dot{X}(\tau))$, where $X(\tau)$ and $Y(\tau)$ are coordinates of the center of a moving vortex. We found that two contributions to the prefactor $A_{\rm vort} = A_{\rm vort,1} + A_{\rm vort,2}$. One comes from fermionic states far away from the vortex core and is the same as in the long-wavelength action: $A_{\rm vort,1} = n/2$. Another comes from delocalized (continuous) fermionic states inside the vortex core. For this second contribution, we obtained $A_{\rm vort,2} = -n_0/2$, where n_0 is the fermionic density at the vortex core (it is equal to the fermionic density in the normal state, but for the same chemical potential μ as in the superconducting state). In physical terms, the long-wavelength contribution $n/2$ represents a Magnus force acting on a moving vortex, while the offset term $-n_0/2$ represents a reaction force from normal fermions at the vortex core. The total $A_{\rm vort} = (n - n_0)/2$. In the BCS limit, $n - n_0 \ll n$, i.e., the total transverse Lorentz-like force acting on the vortex is much smaller than the Magnus force. In the BEC limit, $n_0 = 0$ because the effective $\mu < 0$, and there are no normal state fermions at the vortex core. Then one recovers the result that the total transverse force equals to the Magnus force. We argued that the result for $A_{\rm vort}$ remains valid in the presence of impurity scattering. Finally, we found that in the presence of an external superflow the vortex dynamics obeys Galilean (translational) invariance principle: the vortices move together with the superflow.

The result $A_{\rm vort} = (n - n_0)/2$ agrees with earlier works [34,37,39,40], which obtained this $A_{\rm vort}$ neglecting the quantization of fermionic states in the vortex core. In agreement with these results, we found that in our approach the contribution from the states near the vortex core comes only from the part of the action at $\Delta \to 0$, when the spacing between the states in the vortex core vanishes, and the low-energy states become continuous. Earlier works [26,29,31,34,37,39] speculated that there should be another contribution to $A_{\rm vort}$ from discrete states in the vortex core. We did not find such contribution in our analysis of the effective action. This term may emerge once one moves beyond our approach, based on the evaluation of the effective action for the vortex motion.

Our results for the expansion of the effective action in terms of time derivatives of slowly varying order parameter [Eqs. (32), (38), and (39)] can be straightforwardly extended to other symmetries of the order parameter and to non-Galilean-invariant dispersion, as long as adiabatic approximation is applicable. We note in this regard that the topological term in the action plays a special role in superconductors with the nodes in the order parameter, e.g., it determines the magnitude of the orbital momentum in the A phase of a p-wave superconductor, like ^{3}He-A (Ref. [54]). The terms of higher orders in powers of $\partial_\tau\Delta$ or in higher derivatives of Δ can be obtained from Eqs. (17) and (18), though in practice such calculation is likely to be rather cumbersome.

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APPENDIX: DERIVATION OF THE ACTION TO SECOND ORDER IN TIME DERIVATIVE

Substituting Eq. (28) into Eq. (20), we get two contributions,

$$L^{(2)} = L_1^{(2)} + L_2^{(2)} \equiv \int_0^1 d\lambda \big[l_1^{(2)}(\lambda,\tau) + l_2^{(2)}(\lambda,\tau) \big], \tag{A1}$$

each generated by first and second terms in the RHS of Eq. (28), respectively:

$$l_1^{(2)}(\lambda,\tau) = \int \frac{d\omega}{2\pi} e^{i\omega\epsilon^+} \sum_{n,l} \frac{\langle \chi_{l,\lambda}|\partial_\lambda \hat{\Delta}|\chi_{n,\lambda}\rangle\langle\chi_{n,\lambda}|\partial_\tau^2\hat{\Delta}(\tau)|\chi_{l,\lambda}\rangle}{(i\omega - E_{n,\lambda})^3(i\omega - E_{l,\lambda})}\,, \tag{A2}$$

$$l_2^{(2)}(\lambda,\tau) = \int \frac{d\omega}{2\pi} e^{i\omega\epsilon^+} \sum_{n,m,l} \frac{\langle \chi_{l,\lambda}|\partial_\lambda \hat{\Delta}|\chi_{m,\lambda}\rangle\langle\chi_{m,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle\langle\chi_{n,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\chi_{l,\lambda}\rangle}{(i\omega - E_{n,\lambda})(i\omega - E_{l,\lambda})^2(i\omega - E_{m,\lambda})^2}\,. \tag{A3}$$

