## Band structure engineering and reconstruction in electric circuit networks

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We develop an approach to design, engineer, and measure band structures in a synthetic crystal composed of electric circuit elements. Starting from a nodal analysis of a circuit lattice in terms of currents and voltages, our Laplacian formalism for synthetic matter allows us to investigate arbitrary tight-binding models in terms of wavenumber-resolved Laplacian eigenmodes, yielding an admittance band structure of the circuit. For illustration, we model and measure a honeycomb circuit featuring a Dirac cone admittance bulk dispersion as well as flat band admittance edge modes at its bearded and zigzag terminations. We further employ our circuit band analysis to measure a topological phase transition in the topolectrical Su-Schrieffer-Heeger circuit.

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*Introduction.* Electrons in a periodic lattice potential [1,2] is one of the most central problems in the history of condensed matter research. As our understanding of it progressed over the decades, revolutionary concepts have kept arising, such as, most recently, relativistic particle dispersions in graphene [3] or topologically nontrivial insulators and semimetals [4-6]. In this context, synthetic matter has emerged as a complementary branch to realize lattice potential environments for alternative degrees of freedom. This includes, among others, atoms in optical lattices, exciton polaritons in semiconductor platforms, photons in cavities and waveguides, mechanical and acoustic settings, and several more [7-12]. The common purpose of synthetic matter research is to either accomplish a highly tunable simulator for a given electronic lattice problem, or to establish a framework in which an intricate lattice model can be experimentally realized in the first place.

Electric circuit networks [13,14] naturally present themselves as yet another physical system in which a lattice potential along with tunable lattice connectivity can be realized. While most applications in electrical engineering do not specifically necessitate a translationally invariant arrangement of circuit elements, electric circuit networks still represent a prototypical candidate for such synthetic matter. In the realm of topological matter, it has recently been discovered that a two-dimensional topological crystalline insulator can be built in an electric circuit [15–17], which was subsequently generalized to the prescription for modeling topological insulators, topological semimetals, and higher-order topological states of arbitrary dimension in topolectrical circuits [18,19].

In this Rapid Communication, we develop the framework to build and measure admittance band structures in an electric circuit in a way that allows for a precise translation from a given tight-binding model to its circuit realization. We employ a Laplacian formalism put forward by us [18] to connect the nodewise currents of the circuit with the nodewise voltages measured against the ground. For a translationally invariant system, the circuit Laplacian, whose eigenvalues form the circuit admittance spectrum, then inherits a block diagonal form due to a wave-number component k per periodic direction. As such, the energy band structure from a given abstract tight-binding model translates into an admittance band structure for the circuit derived from the Fourier analysis of site-resolved voltages and currents, lending itself to immediate measurability. The reconstruction of the band structure is thereby straightforwardly accessible in a systematic and scalable measurement in terms of an electrical circuit environment. We illustrate our admittance band engineering for a two-dimensional periodic circuit lattice reminiscent of graphene, its different surface terminations, and the topological phase transition in the Su-Schrieffer-Heeger (SSH) model as a function of the ratio between the intracell and intercell hopping amplitude.

Admittance band analysis. We label each node in the circuit by an index j, where the voltage at that node,  $V_j$ , is measured with respect to the ground. The input current, which defines the current flowing into the circuit at that node from the outside world, is denoted by  $I_j$ . With this, we are able to arrange the components  $I_j$  and  $V_j$  in a vector form linked by using Ohm's and Kirchhoff's law (see Supplemental Material A) [20],

$$\mathbf{I} = J(\omega)\mathbf{V}.\tag{1}$$

*J* denotes the grounded circuit Laplacian and  $\omega$  the ac driving frequency of the excitation current applied to the circuit [18], which takes the role of an external parameter.

The response of the system to a given input current signal is governed by the eigenstates of J. The impedance resonance

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frequencies  $\omega_{\text{res}}^{(n)}$  are the roots of the Laplacian's eigenvalues  $j_n(\omega), n \in \{1, \dots, \dim[J]\}$  [18].

