

Vertex corrections to the dc conductivity in anisotropic multiband systems

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For an isotropic single-band system, it is well known that the semiclassical Boltzmann transport theory within the relaxation time approximation and the Kubo formula with the vertex corrections provide the same result with the $(1 - \cos \theta)$ factor in the inverse transport relaxation time. In anisotropic multiband systems, the semiclassical Boltzmann transport equation is generalized to coupled integral equations relating transport relaxation times at different angles in different bands. Using the Kubo formula, we study the vertex corrections to the dc conductivity in anisotropic multiband systems, and we derive the relation satisfied by the transport relaxation time for both elastic and inelastic scatterings, verifying that the result is consistent with the semiclassical approach.

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I. INTRODUCTION

The semiclassical Boltzmann transport theory within the relaxation time approximation enables us to investigate the transport properties of various materials theoretically. For an isotropic system in which only a single band is involved in scattering, it is well known that the transport relaxation time τ_k^{tr} for a wave vector \mathbf{k} in the relaxation time approximation can be expressed as [1]

$$\frac{1}{\tau_k^{\text{tr}}} = \int \frac{d^d k'}{(2\pi)^d} W_{\mathbf{k}\mathbf{k}'} (1 - \cos \theta_{\mathbf{k}\mathbf{k}'}), \quad (1)$$

where $W_{\mathbf{k}\mathbf{k}'}$ is the transition rate from \mathbf{k} state to \mathbf{k}' state. The inverse relaxation time is a weighted average of the collision probability in which the forward scattering ($\theta_{\mathbf{k}\mathbf{k}'} = 0$) receives reduced weight.

Using a many-body diagrammatic approach, the same result can be obtained from the current-current correlation functions supplemented with the ladder vertex corrections [2,3], as shown in Fig. 1. The single bubble diagram [Fig. 1(a)] captures the Drude conductivity with the quasiparticle lifetime τ_k^{qp} , which does not contain the $(1 - \cos \theta)$ factor, whereas the ladder diagrams [Fig. 1(b)] represent the leading-order corrections to the current vertex from impurity scattering, which give the $(1 - \cos \theta)$ factor replacing τ_k^{qp} by the transport relaxation time τ_k^{tr} in Eq. (1). While there are other sets of diagrams (for example, maximally crossed diagrams), the ladder diagrams are known to be dominant in the limit of small impurity density.

In anisotropic multiband systems, the relaxation time in the semiclassical Boltzmann theory is not simply given by Eq. (1); rather its relation is generalized to coupled integral equations relating the relaxation times at different angles in different bands [4–13]. Many materials, such as nodal line semimetals [14,15], multi-Weyl semimetals [16], and few-layer black phosphorus [17–19], have a Fermi surface that is anisotropic or crosses multiple energy bands. Thus, it is

important to include the effects of the anisotropy and multiple energy bands of the system to describe its transport properties correctly.

There have been numerous theoretical studies on the equivalence between the semiclassical Boltzmann transport equation and the Kubo formula [20–24]. As in the case of isotropic single-band systems, the semiclassical Boltzmann approach and the many-body diagrammatic approach are expected to provide consistent results for anisotropic multiband systems [21,22]. However, most of the previous works have focused on isotropic single-band systems without considering anisotropic Fermi surfaces or multiple energy bands at the Fermi energy, and there has been no systematic study demonstrating the relation between the two approaches in anisotropic multiband systems, which will provide a firm foundation for the transport theory in these systems. Although the equivalence between two approaches has been conjectured by many researchers, the rigorous proof without assuming any single-band or isotropic nature has been challenging.

In this work, we study the vertex corrections to the dc conductivity in anisotropic multiband systems in the weak scattering limit. Using a diagrammatic method and the corresponding Kubo formula, we derive the equivalent results for both elastic and inelastic scatterings obtained from the semiclassical approach. For elastic scattering, we consider randomly distributed impurities in the limit of small impurity density. For inelastic scattering, we consider the electron-phonon interaction. For both cases, we do not assume a specific form of scattering potential. We also prove that the Ward identities are satisfied in anisotropic multiband systems.

The rest of this paper is organized as follows. In Sec. II, we briefly review the semiclassical Boltzmann transport theory in anisotropic multiband systems for both elastic and inelastic scatterings. In Sec. III, using a diagrammatic approach, we develop a theory for the vertex corrections in anisotropic multiband systems, proving that the diagrammatic approach yields the same result as the semiclassical approach. We conclude in Sec. IV with a discussion on the Ward identities. In Appendix A, we present an alternative diagrammatic approach by performing the momentum integral first rather

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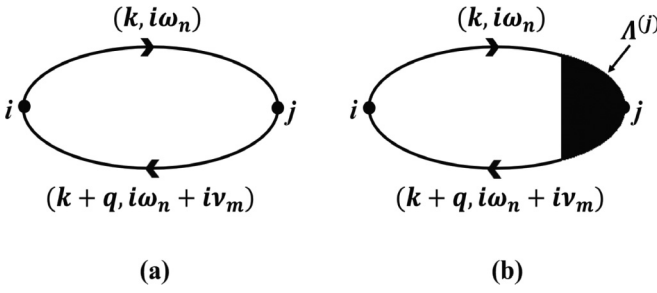


FIG. 1. Schematic diagrams for (a) Drude conductivity without the $(1 - \cos \theta)$ factor, and (b) ladder diagrams giving the $(1 - \cos \theta)$ factor.

than the frequency summation first used in the main text. In Appendixes B and C, we provide detailed derivations for the vertex corrections and the Ward identities, respectively, which were omitted in the main text.

II. SEMICLASSICAL APPROACH

In this section, we provide the semiclassical approach to calculate transport properties. We use the semiclassical Boltzmann theory within the first-order Born approximation, which is known to be valid in the weak scattering limit [25–27].

A. Elastic scattering

In this section, we briefly review the semiclassical Boltzmann theory in d -dimensional anisotropic multiband systems for elastic scattering. In the following derivation, we assume that electrons are scattered from randomly distributed impurities. In this study, we set the reduced Planck constant \hbar to 1 for convenience.

Let $f(\mathbf{r}, \mathbf{k}; t)$ denote the distribution function of an electron at the state \mathbf{k} at position \mathbf{r} at time t . The rate of change of $f(\mathbf{r}, \mathbf{k}; t)$ with respect to time satisfies the following equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{r}} \cdot \mathbf{v}_k + \frac{\partial f}{\partial \mathbf{k}} \cdot \dot{\mathbf{k}} + \frac{\partial f}{\partial t}, \quad (2)$$

where \mathbf{v}_k is the velocity at the state \mathbf{k} . Assuming a homogeneous system without the explicit time dependence in $f(\mathbf{r}, \mathbf{k}; t) \equiv f_k$, Eq. (2) reduces to $\frac{df}{dt} = \frac{\partial f_k}{\partial k} \cdot \dot{\mathbf{k}}$. In the presence of collision, the collision integral is given by

$$\left(\frac{df}{dt}\right)_c = \int \frac{d^d k'}{(2\pi)^d} [W_{kk'} f_{k'} (1 - f_k) - W_{k'k} f_k (1 - f_{k'})], \quad (3)$$

where $W_{kk'}$ is the transition rate from \mathbf{k} to \mathbf{k}' . The first term on the right-hand side of Eq. (3) describes the probability per unit time that an electron is scattered into a state \mathbf{k} , and the second term describes the probability per unit time that an electron in a state \mathbf{k} is scattered out. The Boltzmann transport equation is given by $\left(\frac{df}{dt}\right) = \left(\frac{df}{dt}\right)_c$. Expanding the theory to multiband systems, the Boltzmann transport equation can be generalized as

$$\frac{\partial f_{\alpha,k}}{\partial \mathbf{k}} \cdot \dot{\mathbf{k}} = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} \{W_{\alpha,k;\alpha',k'} f_{\alpha',k'} (1 - f_{\alpha,k}) - W_{\alpha',k';\alpha,k} f_{\alpha,k} (1 - f_{\alpha',k'})\}, \quad (4)$$

where α and α' are band indices.

