

Electron-phonon cooling power in Anderson insulators

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We present a microscopic theory for electron-phonon energy exchange in Anderson insulators at low temperatures. The major contribution to the cooling power $J_{e-ph}(T_{el})$ as a function of electron temperature T_{el} is shown to be directly related to the correlation function of the local density of electron states $K(\omega)$. In Anderson insulators not far from localization transition, the correlation function $K(\omega)$ is enhanced at small ω by the wave function's multifractality and by the presence of Mott's resonant pairs of states. The theory we develop explains a huge enhancement of the cooling power observed in insulating indium oxide films as compared to predictions of the theory previously developed for disordered metals.

DOI: [10.1103/PhysRevB.99.125415](https://doi.org/10.1103/PhysRevB.99.125415)**I. INTRODUCTION**

Energy exchange between electrons and phonons is crucial to many physical properties of Anderson insulators at low temperatures: it determines the relatively slow rate of thermal equilibration. Surprisingly, no theory of such processes seems to be available. On the contrary, the theory of electron-phonon inelastic coupling in disordered metals has been known for a very long time [1–3].

Experimentally, one of the most sensitive methods to study the electron-phonon cooling rate is based on the results of Refs. [4,5] where striking jumps by several orders of magnitude in current-voltage characteristics were observed at low temperatures in insulating indium oxide films. Similar effects were also observed in other insulating systems [6,7]. These jumps in resistance are the signatures of thermal bistability at weak electron-phonon coupling which can be analyzed using the balance between the Joule heat production and the electron cooling power [8], and the temperature dependence of electron-phonon cooling rate can be experimentally obtained [5]. The out-cooling rate at low temperatures of electron system $T = T_{el}$ appeared to be $J(T) = AT^\beta$, where $\beta \approx 6$ agreed well with the theory of electron-phonon cooling presented in [8].

The problem with this result is that the experimentally observed prefactor A is two to three *orders of magnitude* larger than the one predicted by the theory of electron-phonon cooling in strongly disordered metals employed in [8]. At first glance it is also strange that in an insulator the temperature dependence of the cooling is a power law, while the temperature dependence of resistance is exponential or stretch exponential. However, the most surprising fact is that the theory of electron-phonon cooling in Anderson insulators is essentially missing, despite so much effort invested in studying hopping conductivity.

In this paper we present the theory of electron-phonon cooling in insulators in proximity to the Anderson localization transition when the momentum relaxation rate ℓ is of the order of the Fermi wavelength $\lambda_F = 2\pi/k_F$, and the effects of multifractality [9,10] are significant. We show that the temperature dependence of the cooling rate at low temperatures is indeed a power law, since the energy exchange between electron and phonon systems is *local* and does not involve electron transport in space. Therefore it is natural that the additional factor characterizing electron cooling in Anderson insulators obtained in this paper is given by the properly normalized correlation function $K(\omega)$ of the *local* density of states. This correlation function is enhanced due to multifractality of electron wave functions [9,10], which results in an increasing cooling rate both in metals and in insulators close to the Anderson transition (see Fig. 1). Another mechanism of enhancement of the cooling rate [also described by the same correlation function $K(\omega)$] is typical for insulators and is related with the Mott's pairs of resonant states. It is similar to the logarithmic enhancement of the frequency-dependent conductivity in an Anderson insulator [11,12] and efficient at low temperatures. It is because of this effect, enhanced by multifractality, that the enhancement factor shown in Fig. 1 is drastically asymmetric on both sides of the Anderson transition.

At the values of the parameters typical to amorphous indium oxide films used in Refs. [4,5], the total enhancement factor may be as large as 500–800 in the range of electron temperatures 20–100 mK and it decreases very slowly as the system is driven deeper into the insulating phase (see Fig. 1). Moreover, the temperature dependence of the enhancement factor is logarithmic, which makes the effective power β in the out-cooling rate only slightly modified compared to the case of dirty metal [13]. This makes our theory a very plausible

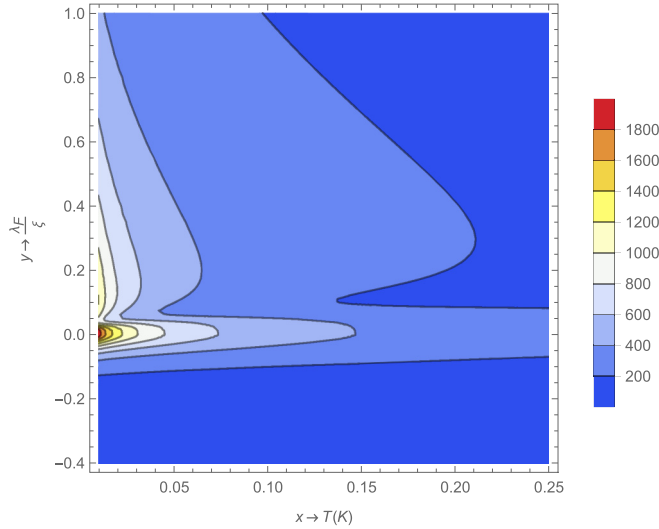


FIG. 1. Enhancement factor for the cooling rate in the Anderson insulator ($y > 0$) compared to the dirty metal ($y < 0$) as a function of temperature and proximity to the Anderson localization transition at $n = n_c$ [parametrized by the ratio of the Fermi wavelength λ_F and the localization ($\xi > 0$) or correlation ($\xi < 0$) length, with $|\xi|/\lambda_F \sim (1 - n/n_c)^{-\nu}$, $\nu \approx 1.6$]. Notice a persistent character of enhancement in insulator at low temperatures even far from the Anderson transition $\lambda_F/\xi = 0$. This enhancement is caused by the pairs of Mott's resonant states with multifractal structure inside localization volume, while the enhancement close to the Anderson transition both in metal and in insulator is caused by multifractality alone.

