Hamiltonian formalism for nonlinear spin wave dynamics under antisymmetric interactions: Application to Dzyaloshinskii-Moriya interaction

Roman Verba,[1] Vasil Tiberkevich,[2] and Andrei Slavin[2]
[1]*Institute of Magnetism, Kyiv 03142, Ukraine*
[2]*Department of Physics, Oakland University, Rochester, Michigan 48309, USA*

(Received 21 March 2019; published 28 May 2019)

A Hamiltonian formalism is applied for the investigation of nonlinear spin wave dynamics under the influence of antisymmetric magnetic interactions. In the framework of this formalism we account not only for symmetric magnetic interactions (exchange, dipole-dipole, magnetocrystalline anisotropy), but also for antisymmetric interactions, such as Dzyaloshinskii-Moriya exchange interaction. The account of antisymmetric exchange, in general, could lead to the appearance of an additive nonreciprocal term in the spin wave dispersion law. We present the generalization of the linear transformation for the diagonalization of quadratic part of the Hamiltonian (so-called "third Holstein-Primakoff transformation") for the antisymmetric case, which allowed us to obtain generalized expressions for the coefficients of the nonlinear three- and four-magnon interactions. As an example, nonlinear spin wave interactions in ultrathin ferromagnetic nanowires and films subjected to interfacial Dzyaloshinskii-Moriya interaction (IDMI) are considered. It was found that three-magnon interaction coefficients in the "Damon-Eshbach" geometry are nonzero only in the case of the noncollinear interacting spin waves, and vanish in the case of the collinear spin waves. It was also found that the nonlinear spin wave frequency shift caused by the four-magnon interaction is nonreciprocal, and has the sign opposite to that of the nonreciprocal term in linear spin wave dispersion, so that the IDMI-induced nonreciprocity of the spin wave spectrum decreases with the increase of the spin wave amplitude.

DOI: 10.1103/PhysRevB.99.174431

I. INTRODUCTION

One of significant advantages of spin waves (SWs) as signal carriers for microwave signal processing is the relatively low amplitude levels at which SWs (or magnons) start to demonstrate nonlinearity, and can be involved in nonlinear processes, including parametric interaction with electromagnetic pumping, three-wave, four-wave, and higher-order magnon-magnon interaction processes [1–3]. Nonlinear SW properties manifest themselves in various phenomena, such as parametric SW instability [4–7], saturation of the ferromagnetic resonance and foldover effect [8–12], nonlinear decay of SWs [13,14], and SW turbulence and chaos [2,15,16]. Exploration of these nonlinear phenomena made possible the development of nonlinear microwave signal processing devices, such as frequency-selective power limiters and signal-to-noise enhancers based on three-wave interaction [1,17,18], nonlinear delay lines based on formation, propagation and manipulation of SW solitons caused by the four-wave interactions [19–25], as well as the nonlinear spin wave switches and logic devices [26–29].

To understand and explore nonlinear SW processes, one needs to know which processes are allowed for a particular group of SWs, and needs to be able to evaluate the efficiency of these processes. Obviously, the understanding of nonlinear SW properties is also necessary for the successful development of *linear* SW devices since the nonlinear processes often limit the power dynamic range of these linear devices. Theoretically, the most general and powerful approach for the quantitative analysis of nonlinear SW interactions and calculation of the efficiencies of multimagnon interaction processes (that is, nonlinear SW coefficients) is the classical Hamiltonian formalism for SW that was originally proposed by Schlömann [30], and, then, developed in Ref. [2]. In the framework of this formalism the components of the dynamical magnetization vector are represented in terms of two scalar canonical Hamiltonian variables $a(\boldsymbol{r}, t)$ and $a^*(\boldsymbol{r}, t)$ [2]. The coefficients of the nonlinear multimagnon interactions are derived using the expansion of the Hamiltonian function for the magnetization dynamics into a series of spatial Fourier harmonics of the above-mentioned canonical variables, while the dynamics is governed by standard Hamiltonian equations [31]. This Hamiltonian formalism for the magnetization dynamics was extensively used for the investigation of nonlinear SW dynamics under parametric pumping (see Refs. [2,32] and references therein) and SW parametric instabilities in different geometries [6,7,14]. More recently, the large-angle magnetization dynamics induced by spin-transfer torque, was explored in Refs. [33,34].

Calculation of nonlinear coefficients in the framework of the above-described Hamiltonian approach is straightforward, but rather cumbersome algebraically when applied to particular geometries (e.g., thin films) [35,36], and cannot be easily generalized. Krivosik and Patton, using effective SW tensor formulation [6], derived explicit general expressions for nonlinear coefficients (up to four-wave coefficients) in a uniformly magnetized sample, assuming that normal modes of the system are plane waves [37]. Their theory is suitable for bulk samples, fundamental modes of ferromagnetic films and nanowires, ferromagnetic resonance of nanostructures, and allows one to take into account *symmetric* magnetic

self-interactions, quadratic in magnetization. Nonuniform exchange, magnetodipolar interaction, and uniaxial anisotropy, which often are the most important interactions, belong to the family of symmetric quadratic interactions, and are described by symmetric tensor operators.

However, not all the quadratic magnetic self-interactions are symmetric. In magnetic multilayers and materials with a specific crystal structure the *antisymmetric* exchange interaction, which is usually called "Dzyaloshinskii-Moriya" interaction (DMI), can appear and play an important role [38,39]. The DMI became of a significant research interest recently when it became possible to fabricate ultrathin ferromagnetic (Fe, permalloy, CoFe, etc.) films on a substrate made of a heavy metal characterized by a large spin-orbit coupling (W, Pt, etc.). In such systems at the interface between the ferromagnet and the heavy metal, the interfacial Dzyaloshinskii-Moriya interaction (IDMI) manifests itself and significantly influences the magnetization dynamics [40–42]. In particular, IDMI can stabilize topologically nontrivial magnetization states [43,44], or lead to the nonreciprocity of the SW spectrum in magnetic films or nanowires magnetized to saturation [41,45–48].

SW nonreciprocity could be very important for microwave signal processing [49–55], and the use of IDMI in thin ferromagnetic films is the most promising way to achieve substantial nonreciprocity of SWs with the wavelength of 100 nm and below. Therefore, it is important to investigate linear and nonlinear SW properties in magnetic materials with IDMI. In particular, it is critical to investigate the variation of SW spectrum at high SW amplitudes and calculate the three-wave nonlinear SW coefficients that determine the threshold of the SW parametric instability. This knowledge is especially important for the successful development of parametric [54] and spin-torque [56–58] IDMI-based devices, in which high amplitudes of SWs are easily realized.

The IDMI is not the only example of antisymmetric magnetic interactions. Different kinds of bulk Dzyaloshinskii-Moriya interactions, which exist in magnetic crystals with specific broken symmetries, are nonsymmetric too [38,39,59,60]. Also, the spin-flexoelectric interaction, which was predicted to manifest itself in ferromagnetic insulators under applied electric field [61,62], also belongs to the family of antisymmetric interactions.

The main aim of our current work is to generalize the existing theory of nonlinear spin wave dynamics based on the Hamiltonian approach [2,35,37] to the case of *any* magnetic self-interactions, quadratic in magnetization. Similarly to Ref. [37] we assume that ferromagnetic sample is in the saturated state, and elementary excitations in this sample are plane SWs. Throughout our current work we keep the notations of Ref. [37], and point out explicitly the main differences which appear due to the presence of nonsymmetric magnetic interactions.

The paper is organized as follows. In Sec. II an overview of the Hamiltonian formalism is given, and magnetic Hamiltonian function is derived in terms of the magnetization components and complex amplitudes of the plane spin waves. The expansion of the Hamiltonian function in a series of Fourier amplitudes of canonic variables $a_{\boldsymbol{k}}(t)$ and $a^*_{\boldsymbol{k}}(t)$ up to the fourth-order terms is presented in Sec. II E. The diagonalization of the quadratic part of the SW Hamiltonian using the generalized third Holstein-Primakoff transformations and expansion of the Hamiltonian function into linear modes are given in Sec. III. In Sec. IV, as an example of application of the developed formalism, we derive general expressions to calculate three-wave splitting efficiency and four-wave nonlinear frequency shift of the SWs in ferromagnetic films subjected to IDMI. Finally, conclusions are given in Sec. V.

II. MAGNETIC HAMILTONIAN FUNCTION

A. Overview of the Hamiltonian formalism

The most important step in the classical Hamiltonian approach is the transformation of the natural dynamic variables of the problem (in our case, components of the dynamical magnetization) into canonical variables a, a^* in which the energy of the studied system becomes a Hamiltonian function. The Hamiltonian function is, commonly, a functional of the full energy of the system E. In the study of ferromagnetic materials, however, it is more convenient to use a reduced Hamiltonian function [30,37]

$$\mathcal{H} = \mathcal{H}[\boldsymbol{M}(\boldsymbol{r},t)] = \frac{\gamma E}{M_s V}, \tag{2.1}$$

where γ is the modulus of the gyromagnetic ratio, M_s is the saturation magnetization, and V is the volume of the ferromagnetic material. The Hamiltonian function (2.1) is measured in the units of frequency. The canonical variables $a(\boldsymbol{r},t)$ and $a^*(\boldsymbol{r},t)$ are introduced as

$$\alpha_\perp(\boldsymbol{r},t) = a(\boldsymbol{r},t)\sqrt{2 - a(\boldsymbol{r},t)a^*(\boldsymbol{r},t)}, \tag{2.2a}$$

$$\alpha_z(\boldsymbol{r},t) = 1 - a(\boldsymbol{r},t)a^*(\boldsymbol{r},t). \tag{2.2b}$$

Here, $\alpha_{x,y,z} = M_{x,y,z}/M_s$ are the normalized magnetization components and $\alpha_\perp = i\alpha_x + \alpha_y$ is the complex dynamic magnetization variable. Equations (2.2) use the convention that static magnetization is directed along the z axis. It is clear that $|\alpha_\perp|^2 + \alpha_z^2 = 1$, i.e., the transformation (2.2) satisfies the condition of conservation of the magnetization vector length.

When Hamiltonian function of the system is expressed in terms of the canonical variables, the dynamical equations for the variables $a(\boldsymbol{r},t)$ and $a^*(\boldsymbol{r},t)$ can be written in a standard Hamiltonian form:

$$i\frac{da(\boldsymbol{r},t)}{dt} = \frac{\delta\mathcal{H}}{\delta a^*(\boldsymbol{r},t)}, \quad -i\frac{da^*(\boldsymbol{r},t)}{dt} = \frac{\delta\mathcal{H}}{\delta a(\boldsymbol{r},t)}. \tag{2.3}$$

The next step in the Hamiltonian formalism for SW is the expansion of $a(\boldsymbol{r},t)$ into a series of plane waves (in the case when the plane waves are the normal modes of the system):

$$a(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{2.4}$$

where new canonical variables $a_{\boldsymbol{k}}(t)$ describe amplitudes of SWs with the wave vector $\boldsymbol{k}$. Since the Fourier transform is canonical, the dynamical equations for $a_{\boldsymbol{k}}(t)$ have the same form as Eq. (2.3), namely,

$$i\frac{da_{\boldsymbol{k}}(t)}{dt} = \frac{\partial\mathcal{H}}{\partial a^*_{\boldsymbol{k}}(t)}, \quad -i\frac{da^*_{\boldsymbol{k}}(t)}{dt} = \frac{\partial\mathcal{H}}{\partial a_{\boldsymbol{k}}(t)}. \tag{2.5}$$

The change of the variational derivative in Eq. (2.3) to the partial derivative in Eq. (2.5) is related with the fact that $\mathcal{H}$ is a functional in terms of $a(\boldsymbol{r},t)$, but becomes a polynomial function in terms of the Fourier amplitudes $a_{\boldsymbol{k}}(t)$.

After the transformation (2.2), the Hamiltonian function can be developed as a series in the SW Fourier amplitudes a_k and a_k^* and, thus, can be represented as $\mathcal{H} \approx \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \cdots$, where the superscripts indicate the number of amplitudes a_k and a_k^* in the corresponding term. The term $\mathcal{H}^{(0)}$ which does not contain any SW amplitudes determines the energy of the ground state, $\mathcal{H}^{(1)}$ describes the linear excitation of SWs by external forces (e.g., by a microwave magnetic field), the term $\mathcal{H}^{(2)}$ determines the "kinetic" energy of the system in the linear regime defined by the linear spectrum of the system SW eigenmodes, and all the higher-order terms describe the nonlinear interactions between the SWs. It has been shown in Ref. [2] that in most cases it is sufficient to consider the expansion of the Hamiltonian function up to the fourth order $\mathcal{H}^{(4)}$ in terms of the variables a_k and a_k^*.

Finally, it is, usually, convenient to diagonalize the quadratic part $\mathcal{H}^{(2)}$ of the SW Hamiltonian using the third Holstein-Primakoff (or u-v Bogoljubov) transformation, and introduce the new (elliptically polarized) variables c_k and c_k^*, which describe the amplitudes of the normal linear SW modes. The derivation of the nonlinear terms $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$ of the Hamiltonian function in terms of the variables c_k and c_k^* is the final step, which gives the coefficients of the nonlinear (three- and four-wave) SW interactions.