For elastic scattering, the transition rate $W_{\alpha',k';\alpha,k}$ is given in the following form:

$$W_{\alpha',k';\alpha,k} = 2\pi n_{\text{imp}} |V_{\alpha',k';\alpha,k}|^2 \delta(\xi_{\alpha,k} - \xi_{\alpha',k'}), \quad (5)$$

where n_{imp} is the impurity density, $V_{\alpha',k';\alpha,k} = \langle \alpha', \mathbf{k}' | V | \alpha, \mathbf{k} \rangle$ is the matrix element of the impurity potential V , which describes a scattering from (α, \mathbf{k}) to (α', \mathbf{k}') , and $\xi_{\alpha,k} \equiv \varepsilon_{\alpha,k} - \mu$ is the energy of an electron at the state (α, \mathbf{k}) measured from the chemical potential μ . Here, the effect of electron-electron interactions can be taken into account through the screening of the impurity potential. Note that $W_{\alpha,k;\alpha',k'} = W_{\alpha',k';\alpha,k}$; thus, Eq. (4) reduces to

$$\frac{\partial f_{\alpha,k}}{\partial \mathbf{k}} \cdot \dot{\mathbf{k}} = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} (f_{\alpha',k'} - f_{\alpha,k}). \quad (6)$$

In the presence of a small external electric field, we assume that $f_{\alpha,k}$ deviates slightly from $f_{\alpha,k}^0$:

$$f_{\alpha,k} = f_{\alpha,k}^0 + \delta f_{\alpha,k}, \quad (7)$$

where $f_{\alpha,k}^0 \equiv f^0(\xi_{\alpha,k}) = [e^{\beta \xi_{\alpha,k}} + 1]^{-1}$ is the Fermi-Dirac distribution function in equilibrium with $\beta = \frac{1}{k_B T}$. We assume that the deviation $\delta f_{\alpha,k}$ can be parametrized up to first order of the electric field \mathbf{E} as follows [11–13]:

$$\delta f_{\alpha,k} = (-e) \sum_i E^{(i)} v_{\alpha,k}^{(i)} \tau_{\alpha,k}^{(i)} S^0(\xi_{\alpha,k}), \quad (8)$$

where $S^0(\xi) = -\frac{\partial f^0(\xi)}{\partial \xi} = \beta f^0(\xi)[1 - f^0(\xi)]$, and $v_{\alpha,k}^{(i)}$ and $\tau_{\alpha,k}^{(i)}$ are the velocity and transport relaxation time along the i th direction at the state (α, \mathbf{k}) , respectively. Inserting Eq. (8) into Eq. (6), we have an integral equation relating the relaxation times at different states,

$$1 = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} \left(\tau_{\alpha,k}^{(i)} - \frac{v_{\alpha',k'}^{(i)}}{v_{\alpha,k}^{(i)}} \tau_{\alpha',k'}^{(i)} \right). \quad (9)$$

Note that, for an isotropic single-band system, Eq. (9) reduces to Eq. (1).

The deviation of the electron distribution function from the equilibrium value gives rise to the current density

$$\mathbf{J}^{(i)} = g \sum_{\alpha} \int \frac{d^d k}{(2\pi)^d} (-e) v_{\alpha,k}^{(i)} \delta f_{\alpha,k} = \sum_j \sigma_{ij} E^{(j)}, \quad (10)$$

where g is the degeneracy factor and σ_{ij} is a matrix element of the conductivity tensor given by

$$\sigma_{ij} = g e^2 \sum_{\alpha} \int \frac{d^d k}{(2\pi)^d} S^0(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \tau_{\alpha,k}^{(j)}. \quad (11)$$

B. Inelastic scattering

For inelastic scattering, such as phonon-mediated scattering, Eq. (9) is no longer valid and the principle of detailed balance should be considered [28]:

$$W_{\alpha',k';\alpha,k} f_{\alpha,k}^0 (1 - f_{\alpha',k'}^0) = W_{\alpha,k;\alpha',k'} f_{\alpha',k'}^0 (1 - f_{\alpha,k}^0). \quad (12)$$

Expanding up to first order of δf , Eq. (4) reduces to

$$\frac{\partial f_{\alpha,k}^0}{\partial \mathbf{k}} \cdot \dot{\mathbf{k}} = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} \times \left(\frac{f_{\alpha,k}^0}{f_{\alpha',k'}^0} \delta f_{\alpha',k'} - \frac{1 - f_{\alpha',k'}^0}{f_{\alpha,k}^0} \delta f_{\alpha,k} \right). \quad (13)$$

Using the parametrization in Eq. (8), we obtain an integral equation for inelastic scattering relating the relaxation times at different states as follows:

$$1 = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} \times \left(\tau_{\alpha,k}^{(i)} - \frac{v_{\alpha',k'}^{(i)}}{v_{\alpha,k}^{(i)}} \tau_{\alpha',k'}^{(i)} \right) \left(\frac{1 - f_{\alpha',k'}^0}{1 - f_{\alpha,k}^0} \right). \quad (14)$$

Note that the integral equation for inelastic scattering is different from that for elastic scattering by the factor $\frac{1 - f_{\alpha',k'}^0}{1 - f_{\alpha,k}^0}$.

For phonon scattering, the transition rate $W_{\alpha',k';\alpha,k}$ is given by [29]

$$W_{\alpha',k';\alpha,k} = 2\pi \sum_{\lambda} |\langle \alpha', \mathbf{k}' | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \times \{ [n_B(\Omega_{\lambda,q}) + 1] \delta(\xi_{\alpha',k'} - \xi_{\alpha,k} + \Omega_{\lambda,q}) + n_B(\Omega_{\lambda,q}) \delta(\xi_{\alpha',k'} - \xi_{\alpha,k} - \Omega_{\lambda,q}) \}, \quad (15)$$

where $n_B(\Omega_{\lambda,q}) = [e^{\beta\Omega_{\lambda,q}} - 1]^{-1}$ is the Bose-Einstein distribution function, and M_{λ} denotes the electron-phonon interaction for the phonon polarization λ . Here, the first (second) term on the right-hand side of Eq. (15) describes the emission (absorption) of a phonon with momentum $\mathbf{q} = \pm(\mathbf{k} - \mathbf{k}')$ and frequency $\Omega_{\lambda,q}$. In this work, umklapp processes are neglected since we are interested in the weak scattering limit where normal processes are dominant [30].

III. DIAGRAMMATIC APPROACH

In this section, using a diagrammatic approach, we develop a theory for the vertex corrections to the dc conductivity for elastic and inelastic scatterings in d -dimensional anisotropic multiband systems, and we verify that the results are consistent with those obtained from the semiclassical Boltzmann equation in Sec. II.

The dc conductivity is obtained by taking the long-wavelength limit and then the static limit as follows [2]:

$$\sigma_{ij}^{\text{dc}} = - \lim_{\nu \rightarrow 0} \frac{1}{\nu} \text{Im} \Pi_{ij}(\mathbf{q} = \mathbf{0}, \nu), \quad (16)$$

where $\Pi_{ij}(\mathbf{q}, \nu)$ is the retarded current-current response function, which is obtained using the analytic continuation $i\nu_m \rightarrow \nu + i0^+$ of the current-current response function $\Pi_{ij}(\mathbf{q}, i\nu_m)$ for an imaginary frequency. First, we consider the single bubble diagram without the vertex corrections [Fig. 1(a)],

$$\Pi_{ij}(i\nu_m) = \frac{ge^2}{\beta\mathcal{V}} \sum_{\alpha, \alpha', \mathbf{k}, i\omega_n} \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n) v_{\alpha, \alpha'}^{(i)}(\mathbf{k}, \mathbf{k}) \times \mathcal{G}_{\alpha'}(\mathbf{k}, i\omega_n + i\nu_m) v_{\alpha', \alpha}^{(j)}(\mathbf{k}), \quad (17)$$

where $\Pi_{ij}(i\nu_m) = \Pi_{ij}(\mathbf{q} = \mathbf{0}, i\nu_m)$, ω_n and ν_m are fermionic and bosonic Matsubara frequencies, respectively, \mathcal{V} is the volume of the system, $\mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n)$ is the interacting Green's function, and $v_{\alpha', \alpha}^{(j)}(\mathbf{k}) = \langle \alpha', \mathbf{k} | \hat{v}^{(j)} | \alpha, \mathbf{k} \rangle$ is the matrix element of the velocity operator $\hat{v}^{(j)} = \frac{\partial \hat{H}}{\partial k_j}$ along the j th direction. The velocity matrix element can be expressed as

$$v_{\alpha', \alpha}^{(j)}(\mathbf{k}) = v_{\alpha, \alpha'}^{(j)}(\alpha', \mathbf{k} | \alpha, \mathbf{k}) + (\varepsilon_{\alpha, \mathbf{k}} - \varepsilon_{\alpha', \mathbf{k}}) \langle \alpha', \mathbf{k} | \frac{\partial}{\partial k_j} | \alpha, \mathbf{k} \rangle. \quad (18)$$

In the $\nu \rightarrow 0$ limit, the second term in Eq. (18) does not contribute to $\Pi_{ij}(i\nu_m)$ as finite-energy transfer between (α', \mathbf{k}) and (α, \mathbf{k}) is not allowed in the single bubble diagram. By choosing an orthonormal basis set, the right-hand side of Eq. (18) simply reduces to $v_{\alpha, \alpha}^{(j)} \delta_{\alpha', \alpha}$, and only diagonal elements of the velocity matrix remain in Eq. (17).