explanation of enhancement of the prefactor A in the cooling rate in the experiments [5].

However, the results of this paper are much more general. They are based on universal properties of random electron wave functions in the *multifractal insulator* [9,10] and are independent of a particular system as well of the presence or absence of superconductivity in it.

The paper is organized as follows. In Sec. II we present a general expression for the out-cooling rate in terms of exact electron wave functions in the presence of strong disorder. In Sec. III and Appendix B we show that the simple random-phase approximation for electron wave functions employed in the theory of Sec. II reproduces all the known results for the electron-phonon cooling obtained earlier using the impurity diagrammatic technique. In Sec. III we modify this random-phase approximation by introducing a nontrivial envelope of oscillating wave functions which accounts for the effects of multifractality and localization. The main result of this section is that the cooling rate is determined by the local density of states correlation function $K(\omega)$. In Sec. IV we review known properties of function $K(\omega)$, in particular the signatures of multifractality and the effect of Mott's resonant pair on it. In Secs. V and VI we compute the enhancement factor for the cooling rate due to these effects for the transverse and longitudinal phonons, respectively. In the Conclusion we formulate the main results of the paper and discuss their implications for low-temperature experiments in Anderson insulators close to localization transitions.

II. GENERAL EXPRESSION FOR THE COOLING RATE

The out-cooling rate $J(T)$ is expressed [14] in terms of the phonon attenuation rate τ_{ph}^{-1} due to electron phonon interaction:

$$J(T) = \int_0^\infty d\omega \omega v_{\text{ph}}(\omega) \frac{B_{\text{ph}}(\omega)}{\tau_{\text{ph}}(\omega)}, \quad (1)$$

where $B_{\text{ph}}(\omega) = \frac{1}{2}[\coth(\omega/2T) - 1]$ is the phonon energy distribution function, and $v_{\text{ph}} = \omega^2/(2\pi^2 v_s^3)$ is the three-dimensional (3D) phonon density of states. The phonon attenuation rate and the sound velocity are different for transverse (t) and longitudinal (l) modes, and the total cooling rate $J_{\text{tot}}(T) = J_l(T) + 2J_t(T)$, each of the contributions being described by (1) with the corresponding $\tau_{\text{ph}}^{(t,l)}$ and sound velocities $v_s^{(t,l)}$.

Thus the primary object of interest is the phonon attenuation rate:

$$\frac{1}{\tau_{\text{ph}}} = \frac{1}{2\rho_i \omega} \text{Im}(\Sigma_\omega^R - \Sigma_\omega^A), \quad (2)$$

where ρ_i is the lattice mass density, and $\Sigma^{R(A)} = \hat{\mathcal{D}} \Pi_{\text{RPA}}^{R(A)} \hat{\mathcal{D}}$ is the (retarded or advanced) phonon self-energy, given by a proper action of the gradient vertex operators $\hat{\mathcal{D}}$ on the random-phase approximation (RPA) polarization bubble $\Pi_{\text{RPA}}^{R(A)}$.

In order to take the localized nature of electron wave functions into account we express the phonon attenuation rate in terms of the exact electron eigenfunctions $\psi_n(\mathbf{r})$ and eigenvalues E_n . To this end we use the reference frame moving locally together with the lattice [3,14]. In this frame the electron-phonon Hamiltonian takes the form [14]

$$\begin{aligned} H_{\text{e-ph}} &= - \sum_{\mathbf{p}, \mathbf{q}} p_\alpha (v_\beta \nabla_\beta u_\alpha)_{\mathbf{q}} \Psi_{\mathbf{p}}^\dagger \Psi_{(\mathbf{p}+\mathbf{q})} \\ &= \frac{1}{m} \int d^d \mathbf{r} [\nabla_\beta u_\alpha(\mathbf{r})] \partial_\alpha \partial'_\beta \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'}, \end{aligned} \quad (3)$$

where $p_\alpha = -i\nabla_\alpha$, $v_\beta = p_\beta/m$ is the electron velocity operator, $\Psi_{\mathbf{p}}$ and $\Psi_{\mathbf{p}+\mathbf{q}}$ are Fourier components of the Fermionic operators $\Psi^\dagger(\mathbf{r})$ and $\Psi(\mathbf{r}')$, m is the electron mass, and u_α is the phonon-induced local shift of the lattice in the laboratory frame. The Greek symbols α, β in Eq. (3) and throughout the paper are the components of 3D vectors, the summation over repeated indexes being assumed. This Hamiltonian should be supplemented by the standard electron interaction with an impurity potential and the electron kinetic energy. The advantage of the comoving frame is that the cross terms with electron-phonon impurity interaction do not appear, which makes calculations much simpler.