B. Terms of the magnetic Hamiltonian function

The first step in the Hamiltonian formalism is the derivation of the Hamiltonian function $\mathcal{H}$ in terms of the magnetization vector $\boldsymbol{M}(\boldsymbol{r},t)$. A generic expression for $\mathcal{H}$ can be written as

$$\mathcal{H} = -\frac{\gamma}{M_s V}\int \boldsymbol{M}(\boldsymbol{r},t)\cdot\boldsymbol{B}_{\mathrm{e}}(\boldsymbol{r},t)d\boldsymbol{r} - \frac{1}{2}\frac{\gamma}{M_s V}\int \boldsymbol{M}(\boldsymbol{r},t)\cdot\boldsymbol{B}_{\boldsymbol{M}}(\boldsymbol{r},t)d\boldsymbol{r} - \cdots . \quad (2.6)$$

Here, the first term corresponds to the Zeeman energy of magnetization in the *external* magnetic field $\boldsymbol{B}_{\mathrm{e}}$, which can be both space and time dependent. The second term represents the interaction of the magnetization with the *internal* field $\boldsymbol{B}_{\boldsymbol{M}}$, which is produced by the magnetization itself. For most common magnetic self-interactions, which are quadratic functionals in terms of magnetization, the field $\boldsymbol{B}_{\boldsymbol{M}}$ can be conveniently expressed as

$$\boldsymbol{B}_{\boldsymbol{M}}(\boldsymbol{r},t) = -\mu_0\int \hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{M}(\boldsymbol{r}',t)d\boldsymbol{r}', \quad (2.7)$$

where $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')$ is the tensor operator describing magnetic self-interactions. It consists of the sum of different contributions, the most important of which are exchange, magnetodipolar, and anisotropy contributions. The isotropic exchange interaction is described by the operator

$$\hat{\boldsymbol{N}}_{\mathrm{ex}}(\boldsymbol{r},\boldsymbol{r}') = -\lambda_{\mathrm{ex}}^2\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{I}}\nabla_r\cdot\nabla_{r'}, \quad (2.8)$$

where $\lambda_{\mathrm{ex}} = \sqrt{2A/\mu_0 M_s^2}$ is the exchange length of the magnetic material, A is the exchange stiffness, $\hat{\boldsymbol{I}}$ is the unit matrix, and subscripts of the nabla operators denote the coordinates ($\boldsymbol{r}$ or $\boldsymbol{r}'$) to which the operator is applied. Magnetodipolar interaction in the magnetostatic approximation, i.e., neglecting retardation effects, is expressed via the magnetostatic Green's function [63]

$$\hat{\boldsymbol{N}}_{\mathrm{dip}} = \frac{1}{4\pi}\nabla_r\left(\nabla_{r'}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right). \quad (2.9)$$

The uniaxial magnetic anisotropy is given by

$$\hat{\boldsymbol{N}}_{\mathrm{an}}(\boldsymbol{r},\boldsymbol{r}') = -\frac{B_{\mathrm{an}}}{\mu_0 M_s}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{e}_\zeta\otimes\boldsymbol{e}_\zeta, \quad (2.10)$$

where $B_{\mathrm{an}} = 2K_{\mathrm{u}}/M_s$ is the anisotropy field, K_{u} is the constant of uniaxial anisotropy, $\boldsymbol{e}_\zeta$ is the unit vector in the direction of the anisotropy axis (ζ direction), and the symbols $\otimes$ are denoting the dyadic product of vectors. Dirac delta function $\delta(\boldsymbol{r}-\boldsymbol{r}')$ in Eqs. (2.8) and (2.10) indicates the local character of the exchange interaction and the crystalline anisotropy, in contrast with the nonlocal long-range character of the magnetodipolar interaction.

In a general case, the Hamiltonian function (2.1) may contain higher-order terms (in respect to the magnetization), in particular, if the other than uniaxial crystalline anisotropy is taken into account. For example, cubic magnetic anisotropy is described by the third order in respect to $\boldsymbol{M}$ term [1]. Below, for simplicity, we skip these possible higher-order contributions, although, if necessary they can be accounted for in a similar way.

The energy of the IDMI in a thin ferromagnetic film or nanowire can be expressed as

$$E_{\mathrm{IDMI}} = \int \frac{\tilde{D}}{M_s^2}[M_z\nabla\cdot\boldsymbol{M} - \boldsymbol{M}\cdot(\nabla M_z)]d\boldsymbol{r}, \quad (2.11)$$

where z axis is a normal to the ferromagnetic–heavy-metal interface, $\tilde{D} = Db/h$ is the effective IDMI parameter, D is the IDMI constant, h is the thickness of the film or nanowire, and b is the thickness of the ferromagnetic monolayer [47]. It is clear that the term representing IDMI in the Hamiltonian function can be also expressed in the form of Eqs. (2.6) and (2.7) with the tensor operator given by

$$\hat{\boldsymbol{N}}_{\mathrm{IDMI}}(\boldsymbol{r},\boldsymbol{r}') = \frac{2\tilde{D}}{\mu_0 M_s^2}\delta(\boldsymbol{r}-\boldsymbol{r}')[\boldsymbol{e}_{z'}\otimes\nabla_{r'} - \nabla_{r'}\otimes\boldsymbol{e}_{z'}] \quad (2.12)$$

or in the explicit matrix form

$$\hat{\boldsymbol{N}}_{\mathrm{IDMI}}(\boldsymbol{r},\boldsymbol{r}') = \frac{2\tilde{D}}{\mu_0 M_s^2}\delta(\boldsymbol{r}-\boldsymbol{r}')\begin{pmatrix} 0 & 0 & -\partial_{x'} \\ 0 & 0 & -\partial_{y'} \\ \partial_{x'} & \partial_{y'} & 0 \end{pmatrix}. \quad (2.13)$$

From Eq. (2.13) one can clearly see the main difference of the IDMI term compared to the other magnetic self-interactions, mentioned above. The tensor operator $\hat{\boldsymbol{N}}_{\mathrm{IDMI}}$ is *antisymmetric*, while the operators of the uniaxial anisotropy, dipolar and exchange interactions are all *symmetric*. The antisymmetric nature of the IDMI is also reflected by the fact that $\hat{\boldsymbol{N}}_{\mathrm{IDMI}}$ changes its sign to the opposite under the space inversion operation $\boldsymbol{r}\to-\boldsymbol{r}$, in contrast to the other self-interaction operators, which remain unchanged in respect to the space inversion.

The bulk Dzyaloshinskii-Moriya interaction, which is present in ferromagnetic crystals with lack of the inversion symmetry, could be also accounted in the same manner as the IDMI, and is represented by an antisymmetric tensor operator [60]. Below, we will not use any specific features of the operator $\hat{\boldsymbol{N}}$, related to the IDMI or other interactions. We assume that tensor operator $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')$ is *nonsymmetric* in a general case, i.e., consists of symmetric and antisymmetric contributions. Thus, the formalism presented below is applicable to *any* magnetic self-interaction, quadratic in magnetization.

C. Conversion to the complex variables

The next step in the Hamiltonian approach is the conversion to the canonic variables $a(\boldsymbol{r},t)$, $a^*(\boldsymbol{r},t)$. This step is just a simple algebraic operation. For convenience and brevity, we, following Ref. [37], introduce the dimensionless complex vector $\boldsymbol{\alpha}(\boldsymbol{r},t)=[\alpha_\perp,\alpha_\perp^*,\alpha_z]$, where $\alpha_\perp(\boldsymbol{r},t)$ and $\alpha_z(\boldsymbol{r},t)$ are related to the canonical variables according to Eq. (2.2). The real magnetization vector $\boldsymbol{M}(\boldsymbol{r},t)$ is connected with $\boldsymbol{\alpha}$ through the relation $\boldsymbol{M}=M_s\hat{\boldsymbol{T}}\cdot\boldsymbol{\alpha}$, where

$$\hat{\boldsymbol{T}}=\frac{1}{2}\begin{pmatrix}-i & i & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}. \tag{2.14}$$

It is also useful to establish the relation $\boldsymbol{\alpha}^*=\hat{\boldsymbol{S}}\cdot\boldsymbol{\alpha}$, where

$$\hat{\boldsymbol{S}}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.15}$$

Here, we should recall that the introduction of the canonical variables in the form of Eq. (2.2) assumes that the static magnetization of a ferromagnetic sample is uniform, and is pointed in the $+z$ direction. Therefore, all the self-interaction operators $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')$ should be derived in this coordinate system. A more complex case of nonuniform static magnetization configurations is considered in a similar way, with the difference that the relation between $\boldsymbol{M}$ and $\boldsymbol{\alpha}$ becomes coordinate dependent, i.e., $\hat{\boldsymbol{T}}=\hat{\boldsymbol{T}}(\boldsymbol{r})$. The examples of application of the Hamiltonian formalism to nonuniform magnetic states can be found in Refs. [64,65].

The direct calculation allows one to obtain the Hamiltonian function in the form

$$\mathcal{H}=-\frac{\gamma}{V}\int\boldsymbol{\alpha}^*(\boldsymbol{r},t)\cdot\tilde{\boldsymbol{B}}(\boldsymbol{r},t)d\boldsymbol{r}+\frac{1}{2V}\iint\boldsymbol{\alpha}^*(\boldsymbol{r},t)\cdot\hat{\boldsymbol{\Gamma}}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{\alpha}(\boldsymbol{r}',t)d\boldsymbol{r}\,d\boldsymbol{r}', \tag{2.16}$$

where

$$\tilde{\boldsymbol{B}}(\boldsymbol{r},t)=\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{T}}^{\mathrm{T}}\cdot\boldsymbol{B}_{\mathrm{e}}(\boldsymbol{r},t) \tag{2.17a}$$

and

$$\hat{\boldsymbol{\Gamma}}(\boldsymbol{r},\boldsymbol{r}')=\omega_M\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{T}}^{\mathrm{T}}\cdot\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')\cdot\hat{\boldsymbol{T}}, \tag{2.17b}$$

with $\omega_M=\gamma\mu_0M_s$. Here, analogously to the complex magnetization vector $\boldsymbol{\alpha}$, we introduce the complex vector field $\tilde{\boldsymbol{B}}(\boldsymbol{r},t)=[\tilde{B}_\perp/\sqrt{2},\tilde{B}_\perp^*/\sqrt{2},B_{\mathrm{e},z}]$, where $\tilde{B}_\perp(\boldsymbol{r},t)=(iB_{\mathrm{e},x}+B_{\mathrm{e},y})/\sqrt{2}$ is the circular component of the external field $\boldsymbol{B}_{\mathrm{e}}$ (the "tilde" sign over the complex vector field is added in order not to mix this vector with the real external field $\boldsymbol{B}_{\mathrm{e}}$). The explicit derivation of the components of the tensor operator $\hat{\boldsymbol{\Gamma}}(\boldsymbol{r},\boldsymbol{r}')$ is not necessary at this step. It is convenient to preserve the vector structure of this expression during the operation of the Fourier transform (next step).

D. Conversion to the plane waves

The goal of the current and following sections in the paper is to represent the Hamiltonian function in terms of the Fourier amplitudes of the SW canonical variables $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*$. It should be noted that the introduction of the SW amplitudes $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*$ in the form of Eq. (2.4) implies that the linear eigenmodes of the considered ferromagnetic sample are plane waves. This statement is correct for the bulk samples and for fundamental SW modes in ferromagnetic films and nanowires, which have almost uniform profiles along the thickness and width coordinate of the sample. In other cases, e.g., when considering quantized modes of a magnetic nanodot, Eq. (2.4) should be modified taking into account the spatial profiles of the quantized modes (see, e.g., Refs. [7,64,65]).

When performing Fourier transform, it is convenient to preserve the vector structure of the Hamiltonian (2.16). Then, all the terms of the Hamiltonian function change to their Fourier images. Namely, the dimensionless complex magnetization vector is represented via its Fourier image as

$$\boldsymbol{\alpha}(\boldsymbol{r},t)=\sum_{\boldsymbol{k}}\boldsymbol{\alpha}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{2.18}$$

where $\boldsymbol{\alpha}_{\boldsymbol{k}}=[\mathcal{F}_{\boldsymbol{k}}[\alpha_\perp(\boldsymbol{r},t)],\mathcal{F}_{\boldsymbol{k}}[\alpha_\perp^*(\boldsymbol{r},t)],\mathcal{F}_{\boldsymbol{k}}[\alpha_z(\boldsymbol{r},t)]]$, and $\mathcal{F}_{\boldsymbol{k}}$ denotes the operator of the Fourier transform. The derivation of the explicit relation between the components of $\boldsymbol{\alpha}_{\boldsymbol{k}}$ and canonical variables $a_{\boldsymbol{k}}$ requires application of the Taylor expansion, and is given in the next subsection. The transformation of the complex vector of the external field is also trivial, and is given by

$$\tilde{\boldsymbol{B}}(\boldsymbol{r},t)=\sum_{\boldsymbol{k}}\tilde{\boldsymbol{B}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{2.19}$$

with $\tilde{\boldsymbol{B}}_{\boldsymbol{k}}(t)=[\tilde{B}_{\perp,\boldsymbol{k}}(t),\tilde{B}_{\perp,\boldsymbol{k}}^*(t),\tilde{B}_{z,\boldsymbol{k}}(t)]$.

It should be noted that the Fourier transformation of the operator of magnetic self-interactions is not so trivial in a general case. Here, we restrict ourselves to the case when the operator $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')$ depends only on the difference of arguments $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')=\hat{\boldsymbol{N}}(\boldsymbol{r}-\boldsymbol{r}')$. This implies that the considered magnetic sample is spatially *homogeneous*, having no spatial variations of the material parameters. In this case, the Fourier transform of the operator $\hat{\boldsymbol{N}}(\boldsymbol{r},\boldsymbol{r}')$ is introduced as

$$\hat{\boldsymbol{N}}_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{r}}=\int\hat{\boldsymbol{N}}(\boldsymbol{r}-\boldsymbol{r}')e^{i\boldsymbol{k}\boldsymbol{r}'}d\boldsymbol{r}'. \tag{2.20}$$

The tensor $\boldsymbol{N}_{\boldsymbol{k}}$ is often called the "effective spin wave tensor" [6]. It is the most important universal characteristic of a ferromagnetic sample for the description of the propagating SWs in it. Together with the direction of the static magnetization, and the value of the external field, it contains all the information about the linear SW spectrum and nonlinear interactions. With certain modifications, the effective spin wave tensor can be also introduced in the problems of spatially nonuniform

propagating SW modes, e.g., for higher thickness or width modes of ferromagnetic films or nanowires [66–68].