The ω integration of Eq. (A2) gives

$$l_1^{(2)}(\lambda,\tau) = \sum_{n,l} \frac{\langle \chi_{l,\lambda}|\partial_\lambda \hat{\Delta}|\chi_{n,\lambda}\rangle\langle\chi_{n,\lambda}|\partial_\tau^2\hat{\Delta}(\tau)|\chi_{l,\lambda}\rangle(\theta_n - \theta_l)}{(E_{n,\lambda} - E_{l,\lambda})^3} = \sum_{n,l} \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\chi_{n,\lambda}|\partial_\tau^2\hat{H}|\chi_{l,\lambda}\rangle(\theta_n - \theta_l)}{(E_{n,\lambda} - E_{l,\lambda})^2}\,, \tag{A4}$$

where in the second equality we have used Eq. (34).

The second integral, e.g., Eq. (A2) is less straightforward. Let us exclude terms with repeated indices from the triple sum in Eq. (A3). Then the integrand has three poles and, after some algebra, we find (the situation when two indices coincide will be considered separately below),

$$\begin{aligned} l_2^{(2)}(\lambda,\tau) = &\sum_{n,m,l} \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle(E_{l,\lambda} - E_{m,\lambda})\theta_n}{(E_{n,\lambda} - E_{l,\lambda})(E_{m,\lambda} - E_{n,\lambda})} \\ &+ \sum_{n,m,l} \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle(E_{n,\lambda} - E_{l,\lambda})(2E_{n,\lambda} - 3E_{m,\lambda} + E_{l,\lambda})\theta_m}{(E_{n,\lambda} - E_{m,\lambda})(E_{m,\lambda} - E_{l,\lambda})^2} \\ &+ \sum_{n,m,l} \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle(E_{m,\lambda} - E_{n,\lambda})(2E_{n,\lambda} - 3E_{l,\lambda} + E_{m,\lambda})\theta_l}{(E_{n,\lambda} - E_{l,\lambda})(E_{m,\lambda} - E_{l,\lambda})^2}, \end{aligned} \tag{A5}$$

where we again used Eqs. (34) and (35).

Next we transform Eq. (A4) by using

$$\langle \chi_{l,\lambda}|\partial_\tau^2\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle = \partial_\tau\langle\chi_{l,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle - \langle\partial_\tau\chi_{l,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle - \langle\chi_{l,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\partial_\tau\chi_{n,\lambda}\rangle\,,$$

and integrating by parts the term containing $\partial_\tau\langle\chi_{l,\lambda}|\partial_\tau\hat{\Delta}(\tau)|\chi_{n,\lambda}\rangle$ [we recall that $l_1^{(2)}$ is under τ integration when substituted in Eqs. (15) and (19)]:

$$\begin{aligned} l_1^{(2)}(\lambda,\tau) = \sum_{n,l} \Bigg\{ &2\frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\partial_\tau(E_{n,\lambda} - E_{l,\lambda})}{(E_{n,\lambda} - E_{l,\lambda})^2} - \frac{\langle \partial_\lambda\partial_\tau\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle}{E_{n,\lambda} - E_{l,\lambda}} - \frac{\langle \partial_\lambda\chi_{l,\lambda}|\partial_\tau\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle}{E_{n,\lambda} - E_{l,\lambda}} \\ &- \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\partial_\tau\hat{H}|\chi_{l,\lambda}\rangle}{(E_{n,\lambda} - E_{l,\lambda})^2} - \frac{\langle \partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\chi_{n,\lambda}|\partial_\tau\hat{H}|\partial_\tau\chi_{l,\lambda}\rangle}{(E_{n,\lambda} - E_{l,\lambda})^2} \Bigg\}(\theta_n - \theta_l)\,. \end{aligned} \tag{A6}$$