This interaction is screened by Coulomb interaction V . In the RPA approximation the screened phonon self-energy is given by

$$\Sigma = \hat{\mathcal{D}} \Pi \hat{\mathcal{D}} + \hat{\mathcal{D}} \Pi \frac{V}{1 - \Pi V} \Pi \hat{\mathcal{D}}, \quad (4)$$

where

$$\hat{D}\Pi = \frac{1}{m} [\nabla_\beta u_\alpha(\mathbf{r})] \partial_\alpha \partial'_\beta \Pi(\mathbf{r}, \mathbf{r}'; \mathbf{r}_1, \mathbf{r}'_1)|_{\mathbf{r}=\mathbf{r}', \mathbf{r}_1=\mathbf{r}'_1}, \quad (5)$$

$$\Pi \hat{D} = \frac{1}{m} [\nabla_\gamma u_\delta(\mathbf{r}_1)] \partial_{1,\gamma} \partial'_{1,\delta} \Pi(\mathbf{r}, \mathbf{r}'; \mathbf{r}_1, \mathbf{r}'_1)|_{\mathbf{r}=\mathbf{r}', \mathbf{r}_1=\mathbf{r}'_1}, \quad (6)$$

and Π is the bare polarization bubble in which all effects of disorder are included but interaction is not.

Note that in the second term in Eq. (4) the fast momenta corresponding to the left vertex of the leftmost Π is completely decoupled from the fast momenta corresponding to the right vertex of the rightmost Π . As a result the second term in Eq. (4) is proportional to $k_F^4 \delta_{\alpha\beta} \delta_{\gamma\delta}$ and thus its contribution vanishes for transverse phonons. This is not the case for the first term in Eq. (4) at distances $|\mathbf{r} - \mathbf{r}_1| < \ell$, where ℓ is the mean free path.

In what follows we first consider the effect of the first term in Eq. (4). Using (4), (3), and (2) one can express the corresponding contribution to τ_{ph} as follows (see Appendix A for details of derivation):

$$\frac{1}{\tau_{ph}^{(1)}} = \pi \frac{q_\beta q_\delta}{m^2} e_\alpha e_\gamma \frac{1}{\rho_i} \int d^d \mathbf{R} e^{i\mathbf{q}\mathbf{R}} K_{\alpha\beta\gamma\delta}(\mathbf{R}, \omega), \quad (7)$$

where e_α is the α component the unit vector of phonon polarization, q_α is the component of the phonon wave vector \mathbf{q} with $|\mathbf{q}| = q = \omega/v_s$, \mathcal{V} is the volume, and the function $K_{\alpha\beta\gamma\delta}(\mathbf{R}; \omega)$ is defined as

$$K_{\alpha\beta\gamma\delta}(\mathbf{R}; \omega) = \left\langle \sum_{nm} [\partial_\alpha \psi_m^*(\mathbf{r})] [\partial_\beta \psi_n(\mathbf{r})] [\partial_\gamma \psi_n^*(\mathbf{r}')] [\partial_\delta \psi_m(\mathbf{r}')] \right. \\ \left. \times \delta(E - E_n) \delta(E' - E_m) \right\rangle. \quad (8)$$

In (8) we denote disorder averaging by $\langle \cdot \rangle$. After such an averaging $K_{\alpha\beta\gamma\delta}(\mathbf{R}, \varepsilon)$ becomes a function of $\mathbf{r} - \mathbf{r}' = \mathbf{R}$ and $E - E' = \varepsilon$ in the bulk of a sample and the spectrum.

III. EFFECTS OF LOCALIZATION AND MULTIFRACTALITY

To further proceed we employ the following ansatz for the electron wave functions:

$$\psi_n(\mathbf{r}) = \int \frac{d\Omega_s}{4\pi} a_s^{(n)}(\mathbf{r}) e^{ik_F \mathbf{s}\mathbf{r}}, \quad (9)$$

where $|\mathbf{s}| = 1$ and $a_s^{(n)}(\mathbf{r})$ is a Gaussian random variable with zero mean and the correlation function

$$\langle a_s^{(n)}(\mathbf{r}) a_{s'}^{(m)}(\mathbf{r}') \rangle = \delta_{nm} \delta_{\mathbf{s}, \mathbf{s}'} e^{-|\mathbf{r}-\mathbf{r}'|/2\ell} \phi_n(\mathbf{r}) \phi_m(\mathbf{r}'). \quad (10)$$

Equations (9) and (10) are essentially a generalization of the semiclassical Berry' ansatz [15] for the case of localization and multifractality. The exponential factor with the momentum relaxation length ℓ in Eq. (10) accounts for the fast randomization of wave-function phases due to elastic scattering, while positive functions $\phi_n(\mathbf{r})$ describe normalized (and smooth at a scale ℓ) envelopes of the wave functions, averaged over fast de Broglie oscillations:

$$[\phi_n(\mathbf{r})]^2 = \overline{\psi_n^2(\mathbf{r})} \mathcal{V}. \quad (11)$$