The properties of the tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ follow from the symmetry of the magnetic self-interactions. Obviously, $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ is self-adjoint, i.e., $(\hat{\boldsymbol{N}}_{\boldsymbol{k}}^{\mathrm{T}})^{*} = \hat{\boldsymbol{N}}_{\boldsymbol{k}}$, which ensures that the Hamiltonian function is real valued. Another general property is $\hat{\boldsymbol{N}}_{-\boldsymbol{k}} = \hat{\boldsymbol{N}}_{\boldsymbol{k}}^{*}$. For symmetric self-interactions, e.g., exchange, dipolar interaction, or anisotropy, the tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ is *symmetric*, $\hat{\boldsymbol{N}}_{\boldsymbol{k}}^{\mathrm{T}} = \hat{\boldsymbol{N}}_{\boldsymbol{k}}$, and is *real*. In a general case, however, magnetic self-interactions are not required to bear these properties. For example, in the case of the IDMI the tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ is *antisymmetric* and *imaginary*. Namely, in the coordinate system with z axis being perpendicular to the ferromagnetic interface we get

$$\hat{\boldsymbol{N}}_{\boldsymbol{k},\mathrm{IDMI}} = \frac{2\tilde{D}}{\mu_0 M_s^2}\begin{pmatrix} 0 & 0 & -ik_x \\ 0 & 0 & -ik_y \\ ik_x & ik_y & 0 \end{pmatrix}. \tag{2.21}$$

Therefore, in a general case we should consider a complex-valued nonsymmetric self-interaction tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$, having a symmetric real part and antisymmetric imaginary part. Mathematically, the appearance of an antisymmetric imaginary contribution of $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ is the only difference from the previous analysis of symmetric interactions, performed in Ref. [37]. However, it leads to significant modifications of all the expressions in the following analysis.

Using the above-defined Fourier images of the complex magnetization vector, the complex vector of external field and the operator of magnetic self-interactions, the Hamiltonian function can be represented in the following form:

$$\mathcal{H} = -\gamma \sum_{\boldsymbol{k}} \boldsymbol{\alpha}_{\boldsymbol{k}}^{*} \cdot \tilde{\boldsymbol{B}}_{\boldsymbol{k}}(t) + \frac{1}{2}\sum_{\boldsymbol{k}} \boldsymbol{\alpha}_{\boldsymbol{k}}^{*}(t) \cdot \hat{\boldsymbol{\Gamma}}_{\boldsymbol{k}} \cdot \boldsymbol{\alpha}_{\boldsymbol{k}}, \tag{2.22}$$

where $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{k}} = \omega_M \hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{T}}^{\mathrm{T}} \cdot \hat{\boldsymbol{N}}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{T}}$. Straightforward calculations give the explicit form of the tensor $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{k}}$:

$$\hat{\boldsymbol{\Gamma}}_{\boldsymbol{k}} = \begin{pmatrix} \frac{1}{2}\mathcal{Q}_{\boldsymbol{k}} & \frac{1}{2}\mathcal{B}_{\boldsymbol{k}} & \frac{1}{\sqrt{2}}\mathcal{D}_{\boldsymbol{k}} \\ \frac{1}{2}\mathcal{B}_{\boldsymbol{k}}^{*} & \frac{1}{2}\mathcal{Q}_{-\boldsymbol{k}} & \frac{1}{\sqrt{2}}\mathcal{D}_{-\boldsymbol{k}}^{*} \\ \frac{1}{\sqrt{2}}\mathcal{D}_{\boldsymbol{k}}^{*} & \frac{1}{\sqrt{2}}\mathcal{D}_{-\boldsymbol{k}} & \Gamma_{zz,\boldsymbol{k}} \end{pmatrix}, \tag{2.23}$$

where

$$\mathcal{Q}_{\boldsymbol{k}} = \frac{\omega_M}{2}(N_{xx,\boldsymbol{k}} + N_{yy,\boldsymbol{k}} - 2\,\mathrm{Im}N_{xy,\boldsymbol{k}}), \tag{2.24}$$

$$\mathcal{B}_{\boldsymbol{k}} = \frac{\omega_M}{2}(-N_{xx,\boldsymbol{k}} + N_{yy,\boldsymbol{k}} + 2i\,\mathrm{Re}N_{xy,\boldsymbol{k}}), \tag{2.25}$$

$$\mathcal{D}_{\boldsymbol{k}} = \frac{\omega_M}{\sqrt{2}}(iN_{xz,\boldsymbol{k}} + N_{yz,\boldsymbol{k}}), \tag{2.26}$$

and

$$\Gamma_{zz,\boldsymbol{k}} = \omega_M N_{zz,\boldsymbol{k}}. \tag{2.27}$$

Using the above-described general properties of the tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$, one can prove that $\mathcal{B}_{\boldsymbol{k}} = \mathcal{B}_{-\boldsymbol{k}}$, $\Gamma_{zz,\boldsymbol{k}} = \Gamma_{zz,-\boldsymbol{k}}$, and that the values $\mathcal{Q}_{\boldsymbol{k}}$ and $\Gamma_{zz,\boldsymbol{k}}$ are real. Simultaneously, in a general case $\mathcal{Q}_{\boldsymbol{k}} \neq \mathcal{Q}_{-\boldsymbol{k}}$ and $\mathcal{D}_{\boldsymbol{k}} \neq \mathcal{D}_{-\boldsymbol{k}}$, while corresponding equalities take place in the case of symmetric self-interactions. These inequalities lead to the appearance of terms with both $\boldsymbol{k}$ and $-\boldsymbol{k}$ in the definition of the tensor $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{k}}$. Another evident difference from the symmetric case is the appearance of real and imaginary parts of $N_{xy,\boldsymbol{k}}$ in the definitions of the coefficients $\mathcal{Q}_{\boldsymbol{k}}$ and $\mathcal{B}_{\boldsymbol{k}}$.

E. Expansion coefficients of the Hamiltonian function

Finally, in order to derive the Hamiltonian function in terms of the canonical variables, one needs to express the Fourier components of the complex magnetization vector $\boldsymbol{\alpha}_{\boldsymbol{k}}$ in terms of the variables $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{*}$. For the component α_z this operation is trivial, and, taking into account Eq. (2.2), we get

$$\mathcal{F}_{\boldsymbol{k}}[\alpha_z(\boldsymbol{r},t)] = \Delta(\boldsymbol{k}) - \sum_{\mathbf{12}} a_{\mathbf{1}}(t)a_{\mathbf{2}}^{*}(t)\Delta(\mathbf{1}-\mathbf{2}-\boldsymbol{k}). \tag{2.28}$$

Here, Δ is the Kronecker delta function, and the abbreviated notations $\mathbf{1} \equiv \boldsymbol{k}_1$, $\mathbf{2} \equiv \boldsymbol{k}_2$, etc., are used. The transformation of the components $\alpha_{\perp}$, $\alpha_{\perp}^{*}$ is not so trivial, as Eq. (2.2a) contains a square root. Therefore, to proceed, one needs to expand it into a Taylor series. Limiting the expansion to the first two terms, we get

$$\alpha_{\perp}(\boldsymbol{r},t) \approx \sqrt{2}a(\boldsymbol{r},t)\big[1 - \tfrac{1}{4}a(\boldsymbol{r},t)a^{*}(\boldsymbol{r},t)\big]. \tag{2.29}$$

The error coming from this approximation is rather small. Indeed, it gives $|\alpha_{\perp}|^2 + \alpha_z^2 = 1 + |a|^6/8$, while the exact value is 1. Even for the precession angles close to 90°, for which $|a| \approx 1$, the expansion (2.29) gives the error not exceeding 12%. The straightforward Fourier transform of Eq. (2.29) and its complex conjugate yields the following relations:

$$\mathcal{F}_{\boldsymbol{k}}[\alpha_{\perp}(\boldsymbol{r},t)] \approx \sqrt{2}\left(a_{\boldsymbol{k}}(t) - \frac{1}{4}\sum_{\mathbf{123}} a_{\mathbf{1}}(t)a_{\mathbf{2}}(t)a_{\mathbf{3}}^{*}(t)\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}-\boldsymbol{k})\right), \tag{2.30a}$$

$$\mathcal{F}_{\boldsymbol{k}}[\alpha_{\perp}^{*}(\boldsymbol{r},t)] \approx \sqrt{2}\left(a_{-\boldsymbol{k}(t)}^{*} - \frac{1}{4}\sum_{\mathbf{123}} a_{\mathbf{1}}^{*}(t)a_{\mathbf{2}}^{*}(t)a_{\mathbf{3}}(t)\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}+\boldsymbol{k})\right). \tag{2.30b}$$

Using the above expansions in Eq. (2.22), it is possible to represent the Hamiltonian function $\mathcal{H}$ in terms of canonical SW amplitudes $a_{\boldsymbol{k}}$, $a_{\boldsymbol{k}}^{*}$. Then, it is easy to collect the terms of the same power in the SW amplitudes, representing, thus, the Hamiltonian function as $\mathcal{H} \approx \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \cdots$, where the superscript denotes the power of the terms respective to the SW amplitudes. Here, we limit the expansion to the fourth-order term $\mathcal{H}^{(4)}$, which corresponds to the four-magnon processes. Usually, this expansion is sufficient as the four-magnon processes are almost never prohibited by the conservation laws, and the higher-order terms in the expansion are rather small [2].

1. Zeroth-order terms

The zeroth-order term of the Hamiltonian function is expressed as

$$\mathcal{H}^{(0)} = -\gamma \tilde{B}_{z,0}(t) + \tfrac{1}{2}\Gamma_{zz,0}. \tag{2.31}$$

This term does not contain any SW amplitudes, and, consequently, does not affect the magnetization dynamics. It

determines the energy of the static magnetization state, which consists of the Zeeman energy of the magnetization in a uniform magnetic field [recall that $\tilde{B}_{z,0}$ is the spatially uniform component of the external field $B_{e,z}(\boldsymbol{r},t)$] and the static demagnetization energy of a uniformly magnetized body.

2. First-order terms

The first-order terms of the Hamiltonian function have the following form:

$$\mathcal{H}^{(1)} = -\sum_{\boldsymbol{k}} \gamma \tilde{B}^{*}_{\perp,\boldsymbol{k}}(t) a_{\boldsymbol{k}} + \mathcal{D}^{*}_{0} a_0 + \text{c.c.}, \tag{2.32}$$

where for brevity we omit the explicit notation of the time dependence of the SW amplitudes $a_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}}(t)$. To analyze these terms it is convenient to split the external field into a static and dynamic components: $\tilde{B}_{\perp,\boldsymbol{k}}(t) = \tilde{B}_{\perp,\boldsymbol{k}} + \tilde{b}_{\perp,\boldsymbol{k}}(t)$. Hereafter, we denote the static part of the external field by a capital symbol, and the time-dependent one (e.g., microwave field) by a lowercase symbol $\tilde{b}(t)$. First, let us look at the static part. Using the relations (2.5), one can derive the dynamic equations for the SW amplitudes, associated with the first-order terms of the Hamiltonian function:

$$\frac{da_{\boldsymbol{k}}}{dt} = -i\frac{\partial \mathcal{H}^{(1)}}{\partial a^{*}_{\boldsymbol{k}}} = i(\gamma \tilde{B}_{\perp,\boldsymbol{k}} - \mathcal{D}_0 \Delta(\boldsymbol{k})). \tag{2.33}$$

If the considered static magnetization state is stable, then, in the absence of a time-dependent field, the time derivative vanishes, $da_{\boldsymbol{k}}/dt = 0$ (note that this is a necessary, but not a sufficient condition). Then, we get the following condition: $\tilde{B}_{\perp,\boldsymbol{k}} = 0$ for $\boldsymbol{k} \neq 0$, i.e., the static field perpendicular to the direction of static magnetization should be spatially uniform. It is an absolutely natural condition for the stability of a uniform magnetization state which is considered here. Simultaneously, the parallel component of the external field $B_{e,z}(\boldsymbol{r})$ can remain spatially nonuniform. From the dynamic equations for a_0 one gets the following condition:

$$\gamma \tilde{B}_{\perp,0} = \mathcal{D}_0. \tag{2.34}$$

Recalling the definitions (2.26) and $\tilde{B}_{\perp}(\boldsymbol{r}) = (iB_{e,x} + B_{e,y})/\sqrt{2}$, this condition can be transformed to the standard conditions of the static equilibrium in a uniformly magnetized sample $B_{e,x} = \mu_0 M_s N_{xz,0}$, $B_{e,y} = \mu_0 M_s N_{yz,0}$.

If the static equilibrium conditions are satisfied, the first-order terms of the Hamiltonian function are simplified to

$$\mathcal{H}^{(1)} = -\sum_{\boldsymbol{k}} \gamma \tilde{b}^{*}_{\perp,\boldsymbol{k}}(t) a_{\boldsymbol{k}} + \text{c.c.} \tag{2.35}$$

These terms describe the interaction of the SWs with the time-dependent magnetic field which is perpendicular to the direction of static magnetization. Such an interaction is responsible for the linear excitation of SWs. In the opposite case, when the equilibrium conditions are not satisfied, one can not follow the formalism below and should find a real equilibrium magnetization state (possibly, nonuniform), and introduce canonical variables on the background of this real equilibrium state.