The circuits we investigate are composed of a repeating minimal set of *M* nodes and conductances, which together we call the circuit unit cell. The nodes of a circuit representing a *D*-dimensional network can be labeled by two indices  $j \equiv (\rho, \alpha)$ , where  $\rho$  is an index denoting the unit cell and  $\alpha \in \{1, \ldots, M\}$  the nodes within a unit cell. *D* specifies the synthetic dimension of circuit lattice periodicity, which is determined by the maximum number of linearly independent Bravais lattice vectors  $\mathbf{R}_{\rho}$ .

Note that, as one central difference from a solid state lattice, there is no fixed length or orientation of the Bravais vectors, as the circuit lattice truly is a graph, and as such solely determined by lattice connectivity. This implies an equivalence class of different choices of Bravais vectors, and thus gives an additional gauge for the circuit lattice network (Supplemental Material B) [20].

Once we fix a Bravais vector gauge  $\{\mathbf{R}_{\rho}\}$ , we can diagonalize a translationally invariant *J* by performing a Fourier transform to *D*-dimensional reciprocal space **k** into *M*-dimensional block matrices,

$$J_{\alpha\beta}(\mathbf{k},\omega) = \sum_{\rho} J_{\alpha\beta}(R_{\rho},\omega) \exp\left[-i\mathbf{R}_{\rho}^{\top}\mathbf{k}\right].$$
 (2)

To find the eigensystem of the Laplacian matrix, and hence the admittance band structure, we diagonalize the block matrices  $J_{\alpha\beta}(\mathbf{k}, \omega)$ .

The Laplacian matrix in reciprocal  $\mathbf{k}$  space forms an irreducible representation of the translation group. The admittance band structure can then be seen as the irreducible representation of the space group incorporating the periodic circuit configuration in graph space.

Admittance band measurement. We apply an input current at one specified node of the circuit and measure the response of the circuit given by the complete voltage vector with respect to that input current. If we apply an input current at node j, we can compute the impedances

$$G_{ij} = V_i^{(j)} / I_j = J_{ij}^{-1}.$$
 (3)

 $V_i^{(j)}$  represents the voltage measured at node *i* when the only input current to the circuit is given by  $I_i$ .

As the matrix G is the inverse of the circuit Laplacian J[18], the complex valued admittance eigenvalues are obtained by inverting the eigenvalues of G. Note that by analogy, such site-resolved measurements are out of reach in generic transport or scattering experiments on physical crystals. For a randomized system of N nodes, the measurement procedure of exciting one node and measuring the whole voltage profile needs to be repeated N times to recreate the matrix G by use of (3), where each of the N measurement processes features an input current at a different node. If we are dealing with a fully periodic system, however, only M nodes are inequivalent. In this case, we thus restrict ourselves to repeating the outlined measurement procedure M times, where each sublattice needs to be supported once (Supplemental Material C) [20]. The data of the voltage and the current vector are then Fourier transformed to reciprocal space, and the  $(M \times M)$  impedance matrix is recovered for each  $\mathbf{k}$  by use of (3). We determine

the complex Laplacian matrix and its eigenvalues for each  $\mathbf{k}$  separately, and thus restore the band structure. The measurement principle readily extends to the case of open boundary conditions for any synthetic circuit dimension.

*Honeycomb circuit.* As introduced in Ref. [18], with a unit cell depicted in the inset of Fig. 1(a), we consider the analog to a honeycomb structure in a circuit network.

We thus have M = 2 and three equivalent capacitive conductances C per node to other nodes,

$$J_{\rm hc}(\mathbf{k}) = i\omega \left[ \left( 3C - \frac{1}{\omega^2 L} \right) \mathbb{1} - C(1 + \cos(k_x) + \cos(k_y))\sigma_x - C(\sin(k_x) + \sin(k_y))\sigma_y \right],$$
(4)

yielding a two-band structure given by

$$j_{\rm hc}^{(\pm)}(\mathbf{k}) = i\omega \left[ \left( 3C - \frac{1}{\omega^2 L} \right) \pm C\sqrt{3 + 2\cos(k_x) + 2\cos(k_x - k_y) + 2\cos(k_y)} \right].$$
(5)

The ac driving with the characteristic resonance frequency  $\omega_0 = 1/\sqrt{3LC}$  eliminates the offset proportional to identity and symmetrizes the honeycomb lattice spectrum around zero admittance. In the absence of dissipative losses such as imposed by serial resistances, the spectrum is purely imaginary.