Such envelopes $\phi_n(\mathbf{r})$ are equal to 1 in the semiclassical Berry approximation $k_F \ell \gg 1$ when both localization and multifractality effects are absent and wave functions are *ergodic*. At $k_F \ell \sim 1$ when multifractality and/or localization is present, these envelope functions have a nontrivial shape which depends on the index n and on the realization of disorder. Thus the averaging in (10) is incomplete. It involves only the random-phase averaging and assumes subsequent disorder averaging of the amplitude. The possibility to separate nearly universal fast wave-function oscillations from the slow envelope that contains information about multifractal behavior was discussed in a different way in Ref. [16]. This idea has been successfully exploited in Ref. [17] in numerical computation of the multifractal spectrum $f(\alpha)$ in order to sort out the effect of nodes which dominate the distribution of small eigenfunction amplitudes.

It is shown in Appendix B that plugging (9) and (10) with $\phi_n(\mathbf{r}) = 1$ in (7) one exactly reproduces at $q\ell \ll 1$ an expression for $\tau_{ph}^{(1)}$ obtained earlier for diffusive metals [3,18,19]:

$$\frac{1}{\tau_{ph}^{(1)}} = \frac{q^2 k_F^4 \ell}{30\pi^2 \rho_i} = \frac{q^2 k_F \ell}{10 \rho_i} n_e, \quad (12)$$

where n_e is the total (two-spin) electron density. The corresponding result for $J_t(T)$ is

$$J_t(T) = \frac{4\pi^4}{630} \frac{(k_F \ell) n_e}{\rho_i [v_s^{(t)}]^5} T^6. \quad (13)$$

Taking now into account $\phi_n(\mathbf{r}) \neq 1$ in (10) one obtains from (7)–(9) the following expression for $\tau_{ph}^{(1)}$:

$$\frac{1}{\tau_{ph}^{(1)}} = \pi v_0^2 k_F^4 \frac{q_\beta q_\delta}{m^2 \rho_i} e_\alpha e_\gamma \int d^3 \mathbf{R} e^{i\mathbf{q}\mathbf{R}} e^{-|\mathbf{R}|/\ell} \int \frac{d\Omega_s}{4\pi} \\ \times \int \frac{d\Omega_{s'}}{4\pi} s_\alpha s'_\beta s'_\gamma e^{-ik_F (\mathbf{s}-\mathbf{s}')\mathbf{R}} K(\omega; \mathbf{R}), \quad (14)$$

where v_0 is the mean density of states at the Fermi level, $\Delta = (v_0 \mathcal{V})^{-1}$ is the mean level spacing in an entire volume \mathcal{V} , and $K(\omega; \mathbf{R}) = \Delta^2 \langle \sum_{n,m} \phi_n(\mathbf{r}) \phi_m(\mathbf{r}') \phi_n(\mathbf{r}') \delta(E - E_n) \delta(E + \omega - E_m) \rangle$. As the exponential factor $e^{-|\mathbf{R}|/\ell}$ makes the main domain of integration over \mathbf{R} in (14) to be $|\mathbf{R}| \lesssim \ell$ and because of the smooth behavior of the envelope functions $\phi_n(\mathbf{r})$ at such scale, one can replace $K(\omega; \mathbf{R}) \rightarrow K(\omega, 0) \equiv K(\omega)$. Then after angular integration over unit vectors \mathbf{s}, \mathbf{s}' and integration over R in (14), one obtains in the limit $|\mathbf{q}|\ell \ll 1$

$$\frac{1}{\tau_{ph}^{(1)}} = \frac{2\pi^2 v_0^2 k_F^2 \ell}{15 m^2 \rho_i} (3q_{||}^2 + q_{\perp}^2) K(\omega), \quad (15)$$

where $q_{||}$ and q_{\perp} are the longitudinal and the transverse components of the phonon wave vector and

$$K(\omega) = \Delta^2 \left\langle \sum_{n,m} \phi_n^2(\mathbf{r}) \phi_m^2(\mathbf{r}) \delta(E - E_n) \delta(E + \omega - E_m) \right\rangle \quad (16)$$

is the local density-of-states correlation function studied in Refs. [9,10].

For transverse phonons ($q_{\parallel} = 0$) Eq. (15) gives the total phonon attenuation rate. It is proportional to the properly normalized electron local density of states correlation function $K(\omega)$ which is coarse grained at a scale ℓ . All the effects of localization and/or multifractality are encoded in this correlation function, while the effects of fast randomization of wave function phases by impurity scattering are taken into account by averaging over momentum directions \mathbf{s}, \mathbf{s}' in Eq. (14).

Equations (15) and (16) are the main result of our paper. Strictly speaking it is valid in $2 + \epsilon$ ($\epsilon \ll 1$) dimensions where the scale separation $k_F \gg \ell^{-1} \gg \xi^{-1}$ holds even in an insulator close to the Anderson transition where the localization length ξ is large. As customary, we extend this result (with the accuracy up to a factor of order 1) for 3D samples and thick films with $k_F \ell \sim 1$.