Incorporating the ladder diagrams, we finally obtain the current-current response function at low frequencies supplemented with the vertex corrections as follows [Fig. 1(b)]:

$$\Pi_{ij}(i\nu_m) = \frac{ge^2}{\beta\mathcal{V}} \sum_{\alpha, \mathbf{k}, i\omega_n} \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n) v_{\alpha, \alpha}^{(i)} \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n + i\nu_m) \times v_{\alpha, \alpha}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m), \quad (19)$$

where $v_{\alpha, \alpha}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m)$ is the vertex corresponding to the current density operator along the j th direction. Note that we only included the diagonal elements of the velocity matrix, as discussed above.

To calculate the dc conductivity using a many-body diagrammatic method, we can either perform the Matsubara frequency summation first or the momentum integral first. Here, we use the former method where the frequency summation is performed first, and we present the other method in Appendix A.

A. Elastic scattering

For elastic scattering, we consider randomly distributed impurities. The effect of impurities can be considered using the disorder-averaged Green's function

$$\mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \xi_{\alpha, \mathbf{k}} - \Sigma_{\alpha}(\mathbf{k}, i\omega_n)}, \quad (20)$$

where $\Sigma_{\alpha}(\mathbf{k}, i\omega_n)$ is the electron self-energy from impurity scattering. The imaginary part of the self-energy can be associated with the quasiparticle lifetime $\tau_{\alpha, \mathbf{k}}^{\text{qp}}$ as $\text{Im} \Sigma_{\alpha}(\mathbf{k}, i\omega_n) = -\frac{1}{2\tau_{\alpha, \mathbf{k}}^{\text{qp}}} \text{sgn}(\omega_n)$. Assuming small impurity density, the inverse of the quasiparticle lifetime is given by

$$\frac{1}{\tau_{\alpha, \mathbf{k}}^{\text{qp}}} = 2\pi n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}|^2 \delta(\xi_{\alpha, \mathbf{k}} - \xi_{\alpha', \mathbf{k}'}) = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}. \quad (21)$$

Note that the integrand on the right-hand side of the first line is identical to $W_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}$ defined in Eq. (5).

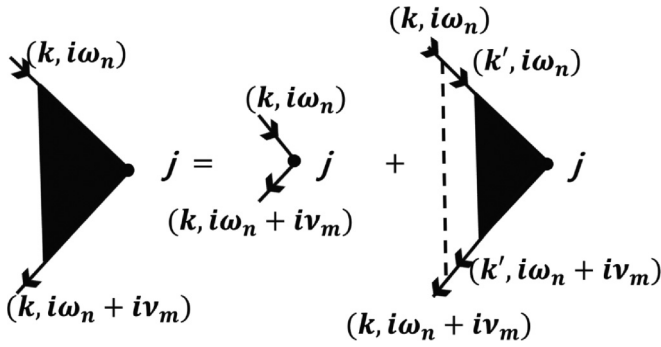


FIG. 2. Diagrams for the ladder vertex corrections for elastic scattering.

Within the ladder approximation, the vertex correction is approximated by a sum of ladder diagrams given by the self-consistent Dyson form as follows (Fig. 2):

$$\begin{aligned} v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + iv_m) \\ = v_{\alpha,k}^{(j)} + \frac{n_{\text{imp}}}{\mathcal{V}} \sum_{\alpha',k'} |V_{\alpha',k';\alpha,k}|^2 \mathcal{G}_{\alpha'}(\mathbf{k}', i\omega_n) \\ \times v_{\alpha',k'}^{(j)} \Lambda_{\alpha'}^{(j)}(\mathbf{k}', i\omega_n, i\omega_n + iv_m) \mathcal{G}_{\alpha'}(\mathbf{k}', i\omega_n + iv_m). \end{aligned} \quad (22)$$

As the form of the self-consistent Dyson's equation in Eq. (22) is analogous to Eq. (9), $\Lambda^{(j)}$ can be related to the transport relaxation time. Here, we derive this relation rigorously.

Let us first calculate the current-current response function. Equation (19) can be expressed as the frequency sum of the following form:

$$\begin{aligned} \Pi_{ij}(iv_m) &\equiv \frac{1}{\beta} \sum_{i\omega_n} P(i\omega_n, i\omega_n + iv_m) \\ &= - \oint_C \frac{dz}{2\pi i} \frac{P(z, z + iv_m)}{e^{\beta z} + 1}, \end{aligned} \quad (23)$$

where $P(i\omega_n, i\omega_n + iv_m)$ is a complex function whose summation can be performed via integration along the contour C shown in Fig. 3. Note that the contour integral in Eq. (23) has poles at $z = i\omega_n$.

After performing the analytic continuation $iv_m \rightarrow \nu + i0^+$ and assuming small impurity density, we finally obtain the dc conductivity as (see Appendix B)

$$\sigma_{ij} = \int \frac{d\xi}{2\pi} S^0(\xi) P^{\text{AR}}(\xi, \xi), \quad (24)$$

where $P^{\text{AR}}(\xi, \xi) = P(\xi - i0^+, \xi + i0^+)$. Henceforth, the superscripts A and R represent advanced and retarded functions, respectively. Note that

$$\begin{aligned} P^{\text{AR}}(\xi, \xi) \\ = \frac{ge^2}{\mathcal{V}} \sum_{\alpha,k} v_{\alpha,k}^{(i)} G_{\alpha}^{\text{A}}(\mathbf{k}, \xi) v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi, \xi) G_{\alpha}^{\text{R}}(\mathbf{k}, \xi) \\ = \frac{2\pi ge^2}{\mathcal{V}} \sum_{\alpha,k} v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \tau_{\alpha,k}^{\text{qp}} \delta(\xi - \xi_{\alpha,k}) \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi, \xi), \end{aligned} \quad (25)$$

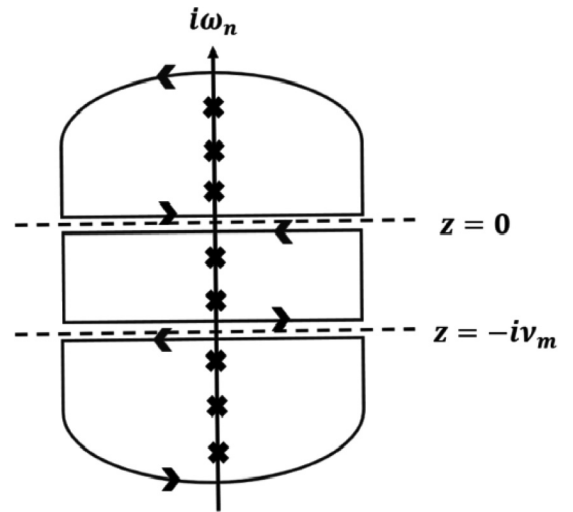


FIG. 3. Contour C used for the contour integral in Eq. (23) having two branch cuts along the axes $z = 0$ and $z = -i\nu_m$.

where $G_{\alpha}^{\text{R,A}}(\mathbf{k}, \xi) = \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n \rightarrow \xi \pm i0^+)$. Here, we used $G_{\alpha}^{\text{A}}(\mathbf{k}, \xi) G_{\alpha}^{\text{R}}(\mathbf{k}, \xi) \rightarrow 2\pi \tau_{\alpha,k}^{\text{qp}} \delta(\xi - \xi_{\alpha,k})$ in the $n_{\text{imp}} \rightarrow 0$ limit.

Therefore, from Eqs. (24) and (25), the dc conductivity is given by

$$\sigma_{ij} = \frac{ge^2}{\mathcal{V}} \sum_{\alpha,k} S^0(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \tau_{\alpha,k}^{\text{qp}} \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi_{\alpha,k}, \xi_{\alpha,k}). \quad (26)$$

For detailed derivations, see Appendix B.

Comparing Eq. (26) with Eq. (11), it is natural to relate $\Lambda^{(j)}$ to the transport relaxation time along the corresponding direction. To determine this relation, let us return to the self-consistent Dyson's equation for the vertex correction in Eq. (22). After analytic continuation, it reduces to

$$\begin{aligned} \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi, \xi) \\ = 1 + \frac{n_{\text{imp}}}{\mathcal{V}} \sum_{\alpha',k'} |V_{\alpha',k';\alpha,k}|^2 \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \\ \times G_{\alpha'}^{\text{A}}(\mathbf{k}', \xi) \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k}', \xi, \xi) G_{\alpha'}^{\text{R}}(\mathbf{k}', \xi) \\ = 1 + \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k'}^{\text{qp}} \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k}', \xi, \xi). \end{aligned} \quad (27)$$

Let us define the transport relaxation time along the j th direction as

$$\tau_{\alpha,k}^{(j)} \equiv \tau_{\alpha,k}^{\text{qp}} \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi_{\alpha,k}, \xi_{\alpha,k}). \quad (28)$$

Subsequently, Eq. (27) can be rewritten as

$$\frac{\tau_{\alpha,k}^{(j)}}{\tau_{\alpha,k}^{\text{qp}}} = 1 + \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k';\alpha,k} \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k'}^{(j)}. \quad (29)$$

Using the definition of the quasiparticle lifetime in Eq. (21), we obtain an integral equation for the transport relaxation time

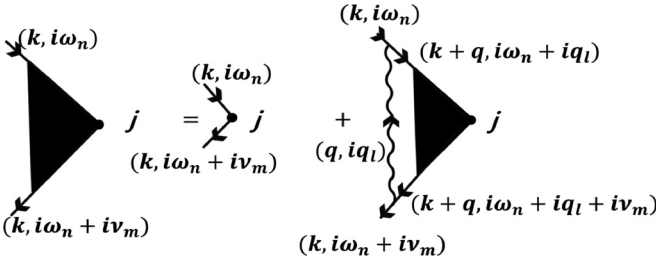


FIG. 4. Dyson's equation for the ladder vertex corrections for inelastic scattering.

for elastic scattering in anisotropic multiband systems as

$$1 = \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha', k'; \alpha, k} \left(\tau_{\alpha, k}^{(j)} - \frac{v_{\alpha', k'}}{v_{\alpha, k}^{(j)}} \tau_{\alpha', k'}^{(j)} \right), \quad (30)$$

which is the same as the semiclassical result in Eq. (9). Furthermore, using the definition of the transport relaxation time, we can easily verify that Eq. (26) is consistent with Eq. (11) obtained from the semiclassical approach.