For the experimental implementation, we devise standard printed circuit boards (PCBs), and fit them with commercially available electronic components (Supplemental Material D) [20]. The PCB modules for the honeycomb circuit are designed to contain  $6 \times 6$  unit cells with the option of selecting specific components at the edge termination. We serially connect the edges in both spatial dimensions to fuse several PCB modules and set the circuit termination to either provide periodic or open boundary conditions. The driver current is fed into a particular sublattice site from the ground.

The measurements of the ac voltages are done by Stanford Research 530 lock-in amplifiers. The driving current is detected as a voltage drop through a shunt resistor. The driving frequency is set to the respective operational resonance frequency, which is identified in the impedance spectrum recorded by a B&K Precision 894 LCR meter. The reconstructed band structure measurement is summarized in Fig. 1. As seen, the data are in good correspondence to the theoretical prediction (5). The deviations of large admittance eigenvalues from theory are greater due to reduced excitation of the corresponding eigenstates (Supplemental Material D) [20]. The red/black data points in Fig. 1(a) correspond to the red/black path taken in the Brillouin zone as shown in Fig. 1(b). To illustrate the Bravais gauge, we have picked the Brillouin zone to take the form of a square (brick wall type) which, upon suitable reciprocal folding, appears as a distorted hexagon (Supplemental Material C) [20]. While the spectrum from (5) is gauge invariant, the map onto wave vector momenta is not, leading to the distorted spectrum for the chosen gauge.

*Open boundary termination.* We adjust the honeycomb circuit PCBs to exhibit open boundary conditions in one brick



FIG. 1. (a) Admittance band structure of the honeycomb circuit (two-node unit cell shown in the upper central inset). Gray dashed lines highlight the continuum theoretical admittance dispersion. In the absence of any fixed length scale entering the circuit system of connected nodes, there is an equivalence class of individually scaled and oriented (reciprocal) lattice vectors as long as lattice connectivity and number of nodes per unit cell is preserved. The red × and black + data points were measured for  $18 \times 18$  unit cells for the red and black trajectory through the Brillouin zone depicted in (b). In contrast to a usual honeycomb reciprocal lattice vector structure, the gauge for the circuit Brillouin zone is chosen to be quadratic (straight white), which upon folding takes the form of a distorted hexagon (dashed white). The heat map of admittance in reciprocal space stresses the dominant low spectral regime around the *K*/*K*' points, hence dominating the impedance read-out.

wall direction while keeping periodic boundary conditions for the other. Due to two sublattice components and two choices of termination of the resulting cylindric geometry, different settings can be investigated. Figure 2 shows the predicted and measured admittance band structure for different choices of termination, where we put an emphasis on those exhibiting flat surface admittance modes. Viewed together, Figs. 2(a) and 2(b) display one complete flat band of admittance eigenvalues, which is doubly degenerate because of the two identical edges. The flat band splits into a regime  $|k| > 2\pi/3$  and  $|k| < 2\pi/3$  between the A-B zigzag and bearded termination, respectively. For the B-B bearded/zigzag termination, Fig. 2(c) displays a nondegenerate flat band where, if it were resolved with respect to the two edges, the same distribution between the bearded and zigzag edge would be observed as for Figs. 2(a) and 2(b).



FIG. 2. Admittance band analysis of different graphene circuit edge terminations given by (a) A-B zigzag, (b) B-A bearded, and (c) B-B bearded/zigzag termination, where A (green) and B (red) used in the schematic honeycomb geometry label the different nodes of the circuit unit cell. The admittance band structure data (black × points) are derived from an cylindric circuit geometry with 18 unit cells along the periodic and 5 along the open direction, where the gray dashed lines highlight the theoretical expectation for the bands. Depending on the termination, flattened spectral admittance features are visible whose eigenstates localize at the termination. For the Brillouin zone gauge chosen in Fig. 1, (a) yields a flat spectrum at zero admittance for  $|k| > 2\pi/3$  while (b) shows the complementary flattening for  $|k| < 2\pi/3$ . Independent of the Brillouin zone gauge, the B-B terminated circuit in (c) exhibits a flat admittance band.