3. Second-order terms

After collecting all the terms, the quadratic part $\mathcal{H}^{(2)}$ of the Hamiltonian function can be expressed as

$$\mathcal{H}^{(2)} = \sum_{\boldsymbol{k}} \left[\mathcal{A}_{\boldsymbol{k}} a^{*}_{\boldsymbol{k}} a_{\boldsymbol{k}} + \left(\frac{\mathcal{B}_{\boldsymbol{k}}}{2} a^{*}_{\boldsymbol{k}} a^{*}_{-\boldsymbol{k}} + \text{c.c.} \right) \right] + \sum_{\boldsymbol{1}\neq\boldsymbol{2}} \gamma \tilde{B}_{z,\boldsymbol{2}-\boldsymbol{1}} a_{\boldsymbol{1}} a^{*}_{\boldsymbol{2}} + \sum_{\boldsymbol{1},\boldsymbol{2}} \gamma \tilde{b}_{z,\boldsymbol{2}-\boldsymbol{1}}(t) a_{\boldsymbol{1}} a^{*}_{\boldsymbol{2}}. \tag{2.36}$$

The first sum is familiar from the linear SW theory. It involves pairs of SWs with the same wave vectors, and coupling of the SWs with opposite wave vectors, which can be present in a general case. In fact, this coupling reflects the fact that the magnetization precession is not circular, but is elliptical, as it is shown in the next section. The coefficient $\mathcal{A}_{\boldsymbol{k}}$ is equal to

$$\begin{aligned} \mathcal{A}_{\boldsymbol{k}} &= \gamma B_{z,0} - \Gamma_{zz,0} + \mathcal{Q}_{\boldsymbol{k}} \\ &= \omega_H + \frac{\omega_M}{2}(N_{xx,\boldsymbol{k}} + N_{yy,\boldsymbol{k}} - 2\,\text{Im}N_{xy,\boldsymbol{k}}), \end{aligned} \tag{2.37}$$

where $\omega_H = \gamma(B_{z,0} - \mu_0 M_s N_{zz,0})$. A general assumption of the existence of nonsymmetric magnetic self-interactions used in this work results in the appearance of the last term $\text{Im}N_{xy,\boldsymbol{k}}$ in the definition of $\mathcal{A}_{\boldsymbol{k}}$. As a consequence, $\mathcal{A}_{\boldsymbol{k}}$ loses the symmetry in respect to the wave-vector inversion $\mathcal{A}_{\boldsymbol{k}} \neq \mathcal{A}_{-\boldsymbol{k}}$, while such a symmetry is valid for the symmetric magnetic self-interactions. As it will be shown below, this inequality qualitatively affects the procedure of diagonalization of the quadratic part of the Hamiltonian function.

The second term in Eq. (2.36) describes the coupling of plane waves with arbitrary unequal wave vectors. This coupling is present only in the case of a spatially inhomogeneous external field $B_{e,z}(\boldsymbol{r})$, and the coupling strength is proportional to the Fourier component of the external field at nonzero $\boldsymbol{k}$. In fact, it means that in the case of an inhomogeneous field the plane waves having a definite $\boldsymbol{k}$ cease to be the normal modes of a ferromagnetic body. Instead, normal modes are formed by the sums of plane waves, which are finite in the case of harmonic (e.g., sinelike or cosinelike) field, and are infinite otherwise. In the following, we will not address this case and will assume that the static external field is spatially uniform.

Finally, the last term represents the coupling of the SW pairs having, in general, different wave vectors with the time-dependent external field. It is a well-known "parametric" coupling in the so-called "parallel pumping geometry" [1,2], which can be understood as a "three-quasiparticle" interaction when one microwave photon splits into two magnons. The spatially uniform parametric pumping couples with the SWs having opposite wave vectors, while a spatially nonuniform pumping can lead to a coupling of SWs with arbitrary wave vectors [69]. The different condition of the summation in the second and last terms ($\boldsymbol{1} \neq \boldsymbol{2}$ in the second term) appears because the term $\tilde{B}_{z,0} a_{\boldsymbol{1}} a^{*}_{\boldsymbol{1}}$ is already accounted in the first sum $\sum_{\boldsymbol{k}} \mathcal{A}_{\boldsymbol{k}} a_{\boldsymbol{k}} a^{*}_{\boldsymbol{k}}$.

4. Third-order terms

The part of the Hamiltonian, which includes three SW amplitudes a_k, can be written as

$$\mathcal{H}^{(3)} = -\frac{1}{2}\sum_{\mathbf{123}}[(\mathcal{D}_{\mathbf{1}}^{*}+\mathcal{D}_{\mathbf{2}}^{*})a_{\mathbf{1}}a_{\mathbf{2}}a_{\mathbf{3}}^{*}+\text{c.c.}]\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}) + \frac{1}{4}\sum_{\mathbf{123}k}[\gamma\tilde{b}_{\perp,k}^{*}(t)a_{\mathbf{1}}a_{\mathbf{2}}a_{\mathbf{3}}^{*}+\text{c.c.}]\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}-k). \tag{2.38}$$

In the derivation of this third-order contribution we took into account the stability conditions (2.34). The expression for $\mathcal{H}^{(3)}$ is absolutely the same as in the case of symmetric magnetic self-interactions [37]. However, one should remember that the presence of antisymmetric interactions changes the properties of the coefficients $\mathcal{D}_k$ (see above).

The first term in Eq. (2.38) represents a pure three-magnon scattering process, namely, splitting of a magnon **3** into a pair of magnons **1** and **2**, and the opposite process. These scattering processes are allowed only when the momentum and energy conservation conditions are satisfied, which depends on the sample geometry and material, and on the external field. It should be noted that if the resonant three-magnon processes are not allowed (i.e., if the conservation laws for the three-magnon process are not satisfied), but the three-magnon interaction efficiency ($\mathcal{D}_{\mathbf{1}}^{*}+\mathcal{D}_{\mathbf{2}}^{*}$) is nonzero, these "prohibited" processes may affect the strength of the nonlinear processes of higher orders (see Sec. III C).

The second term in Eq. (2.38) describes the scattering of a microwave photon and a magnon into two other magnons. Such a process is, typically, weak, compared to the common parametric interaction of SW pairs by a microwave photon and the three-magnon splitting processes. Nevertheless, in certain cases such a process can be involved in the limitation of the FMR amplitude at high excitation fields [6].

5. Fourth-order terms

The highest-order nonlinear SW processes, which we consider in this work, are the fourth-order processes. The importance of these processes is related with the fact that some of them are always allowed, as the energy and momentum conservation laws for such processes can be satisfied for any SW spectrum. For example, a scattering of a pair of magnons with wave vectors **1** and **2** into a pair of magnons with the same wave vectors $(\mathbf{1},\mathbf{2})\rightarrow(\mathbf{1},\mathbf{2})$, but, possibly, different phases, is always allowed. Obviously, the process $(\mathbf{1},\mathbf{1})\rightarrow(\mathbf{1},\mathbf{1})$ is also allowed in any case. While this kind of the four-magnon processes does not change the number of magnons, these processes affect the magnon phase, being, in particular, responsible for the nonlinear frequency shift (process $(\mathbf{1},\mathbf{1})\rightarrow(\mathbf{1},\mathbf{1})$), or the "phase mechanism" of the limitation of the parametric instability growth [process $(\mathbf{1},\mathbf{2})\rightarrow(\mathbf{1},\mathbf{2})$] [2].

In a general case, the fourth-order terms of the Hamiltonian function can be written as

$$\mathcal{H}^{(4)} = \frac{1}{2}\sum_{\mathbf{1234}}\Psi_{\mathbf{12},(-\mathbf{3})(-\mathbf{4})}a_{\mathbf{1}}a_{\mathbf{2}}a_{\mathbf{3}}^{*}a_{\mathbf{4}}^{*}\Delta(\mathbf{1}+\mathbf{2}-\mathbf{3}-\mathbf{4}) + \frac{1}{3}\sum_{\mathbf{1234}}[\Phi_{\mathbf{123},\mathbf{4}}a_{\mathbf{1}}a_{\mathbf{2}}a_{\mathbf{3}}a_{\mathbf{4}}^{*}+\text{c.c.}]\Delta(\mathbf{1}+\mathbf{2}+\mathbf{3}-\mathbf{4}), \tag{2.39}$$

where the coefficients are equal to

$$\Psi_{\mathbf{12},\mathbf{34}} = -\tfrac{1}{4}(\mathcal{Q}_{\mathbf{1}}+\mathcal{Q}_{\mathbf{2}}+\mathcal{Q}_{-\mathbf{3}}+\mathcal{Q}_{-\mathbf{4}}) + \tfrac{1}{4}(\Gamma_{zz,\mathbf{1}+\mathbf{3}}+\Gamma_{zz,\mathbf{1}+\mathbf{4}}+\Gamma_{zz,\mathbf{2}+\mathbf{3}}+\Gamma_{zz,\mathbf{2}+\mathbf{4}}) \tag{2.40}$$

and

$$\Phi_{\mathbf{123},\mathbf{4}} = -\tfrac{1}{4}(\mathcal{B}_{\mathbf{1}}+\mathcal{B}_{\mathbf{2}}+\mathcal{B}_{\mathbf{3}}). \tag{2.41}$$

In the notations of the above-presented coefficients we used a common convention, when the indices, which can be interchanged without any effect on the coefficient, are not separated by a comma, while the indices (or groups of indices) separated by a comma can not be interchanged. For example, $\Psi_{\mathbf{12},\mathbf{34}}=\Psi_{\mathbf{21},\mathbf{34}}$, but $\Psi_{\mathbf{12},\mathbf{34}}\neq\Psi_{\mathbf{13},\mathbf{24}}$, as one can check from the definition (2.40).

The properties of the four-magnon coefficients follow from Eqs. (2.24), (2.25), and (2.27). Namely, the coefficient $\Psi_{\mathbf{12},\mathbf{34}}$ is real valued: $\Psi_{\mathbf{12},\mathbf{34}}=\Psi_{(-\mathbf{3})(-\mathbf{4}),(-\mathbf{1})(-\mathbf{2})}$. At the same time, $\Psi_{\mathbf{12},\mathbf{34}}\neq\Psi_{(-\mathbf{1})(-\mathbf{2}),(-\mathbf{3})(-\mathbf{4})}$ in the general case, while the equality in this equation is fulfilled in the case of symmetric magnetic self-interactions.

III. CONVERSION TO THE LINEAR NORMAL MODES

A. Diagonalization of the quadratic part of the SW Hamiltonian function

The Hamiltonian expansion presented in the previous section gives, in principle, the full description of the linear and nonlinear SW processes up to the fourth order. However, this description is rather cumbersome since SWs with different wave numbers remain coupled even in the second-order terms of the Hamiltonian, which describe the *linear* SW dynamics. This demonstrates that *circularly polarized* plane waves are not the normal modes (or eigenmodes) of a considered ferromagnetic sample. It turns out that the normal SW modes have the elliptical polarization because of the presence of the anisotropy in a magnetic material and because of the anisotropic nature of the dipolar interaction. Only in some specific cases, and in the limit of purely exchange SWs, the polarization of SWs becomes circular.

The transformation from the circularly polarized plane waves to elliptically polarized linear SW normal modes is called the diagonalization of the quadratic part of the SW Hamiltonian, and is performed using a canonical linear transformation. In the SW theory, this transformation was introduced by Hostein and Primakoff [70], and is often called "third Holstein-Primakoff transformation." The similar transformation was later introduced by Bogoliubov and Valatin in the theory of superconductivity [71,72]. By this transformation, new SW variables c_k [73] are introduced, and in terms

of these variables the quadratic part of the SW Hamiltonian function assumes a diagonal form

$$\mathcal{H}^{(2)} = \sum_{k} \omega_k c_k c_k^*. \tag{3.1}$$

Then, it becomes clear that in terms of the variables c_k [73], the Hamiltonian equations of motion, in which only a quadratic part of the Hamiltonian function is retained, become uncoupled from each other and assume the simple form $dc_k/dt = -i\omega_k c_k$, which demonstrates that c_k are the linear eigenmodes of the system. Naturally, the quantity ω_k has the meaning of an eigenfrequency of a linear SW mode c_k.

In the case of a uniform external field, when the second sum in Eq. (2.36) is identically zero, the relation between the new variables c_k and the old variables a_k is given by

$$\begin{aligned} a_k(t) &= u_k c_k(t) + v_k c_{-k}^*, \\ a_{-k}^*(t) &= v_k^* c_k(t) + u_k c_{-k}^*, \end{aligned} \tag{3.2}$$

where the coefficients are equal to

$$u_k = \sqrt{\frac{\mathcal{A}_k + \omega_k}{2\omega_k}}, \quad v_k = -\frac{\mathcal{B}_k}{|\mathcal{B}_k|}\sqrt{\frac{\mathcal{A}_k - \omega_k}{2\omega_k}}, \tag{3.3}$$

and the SW frequency is defined as

$$\omega_k = \sqrt{\mathcal{A}_k^2 - |\mathcal{B}_k|^2}. \tag{3.4}$$

It is important to note that the third Holstein-Primakoff transformation (3.2) was derived in the case of *symmetric* magnetic self-interactions, and is not applicable in our more general case of nonsymmetric interactions. Using the mathematical procedure of a matrix diagonalization, we found that the quadratic part of the Hamiltonian function in the presence of antisymmetric interactions can be diagonalized by the transformation (3.2), but with the coefficients defined by the following relations:

$$\begin{aligned} u_k &= \sqrt{\frac{\mathcal{A}_{-k} + \omega_k}{2\omega_k + \mathcal{A}_{-k} - \mathcal{A}_k}}, \\ v_k &= -\frac{\mathcal{B}_k}{|\mathcal{B}_k|}\sqrt{\frac{\mathcal{A}_k - \omega_k}{2\omega_k + \mathcal{A}_{-k} - \mathcal{A}_k}}. \end{aligned} \tag{3.5}$$

The SW frequency in this case is given by

$$\omega_k = \frac{\mathcal{A}_k - \mathcal{A}_{-k}}{2} + \sqrt{\left(\frac{\mathcal{A}_k + \mathcal{A}_{-k}}{2}\right)^2 - |\mathcal{B}_k|^2}. \tag{3.6}$$

The above-derived transformation (3.5) represents an important result of this work, which allows us to derive explicit expressions for the coefficients of nonlinear interactions of *normal SW modes*.