B. Inelastic scattering

As in the case of elastic scattering, we develop a theory for the vertex corrections for inelastic scattering. Here, we specifically consider phonon-mediated scattering, which yields intrinsic resistivity in a metal.

The self-consistent Dyson's equation of the vertex part for phonon scattering is given by (Fig. 4)

$$\begin{aligned} & v_{\alpha, k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + iv_m) \\ &= v_{\alpha, k}^{(j)} - \frac{1}{\beta \mathcal{V}} \sum_{\alpha', q, iq_l, \lambda} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 D_{\lambda}(\mathbf{q}, iq_l) \\ & \quad \times \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + iq_l) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + iq_l + iv_m) \\ & \quad \times v_{\alpha', \mathbf{k} + \mathbf{q}}^{(j)} \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, i\omega_n + iq_l, i\omega_n + iq_l + iv_m), \quad (31) \end{aligned}$$

where q_l is a bosonic Matsubara frequency and $D_{\lambda}(\mathbf{q}, iq_l) = \frac{2\Omega_{\lambda, q}}{(iq_l)^2 - \Omega_{\lambda, q}^2}$ is the noninteracting phonon Green's function with the renormalized phonon frequency $\Omega_{\lambda, q}$. Equation (31) can be rewritten as

$$\begin{aligned} & \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + iv_m) \\ &= 1 - \sum_{\alpha', \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \frac{v_{\alpha', \mathbf{k} + \mathbf{q}}^{(j)}}{v_{\alpha, k}^{(j)}} \\ & \quad \times \frac{1}{\beta} \sum_{iq_l} Q(iq_l + i\omega_n, iq_l + i\omega_n + iv_m), \quad (32) \end{aligned}$$

where

$$\begin{aligned} & Q(iq_l + i\omega_n, iq_l + i\omega_n + iv_m) \\ & \equiv \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, iq_l + i\omega_n) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, iq_l + i\omega_n + iv_m) \\ & \quad \times D_{\lambda}(\mathbf{q}, iq_l) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, iq_l + i\omega_n, iq_l + i\omega_n + iv_m). \quad (33) \end{aligned}$$

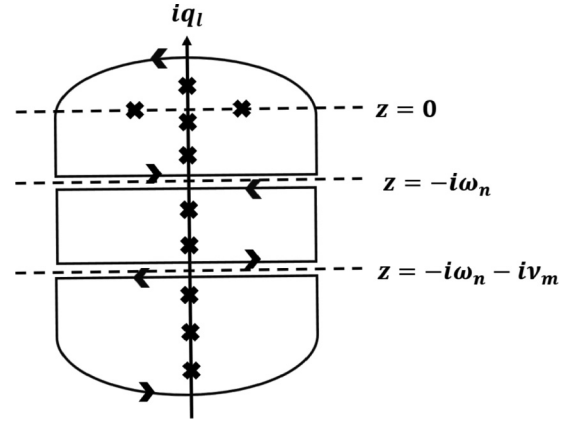


FIG. 5. Contour C' used for the contour integral in Eq. (34) having two branch cuts along the axes $z = -i\omega_n$ and $z = -i\omega_n - iv_m$.

The summation of $Q(iq_l + i\omega_n, iq_l + i\omega_n + iv_m)$ over the bosonic Matsubara frequency q_l can be performed with the aid of a contour integral along the contour C' shown in Fig. 5:

$$\oint_{C'} \frac{dz}{2\pi i} \frac{Q(z + i\omega_n, z + i\omega_n + iv_m)}{e^{\beta z} - 1}. \quad (34)$$

Note that the contour integral in Eq. (34) has poles at $z = iq_l$ as well as $z = \pm \Omega_{\lambda, q}$.

According to Eq. (26), the only vertex part contributing to the dc conductivity is $\Lambda^{(j)\text{AR}}(\mathbf{k}, \xi_{\alpha, k}, \xi_{\alpha, k})$. After performing the analytic continuation $i\omega_n \rightarrow \xi - i0^+$ and $i\omega_n + iv_m \rightarrow \xi + v + i0^+$ in Eq. (32), and assuming weak scattering, the self-consistent Dyson's equation at $v = 0$ reduces to

$$\begin{aligned} & \Lambda_{\alpha}^{(j)\text{AR}}(\mathbf{k}, \xi, \xi) \\ &= 1 + 2\pi \sum_{\alpha', \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \frac{v_{\alpha', \mathbf{k} + \mathbf{q}}^{(j)}}{v_{\alpha, k}^{(j)}} \\ & \quad \times \tau_{\alpha', \mathbf{k} + \mathbf{q}}^{\text{qp}} \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k} + \mathbf{q}, \xi_{\alpha', \mathbf{k} + \mathbf{q}}, \xi_{\alpha', \mathbf{k} + \mathbf{q}}) \\ & \quad \times \{ [n_{\text{B}}(\Omega_{\lambda, q}) + f^0(\xi + \Omega_{\lambda, q})] \delta(\xi + \Omega_{\lambda, q} - \xi_{\alpha', \mathbf{k} + \mathbf{q}}) \\ & \quad + [n_{\text{B}}(\Omega_{\lambda, q}) + 1 - f^0(\xi - \Omega_{\lambda, q})] \delta(\xi - \Omega_{\lambda, q} - \xi_{\alpha', \mathbf{k} + \mathbf{q}}) \}. \quad (35) \end{aligned}$$

For detailed derivations, see Appendix B.

Finally, let us replace $\Lambda^{(j)}$ by the transport relaxation time defined in Eq. (28). The quasiparticle lifetime in the presence of phonon scattering is given by [29]

$$\begin{aligned} & \frac{1}{\tau_{\alpha, k}^{\text{qp}}} = 2\pi \sum_{\alpha', \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \\ & \quad \times \{ [1 + n_{\text{B}}(\Omega_{\lambda, q}) - f_{\alpha', \mathbf{k} + \mathbf{q}}^0] \delta(\xi_{\alpha, k} - \xi_{\alpha', \mathbf{k} + \mathbf{q}} - \Omega_{\lambda, q}) \\ & \quad + [n_{\text{B}}(\Omega_{\lambda, q}) + f_{\alpha', \mathbf{k} + \mathbf{q}}^0] \delta(\xi_{\alpha, k} - \xi_{\alpha', \mathbf{k} + \mathbf{q}} + \Omega_{\lambda, q}) \}. \quad (36) \end{aligned}$$

Here, we replaced $f^0(\xi_{\alpha', \mathbf{k} + \mathbf{q}})$ by $f_{\alpha', \mathbf{k} + \mathbf{q}}^0$.

Notably, for the phonon emission process ($\xi_{\alpha,k} = \xi_{\alpha',k+q} + \Omega_{\lambda,q}$),

$$1 + n_B(\Omega_{\lambda,q}) - f_{\alpha',k+q}^0 = [1 + n_B(\Omega_{\lambda,q})] \left(\frac{1 - f_{\alpha',k+q}^0}{1 - f_{\alpha,k}^0} \right), \quad (37)$$

whereas for the phonon absorption process ($\xi_{\alpha,k} = \xi_{\alpha',k+q} - \Omega_{\lambda,q}$),

$$n_B(\Omega_{\lambda,q}) + f_{\alpha',k+q}^0 = n_B(\Omega_{\lambda,q}) \left(\frac{1 - f_{\alpha',k+q}^0}{1 - f_{\alpha,k}^0} \right). \quad (38)$$

Thus, Eq. (36) can be rewritten as

$$\frac{1}{\tau_{\alpha,k}^{\text{qp}}} = \sum_{\alpha'} \int \frac{d^d q}{(2\pi)^d} W_{\alpha',k+q;\alpha,k} \left(\frac{1 - f_{\alpha',k+q}^0}{1 - f_{\alpha,k}^0} \right), \quad (39)$$

where $W_{\alpha',k+q;\alpha,k}$ is the transition rate defined in Eq. (15).