IV. FUNCTION $K(\omega)$ CLOSE TO LOCALIZATION TRANSITION

The behavior of the correlation function $K(\omega)$ was studied in detail in Refs. [9,10]. It was shown that for $E_0 \gg \omega \gg \delta_{\xi}$, where $\delta_{\xi} = (\nu_0 \xi^3)^{-1}$ is the level spacing in the volume characterized by the correlation/localization length ξ , and E_0 is of the order of total bandwidth of conduction band, the effects of multifractality lead to the power-law enhancement of $K(\omega) = (E_0/\omega)^{\gamma}$, where $\gamma = 1 - d_2/3 \approx 0.59$ is determined by the fractal dimension $d_2 \approx 1.24 \pm 0.03$ [20]. This effect is due to the nonergodicity of wave functions which do not occupy all the available volume causing the enhancement of their amplitude by normalization. Furthermore, the support sets of different wave functions are strongly correlated thus giving rise to enhancement of the overlap function $K(\omega)$. Albeit analysis in [10] concerned the case of noninteracting electrons, the subsequent study [21] has shown that localization transition and multifractality survive almost unchanged when Coulomb interaction is taken into account.

As ω decreases below δ_{ξ} the behavior of $K(\omega)$ starts to depend on whether the system is insulating or metallic. In the latter case $K(\omega) \sim (E_0/\delta_{\xi})^{\gamma}$ saturates at its value for $\omega = \delta_{\xi}$. However, in the insulator $K(\omega) \sim (E_0/\delta_{\xi})^{\gamma} \ln^{d-1}(\delta_{\xi}/\omega)$ increases further upon ω decrease [9,10]. This logarithmic enhancement is due to the Mott's pairs of resonant levels which results in a well-known [11,12] logarithmic enhancement of frequency-dependent conductivity $\sigma(\omega) \sim \omega^2 \ln^{d+1}(\omega)$ in an insulator. The difference in the power of logarithm in $K(\omega)$ and $\sigma(\omega)$ is due to the square of the dipole moment matrix element entering the conductivity. Both limiting cases in a 3D insulator can be combined in one interpolating expression [9]:

$$K(\omega) = \frac{(E_0/\delta_{\xi})^{\gamma} \ln^2(\delta_{\xi}/\omega)}{c + (\omega/\delta_{\xi})^{\gamma} \ln^2(\delta_{\xi}/\omega)}, \quad (c \sim 1). \quad (17)$$

V. ENHANCEMENT OF COOLING IN A WEAK INSULATOR

Because of the strong dependence of the cooling power $J \propto v_s^{-5}$ on the sound velocity v_s , the cooling is usually dominated by the transverse phonons whose sound velocity is typically smaller by a factor of about 2. Then neglecting the

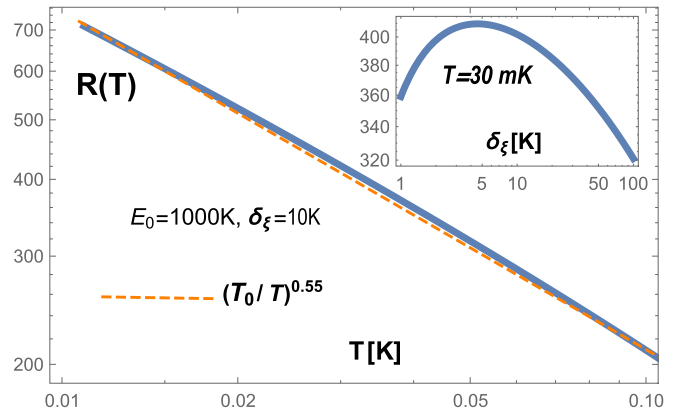


FIG. 2. Enhancement factor $R(T)$ in Eqs. (18) and (20) as a function of temperature for $E_0 = 1000$ K and $\delta_{\xi} = 10$ K. The dashed line represents a power law $(T_0/T)^{0.55}$ with $T_0 = 1710$ K. Both the value of $R(T)$ and the exponent in the apparent power-law dependence are in agreement with experiment Refs. [5,22]. Inset: the enhancement factor R as a function of δ_{ξ} for $T_{\text{el}} = 30$ mK, $E_0 = 1000$ K.

contribution of longitudinal phonons to cooling one obtains from (1)

$$J_{\ell}(T_{\text{el}}) = \frac{8}{5\pi^2} \frac{(k_F \ell) n_e}{\rho_i [v_s^{(\ell)}]^5} T_{\text{el}}^6 R(T_{\text{el}}), \quad (18)$$

where T_{el} is the temperature of electron system and the function

$$R(T) = \int_0^{\infty} dx x^5 [\coth(x) - 1] K(2Tx). \quad (19)$$

Actually the integral in (19) is strongly peaked at $2x \approx 5$, thus the ratio $J(T_{\text{el}})/T_{\text{el}}^6$ is proportional to $K(5T_{\text{el}})$. In a limited interval of electron system temperatures $T_{\text{el}} = 10$ – 100 mK the enhancement factor $R(T_{\text{el}})$ for typical parameters of indium oxide films $E_0 = 1000$ K, $\delta_{\xi} = 10$ K, $c = 1$ is well approximated by the power law $R(T) \approx (T_0/T_{\text{el}})^{0.55}$ with $T_0 \approx 1700$ K; see Fig. 2. Thus the effective power of temperature in $J(T_{\text{el}})$ should be $\beta_{\text{eff}} \approx 5.5$ rather than 6.0, in agreement with Ref. [22]. The overall enhancement factor for these values of parameters varies from 700 to 200 at $T_{\text{el}} = 10$ – 100 mK which is consistent with experiment [5]. The dependence of the $R(T)$ factor on the local level spacing δ_{ξ} is rather weak; see inset to Fig. 2.