The last three terms in the of Eq. (A6) can be combined with Eq. (A5) when we insert $\sum_m |\chi_{m,\lambda}\rangle\langle\chi_{m,\lambda}|$ in the matrix elements in the terms that contain double derivatives (with respect to τ in the fourth and fifth terms and with respect λ and τ in the third term). In doing so we treat separately the terms with $m \neq n$ and $m \neq l$ and the terms with $m = n$ and $m = l$. Using the relation

$$\langle \chi_{m,\lambda}|\partial_\tau\hat{H}|\chi_{m,\lambda}\rangle = \partial_\tau\langle\chi_{m,\lambda}|\hat{H}|\chi_{m,\lambda}\rangle = \partial_\tau E_{m,\lambda}\,, \tag{A7}$$

for the diagonal matrix elements, as well as $\langle\partial_\tau\chi_{m,\lambda}|\chi_{m,\lambda}\rangle = -\langle\chi_{m,\lambda}|\partial_\tau\chi_{m,\lambda}\rangle$, we obtain after some algebra

$$\begin{aligned}[\text{last three terms of Eq. (A6)}] = &\sum_{n,m,l}\langle\partial_\lambda\chi_{l,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\left[\frac{\theta_n-\theta_l}{(E_{n,\lambda}-E_{l,\lambda})}-\frac{(2E_{n,\lambda}-E_{l,\lambda}-E_{m,\lambda})(\theta_m-\theta_l)}{(E_{m,\lambda}-E_{l,\lambda})^2}\right]\\ &+\sum_{n,l}\frac{\langle\partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\partial_\tau(E_{n,\lambda}-E_{l,\lambda})(\theta_n-\theta_l)}{(E_{n,\lambda}-E_{l,\lambda})^2}\\ &-\sum_{n,l}\frac{\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\langle\partial_\tau\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\lambda\chi_{l,\lambda}|\chi_{l,\lambda}\rangle+\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\langle\partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{l,\lambda}|\chi_{l,\lambda}\rangle}{E_{n,\lambda}-E_{l,\lambda}}\\ &\times(\theta_n-\theta_l).\end{aligned}\tag{A8}$$

In the triple sum, i.e., in the first term in the RHS of Eq. (A8), $m\neq n\neq l$, while the last term in Eq. (A8) arises due to $m=n$ and $m=l$ terms. Adding this triple sum with $l_2^{(2)}(\lambda,\tau)$ from Eq. (A5), we obtain

$$\sum_{n,m,l}\frac{\langle\partial_\lambda\chi_{l,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)\,.\tag{A9}$$

The triple sum in Eq. (A9) can be transformed into the double sum by using a completeness relation, $\sum_l|\chi_{l,\lambda}\rangle\langle\chi_{l,\lambda}|=1$,

$$\begin{aligned}&-\sum_{n,m}\frac{\langle\partial_\tau\chi_{n,\lambda}|\partial_\lambda\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)+\sum_{n,m}\frac{\langle\chi_{m,\lambda}|\partial_\lambda\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)\\ &+\sum_{n,m}\frac{\langle\chi_{n,\lambda}|\partial_\lambda\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{n,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)\,,\end{aligned}\tag{A10}$$

where the last two terms in Eq. (A10) correspond to $l=n$ and $l=m$ terms, which are omitted in the triple sum in Eq. (A9). The last two terms in Eq. (A10) cancel the last term in Eq. (A8). Using this, we finally obtain that

$$\begin{aligned}[\text{last three terms of Eq. (A6)}]+l_2^{(2)} = &-\sum_{n,m}\frac{\langle\partial_\tau\chi_{n,\lambda}|\partial_\lambda\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)\\ &+\sum_{n,l}\frac{\langle\partial_\lambda\chi_{l,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{l,\lambda}\rangle\partial_\tau(E_{n,\lambda}-E_{l,\lambda})(\theta_n-\theta_l)}{(E_{n,\lambda}-E_{l,\lambda})^2}.\end{aligned}\tag{A11}$$