FIG. 3. (a) Absolute value of impedance as a function of ac frequency  $\omega$  for the open boundary SSH circuit (unit cell depicted at the upper right inset) at the critical value t = 1 (blue, 10 unit cells), t = 1.7 (red, 19 unit cells) contained in the topologically trivial regime t > 1, and t = 0.59 (black, 20 unit cells) contained in the nontrivial regime t < 1. At  $\omega = \omega_0$ , the topolectrical boundary resonance (TBR) related to the topological SSH midgap states is resolved. (b) Admittance band structure measured for the periodic SSH circuit. For t = 1 (+ sign, continuum theory curve in dashed blue), the band structure appears critical at  $k = \pi$ . In the periodic case, t = 1.7 and t = 0.59 yield the same bands (× sign, continuum theory curve in dashed red) due to spectral self-similarity under  $t \rightarrow 1/t$ , with an admittance gap at  $k = \pi$ .

Topological phase transition. The admittance band measurement we propose also allows us to track the bulk topological phase transition of a topolectrical circuit. As its most elementary representative, we study the Su-Schrieffer-Heeger (SSH) circuit, an M = 2 one-dimensionally connected circuit whose admittance band structure corresponds to that of the SSH tight-binding model for polyacetylene [18,21,22]. The conductances are given by capacitors with capacitance  $C_1$ inside the unit cell (intracell) and  $C_2$  between adjacent unit cells (intercell) described by the paramter  $t = C_1/C_2$ . Each node is also connected to the ground by an inductor with inductance L [see the inset in Fig. 3(a)]. The SSH PCBs are designed to contain ten unit cells, with the option to have different edge terminations and to stack several circuit boards by connecting them in a series. The circuit Laplacian is given by [18]

$$J_{\text{SSH}}(k) = i\omega \left[ \left( C_1 + C_2 - \frac{1}{\omega^2 L} \right) \mathbb{1} - [C_1 + C_2 \cos(k)] \sigma_x - [C_2 \sin(k)] \sigma_y \right],$$
(6)

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$$\begin{aligned} {}^{(\pm)}_{\text{SSH}}(k) = &i\omega \bigg[ \left( C_1 + C_2 - \frac{1}{\omega^2 L} \right) \\ \pm \sqrt{C_1^2 + C_2^2 + 2C_1 C_2 \cos(k)} \bigg]. \end{aligned}$$
(7)

Figure 3(a) depicts an open boundary impedance measurement for the parameters t = 1 as well as t = 0.59 and its inverse t = 1.7. In the topologically nontrivial regime t < 1, at  $\omega_0 = 1/\sqrt{L(C_1 + C_2)}$ , the circuit exhibits an admittance midgap state at the boundary, which manifests as an impedance peak. In the dissipationless limit and for an exact zero admittance SSH midgap state, this peak would be divergent, but in reality becomes damped due to serial circuit resistance and component disorder [18]. This peak is absent for t > 1. Figure 3(b) shows the reconstructed bulk admittance band structures. Because of the duality under  $t \rightarrow 1/t$ , the bulk spectrum is identical for t = 0.59 and its inverse, showing a bulk admittance gap. The phase transition occurs at t = 1, where the admittance gap closes at  $k = \pi$ .

Conclusions and outlook. Electric circuit networks, together with the admittance band measurement protocol developed in our work, establish a promising platform for the design, engineering, and measurement of tight-binding models. In comparison to alternative frameworks of synthetic matter, electric circuits offer unique advantages. First, electric circuits are placed in the infinite tight-binding limit, and as such are arbitrarily scalable. Second, the circuit boundary conditions can be conveniently switched between open and periodic, allowing us to investigate bulk band properties and edge states in the same experimental sample. Third, while we have not yet exploited it in this work, arbitrary longer ranged hopping can be straightforwardly considered, along with realizing lattices of arbitrary dimension and connectivity. Here, the graph property of electric circuits will allow for the implementation of symmetries independent of the physical embedding space which are in part inaccessible to physical crystals. Together with their feasibility and accessibility, electric circuit networks promise to yield fundamental insights into topological band structures [16-19,23,24] and beyond.

*Note added.* Recently, we became aware of a contemporaneous work [25] providing an experimental realization for a Weyl circuit [18]. An inductive nodal measurement is performed to reconstruct energy band dispersion which is not rigid, i.e., sensitive to the energy offset, while we reconstruct rigid admittance bands, i.e., insensitive to the grounding adjustment of admittance.

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