One can easily verify that these transformations are canonical, as $(u_k^2 - |v_k|^2) = 1$, that allows to fulfill the conditions of the canonical transformations: $\{c_k, c_{k'}\} = 0$, $\{c_{-k}^*, c_{-k'}^*\} = 0$, and $\{c_k, c_{-k'}^*\} = \Delta(\boldsymbol{k} - \boldsymbol{k}')$, where $\{f, g\}$ denotes the Poisson's brackets respective to a_k, a_{-k}^*. It is also clear that in the case of symmetric magnetic self-interactions, when $\mathcal{A}_k = \mathcal{A}_{-k}$, the expressions (3.5) are reduced to the standard Holstein-Primakoff transformations (3.3).

It is important to note that in a general case $\omega_k \neq \omega_{-k}$, which means that the SW spectrum can be nonreciprocal. We also have $u_k = u_{-k}$ and $v_k = v_{-k}$, meaning that the SW structure (ellipticity) does not change with the reversal of the propagation direction. Moreover, using the definition (2.37) of $\mathcal{A}_k$, one can find that the coefficients u_k and v_k are completely independent of the term $\mathrm{Im}N_{xy,k}$, which is the only term in $\mathcal{A}_k$ and $\mathcal{B}_k$ reflecting the presence of the antisymmetric interactions. Thus, we can conclude that the presence of antisymmetric magnetic self-interactions affects the SW dispersion relation ω_k, but does not affect SW polarization properties. For the case of IDMI, this fact has been previously pointed in Refs. [55,58].

B. Transformation of the nonlinear coefficients

Now, using the transformations (3.2) and (3.5) in the SW Hamiltonian function, we can represent it in terms of the amplitudes of the SW normal modes $c_k(t)$. Although this action is straightforward, it is a rather tedious and cumbersome algebraic operation. Below, we present the general expressions for all the third- and fourth-order nonlinear coefficients in the Taylor expansion of the SW Hamiltonian function. These general expressions for the nonlinear interaction coefficients of SWs having arbitrary wave vectors are rather cumbersome, but in many important particular cases they could be significantly simplified due to a symmetry of the considered nonlinear processes. For example, among the fourth-order nonlinear processes, the most important are the processes $(\mathbf{1}, \mathbf{1}) \rightarrow (\mathbf{1}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2}) \rightarrow (\mathbf{1}, \mathbf{2})$.

The zeroth-order term of the SW Hamiltonian function, naturally, remains unchanged. The first-order term (2.35) is transformed to the following form:

$$\mathcal{H}^{(1)} = -\sum_{k} \gamma[u_k \tilde{b}_{\perp,k}^*(t) + v_k^* \tilde{b}_{\perp,-k}] c_k + \text{c.c.} \tag{3.7}$$

The term in the brackets here describes the effect of the precession ellipticity on the interaction of a SW with an external magnetic field.

The second-order term, including the interaction of magnons with dynamic magnetic fields, can be written as

$$\begin{aligned} \mathcal{H}^{(2)} = &\sum_{k} \omega_k c_k c_k^* + \sum_{\mathbf{12}} \tilde{b}_{z,\mathbf{2-1}}(u_\mathbf{1} u_\mathbf{2} + v_\mathbf{1}^* v_\mathbf{2}) c_\mathbf{1} c_\mathbf{2}^* \\ &+ \sum_{\mathbf{12}} \left[\frac{1}{2}\tilde{b}_{\mathbf{-1-2}}(u_\mathbf{1} v_\mathbf{2}^* + v_\mathbf{1}^* u_\mathbf{2}) c_\mathbf{1} c_\mathbf{2} + \text{c.c.}\right]. \end{aligned} \tag{3.8}$$

The first term here, as discussed above, defines the eigenfrequencies of the SW normal modes. The second term describes interaction of a microwave photon with two different magnons. This interaction, usually, is not very strong since the resonance condition for such a process is rather difficult to satisfy because it requires a highly spatially nonuniform distribution of the microwave magnetic field. The last term corresponds to the familiar (and much more efficient) parametric excitation of SWs under parallel microwave pumping [1,2].

In the consideration of the third-order terms in the SW Hamiltonian function, we limit ourselves to the pure nonlinear magnon-magnon interactions, and neglect the processes of the

photon-stimulated three-magnon interactions caused by the the second term in Eq. (2.38). Then, the third-order term in the Hamiltonian function describing only the magnon-magnon interactions in terms of the amplitudes of the normal SW modes is expressed as

$$\mathcal{H}^{(3)} = \frac{1}{3}\sum_{\mathbf{123}}[U^*_{\mathbf{123}}c_1c_2c_3 + \text{c.c.}]\Delta(\mathbf{1+2+3}) + \sum_{\mathbf{123}}[V^*_{\mathbf{12},3}c_1c_2c^*_3 + \text{c.c.}]\Delta(\mathbf{1+2-3}). \quad (3.9)$$

The coefficients of the three-magnon interaction are

$$U_{\mathbf{123}} = -\tfrac{1}{2}[(\mathcal{D}_1u_1 + \mathcal{D}^*_{-1}v_1)(u_2v_3 + v_2u_3) + (\mathcal{D}_2u_2 + \mathcal{D}^*_{-2}v_2)(u_1v_3 + v_1u_3) + (\mathcal{D}_3u_3 + \mathcal{D}^*_{-3}v_3)(u_1v_2 + v_1u_2)] \quad (3.10)$$

and

$$V_{\mathbf{12},3} = -\tfrac{1}{2}[(\mathcal{D}_1u_1 + \mathcal{D}^*_{-1}v_1)(u_2u_3 + v_2v^*_3) + (\mathcal{D}_2u_2 + \mathcal{D}^*_{-2}v_2)(u_1u_3 + v_1v^*_3) + (\mathcal{D}^*_3u_3 + \mathcal{D}_{-3}v^*_3)(u_1v_2 + v_1u_2)]. \quad (3.11)$$

Although these expressions appear to be cumbersome, they have a clear structure, if one recalls the above introduced rules of indices interchange in the SW nonlinear interaction coefficients.

It should be noted that the ellipticity of the magnetization precession results in the appearance of qualitatively new term in the expansion of the SW Hamiltonian function: the first term in Eq. (3.9). From a formal point of view, this term and its complex conjugate correspond to the appearance of three magnons from "vacuum" or to the annihilation of three magnons. Such a term, however, can not become resonant since, although the momentum conservation $\mathbf{1+2+3}=0$ can be satisfied, the energy conservation condition $\omega_1+\omega_2+\omega_3=0$ can not because for any stable magnetization configuration $\omega_k \geqslant 0$. Nevertheless, these nonresonant three-magnon processes can play a role in the resonant nonlinear processes of a higher (fourth) order (see explanation presented in the next subsection).

The fourth-order terms in the expansion of the Hamiltonian function after the transformation to normal SW modes acquire the following form:

$$\mathcal{H}^{(4)} = \frac{1}{2}\sum_{\mathbf{1234}} W_{\mathbf{12},\mathbf{34}}c_1c_2c^*_3c^*_4\Delta(\mathbf{1+2-3-4}) + \sum_{\mathbf{1234}}[G^*_{\mathbf{123},4}c_1c_2c_3c^*_4 + \text{c.c.}]\Delta(\mathbf{1+2+3-4}) + \frac{1}{4}\sum_{\mathbf{1234}}[R^*_{\mathbf{1234}}c_1c_2c_3c_4 + \text{c.c.}]\Delta(\mathbf{1+2+3+4}). \quad (3.12)$$

As in the case of the third-order terms, the SW ellipticity results in the appearance of new terms in the Hamiltonian expansion, in particular, the last sum in the above-presented expression. These terms are always nonresonant and, therefore, can effectively contribute only to the five-wave and higher-order magnon-magnon interaction processes. For this reason, we do not present here the explicit expression for the corresponding nonlinear coefficient $R_{\mathbf{1234}}$.

The remaining coefficients of four-magnon nonlinear interaction are given by the following expressions:

$$\begin{aligned} W_{\mathbf{12},\mathbf{34}} = {} & \Psi_{\mathbf{12},(-\mathbf{3})(-\mathbf{4})}u_1u_2u_3u_4 + \Psi_{(-\mathbf{1})(-\mathbf{2}),\mathbf{34}}v^*_1v^*_2v_3v_4 \\ & + \Psi_{\mathbf{2}(-\mathbf{3}),\mathbf{1}(-\mathbf{4})}v^*_1u_2v_3u_4 + \Psi_{(-\mathbf{2})\mathbf{3},(-\mathbf{1})\mathbf{4}}u_1v^*_2u_3v_4 \\ & + \Psi_{\mathbf{1}(-\mathbf{3}),\mathbf{2}(-\mathbf{4})}u_1v^*_2v_3u_4 + \Psi_{(-\mathbf{1})\mathbf{3},(-\mathbf{2})\mathbf{4}}v^*_1u_2u_3v_4 \\ & + \Phi_{\mathbf{123},\mathbf{4}}v^*_1v^*_2u_3v_4 + \Phi^*_{\mathbf{123},\mathbf{4}}u_1u_2v_3u_4 \\ & + \Phi_{\mathbf{412},\mathbf{3}}v^*_1v^*_2v_3u_4 + \Phi^*_{\mathbf{412},\mathbf{3}}u_1u_2u_3v_4 \\ & + \Phi_{\mathbf{341},\mathbf{2}}v^*_1u_2u_3u_4 + \Phi^*_{\mathbf{341},\mathbf{2}}u_1v^*_2v_3v_4 \\ & + \Phi_{\mathbf{234},\mathbf{1}}u_1v^*_2u_3u_4 + \Phi^*_{\mathbf{234},\mathbf{1}}v^*_1u_2v_3v_4 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} G_{\mathbf{123},\mathbf{4}} = {} & \tfrac{1}{3}[\Psi_{\mathbf{12},\mathbf{3}(-\mathbf{4})}u_1u_2v_3u_4 + \Psi_{(-\mathbf{1})(-\mathbf{2}),(-\mathbf{3})\mathbf{4}}v_1v_2u_3v^*_4 \\ & + \Psi_{\mathbf{23},\mathbf{1}(-\mathbf{4})}v_1u_2u_3u_4 + \Psi_{(-\mathbf{2})(-\mathbf{3}),\mathbf{1}(-\mathbf{4})}u_1v_2v_3v^*_4 \\ & + \Psi_{\mathbf{13},\mathbf{2}(-\mathbf{4})}u_1v_2u_3u_4 + \Psi_{(-\mathbf{1})(-\mathbf{3}),(-\mathbf{2})\mathbf{4}}v_1u_2v_3v^*_4 \\ & + \Phi_{\mathbf{123},\mathbf{4}}u_1u_2u_3u_4 + \Phi^*_{\mathbf{123},\mathbf{4}}v_1v_2v_3v^*_4 \\ & + \Phi_{\mathbf{412},\mathbf{3}}u_1u_2v_3v^*_4 + \Phi^*_{\mathbf{412},\mathbf{3}}v_1v_2u_3u_4 \\ & + \Phi_{\mathbf{341},\mathbf{2}}u_1v_2u_3v^*_4 + \Phi^*_{\mathbf{341},\mathbf{2}}v_1u_2v_3u_4 \\ & + \Phi_{\mathbf{234},\mathbf{1}}v_1u_2u_3v^*_4 + \Phi^*_{\mathbf{234},\mathbf{1}}u_1v_2v_3u_4]. \end{aligned} \quad (3.14)$$

One can easily derive these expressions by direct substitution of the transformations (3.2) to (2.39), accounting for the symmetry properties of the coefficients $\Psi_{\mathbf{12},\mathbf{34}}$ and $\Phi_{\mathbf{123},\mathbf{4}}$.

Using the above-presented expressions, one can calculate the efficiency of any relevant nonlinear SW interaction up to the fourth order. In the limit of symmetric magnetic self-interactions, when $\mathcal{D}_k = \mathcal{D}_{-k}$ and $\Psi_{\mathbf{12},\mathbf{34}} = \Psi_{(-\mathbf{1})(-\mathbf{2}),(-\mathbf{3})(-\mathbf{4})}$, these expressions are reduced to Eqs. (60)–(65) from Ref. [37]. Also, in the case of a circular polarization of SWs (e.g., in the limit of purely exchange SWs), when $u_k = 1$ and $v_k = 0$, one can find $U_{\mathbf{123}} = 0$, $V_{\mathbf{12},3} = -(\mathcal{D}_1 + \mathcal{D}_2)/2$, $W_{\mathbf{12},\mathbf{34}} = \Psi_{\mathbf{12},(-\mathbf{3})(-\mathbf{4})}$, $G_{\mathbf{123},\mathbf{4}} = \Phi_{\mathbf{123},\mathbf{4}}/3$, and $R_{\mathbf{1234}} = 0$, in full agreement with Eqs. (2.38) and (2.39).