VI. COOLING BY LONGITUDINAL PHONONS

Considering the contribution of longitudinal phonons to the cooling rate, one has to take into account the screening given by the second term in Eq. (4). The simplest case is the universal limit of screening when $V(q)\Pi \gg 1$ which is always the case in a 3D metal in the limit $q \rightarrow 0$ due to long-ranged Coulomb interaction $V(q) \propto 1/q^2$. In Anderson insulators this limit is approximately controlled by the large value of the dielectric constant close to the localization transition [23]. In this limit the electroneutrality condition is strictly enforced and the second term in Eq. (4) takes the universal form $-(\hat{D}\Pi)\Pi^{-1}(\Pi\hat{D})$. One can approximate $\hat{D}\Pi \approx \frac{\nabla \mathbf{u}}{m} k_F^2 \delta_{\alpha\beta} \Pi$, and $\Pi\hat{D} \approx \frac{\nabla \mathbf{u}}{m} k_F^2 \delta_{\gamma\delta} \Pi$. Now proceeding in the

same way as above using (9) and (10) and taking into account also the longitudinal part of (15) we obtain the contribution of the longitudinal phonons to electron cooling:

$$J_l(T_{\text{el}}) = \frac{24}{5\pi^2} \frac{(k_F \ell) n_e}{\rho_i [v_s^{(l)}]^5} T_{\text{el}}^6 R(T_{\text{el}}). \quad (20)$$

As in Eq. (18), this result differs only by a factor $R(T_{\text{el}})$ from that for a disordered metal [3,18].

Note that the above method of calculation using the ansatz (10) is valid only for local contributions, as it completely ignores a possibility of building a density-density propagator, the “diffuson.” However, in the universal limit of screening the diffusion cannot be excited, as it is forbidden by electroneutrality. The effect of incomplete screening on the longitudinal phonon decay rate and cooling is much more involved (see, e.g., Ref. [14]). It may play some role in low-dimensional cases where the effects of incomplete Coulomb screening are stronger.

VII. CONCLUSIONS

The main result of this paper is given by Eqs. (15) and (16) which relates the phonon decay rate $1/\tau(\omega)$ due to inelastic interaction with electrons, and the correlation function of the local density of states $K(\omega)$ characterizing electron wave functions near the Anderson mobility edge. A direct consequence of this relation is a strong enhancement of the electron-phonon cooling power in weak insulators, in comparison with usual diffusive metals, as demonstrated by Eqs. (18) and (19) and Figs. 1 and 2. For the case of insulating indium-oxide films, studied in Ref. [5], this enhancement is estimated to be in the range of 500–1000, in agreement with the experimental data. In general, our results suggest that measurements of the cooling rate Eq. (18) or ultrasound attenuation rate Eq. (15) provide a direct access to the electronic local density of states (LDOS) correlation function $K(\omega)$.

On a more technical side, we expect that the same relation (15) can be obtained by means of functional “sigma-model” approach like the one developed in [24].

The above results are general and valid for any 3D Anderson insulator with long localization length and relatively weak Coulomb interaction (a slight modification of our formulas will also work for 2D Anderson insulators). In particular, one can use this approach to analyze the data on bistability of I - V characteristics and switching between high-resistance and low-resistance branches as a function of applied voltage, as reported for a number of various semiconductors or insulators; see Refs. [25–28]. However, one should keep in mind that in insulators with strong Coulomb interaction it might be difficult to disentangle Coulomb correlation effects from purely localization effects. In such a case an effective correlation function $K_{\text{eff}}(\omega)$ may differ from its noninteracting version given in Eq. (17).

Our results for the electron-phonon cooling power make it possible to establish conditions for the observation of a many-body localization transition in electronic insulators, predicted theoretically more than a decade ago [29] but not yet observed. One of the crucial problems to be solved in this respect is to find an insulator with an extremely low thermal coupling between electrons and phonons, yet with measurable

electric conductance. Our theory will be instrumental to solve this important issue.

The behavior of the cooling power very similar to our prediction has been recently seen in the resistive state of moderately disordered superconducting indium oxide films at strong magnetic field and low temperatures; see Sec. IV of the Supplemental Material to Ref. [22], where $J(T_{\text{el}}) \propto T_{\text{el}}^{5.5}$ was observed. An enhancement (compared to the prediction for dirty metals with $k_F \ell \sim 0.3$) by a factor 400–800 of cooling power *per carrier* in insulating $\text{Nb}_x\text{Si}_{1-x}$ can also be extracted from the results of Ref. [30].