Using Eqs. (37) and (38), and replacing $\Lambda^{(j)}$ by the transport relaxation time, we can rewrite Eq. (35) as

$$\begin{aligned} \frac{\tau_{\alpha,k}^{(j)}}{\tau_{\alpha,k}^{\text{qp}}} &= 1 + \sum_{\alpha'} \int \frac{d^d q}{(2\pi)^d} W_{\alpha',k+q;\alpha,k} \\ &\times \frac{v_{\alpha',k+q}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k+q}^{(j)} \left(\frac{1 - f_{\alpha',k+q}^0}{1 - f_{\alpha,k}^0} \right). \end{aligned} \quad (40)$$

Using the definition of quasiparticle lifetime in Eq. (39), we finally obtain an integral equation for the transport relaxation time for inelastic scattering in anisotropic multiband systems given by

$$\begin{aligned} 1 &= \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} W_{\alpha',k+q;\alpha,k} \\ &\times \left(\tau_{\alpha,k}^{(j)} - \frac{v_{\alpha',k+q}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k+q}^{(j)} \right) \left(\frac{1 - f_{\alpha',k+q}^0}{1 - f_{\alpha,k}^0} \right), \end{aligned} \quad (41)$$

which is consistent with the semiclassical result in Eq. (14).

IV. DISCUSSION AND SUMMARY

The validity of the diagrammatic approach for the vertex corrections in this work can be verified by testing the Ward identity [31,32]. The Ward identity is the exact relationship between the self-energy and the vertex correction arising from the continuity equation, which must hold if the corresponding diagrams are properly included both in the self-energy and the vertex correction. For both elastic and inelastic scatterings, we demonstrate that the Ward identity is satisfied in anisotropic multiband systems as follows (see Appendix C):

$$v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n) = v_{\alpha,k}^{(j)} + \frac{\partial \Sigma_{\alpha}(\mathbf{k}, i\omega_n)}{\partial k^{(j)}}, \quad (42)$$

which indicates that we have employed the proper vertex part $\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + iv_m)$ corresponding to the ladder self-energy diagrams.

The ladder approximation is known to be valid in the limit of small impurity density or weak scattering. As the impurity density increases, further terms called the maximally crossed diagrams, which arise from the coherent interference between electron wave functions, should be incorporated into the vertex corrections, leading to the quantum corrections to the conductivity called the weak localization.

In summary, using a many-body diagrammatic approach, we studied the vertex corrections to the dc conductivity in anisotropic multiband systems, demonstrating that the diagrammatic approach provides an equivalent result to that obtained from the semiclassical Boltzmann approach for both elastic and inelastic scatterings. Our work provides a many-body justification for the generalized Boltzmann transport theory given by coupled integral equations for anisotropic multiband systems, which is essential to capture the effects of the anisotropy and multiple energy bands in transport correctly.

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APPENDIX A: ALTERNATIVE DERIVATIONS FOR THE VERTEX CORRECTIONS

In this Appendix, following Chap. 10 of Coleman [3], we derive the vertex corrections to the dc conductivity for elastic impurity scattering by performing the momentum integral first. Let us first consider the self-consistent Dyson's equation in Eq. (22). The electrons on the Fermi surface mainly contribute to the dc conductivity. Therefore, we focus on calculating the vertex corrections for electrons at the Fermi energy. As the two Green's functions on the right-hand side of Eq. (22) become appreciable near the Fermi energy at low frequencies, we separate the two terms from the rest. Thus, Eq. (22) reduces to

$$\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + iv_m) \approx 1 + n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha',k';\alpha,k}|^2 \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \Lambda_{\alpha'}^{(j)}(\mathbf{k}', i\omega_n, i\omega_n + iv_m) \delta(\xi_{\alpha',k'}) \int d\xi \mathcal{G}(\xi, i\omega_n) \mathcal{G}(\xi, i\omega_n + iv_m), \quad (A1)$$

where

$$\mathcal{G}(\xi, i\omega_n) = \frac{1}{i\omega_n - \xi + i \text{sgn}(\omega_n) \frac{1}{2\tau_{\alpha,k}^{\text{qp}}}} \quad (A2)$$

is the disorder-averaged Green's function up to the first-order Born approximation [see Eq. (20) in the main text]. Note that impurity scattering of electrons at the Fermi energy provides a constant contribution to the real part of the self-energy, which can be absorbed in the chemical potential.

Let us calculate the energy integral first. We can compute the energy integral in Eq. (A1) with the aid of a contour integral method as follows:

$$\int d\xi \mathcal{G}(\xi, i\omega_n) \mathcal{G}(\xi, i\omega_n + i\nu_m) = \frac{2\pi \Theta(\nu_m, \omega_n)}{\nu_m + \frac{1}{\tau_{\alpha',k}^{\text{qp}}}}, \quad (\text{A3})$$

where

$$\Theta(\nu_m, \omega_n) = \begin{cases} 1 & \text{for } -\nu_m < \omega_n < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A4})$$

Here, we assumed $\nu_m > 0$. Thus, Eq. (A1) reduces to

$$\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m) \approx 1 + n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} \delta(\xi_{\alpha',k'}) |V_{\alpha',k';\alpha,k}|^2 \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \Lambda_{\alpha'}^{(j)}(\mathbf{k}', i\omega_n, i\omega_n + i\nu_m) \frac{2\pi \Theta(\nu_m, \omega_n)}{\nu_m + \frac{1}{\tau_{\alpha',k'}^{\text{qp}}}}. \quad (\text{A5})$$

Here, the integral provides a nonzero value only if the poles of the two Green's functions are on the opposite sides with respect to the real axis in frequency space.

Note that, because of the $\Theta(\nu_m, \omega_n)$ term, $\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m)$ has a value independent of ω_n within the range $-\nu_m < \omega_n < 0$, and otherwise 1. Thus, $\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m)$ can be expressed as

$$\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m) = \begin{cases} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\nu_m) & \text{for } -\nu_m < \omega_n < 0, \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A6})$$

Here, we assumed $\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m) = \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\nu_m)$ for $-\nu_m < \omega_n < 0$.

An alternative expression of the dc conductivity in the imaginary-time formalism is given by

$$\sigma_{ij}(i\nu_m) = \frac{1}{\nu_m} [\Pi_{ij}(i\nu_m) - \Pi_{ij}(0)]. \quad (\text{A7})$$

Note that electrons near the Fermi surface mostly contribute to the difference between the current-current response functions at ν_m and $\nu_m = 0$ [3]. Therefore, the dc conductivity can be obtained as

$$\begin{aligned} \sigma_{ij}(i\nu_m) &= \frac{ge^2}{\beta\nu_m\nu} \sum_{\alpha,\mathbf{k},i\omega_n} [v_{\alpha,\mathbf{k}}^{(i)} v_{\alpha,\mathbf{k}}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m) \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n) \mathcal{G}_{\alpha}(\mathbf{k}, i\omega_n + i\nu_m) - (i\nu_m \rightarrow 0)] \\ &\approx \frac{ge^2}{\beta\nu_m} \sum_{\alpha,i\omega_n} \int \frac{d^d k}{(2\pi)^d} \delta(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \left[\Lambda_{\alpha}^{(j)}(i\omega_n, i\omega_n + i\nu_m) \int d\xi \mathcal{G}(\xi, i\omega_n) \mathcal{G}(\xi, i\omega_n + i\nu_m) - (i\nu_m \rightarrow 0) \right] \\ &= \frac{ge^2}{\beta\nu_m} \sum_{\alpha} \int \frac{d^d k}{(2\pi)^d} \delta(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \sum_{i\omega_n} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n + i\nu_m) \frac{2\pi}{\nu_m + \frac{1}{\tau_{\alpha,k}^{\text{qp}}}} \Theta(\nu_m, \omega_n) \\ &= ge^2 \sum_{\alpha} \int \frac{d^d k}{(2\pi)^d} \delta(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \frac{\Lambda_{\alpha}^{(j)}(\mathbf{k}, i\nu_m)}{\nu_m + \frac{1}{\tau_{\alpha,k}^{\text{qp}}}}. \end{aligned} \quad (\text{A8})$$

Here, we used $\frac{1}{\beta} \sum_{\omega_n} \Theta(\nu_m, \omega_n) = \frac{\nu_m}{2\pi}$. Therefore, by defining the transport relaxation time along the j th direction as

$$\tau_{\alpha,k}^{(j)} \equiv \lim_{\nu_m \rightarrow 0} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\nu_m) \tau_{\alpha,k}^{\text{qp}}, \quad (\text{A9})$$

we obtain a result consistent with the dc conductivity obtained through the semiclassical approach in Eq. (11).