C. Elimination of the nonresonant three-wave terms

Using the above-derived expansion of the SW Hamiltonian function in terms of the normal mode amplitudes c_k, one can investigate the nonlinear SW dynamics. Naturally, in practical particular cases it is not necessary to take into account all the existing nonlinear magnon-magnon interactions as, typically, only the resonant ones, which satisfy both the momentum and energy conservation laws, play a significant role in the interaction outcome. In real magnetic systems, due to the effect of dissipation, the frequency (energy) conservation laws can be satisfied only approximately to the accuracy of the SW damping rate. Therefore, in most practical cases it is sufficient to take into account only the "resonant" nonlinear processes since the influence of other processes, which are far from the resonance conditions, is typically negligible.

However, in certain cases the nonresonant processes cannot be simply neglected. It is known that the nonresonant nonlinear processes of a lower order can significantly influence the resonant processes of a higher order. In particular, the

nonresonant three-wave processes can contribute to the intensity of the resonant four-wave processes, as it was pointed out by Zakharov [74]. This contribution can be understood as a four-wave process, mediated by the creation and annihilation of a "virtual" magnon. For example, a four-magnon process $(\mathbf{1},\mathbf{2}) \to (\mathbf{3},\mathbf{4})$ can be a combination of two subsequent three-magnon processes $(\mathbf{1},\mathbf{2}) \to \mathbf{5}$ and $\mathbf{5} \to (\mathbf{3},\mathbf{4})$, mediated by a "virtual" magnon $\mathbf{5}$. To account for the effect of such nonresonant interaction processes, one needs to perform an additional transformation of variables, which is nonlinear and, strictly speaking, only approximately canonical. The presence of the antisymmetric interactions does not lead to any changes in this additional transformation, and we will not reproduce it here. The complete description of this transformation can be found, e.g., in Refs. [7,75,76].

IV. APPLICATION: THREE-MAGNON SPLITTING AND NONLINEAR FREQUENCY SHIFT OF SPIN WAVES SUBJECTED TO IDMI

A. Ferromagnetic nanowire

In this section we apply the above-developed formalism to the investigation of nonlinear SW interaction in a magnetic sample subjected to IDMI. Specifically, we study a nanowire made of a ferromagnetic–heavy-metal bilayer (e.g., CoFeB-Pt), having the width w_x and the ferromagnetic layer thickness h, as shown in Fig. 1. The nanowire is magnetized in its plane by an external field $\boldsymbol{B}_e$, so that the static magnetization $\boldsymbol{M}_0$ makes the angle ϕ_M with the direction of the SW propagation (axis of the nanowire). The spectrum of a nanowire, in general, contains a set of SW modes with different width profiles. Here, we restrict our analysis to the case of a quasiuniform width mode. If the nanowire is sufficiently narrow (the width is less than 100–200 nm, typically [67]), the quasiuniform mode is the fundamental mode of the nanowire, being the lowest in frequency. Also, in a certain frequency range this mode is not frequency degenerate with any other width mode of the nanowire, meaning that it can be the only one excited, and that the magnetization dynamics in this frequency range is determined by the fundamental mode only.

To apply the above-developed formalism, one needs to derive expressions for effective SW tensor $\hat{\boldsymbol{N}}_{\boldsymbol{k}}$ for the nanowire sample. It is convenient, first, to derive the tensor $\hat{\boldsymbol{N}}$ in a standard coordinate system, having axes aligned with the axes of the nanowire [coordinate system (xyz) in Fig. 1]. The

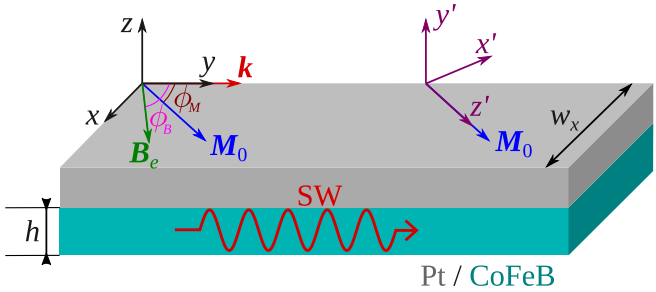

FIG. 1. A sketch of the considered bilayer nanowire, showing the directions of the bias magnetic field $\boldsymbol{B}_e$, static magnetization $\boldsymbol{M}_0$, and the principal and auxiliary coordinate systems (see text).

contribution of the IDMI in this "principal" coordinate system is described by the expression (2.21). Since in our case the SW wave vector is always parallel to the y axis, $\boldsymbol{k} = k_y \boldsymbol{e}_y$, in Eq. (2.21) we set $k_x = 0$. The exchange interaction is described by the diagonal tensor $\hat{\boldsymbol{N}}_{\mathrm{ex},k} = \lambda_{\mathrm{ex}}^2 k_y^2 \hat{\boldsymbol{I}}$ [see Eqs. (2.8) and (2.20)]. The dipolar interaction in the considered case of quasiuniform mode is described by the tensor [66,67]

$$\hat{N}_{\mathrm{dip},\boldsymbol{k}} \equiv \hat{\boldsymbol{F}}_{\boldsymbol{k}} = \frac{1}{2\pi w_x} \int_{-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{k_x w_x}{2}\right) \hat{\tilde{N}}_{\mathrm{dip},\boldsymbol{K}} dk_x, \quad (4.1)$$

where

$$\hat{\tilde{N}}_{\mathrm{dip},\boldsymbol{K}} = \begin{pmatrix} \frac{k_x^2}{K^2} f(Kh) & \frac{k_x k_y}{K^2} f(Kh) & 0 \\ \frac{k_x k_y}{K^2} f(Kh) & \frac{k_y^2}{K^2} f(Kh) & 0 \\ 0 & 0 & 1 - f(Kh) \end{pmatrix}, \quad (4.2)$$

with $K = \sqrt{k_x^2 + k_y^2}$ and $f(x) = 1 - (1 - e^{-|x|})/|x|$. In fact, the integration in Eq. (4.1) yields identical zeros for all the off-diagonal components, so the tensor $\hat{\boldsymbol{F}}_{\boldsymbol{k}}$ has only three nonzero components $F_{xx,\boldsymbol{k}}$, $F_{yy,\boldsymbol{k}}$, and $F_{zz,\boldsymbol{k}}$. Finally, we also need to take into account the perpendicular surface magnetic anisotropy, which is especially important for ultrathin magnetic films. According to Eqs. (2.10) and (2.20), the corresponding contribution to the SW tensor has only one nonzero component $(\hat{\boldsymbol{N}}_{\mathrm{an},\boldsymbol{k}})_{zz} = n_{\mathrm{an}} = -2K_s/(\mu_0 M_s^2 h)$, where K_s is the constant of the surface magnetic anisotropy.

In the Hamiltonian formalism the effective SW tensor should be expressed in the coordinate system having axis z' parallel to the static magnetization ($x'y'z'$ system in Fig. 1). The rotation of the coordinate system is expressed via the rotation tensor

$$\hat{\boldsymbol{T}} = \begin{pmatrix} -\cos\phi_M & \sin\phi_M & 0 \\ 0 & 0 & 1 \\ \sin\phi_M & \cos\phi_M & 0 \end{pmatrix}. \quad (4.3)$$

Then, the effective SW tensor in the new (auxiliary) coordinate system is expressed as $\hat{\boldsymbol{N}}_{\boldsymbol{k}}^{(x'y'z')} = \hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{N}}_{\boldsymbol{k}}^{(xyz)} \cdot \hat{\boldsymbol{T}}^{-1}$. By direct calculations, one finds the following expressions for the components of the effective SW tensor:

$$\begin{aligned}
N_{x'x',\boldsymbol{k}} &= \lambda_{\mathrm{ex}}^2 k_y^2 + F_{xx,\boldsymbol{k}} \cos^2\phi_M + F_{yy,\boldsymbol{k}} \sin^2\phi_M, \\
N_{x'y',\boldsymbol{k}} &= -N_{y'x',\boldsymbol{k}} = -i k_y \tilde{d} \sin\phi_M, \\
N_{x'z',\boldsymbol{k}} &= N_{z'x',\boldsymbol{k}} = (F_{yy,\boldsymbol{k}} - F_{xx,\boldsymbol{k}}) \sin\phi_M \cos\phi_M, \\
N_{y'y',\boldsymbol{k}} &= \lambda_{\mathrm{ex}}^2 k_y^2 + F_{zz,\boldsymbol{k}} - n_{\mathrm{an}}, \\
N_{y'z',\boldsymbol{k}} &= -N_{z'y',\boldsymbol{k}} = i k_y \tilde{d} \cos\phi_M, \\
N_{z'z',\boldsymbol{k}} &= \lambda_{\mathrm{ex}}^2 k_y^2 + F_{xx,\boldsymbol{k}} \sin^2\phi_M + F_{yy,\boldsymbol{k}} \cos^2\phi_M, \quad (4.4)
\end{aligned}$$

where we used the short notation $\tilde{d} = 2\tilde{D}/(\mu_o M_s^2)$. Using these expressions for the components of the effective SW tensor, we derived the following expressions for the coefficients (2.24)–(2.27):

$$\begin{aligned}
\mathcal{Q}_{\boldsymbol{k}} &= \frac{\omega_M}{2}\big(2\lambda_{\mathrm{ex}}^2 k_y^2 + F_{xx,\boldsymbol{k}} \cos^2\phi_M + F_{yy,\boldsymbol{k}} \sin^2\phi_M \\
&\quad + F_{zz,\boldsymbol{k}} - n_{\mathrm{an}} + 2 k_y \tilde{d} \sin\phi_M\big), \\
\mathcal{B}_{\boldsymbol{k}} &= \frac{\omega_M}{2}(F_{zz,\boldsymbol{k}} - n_{\mathrm{an}} - F_{xx,\boldsymbol{k}} \cos^2\phi_M - F_{yy,\boldsymbol{k}} \sin^2\phi_M),
\end{aligned}$$

$$\mathcal{D}_{\boldsymbol{k}} = \frac{i\omega_M}{\sqrt{2}}[(F_{yy,\boldsymbol{k}} - F_{xx,\boldsymbol{k}})\sin\phi_M + k_y\tilde{d}]\cos\phi_M,$$

$$\Gamma_{z'z',\boldsymbol{k}} = \omega_M\left(\lambda_{\rm ex}^2 k_y^2 + F_{xx,\boldsymbol{k}}\sin^2\phi_M + F_{yy,\boldsymbol{k}}\cos^2\phi_M\right). \quad (4.5)$$

The equilibrium condition (2.34) in our case is reduced to $\mu_0 M_s F_{xx,0}\sin\phi_M\cos\phi_M = B_e\sin(\phi_B - \phi_M)$, which is a pretty standard condition for a ferromagnetic nanowire. Naturally, this condition is not affected by the IDMI. For the derivation of this condition we used the identities $F_{yy,0} = 0$ and $\tilde{B}_{\perp,0} = iB_{e,x'}/\sqrt{2} = -iB_e\sin(\phi_B - \phi_M)/\sqrt{2}$.

According to Eq. (2.37), the coefficient $\mathcal{A}_{\boldsymbol{k}} = \omega_H + \mathcal{Q}_{\boldsymbol{k}}$, where $\omega_H = \gamma B_e\cos(\phi_B - \phi_M) - \omega_M F_{xx,0}\sin^2\phi_M$. Using these expressions in the general equation (3.6), we can directly calculate the dispersion relation of the linear SWs propagating in the nanowire:

$$\omega_{\boldsymbol{k}} = \sqrt{\omega_H + \omega_M\left(\lambda_{\rm ex}^2 k_y^2 + F_{xx,\boldsymbol{k}}\cos^2\phi_M + F_{yy,\boldsymbol{k}}\sin^2\phi_M\right)} \times\sqrt{\omega_H + \omega_M\left(\lambda_{\rm ex}^2 k_y^2 + F_{zz,\boldsymbol{k}} - n_{\rm an}\right)} + \omega_M k_y\tilde{d}\sin\phi_M. \quad (4.6)$$

A similar SW dispersion equation in different particular cases was previously derived in Refs. [46,47,55,60]. The influence of the IDMI results in the appearance of the last term, which is linear in the SW wave number, and is *nonreciprocal*. Note that this peculiarity is general. The nonreciprocity coming from an antisymmetric magnetic self-interaction always appears as an additive term in the dispersion relation for SWs, as can be seen from Eq. (3.6).

Next, let us look at the three-wave terms of the Hamiltonian. The coefficients of the three-magnon interaction $V_{\mathbf{12,3}}$ and $U_{\mathbf{123}}$ are proportional to the values $\mathcal{D}_{\mathbf{1}}$, $\mathcal{D}_{\mathbf{2}}$, $\mathcal{D}_{\mathbf{3}}$. The $\mathcal{D}_i$ values are proportional to $\mathcal{D}_k \sim \cos\phi_M$, independently of the length of the SW wave vector. Thus, in the case $\phi_M = 90^{\rm o}$ (often called the "Damon-Eshbach geometry"), i.e., when the nanowire is magnetized in its plane perpendicularly to the nanowire axis, the three-magnon interaction efficiency is identically zero for all the SWs independently of the magnitude of their wave vector. Thus, the fundamental SW mode of a transversely magnetized nanowire cannot split into two other SWs of the fundamental SW branch (notes on the splitting of a propagating fundamental SW into *different* SW branches are given below).

This feature could be very useful, as the three-magnon splitting is often an undesirable process, which limits the maximum power at which the SW propagation is stable. It is also important that the three-magnon splitting is prohibited for the propagation angle $\phi_M = 90^{\rm o}$, at which the nonreciprocity of a linear SW dispersion is maximum. Thus, the nanowires made of the ferromagnetic–heavy-metal bilayers with a proper magnetization direction can support the propagation of stable nonreciprocal SWs of a relatively large amplitude.