Finally, we note that the obtained results are not expected to hold for pseudogapped insulators where single-electron DOS is strongly suppressed due to local pairing [9]. Indeed, the electron-phonon cooling rate in the insulating state of indium-oxide realized at relatively low magnetic field is known [31] to be much lower (and follow much faster temperature dependence) than the high-field data reported in Ref. [5]. The reason for that difference is that strong magnetic field (above approximately 10 T) destroys local pairs and makes the electron spectrum gapless.

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APPENDIX A: GENERAL EXPRESSION FOR PHONON ATTENUATION RATE IN TERMS OF ELECTRON WAVE FUNCTIONS

In order to take the localized nature of electron wave functions into account we express the phonon attenuation rate in terms of the exact electron eigenfunctions $\psi_n(\mathbf{r})$ and eigenvalues E_n . Using (2) and (3) of the main text one can express the contribution to τ_{ph} from the first term of (4) as follows:

$$\begin{aligned} \frac{1}{\tau_{\text{ph}}^{(1)}} &= \frac{\pi}{2} \frac{q_\beta q_\delta}{m^2} e_\alpha e_\gamma \frac{1}{\rho_i \omega} \frac{1}{\mathcal{V}} \int dE \int dE' \\ &\times \int d^d \mathbf{r} \int d^d \mathbf{r}' e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')} F_{E,E'}(\omega) K_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}'; E, E'), \end{aligned} \quad (A1)$$

where e_α is the α component of the unit vector of phonon polarization, q_α is the component of the phonon wave vector \mathbf{q} with $|\mathbf{q}| = q = \omega/v_s$, \mathcal{V} is the volume, $F_{E,E'}(\omega) = [\tanh(\frac{E'+\omega}{2T}) - \tanh(\frac{E'}{2T})] \delta(E' - E + \omega)$ is the Fermi distribution factor and the function $K_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}'; E, E')$ is defined as

$$\begin{aligned} K_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}'; E, E') &= \left\langle \sum_{nm} [\partial_\alpha \psi_m^*(\mathbf{r})] [\partial_\beta \psi_n(\mathbf{r})] [\partial_\gamma \psi_n^*(\mathbf{r}')] \right. \\ &\quad \left. \times [\partial_\delta \psi_m(\mathbf{r}')] \delta(E - E_n) \delta(E' - E_m) \right\rangle. \end{aligned} \quad (A2)$$

In (A2) we denote disorder averaging by $\langle \cdot \rangle$. After such an averaging $K_{\alpha\beta\gamma\delta}(\mathbf{r}, \mathbf{r}'; E, E') = K_{\alpha\beta\gamma\delta}(\mathbf{R}, \varepsilon)$ becomes a function of $\mathbf{r} - \mathbf{r}' = \mathbf{R}$ and $E - E' = \varepsilon$ in the bulk of the spectrum. One can use the (approximate) translation invariance in the energy space and perform integration over E' :

$$\int dE' F_{E,E'}(\omega) = 2\omega \delta(\varepsilon - \omega). \quad (\text{A3})$$

Now the general expression for $\tau_{\text{ph}}^{(1)}$ takes the following form:

$$\frac{1}{\tau_{\text{ph}}^{(1)}} = \pi \frac{q_\beta q_\delta}{m^2} e_\alpha e_\gamma \frac{1}{\rho_i} \int d^d \mathbf{R} e^{i\mathbf{q}\mathbf{R}} K_{\alpha\beta\gamma\delta}(\mathbf{R}, \omega). \quad (\text{A4})$$

APPENDIX B: PHONON ATTENUATION RATE IN DISORDERED METALS

At $k_F \ell \gg 1$ when the effects of multifractality can be neglected the correlation function $K(\omega) \approx 1$ at $\omega \gg \Delta$. Here we consider this limit in order to show that our approach based on Eqs. (11) and (12) of the main text [in which $\phi_n(r) = 1$] reproduces the well-known results of Refs. [18,19] where the diagrammatic approach was adopted.

We start by evaluating the angular integrals over unit vectors \mathbf{s} and \mathbf{s}' in Eq. (14) of the main text. The result should have the following form:

$$I_1 \delta_{\alpha\delta} \delta_{\beta\gamma} + I_2 (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta}), \quad (\text{B1})$$

as the integrals do not contain any preferential direction.