The first term in the RHS of Eq. (A11) can be combined with the second term in Eq. (A6) as

$$\begin{aligned}&-\sum_{n,m}\frac{\langle\partial_\lambda\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle+\langle\partial_\tau\chi_{n,\lambda}|\partial_\lambda\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)\\ &=-\sum_{n,m}\frac{(\partial_\lambda\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle)\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n)=-\frac{1}{2}\sum_{n,m}\frac{\partial_\lambda(\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle)}{E_{m,\lambda}-E_{n,\lambda}}(\theta_m-\theta_n).\end{aligned}\tag{A12}$$

Using this, we find that

$$l_1^{(2)}+l_{2\,(m\neq n\neq l)}^{(2)}=\sum_{n,m}\left\{\frac{1}{2}\frac{\partial_\lambda(\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle)}{E_{m,\lambda}-E_{n,\lambda}}+3\frac{\langle\partial_\lambda\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle\partial_\tau(E_{n,\lambda}-E_{m,\lambda})}{(E_{n,\lambda}-E_{m,\lambda})^2}\right\}(\theta_n-\theta_m).\tag{A13}$$

Equation (A13) only accounts for $m\neq n\neq l$ terms in Eq. (A3) and should be added with $l=m\neq n$, $m=n\neq l$, and $n=l\neq m$ terms. To consider these terms we need to return to the evaluation of the ω integral in Eq. (A3). For $l=m\neq n$ terms, we obtain, using Eq. (31),

$$l_{2\,(m=l)}^{(2)}=\sum_{n,m}\frac{(\partial_\lambda E_{m,\lambda})\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle(\theta_m-\theta_n)}{(E_{m,\lambda}-E_{n,\lambda})^2}=\frac{1}{2}\sum_{n,m}\frac{\partial_\lambda(E_{m,\lambda}-E_{n,\lambda})\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle(\theta_m-\theta_n)}{(E_{m,\lambda}-E_{n,\lambda})^2}.\tag{A14}$$

Now $l_{2\,(m=l)}^{(2)}$ in Eq. (A14) nicely combines with the first term in Eq. (A13) to give the full derivative,

$$\frac{1}{2}\sum_{n\neq m}\frac{\partial}{\partial\lambda}\left[\frac{\langle\partial_\tau\chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau\chi_{n,\lambda}|\chi_{m,\lambda}\rangle}{(E_{m,\lambda}-E_{n,\lambda})}\right](\theta_m-\theta_n)\,.\tag{A15}$$

The $m = n \neq l$ and $n = l \neq m$ cases are considered similarly, by reevaluating ω integrals in Eq. (A3). It is easy to see that these two contributions cancel the last term in Eq. (A13) and we finally obtain

$$L^{(2)} = \frac{1}{2}\sum_{n\neq m}\int_0^1 d\lambda \frac{\partial}{\partial\lambda}\left[\frac{\langle\partial_\tau \chi_{m,\lambda}|\chi_{n,\lambda}\rangle\langle\partial_\tau \chi_{n,\lambda}|\chi_{m,\lambda}\rangle}{(E_{m,\lambda} - E_{n,\lambda})}\right](\theta_m - \theta_n)\,. \tag{A16}$$

Because the integrand is a full derivative over λ, the value of the integral is the difference of this function at the end points, at $\lambda = 1$ and at $\lambda = 0$. Using this, we arrive at Eq. (39).

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