Now, we consider the influence of the IDMI on the nonlinear frequency shift of SWs propagating in a ferromagnetic nanowire. This nonlinear frequency shift is a result of the four-magnon interaction of the type $(\boldsymbol{k},\boldsymbol{k}) \to (\boldsymbol{k},\boldsymbol{k})$, and leads to the following power-dependent modification of the SW dispersion:

$$\omega_k(c_{\boldsymbol{k}}) = \omega_{\boldsymbol{k}}(0) + T_{\boldsymbol{k}}|c_{\boldsymbol{k}}|^2, \quad (4.7)$$

where $\omega_{\boldsymbol{k}}(0)$ describes the linear SW dispersion, given by Eq. (4.6), and the coefficient $T_{\boldsymbol{k}} \equiv W_{\boldsymbol{kk,kk}}$. For this particular four-wave process, the general expression (3.13) is greatly simplified, and yields

$$W_{\boldsymbol{kk,kk}} = \Psi_{\boldsymbol{k},(-\boldsymbol{k})(-\boldsymbol{k})}u_{\boldsymbol{k}}^4 + 4\Psi_{\boldsymbol{k}(-\boldsymbol{k}),\boldsymbol{k}(-\boldsymbol{k})}u_{\boldsymbol{k}}^2|v_{\boldsymbol{k}}|^2 + \Psi_{(-\boldsymbol{k})(-\boldsymbol{k}),\boldsymbol{kk}}|v_{\boldsymbol{k}}|^4 + 2\left[\Phi_{\boldsymbol{kkk},\boldsymbol{k}}u_{\boldsymbol{k}}v_{\boldsymbol{k}}^*\left(u_{\boldsymbol{k}}^2 + |v_{\boldsymbol{k}}|^2\right) + \text{c.c.}\right]. \quad (4.8)$$

For simplicity, we consider here only the case of the Damon-Eshbach geometry, i.e., $\phi_M = \phi_B = 90^{\rm o}$. As it was pointed out above, in this geometry all the three-magnon splitting processes have zero efficiency, and, therefore, one does not need to calculate the contribution from the nonresonant three-magnon processes to the four-magnon scattering efficiency.

Calculating the values of the coefficients $\Psi_{\mathbf{12,34}}$ according to Eq. (2.24), and using the definitions (3.5), we finally arrive to the following compact expression for the coefficient of the nonlinear frequency shift:

$$T_{\boldsymbol{k}} = (\gamma B_e - \bar{\mathcal{A}}_{\boldsymbol{k}}) + \frac{\mathcal{B}_{\boldsymbol{k}}^2}{2\bar{\omega}_{\boldsymbol{k}}^2}\left(\omega_M\left[4\lambda_{\rm ex}^2 k_y^2 + F_{xx,2\boldsymbol{k}} - F_{xx,0}\right] + 3\gamma B_e\right) - k_y\tilde{d}\frac{\omega_M\bar{\mathcal{A}}_{\boldsymbol{k}}}{\bar{\omega}_{\boldsymbol{k}}}. \quad (4.9)$$

Here, for brevity, we introduce $\bar{\mathcal{A}}_{\boldsymbol{k}} \equiv (\mathcal{A}_{\boldsymbol{k}} + \mathcal{A}_{-\boldsymbol{k}})/2$ and $\bar{\omega}_{\boldsymbol{k}} = (\omega_{\boldsymbol{k}} + \omega_{-\boldsymbol{k}})/2$, which are, in fact, the values of $\mathcal{A}_{\boldsymbol{k}}$ and $\omega_{\boldsymbol{k}}$ in the absence of the IDMI. The first two terms in the expression (4.9) are standard ones for a ferromagnetic nanowire without the IDMI [77]. The influence of the IDMI is reflected in the appearance of the last term, which is *nonreciprocal*. It is important that this nonreciprocal contribution to the nonlinear frequency shift has the sign that is opposite to the sign of the IDMI contribution to the linear SW dispersion [see Eqs. (4.6) and (4.9)]. This means that with the increase of the SW amplitude, the nonlinear nonreciprocal term in the dispersion law will partly compensate the linear one. Thus, the nonreciprocal splitting of the SW dispersion will decrease with the increased SW power, and the dispersion relation of a nonlinear SW becomes less nonreciprocal with the increase of the wave amplitude.

To illustrate this effect, we calculated the dispersion relation of SWs having different amplitudes for the example of a CoFeB/Pt bilayer nanowire of the width $w_x = 50$ nm, and CoFeB thickness $h = 1.5$ nm. The material parameters used in calculations are [56] saturation magnetization $\mu_0 M_s = 1.28$ T, exchange stiffness $A = 2\times10^{-11}$ J/m (corresponding exchange length $\lambda_{\rm ex} = 5.5$ nm), constant of the surface magnetic anisotropy $K_s = 5.5\times10^{-4}$ J/m^2, and the effective IDMI strength per 1.5 nm film is $Db/h = 6.6\times10^{-4}$ J/m (corresponding value $\tilde{d} = 1$). The value of the external bias magnetic field is assumed to be $B_e = 0.2$ T.

The calculated SW spectra are shown in Fig. 2. For all the SW wave vectors the nonlinear frequency shift is negative (which is typical for the in-plane magnetized ferromagnetic films and nanowires), and increases with the SW wave number. It is also clearly seen that the nonreciprocity of the SW spectrum decreases with the increase of the SW amplitude. While for small-amplitude linear SWs the spectrum is clearly nonreciprocal, having the nonreciprocal linear frequency shift

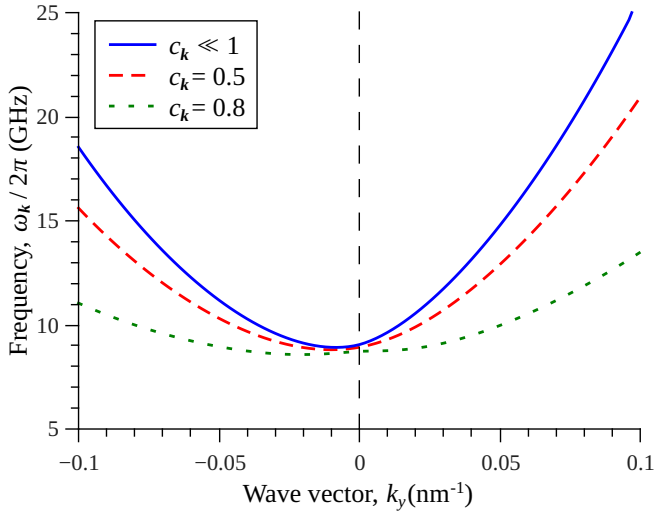

FIG. 2. SW spectrum of a CoFeB/Pt bilayer nanowire for different dimensionless amplitudes c_k of the propagating SWs. The calculation parameters are given in the text.

of $(\omega_{\boldsymbol{k}} - \omega_{-\boldsymbol{k}})/2\pi = 7.2$ GHz at $|k_y| = 0.1$ nm^{-1}, the spectrum of nonlinear SWs with the amplitude $c_{\boldsymbol{k}} = 0.8$ looks much more reciprocal, and the corresponding nonreciprocal spectral shift at the same value of the SW wave number is only 2.4 GHz. This effect can be useful for the development of power-dependent nonreciprocal devices.

B. Ferromagnetic film

The case of a ferromagnetic film, magnetized in its plane, is considered in a similar way. All the expressions (4.4) for the effective SW tensor remain the same with only the change of the dynamic dipolar contribution, namely, $F_{xx,k} = 0$, $F_{yy,k} = f(kh)$, and $F_{zz,k} = 1 - f(kh)$ [note that $f(x) = 1 - (1 - e^{-|x|})/|x|$]. Making this substitution in Eq. (4.6), one gets a well-known dispersion relation for the linear SWs in a thin ferromagnetic film [46,56].

Considering three-magnon splitting, we arrive at the same conclusion, that an SW, which propagates perpendicular to the static magnetization, cannot split into two SWs, propagating *in the same direction*. Indeed, in this case we get $\mathcal{D}_{\mathbf{1}} = \mathcal{D}_{\mathbf{2}} = \mathcal{D}_{\mathbf{3}} = 0$ and, consequently, $V_{\mathbf{12,3}} = 0$.

However, in the case of a film that is unrestricted in its plane, the SWs can propagate at an arbitrary angle to the static magnetization, and the three-magnon splitting into *noncollinear* SWs is allowed. To analyze this case, it is convenient to use the components of the SW wave vectors instead of the angles ϕ in respect to the static magnetization direction. The reference coordinate system is shown in Fig. 3(a). Then, the coefficient $\mathcal{D}_{\boldsymbol{k}}$ for an arbitrary SW can be obtained from Eqs. (4.5) in the form

$$\mathcal{D}_{\boldsymbol{k}} = \frac{i\omega_M}{\sqrt{2}}\left(\frac{k_x k_y}{k^2} f(kh) + k_x \tilde{d}\right). \tag{4.10}$$

Denoting the initial SW as the third one, and the secondary SWs as the first and second [i.e., considering the three-wave splitting process $\boldsymbol{k}_3 \to (\boldsymbol{k}_1, \boldsymbol{k}_2)$], we get $\mathcal{D}_{k_3} = 0$. Additionally, the momentum conservation rule requires that $\boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$ or, in terms of the wave-vector components, $k_{y,3} = k_{y,1} + k_{y,2}$ and $k_{x,1} = -k_{x,2}$.

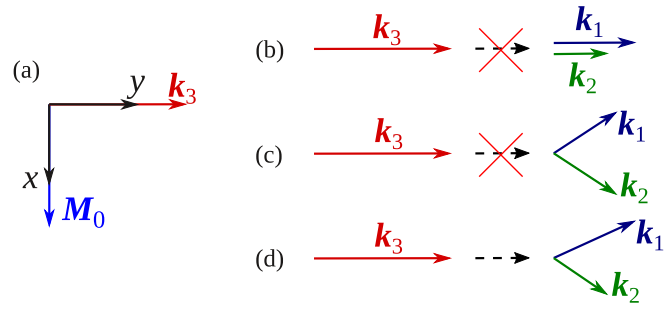

FIG. 3. Three-magnon splitting in a ferromagnetic–heavy-metal bilayer film: (a) considered geometry of the static magnetization and initial SW propagation direction, (b), (c) prohibited splitting processes, (d) allowed splitting process.

Substituting these expressions into the definition of the three-magnon scattering coefficient (3.11), we found, that in a general case, the splitting coefficient $V_{12,3}$ is not required to be zero, and the three-magnon splitting processes are allowed. The are only two exceptions: (i) splitting of an SW into two collinear SWs, and (ii) a symmetric splitting, when the wave vectors of the resulting waves possess a mirror symmetry relative to the initial SW, i.e., when $k_{y,1} = k_{y,2}$. Thus, in a ferromagnetic–heavy-metal bilayer film, three-magnon splitting can occur even in the Damon-Eshbach geometry if, of course, the resonance conditions are satisfied. A schematic illustration of the prohibited and allowed splitting processes is shown in Fig. 3.

These results can be generalized qualitatively to the case of a relatively wide ferromagnetic nanowire, in which many different width modes are degenerate, and the three-magnon splitting resonance condition $\omega_3 = \omega_1 + \omega_2$ can be satisfied for the SWs belonging to different branches of the SW spectrum (i.e., to the modes having different width profiles).

If we consider a higher-order SW width mode as a superposition of a plane wave with a transverse component of the wave vector k_x and $-k_x$, it becomes clear that in the Damon-Eshbach geometry a fundamental (uniform) SW mode cannot scatter into higher-order SWs of the same SW branch. At the same time, the scattering into SW modes having different width profiles could be allowed. The SW property that in a symmetric system a symmetric three-magnon SW splitting is prohibited, while a nonsymmetric one is allowed, is not unique, and was described for the SWs in bulk ferromagnets [2] and for SW modes of a magnetic vortex [65].

Finally, we note that the nonlinear frequency shift in the case of a bilayer film and magnetization angle $\phi_M = 90^\circ$ is also expressed by Eq. (4.9) with corresponding substitution of the dipolar tensor components. Indeed, as it is shown in Refs. [2,75], only the nonresonant three-magnon processes involving the SWs with wave vectors $\pm\boldsymbol{k}$, $\pm 2\boldsymbol{k}$, and $\boldsymbol{k} = 0$ contribute to the renormalization of the four-magnon nonlinear coefficients. In the considered case, all the three-magnon processes that involve the SWs, which are collinear, and propagate perpendicular to the static magnetization, have zero efficiency and, therefore, it is not necessary to calculate

corrections caused by these three-wave processes to the expression (4.9).

V. SUMMARY

In this work we presented a generalization of the theory of nonlinear spin wave dynamics based on the Hamiltonian approach to the case when the antisymmetric magnetic interactions are present. The developed formalism allows one to calculate the linear SW dispersion and the coefficients of nonlinear SW interactions for propagating SWs in a uniformly magnetized sample with arbitrary symmetric and antisymmetric magnetic self-interactions, quadratic in magnetization. In particular, it allows to account for the various types of a bulk and interfacial Dzyaloshinskii-Moriya interactions, spin flexoelectric interaction, etc.