Finally, let us perform $\lim_{\nu_m \rightarrow 0} \frac{1}{\beta} \sum_{\omega_n} \Theta(\nu_m, \omega_n)$ on both sides of Eq. (A5), followed by multiplication by $\frac{2\pi}{\nu_m}$. Thus, we have

$$\frac{\tau_{\alpha,k}^{(j)}}{\tau_{\alpha,k}^{\text{qp}}} = 1 + 2\pi n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha',k';\alpha,k}|^2 \delta(\xi_{\alpha,k} - \xi_{\alpha',k'}) \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k'}^{(j)}. \quad (\text{A10})$$

Here, we assumed $\xi_{\alpha,k} = \xi_{\alpha',k'} \approx 0$. Therefore, we have an integral equation relating the transport relaxation times as follows:

$$1 = 2\pi n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha',k';\alpha,k}|^2 \delta(\xi_{\alpha,k} - \xi_{\alpha',k'}) \left(\tau_{\alpha,k}^{(j)} - \frac{v_{\alpha',k'}^{(j)}}{v_{\alpha,k}^{(j)}} \tau_{\alpha',k'}^{(j)} \right), \quad (\text{A11})$$

which is consistent with the semiclassical result in Eq. (9).

APPENDIX B: DETAILED DERIVATIONS FOR THE VERTEX CORRECTIONS

1. Elastic scattering

In this Appendix, following Chap. 8 of Mahan [2], we present detailed derivations for the vertex corrections for elastic scattering. Let us first consider the contour integral in Eq. (23):

$$\begin{aligned}\Pi_{ij}(i\nu_m) &= -\oint_C \frac{dz}{2\pi i} f^0(z) P(z, z + i\nu_m) \\ &= \int \frac{d\xi}{2\pi i} f^0(\xi) [-P(\xi + i0^+, \xi + i\nu_m) + P(\xi - i0^+, \xi + i\nu_m) - P(\xi - i\nu_m, \xi + i0^+) + P(\xi - i\nu_m, \xi - i0^+)].\end{aligned}\quad (\text{B1})$$

After performing the analytic continuation ($i\nu_m \rightarrow \nu + i0^+$), we have

$$\Pi_{ij}^R(\nu) = \int \frac{d\xi}{2\pi i} \{ [f^0(\xi) - f^0(\xi + \nu)] P^{\text{AR}}(\xi, \xi + \nu) - f^0(\xi) P^{\text{RR}}(\xi, \xi + \nu) + f^0(\xi + \nu) P^{\text{AA}}(\xi, \xi + \nu) \}.\quad (\text{B2})$$

Thus, in the $\nu \rightarrow 0$ limit, the dc conductivity can be rewritten as

$$\sigma_{ij} = \frac{ge^2}{2\pi} \int d\xi S^0(\xi) [P^{\text{AR}}(\xi, \xi) - \text{Re} P^{\text{RR}}(\xi, \xi)],\quad (\text{B3})$$

which includes the $P^{\text{AR}}(\xi, \xi)$ and $P^{\text{RR}}(\xi, \xi)$ terms in the integrand.

Here, we show that only the first term on the right-hand side contributes to the dc conductivity, whereas the second term becomes negligible in the limit of small impurity density. Before computing each term, we note several useful formulas pertaining to the spectral function $A_\alpha(\mathbf{k}, \xi) = -2\text{Im}G_\alpha^R(\mathbf{k}, \xi)$:

$$\lim_{\Delta_{\alpha,k} \rightarrow 0} A_\alpha(\mathbf{k}, \xi) = 2\pi \delta(\xi - \xi_{\alpha,k}),\quad (\text{B4a})$$

$$\lim_{\Delta_{\alpha,k} \rightarrow 0} A_\alpha^2(\mathbf{k}, \xi) = \frac{2\pi \delta(\xi - \xi_{\alpha,k})}{\Delta_{\alpha,k}},\quad (\text{B4b})$$

where $\Delta_{\alpha,k} \equiv \frac{1}{2\tau_{\alpha,k}^{\text{qp}}}$. Note that, in the $\Delta_{\alpha,k} \rightarrow 0$ limit, or equivalently in the $n_{\text{imp}} \rightarrow 0$ limit, the spectral function reduces to a δ function.

First, let us evaluate the contribution of the $P^{\text{RR}}(\xi, \xi)$ term to the dc conductivity:

$$P^{\text{RR}}(\xi, \xi) = \frac{ge^2}{\mathcal{V}} \sum_{\alpha,k} v_{\alpha,k}^{(i)} G_\alpha^R(\mathbf{k}, \xi) v_{\alpha,k}^{(j)} \Lambda_\alpha^{(j)\text{RR}}(\mathbf{k}, \xi, \xi) G_\alpha^R(\mathbf{k}, \xi).\quad (\text{B5})$$

In the $n_{\text{imp}} \rightarrow 0$ limit, the product of the two Green's functions vanishes and the contribution of $P^{\text{RR}}(\xi, \xi)$ to the dc conductivity becomes negligible [2].

Subsequently, let us compute the $P^{\text{AR}}(\xi, \xi)$ term as follows:

$$P^{\text{AR}}(\xi, \xi) = \frac{ge^2}{\mathcal{V}} \sum_{\alpha,k} v_{\alpha,k}^{(i)} G_\alpha^A(\mathbf{k}, \xi) v_{\alpha,k}^{(j)} \Lambda_\alpha^{(j)\text{AR}}(\mathbf{k}, \xi, \xi) G_\alpha^R(\mathbf{k}, \xi) = \frac{2\pi ge^2}{\mathcal{V}} \sum_{\alpha,k} v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \tau_{\alpha,k}^{\text{qp}} \delta(\xi - \xi_{\alpha,k}) \Lambda_\alpha^{(j)\text{AR}}(\mathbf{k}, \xi, \xi).\quad (\text{B6})$$

Therefore, the dc conductivity can be simplified as

$$\sigma_{ij} = \frac{1}{2\pi} \int d\xi S^0(\xi) P^{\text{AR}}(\xi, \xi) = \frac{ge^2}{\mathcal{V}} \sum_{\alpha,k} S^0(\xi_{\alpha,k}) v_{\alpha,k}^{(i)} v_{\alpha,k}^{(j)} \tau_{\alpha,k}^{\text{qp}} \Lambda_\alpha^{(j)\text{AR}}(\mathbf{k}, \xi_{\alpha,k}, \xi_{\alpha,k}),\quad (\text{B7})$$

thus yielding Eq. (26) in the main text.

2. Inelastic scattering

In this section, we present detailed derivations for the vertex corrections for inelastic scattering. Let us first consider Eq. (34) in the main text. The contour integral along C' has two types of poles: $z = iq_l$ and $z = \pm\Omega_{\lambda,q}$. Therefore, the summation $S(i\omega_n, i\omega_n + i\nu_m) \equiv \frac{1}{\beta} \sum_{iq_l} Q(iq_l + i\omega_n, iq_l + i\omega_n + i\nu_m)$ can be rewritten as follows:

$$\begin{aligned}S(i\omega_n, i\omega_n + i\nu_m) &= \oint_{C'} \frac{dz}{2\pi i} n_B(z) Q(z + i\omega_n, z + i\omega_n + i\nu_m) - n_B(\Omega_{\lambda,q}) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + \Omega_{\lambda,q} + i\nu_m) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + \Omega_{\lambda,q}) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, i\omega_n \\ &\quad + \Omega_{\lambda,q}, i\omega_n + \Omega_{\lambda,q} + i\nu_m) - [n_B(\Omega_{\lambda,q}) + 1] \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n - \Omega_{\lambda,q} + i\nu_m) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n - \Omega_{\lambda,q}) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, i\omega_n \\ &\quad - \Omega_{\lambda,q}, i\omega_n - \Omega_{\lambda,q} + i\nu_m),\end{aligned}\quad (\text{B8})$$

where the contour integral can be decomposed as

$$\begin{aligned}
 & \oint_{C'} \frac{dz}{2\pi i} n_B(z) Q(z + i\omega_n, z + i\omega_n + iv_m) \\
 &= - \int \frac{d\xi'}{2\pi i} f^0(\xi') \frac{2\Omega_{\lambda,q}}{(\xi' - i\omega_n)^2 - \Omega_{\lambda,q}^2} \{ \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' + i0^+) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' + iv_m) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, \xi' + i0^+, \xi' + iv_m) \\
 & \quad - \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' - i0^+) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' + iv_m) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, \xi' - i0^+, \xi' + iv_m) \} \\
 & \quad - \int \frac{d\xi'}{2\pi i} f^0(\xi') \frac{2\Omega_{\lambda,q}}{(\xi' - i\omega_n - iv_m)^2 - \Omega_{\lambda,q}^2} \{ \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' - iv_m) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' + i0^+) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, \xi' - iv_m, \xi' + i0^+) \\
 & \quad - \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' - iv_m) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, \xi' - i0^+) \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, \xi' - iv_m, \xi' - i0^+) \}. \tag{B9}
 \end{aligned}$$

To compute $\Lambda^{(j)\text{AR}}(\mathbf{k}, \xi, \xi)$, let us perform the analytic continuation $i\omega_n \rightarrow \xi - i0^+$ and $i\omega_n + iv_m \rightarrow \xi + \nu + i0^+$. Thus, Eq. (B8) at $\nu = 0$ reduces to

$$\begin{aligned}
 \mathcal{S}^{\text{AR}}(\xi, \xi) &= -n_B(\Omega_{\lambda,q}) |G_{\alpha'}^{\text{R}}(\mathbf{k} + \mathbf{q}, \xi + \Omega_{\lambda,q})|^2 \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k} + \mathbf{q}, \xi + \Omega_{\lambda,q}, \xi + \Omega_{\lambda,q}) \\
 & \quad - [1 + n_B(\Omega_{\lambda,q})] |G_{\alpha'}^{\text{R}}(\mathbf{k} + \mathbf{q}, \xi - \Omega_{\lambda,q})|^2 \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k} + \mathbf{q}, \xi - \Omega_{\lambda,q}, \xi - \Omega_{\lambda,q}) \\
 & \quad - \int \frac{d\xi'}{2\pi i} f^0(\xi') |G_{\alpha'}^{\text{R}}(\mathbf{k} + \mathbf{q}, \xi')|^2 \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k} + \mathbf{q}, \xi', \xi') \left[\frac{2\Omega_{\lambda,q}}{(\xi' - \xi - i0^+)^2 - \Omega_{\lambda,q}^2} - \frac{2\Omega_{\lambda,q}}{(\xi' - \xi + i0^+)^2 - \Omega_{\lambda,q}^2} \right]. \tag{B10}
 \end{aligned}$$

The last integration over ξ' can be performed with the aid of the Cauchy principal value

$$\frac{2\Omega_{\lambda,q}}{(\xi' - \xi - i0^+)^2 - \Omega_{\lambda,q}^2} - \frac{2\Omega_{\lambda,q}}{(\xi' - \xi + i0^+)^2 - \Omega_{\lambda,q}^2} = 2\pi i [\delta(\xi' - \xi - \Omega_{\lambda,q}) - \delta(\xi' - \xi + \Omega_{\lambda,q})]. \tag{B11}$$

Therefore, in the weak-scattering limit, the self-consistent Dyson's equation for inelastic scattering [Eq. (31)] can be rewritten as

$$\begin{aligned}
 & \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k}, \xi, \xi) \\
 &= 1 - \sum_{\alpha', \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \frac{v_{\alpha', \mathbf{k} + \mathbf{q}}^{(j)}}{v_{\alpha, \mathbf{k}}^{(j)}} \mathcal{S}^{\text{AR}}(\xi, \xi) \\
 &= 1 + 2\pi \sum_{\alpha', \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \frac{v_{\alpha', \mathbf{k} + \mathbf{q}}^{(j)}}{v_{\alpha, \mathbf{k}}^{(j)}} \tau_{\alpha', \mathbf{k} + \mathbf{q}}^{\text{qp}} \Lambda_{\alpha'}^{(j)\text{AR}}(\mathbf{k} + \mathbf{q}, \xi_{\alpha', \mathbf{k} + \mathbf{q}}, \xi_{\alpha', \mathbf{k} + \mathbf{q}}) \\
 & \quad \times \{ [n_B(\Omega_{\lambda,q}) + f^0(\xi + \Omega_{\lambda,q})] \delta(\xi + \Omega_{\lambda,q} - \xi_{\alpha', \mathbf{k} + \mathbf{q}}) + [n_B(\Omega_{\lambda,q}) + 1 - f^0(\xi - \Omega_{\lambda,q})] \delta(\xi - \Omega_{\lambda,q} - \xi_{\alpha', \mathbf{k} + \mathbf{q}}) \}, \tag{B12}
 \end{aligned}$$

thus yielding Eq. (35) in the main text.

APPENDIX C: WARD IDENTITIES

1. Elastic scattering

The self-energy diagrams for an impurity-scattered electron are given by Fig. 6. The diagrams in the third line are negligible because the crossing induces the momentum restriction, which provides a reduction factor of the order $\frac{1}{k_{\text{F}} l}$, where l is the mean free path of the electrons. The first diagram in the second equality with the lowest-order perturbation only provides a constant energy shift, which can be absorbed in the chemical potential. As we only include the next lowest-order diagrams in the limit of $n_{\text{imp}} \rightarrow 0$ into the ladder vertex correction, the proper self-energy reduces to the second diagram in the second equality given by

$$\Sigma_{\alpha}(\mathbf{k}, i\omega_n) = n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}|^2 \mathcal{G}_{\alpha'}(\mathbf{k}', i\omega_n). \tag{C1}$$

The Ward identity can be obtained by subtracting $\Sigma_{\alpha}(\mathbf{k}, i\omega_n)$ from $\Sigma_{\alpha}(\mathbf{k} + \mathbf{p}, i\omega_n + ip_r)$ [2]:

$$\begin{aligned}
 & \Sigma_{\alpha}(\mathbf{k} + \mathbf{p}, i\omega_n + ip_r) - \Sigma_{\alpha}(\mathbf{k}, i\omega_n) \\
 &= n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}|^2 \mathcal{G}_{\alpha'}(\mathbf{k}' + \mathbf{p}, i\omega_n + ip_r) \\
 & \quad \times \mathcal{G}_{\alpha'}(\mathbf{k}', i\omega_n) [\xi_{\alpha', \mathbf{k}' + \mathbf{p}} - \xi_{\alpha', \mathbf{k}'} + \Sigma_{\alpha'}(\mathbf{k}' + \mathbf{p}, i\omega_n + ip_r) \\
 & \quad - \Sigma_{\alpha'}(\mathbf{k}', i\omega_n) - ip_r]. \tag{C2}
 \end{aligned}$$

Let us set ip_r to zero and take the limit $\mathbf{p} \rightarrow \mathbf{0}$:

$$\begin{aligned}
 & \left[\frac{\partial \Sigma_{\alpha}(\mathbf{k}'', i\omega_n)}{\partial \mathbf{k}''} \right]_{\mathbf{k}'' = \mathbf{k}} \cdot \mathbf{p} \\
 &= n_{\text{imp}} \sum_{\alpha'} \int \frac{d^d k'}{(2\pi)^d} |V_{\alpha', \mathbf{k}'; \alpha, \mathbf{k}}|^2 \mathcal{G}_{\alpha'}(\mathbf{k}', i\omega_n)^2 \\
 & \quad \times \left(\mathbf{v}_{\alpha', \mathbf{k}'} + \left[\frac{\partial \Sigma_{\alpha'}(\mathbf{k}'', i\omega_n)}{\partial \mathbf{k}''} \right]_{\mathbf{k}'' = \mathbf{k}'} \right) \cdot \mathbf{p}. \tag{C3}
 \end{aligned}$$

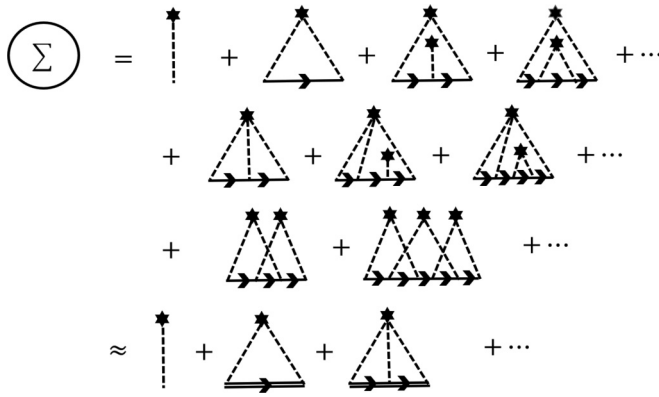


FIG. 6. Self-energy diagrams for an impurity-scattered electron.

Subsequently, let us take the limit $i\nu_m \rightarrow 0$ in Eq. (22). As the integral equations in Eq. (C3) and the self-consistent Dyson's equation describing the vertex correction have the same form, we obtain the Ward identity given by

$$v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n) = v_{\alpha,k}^{(j)} + \frac{\partial \Sigma_{\alpha}(\mathbf{k}, i\omega_n)}{\partial k^{(j)}}, \quad (\text{C4})$$

proving that the Ward identity still holds in anisotropic multi-band systems for elastic scattering.