It was shown that the presence of antisymmetric magnetic self-interactions reduces the symmetry of the effective SW tensor and, consequently, the symmetry of the coefficients of the Hamiltonian function expansion, both for quadratic terms, three-magnon, four-magnon, and higher-order terms. We derived the generalized third Holstein-Primakoff transformation, which diagonalizes the quadratic part of the SW Hamiltonian function in a general case, as well as the explicit expressions for the three- and four-magnon interaction coefficients. At the same time, it was shown that the antisymmetric interactions can lead to the frequency nonreciprocity of the SW spectrum, and could affect the nonlinear SW processes. Also, it turned out that the structure (ellipticity) of the linear SWs is not affected by the antisymmetric interactions, and the SWs propagating in opposite directions have the same ellipticity (in the case of a uniform SW spatial profile).

As an example of application of the developed generalized formalism, we considered nonlinear SW interactions in ferromagnetic–heavy-metal bilayer nanowires and films, subjected to the IDMI. It was shown that three-magnon splitting, that is often undesirable in practical signal-processing applications, can be completely avoided in a nanowire, which is in-plane magnetized perpendicularly to the nanowire axis.

In the case of a magnetic film which is unrestricted in plane, however, the three-magnon splitting for noncollinear SWs could be allowed. Thus, the three-magnon splitting into noncollinear SWs can occur for any magnetization direction if, of course, the resonance conditions for this splitting are satisfied.

It was also shown that the nonlinear frequency shift, which is caused by the four-magnon interaction processes, is nonreciprocal, and the sign of the nonreciprocal term in the nonlinear frequency shift is opposite to the sign of the term describing the frequency nonreciprocity of linear (small-amplitude) SWs. Consequently, the nonreciprocal shift of the SW dispersion decreases with the increase of the SW amplitude. This fact can be used for the development of power-dependent nonreciprocal devices.

ACKNOWLEDGMENTS

This work at the Oakland University was supported by the National Science Foundation of the USA (Grants No. EFMA-1641989 and No. ECCS-1708982), and by the Oakland University Foundation. R.V. acknowledges support form the State Fund for Fundamental Research of Ukraine (President's of Ukraine grant for young scientists Grant No. F75/151-2018), and from the Ministry of Education and Sciences of Ukraine (Project No. 0118U004007).

[1] A. G. Gurevich and G. A. Melkov, *Magnetization Oscillations and Waves* (CRC Press, New York, 1996), p. 464.
[2] V. S. L'vov, *Wave Turbulence under Parametric Excitation* (Springer, New York, 1994).
[3] G. Bertotti, I. Mayergoyz, and C. Serpico, *Nonlinear Magnetization Dynamics in Nanosystems* (Elsevier, Oxford, 2009).
[4] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).
[5] C. E. Patton, Phys. Status Solidi B **92**, 211 (1979).
[6] A. Nazarov, C. Patton, R. Cox, L. Chen, and P. Kabos, J. Magn. Magn. Mater. **248**, 164 (2002).
[7] K. L. Livesey, M. P. Kostylev, and R. L. Stamps, Phys. Rev. B **75**, 174427 (2007).
[8] H. Suhl, J. Appl. Phys. **31**, 935 (1960).
[9] P. Gottlieb and H. Suhl, J. Appl. Phys. **33**, 1508 (1962).
[10] W. Chen, G. de Loubens, J. M. L. Beaujour, J. Z. Sun, and A. D. Kent, Appl. Phys. Lett. **95**, 172513 (2009).
[11] Y. S. Gui, A. Wirthmann, N. Mecking, and C.-M. Hu, Phys. Rev. B **80**, 060402(R) (2009).
[12] G. A. Melkov, D. V. Slobodianiuk, V. S. Tiberkevich, G. de Loubens, O. Klein, and A. N. Slavin, IEEE Magn. Lett. **4**, 4000504 (2013).
[13] A. D. Boardman and S. A. Nikitov, Phys. Rev. B **38**, 11444 (1988).
[14] A. Y. Dobin and R. H. Victora, Phys. Rev. Lett. **90**, 167203 (2003).
[15] G. Gibson and C. Jeffries, Phys. Rev. A **29**, 811 (1984).
[16] F. M. de Aguiar and S. M. Rezende, Phys. Rev. Lett. **56**, 1070 (1986).
[17] S. N. Stitzer and P. R. Emtage, Circuits Syst. Signal Process. **4**, 227 (1985).
[18] W. S. Ishak, Proc. IEEE **76**, 171 (1988).
[19] B. A. Kalinikos, N. G. Kovshikov, and A. N. Slavin, Zh. Eksp. Teor. Fiz. **94**, 159 (1988) [Sov. Phys.–JETP **67**, 303 (1988)].
[20] M. Chen, M. A. Tsankov, J. M. Nash, and C. E. Patton, Phys. Rev. Lett. **70**, 1707 (1993).
[21] B. A. Kalinikos and M. P. Kostylev, IEEE Trans. Magn. **33**, 3445 (1997).
[22] G. A. Melkov, Y. V. Kobljanskyj, A. A. Serga, V. S. Tiberkevich, and A. N. Slavin, J. Appl. Phys. **89**, 6689 (2001).
[23] A. Slavin and V. Tiberkevich, Phys. Rev. Lett. **95**, 237201 (2005).
[24] S. Bonetti, V. Tiberkevich, G. Consolo, G. Finocchio, P. Muduli, F. Mancoff, A. Slavin, and J. Åkerman, Phys. Rev. Lett. **105**, 217204 (2010).
[25] O. R. Sulymenko, O. V. Prokopenko, V. S. Tyberkevych, A. N. Slavin, and A. A. Serga, Low Temp. Phys. **44**, 602 (2018).

[26] A. V. Sadovnikov, S. A. Odintsov, E. N. Beginin, S. E. Sheshukova, Y. P. Sharaevskii, and S. A. Nikitov, Phys. Rev. B **96**, 144428 (2017).

[27] M. Morozova, D. Romanenko, O. Matveev, S. Grishin, Y. Sharaevskii, and S. Nikitov, J. Magn. Magn. Mater. **466**, 119 (2018).

[28] T. Meyer, T. Brächer, F. Heussner, A. A. Serga, H. Naganuma, K. Mukaiyama, M. Oogane, Y. Ando, B. Hillebrands, and P. Pirro, IEEE Magn. Lett. **9**, 1 (2018).

[29] Q. Wang, R. Verba, T. Brächer, P. Pirro, and A. V. Chumak, Integrated magnonic half-adder, arXiv:1902.02855.

[30] E. Schlömann, Phys. Rev. **116**, 828 (1959).

[31] L. D. Landau and E. M. Lifshitz, *Mechanics (Vol. 1 of Course ot Theoretical Physics)* (Pergamon, Oxford, 1960).

[32] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Uspekhi Fiz. Nauk. **114**, 609 (1974) [Sov. Phys.–Usp. **17**, 896 (1975)].

[33] A. N. Slavini and P. Kabos, IEEE Trans. Magn. **41**, 1264 (2005).

[34] A. Slavin and V. Tiberkevich, IEEE Trans. Magn. **45**, 1875 (2009).

[35] A. N. Slavin and I. V. Rojdestvenski, IEEE Trans. Magn. **30**, 37 (1994).

[36] O. Dzyapko, I. Lisenkov, P. Nowik-Boltyk, V. E. Demidov, S. O. Demokritov, B. Koene, A. Kirilyuk, T. Rasing, V. Tiberkevich, and A. Slavin, Phys. Rev. B **96**, 064438 (2017).

[37] P. Krivosik and C. E. Patton, Phys. Rev. B **82**, 184428 (2010).

[38] I. E. Dzialoshinskii, Zh. Eksp. Teor. Fiz. **32**, 1547 (1957) [Sov. Phys.–JETP **5**, 1259 (1957)].

[39] T. Moriya, Phys. Rev. **120**, 91 (1960).

[40] M. Bode, M. Heide, K. von Bergmann, P. Ferriani, S. Heinze, G. Bihlmayer, A. Kubetzka, O. Pietzsch, S. Blügel, and R. Wiesendanger, Nature (London) **447**, 190 (2007).

[41] L. Udvardi and L. Szunyogh, Phys. Rev. Lett. **102**, 207204 (2009).

[42] K.-S. Ryu, L. Thomas, S.-H. Yang, and S. Parkin, Nat. Nanotechnol. **8**, 527 (2013).

[43] M. Heide, G. Bihlmayer, and S. Blügel, Phys. Rev. B **78**, 140403(R) (2008).

[44] K.-W. Kim, H.-W. Lee, K.-J. Lee, and M. D. Stiles, Phys. Rev. Lett. **111**, 216601 (2013).

[45] K. Zakeri, Y. Zhang, J. Prokop, T.-H. Chuang, N. Sakr, W. X. Tang, and J. Kirschner, Phys. Rev. Lett. **104**, 137203 (2010).

[46] J.-H. Moon, S.-M. Seo, K.-J. Lee, K.-W. Kim, J. Ryu, H.-W. Lee, R. D. McMichael, and M. D. Stiles, Phys. Rev. B **88**, 184404 (2013).

[47] M. Kostylev, J. Appl. Phys. **115**, 233902 (2014).

[48] V. L. Zhang, K. Di, H. S. Lim, S. C. Ng, M. H. Kuok, J. Yu, J. Yoon, X. Qiu, and H. Yang, Appl. Phys. Lett. **107**, 022402 (2015).

[49] G. A. Melkov, V. I. Vasyuchka, V. V. Lazovskiy, V. S. Tiberkevich, and A. N. Slavin, Appl. Phys. Lett. **89**, 252510 (2006).

[50] B. K. Kuanr, V. Veerakumar, R. Marson, S. R. Mishra, R. E. Camley, and Z. Celinski, Appl. Phys. Lett. **94**, 202505 (2009).

[51] T. J. Fal and R. E. Camley, J. Appl. Phys. **110**, 053912 (2011).

[52] J. Wu, X. Yang, S. Beguhn, J. Lou, and N. X. Sun, IEEE Trans. Microwave Theory Tech. **60**, 3959 (2012).

[53] M. Mruczkiewicz, M. Krawczyk, G. Gubbiotti, S. Tacchi, Y. A. Filimonov, D. V. Kalyabin, I. V. Lisenkov, and S. A. Nikitov, New J. Phys. **15**, 113023 (2013).

[54] R. Verba, V. Tiberkevich, and A. Slavin, Appl. Phys. Lett. **107**, 112402 (2015).

[55] T. Brächer, O. Boulle, G. Gaudin, and P. Pirro, Phys. Rev. B **95**, 064429 (2017).

[56] A. Giordano, R. Verba, R. Zivieri, A. Laudani, V. Puliafito, G. Gubbiotti, R. Tomasello, G. Siracusano, B. Azzerboni, M. Carpentieri, A. Slavin, and G. Finocchio, Sci. Rep. **6**, 36020 (2016).

[57] L. Yang, P. Dürrenfeld, X. Ruan, W. Liu, L. He, Y. Xu, and Y. Zhou, IEEE Magn. Lett. **8**, 1 (2017).

[58] R. Zivieri, A. Giordano, R. Verba, B. Azzerboni, M. Carpentieri, A. N. Slavin, and G. Finocchio, Phys. Rev. B **97**, 134416 (2018).

[59] R. L. Melcher, Phys. Rev. Lett. **30**, 125 (1973).

[60] D. Cortés-Ortuño and P. Landeros, J. Phys.: Condens. Matter **25**, 156001 (2013).

[61] D. L. Mills and I. E. Dzyaloshinskii, Phys. Rev. B **78**, 184422 (2008).

[62] T. Liu and G. Vignale, Phys. Rev. Lett. **106**, 247203 (2011).

[63] K. Y. Guslienko and A. N. Slavin, J. Magn. Magn. Mater. **323**, 2418 (2011).

[64] A. Y. Galkin, B. A. Ivanov, and C. E. Zaspel, Phys. Rev. B **74**, 144419 (2006).

[65] K. Schultheiss, R. Verba, F. Wehrmann, K. Wagner, L. Körber, T. Hula, T. Hache, A. Kákay, A. A. Awad, V. Tiberkevich, A. N. Slavin, J. Fassbender, and H. Schultheiss, Phys. Rev. Lett. **122**, 097202 (2019).

[66] R. Verba, M. Carpentieri, G. Finocchio, V. Tiberkevich, and A. Slavin, in *Spin Wave Confinement: Propagating Waves*, 2nd ed., edited by S. O. Demokritov (Pan Stanford Publishing Pte. Ltd., Singapore, 2017), pp. 385–426.

[67] Q. Wang, B. Heinz, R. Verba, M. Kewenig, P. Pirro, M. Schneider, T. Meyer, B. Lägel, C. Dubs, T. Brächer, and A. V. Chumak, Spin pinning and spin-wave dispersion in nanoscopic ferromagnetic waveguides [Phys. Rev. Lett. (to be published)], arXiv:1807.01358.

[68] In Refs. [66,67] the effective spin wave tensor is denoted by the symbol $\hat{F}_k$.

[69] G. A. Melkov, A. A. Serga, V. S. Tiberkevich, Y. V. Kobljanskij, and A. N. Slavin, Phys. Rev. E **63**, 066607 (2001).

[70] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[71] N. N. Bogoljubov, Nuovo Cimento **7**, 794 (1958).

[72] J. G. Valatin, Nuovo Cimento **7**, 843 (1958).

[73] In the most of papers these new variables are denoted as b_k. Here, we use notations c_k, in order to not confuse with the notations of microwave external field.

[74] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 1107 (1967) [Sov. Phys.–JETP **24**, 740 (1967)].

[75] V. L. Safonov, *Nonequilibrium Magnons:Theory, Experiment and Applications* (Wiley, Weinheim, 2013).

[76] Q. Shi, V. L. Safonov, M. Mino, and H. Yamazaki, Phys. Lett. A **238**, 258 (1998).

[77] R. Verba, M. Carpentieri, G. Finocchio, V. Tiberkevich, and A. Slavin, Sci. Rep. **6**, 25018 (2016).