2. Inelastic scattering

As in the case of elastic impurity scattering, we show that the Ward identity holds for inelastic phonon scattering. The self-energy diagrams for a phonon-scattered electron are given by Fig. 7. The second diagram in the first equality can be regarded as the vertex correction to the first diagram, and such terms can be ignored according to Migdal's theorem [33]. Thus, the self-energy can be approximated as the diagram with self-consistency shown in the third equality given by

$$\Sigma_{\alpha}(\mathbf{k}, i\omega_n) = -\frac{1}{\beta} \sum_{\alpha', \lambda} \sum_{i q_l} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \times D(\mathbf{q}, i q_l) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + i q_l). \quad (\text{C5})$$

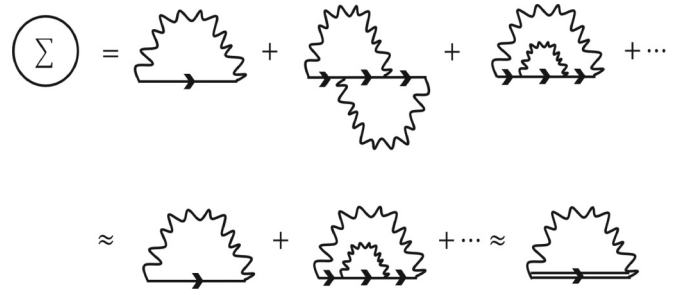


FIG. 7. Self-energy diagrams for a phonon-scattered electron.

Repeating similar steps from Eq. (C1) to Eq. (C4) for impurity scattering, we have

$$\left[\frac{\partial \Sigma_{\alpha}(\mathbf{k}', i\omega_n)}{\partial \mathbf{k}'} \right]_{\mathbf{k}'=\mathbf{k}} \cdot \mathbf{p} = -\frac{1}{\beta} \sum_{\alpha', \lambda} \sum_{i q_l} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \times D(\mathbf{q}, i q_l) \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + i q_l)^2 \times \left(\mathbf{v}_{\alpha', \mathbf{k}+\mathbf{q}} + \left[\frac{\partial \Sigma_{\alpha'}(\mathbf{k}', i\omega_n)}{\partial \mathbf{k}'} \right]_{\mathbf{k}'=\mathbf{k}+\mathbf{q}} \right) \cdot \mathbf{p} \quad (\text{C6})$$

and

$$v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n) = v_{\alpha,k}^{(j)} - \frac{1}{\beta} \sum_{\alpha', i q_l, \lambda} \int \frac{d^d q}{(2\pi)^d} |\langle \alpha', \mathbf{k} + \mathbf{q} | M_{\lambda} | \alpha, \mathbf{k} \rangle|^2 \times D(\mathbf{q}, i q_l) v_{\alpha', \mathbf{k}+\mathbf{q}}^{(j)} \Lambda_{\alpha'}^{(j)}(\mathbf{k} + \mathbf{q}, i\omega_n + i q_l, i\omega_n + i q_l) \times \mathcal{G}_{\alpha'}(\mathbf{k} + \mathbf{q}, i\omega_n + i q_l)^2. \quad (\text{C7})$$

Thus, by comparing Eqs. (C6) and (C7), we obtain

$$v_{\alpha,k}^{(j)} \Lambda_{\alpha}^{(j)}(\mathbf{k}, i\omega_n, i\omega_n) = v_{\alpha,k}^{(j)} + \frac{\partial \Sigma_{\alpha}(\mathbf{k}, i\omega_n)}{\partial k^{(j)}}, \quad (\text{C8})$$

proving that the Ward identity still holds in anisotropic multi-band systems for inelastic scattering.

[1] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Brooks Cole, Pacific Grove, CA, 1976).
 [2] G. D. Mahan, *Many-particle Physics*, 3rd ed. (Springer, Berlin, 2000).
 [3] P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, Cambridge, 2016).
 [4] E. D. Siggia and P. C. Kwok, Properties of electrons in semiconductor inversion layers with many occupied electric subbands. I. Screening and impurity scattering, *Phys. Rev. B* **2**, 1024 (1970).
 [5] K. Výborný, A. A. Kovalev, J. Sinova, and T. Jungwirth, Semiclassical framework for the calculation of transport anisotropies, *Phys. Rev. B* **79**, 045427 (2009).
 [6] M. Breitkreiz, P. M. R. Brydon, and C. Timm, Transport anomalies due to anisotropic interband scattering, *Phys. Rev. B* **88**, 085103 (2013).

[7] Y. Liu, T. Low, and P. P. Ruden, Mobility anisotropy in monolayer black phosphorus due to scattering by charged impurities, *Phys. Rev. B* **93**, 165402 (2016).
 [8] C. Xiao, D. Li, and Z. Ma, Unconventional thermoelectric behaviors and enhancement of figure of merit in Rashba spintronics systems, *Phys. Rev. B* **93**, 075150 (2016).
 [9] V. Brosco, L. Benfatto, E. Cappelluti, and C. Grimaldi, Unconventional dc Transport in Rashba Electron Gases, *Phys. Rev. Lett.* **116**, 166602 (2016).
 [10] C. Xiao, D. Li, and Z. Ma, Role of band-index-dependent transport relaxation times in anomalous Hall effect, *Phys. Rev. B* **95**, 035426 (2017).
 [11] S. Park, S. Woo, E. J. Mele, and H. Min, Semiclassical Boltzmann transport theory for multi-Weyl semimetals, *Phys. Rev. B* **95**, 161113(R) (2017).

- [12] S. Woo, E. H. Hwang, and H. Min, Large negative differential transconductance in multilayer graphene: The role of intersubband scattering, *2D Mater.* **4**, 025090 (2017).
- [13] S. Park, S. Woo, and H. Min, Semiclassical Boltzmann transport theory of few-layer black phosphorous in various phases, [arXiv:1811.03903](https://arxiv.org/abs/1811.03903).
- [14] A. A. Burkov, M. D. Hook, and L. Balents, Topological nodal semimetals, *Phys. Rev. B* **84**, 235126 (2011).
- [15] C. Fang, H. Weng, X. Dai, and Z. Fang, Topological nodal line semimetals, *Chin. Phys. B* **25**, 117106 (2016).
- [16] C. Fang, M. J. Gilbert, X. Dai, and B. AndreiBernevig, Multi-Weyl Topological Semimetals Stabilized by Point Group Symmetry, *Phys. Rev. Lett.* **108**, 266802 (2012).
- [17] R. Fei, V. Tran, and L. Yang, Topologically protected Dirac cones in compressed bulk black phosphorus, *Phys. Rev. B* **91**, 195319 (2015).
- [18] A. Carvalho, M. Wang, X. Zhu, A. S. Rodin, H. Su, and A. H. Castro Neto, Phosphorene: From theory to applications, *Nat. Rev. Mater.* **1**, 16061 (2016).
- [19] G. Rui, S. Zdenek, and P. Martin, Black phosphorus rediscovered: From bulk material to monolayers, *Angew. Chem. Int. Ed.* **56**, 8052 (2017).
- [20] J. S. Langer, Theory of impurity resistance in metals, *Phys. Rev.* **120**, 714 (1960).
- [21] R. E. Prange and L. P. Kadanoff, Transport theory for electron-phonon interactions in metals, *Phys. Rev.* **134**, A566 (1964).
- [22] T. Holstein, Theory of transport phenomena in an electron-phonon gas, *Ann. Phys.* **29**, 410 (1964).
- [23] W. Hänsch and G. D. Mahan, Transport equations for many-particle systems, *Phys. Rev. B* **28**, 1902 (1983).
- [24] E. Cappelluti and L. Benfatto, Vertex renormalization in dc conductivity of doped chiral graphene, *Phys. Rev. B* **79**, 035419 (2009).
- [25] W. Kohn and J. M. Luttinger, Quantum theory of electrical transport phenomena, *Phys. Rev.* **108**, 590 (1957).
- [26] T. Ando, A. B. Fowler, and F. Stern, Electronic properties of two-dimensional systems, *Rev. Mod. Phys.* **54**, 437 (1982).
- [27] D. K. Ferry, S. M. Goodnick, and J. Bird, *Transport in Nanostructures*, 2nd ed. (Cambridge University Press, Cambridge, England, 2009).
- [28] T. Kawamura and S. Das Sarma, Phonon-scattering-limited electron mobilities in $\text{Al}_x\text{Ga}_{1-x}\text{As}/\text{GaAs}$ heterojunctions, *Phys. Rev. B* **45**, 3612 (1992).
- [29] J. M. Ziman, *Electrons and Phonons* (Oxford University Press, New York, 1963).
- [30] H. Bruus and K. Flesberg, *Many-body Quantum Theory in Condensed Matter Physics* (Oxford University Press, Oxford, 2004).
- [31] J. C. Ward, An identity in quantum electrodynamics, *Phys. Rev.* **78**, 182 (1950).
- [32] S. Engelsberg and J. R. Schrieffer, Coupled electron-phonon system, *Phys. Rev.* **131**, 993 (1963).
- [33] A. B. Migdal, Interaction between electrons and lattice vibrations in a normal metal, *Sov. Phys. JETP* **34**, 996 (1958).