The quantities I_1 and I_2 can be found from the following equations:

$$9I_1 + 6I_2 = \int d^3 \mathbf{R} e^{-R/\ell} \left| \int \frac{d\Omega_{\mathbf{s}}}{4\pi} e^{ik_F \mathbf{s}\mathbf{R}} \right|^2, \quad (\text{B2})$$

$$3I_1 + 12I_2 = \int d^3 \mathbf{R} e^{-R/\ell} \int \frac{d\Omega_{\mathbf{s}}}{4\pi} \int \frac{d\Omega_{\mathbf{s}'}}{4\pi} e^{ik_F(\mathbf{s}-\mathbf{s}')\mathbf{R}} (\mathbf{s}\mathbf{s}')^2. \quad (\text{B3})$$

The integral under the absolute value sign in Eq. (B2) is nothing but the Friedel oscillation in 3D space:

$$\Psi_{\text{Fried}} = \left(\frac{\sin(k_F R)}{k_F R} \right). \quad (\text{B4})$$

Thus the right-hand side of Eq. (B2) reduces to

$$\int d^3 \mathbf{R} \left(\frac{\sin(k_F R)}{k_F R} \right)^2 e^{-R/\ell} = \frac{8\pi \ell^3}{1 + 4(k_F \ell)^2}. \quad (\text{B5})$$

The double angular integral in Eq. (B3) can also be expressed in terms of Ψ_{Fried} and its second derivative:

$$\begin{aligned} \frac{1}{2} \left(\frac{\sin y}{y} \right)^2 e^{-y/d} + \left(\frac{\sin y}{y} \right) \partial_y^2 \left(\frac{\sin y}{y} \right) e^{-y/d} \\ + \frac{3}{2} \left[\partial_y^2 \left(\frac{\sin y}{y} \right) \right]^2 e^{-y/d}, \end{aligned} \quad (\text{B6})$$

where $y = k_F R$ and $d = k_F \ell$.

Now doing the \mathbf{R} integral in Eq. (14) of the main text we obtain

$$\int d^3 \mathbf{R} e^{-R/\ell} \int \frac{d\Omega_{\mathbf{s}}}{4\pi} \int \frac{d\Omega_{\mathbf{s}'}}{4\pi} e^{ik_F(\mathbf{s}-\mathbf{s}')\mathbf{R}} (\mathbf{s}\mathbf{s}')^2 = 4\pi \ell^3 Y(d), \quad (\text{B7})$$

where the function $Y(d)$ is

$$Y(d) = \frac{2}{1 + 4d^2} + \frac{1}{d^4} - \frac{1 + 2d^2}{4d^6} \ln(1 + 4d^2). \quad (\text{B8})$$

In the limit $d = k_F \ell \gg 1$ one obtains from Eqs. (B1)–(B3) and (B7):

$$I_2 \approx I_1 = \frac{2}{15} \pi \frac{\ell}{k_F^2}. \quad (\text{B9})$$

so that the combination of delta symbols in Eq. (B1) is totally symmetric.

Now plugging this result into Eq. (14) of the main text one obtains the transverse phonon attenuation rate:

$$\frac{1}{\tau_{\text{ph}}^{(t)}} = \frac{q_\perp^2}{30\pi^2} \frac{k_F^4 \ell}{\rho_i} = \frac{q_\perp^2}{10} \frac{k_F \ell}{\rho_i} n_e, \quad (\text{B10})$$

where n_e is the total electron density (with both spin directions).

Correspondingly, the result for the out-cooling rate is

$$J_t(T) = \frac{4\pi^4}{630} \frac{1}{[v_s^{(t)}]^5 \rho_i} (k_F \ell) n_e T^6, \quad (\text{B11})$$

which coincides with the result of Refs. [18,19].

For longitudinal phonons Eq. (14) of the main text gives the result which is by a factor of 3 larger than in (B10):

$$\frac{1}{\tau_{\text{ph}}^{(1,l)}} = \frac{q_\parallel^2}{10\pi^2} \frac{k_F^4 \ell}{\rho_i}, \quad (\text{B12})$$

However, in order to compute the attenuation rate of longitudinal phonons one has to take into account also the second term in Eq. (4) of the main text. At a complete screening, this term has an opposite sign compared to (B10) and thus the phonon attenuation rate for longitudinal phonons is smaller than in Eq. (B12).

The additional negative contribution should be found from the expression similar to Eq. (7) of the main text:

$$\frac{1}{\tau_{\text{ph}}^{(1)}} = \pi \frac{q_\beta q_\delta}{m^2} e_\alpha e_\gamma \frac{1}{\rho_i} \int d^d \mathbf{R} e^{i\mathbf{q}\mathbf{R}} K_{\alpha\beta\gamma\delta}^{(2)}(\mathbf{R}, \omega), \quad (\text{B13})$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\omega = E - E'$, and the correlation function $K_{\alpha\beta\gamma\delta}^{(2)}(\mathbf{R}, \omega)$ is

$$\begin{aligned} K_{\alpha\beta\gamma\delta}^{(2)}(\mathbf{R}, \omega) = -\frac{1}{9} \delta_{\alpha\beta} \delta_{\gamma\delta} k_F^4 \left\langle \sum_{nm} \Psi_m^*(\mathbf{r}) \Psi_n(\mathbf{r}) \right. \\ \left. \times \Psi_n^*(\mathbf{r}') \Psi_m(\mathbf{r}') \delta(E - E_n) \delta(E' - E_m) \right\rangle. \end{aligned} \quad (\text{B14})$$

Now substituting Eqs. (9) and (10) of the main text into (B14) we obtain in the limit $q\ell \ll 1$

$$\frac{1}{\tau_{\text{ph}}^{(2)}} = -\frac{q_\parallel^2}{18\pi^2} \frac{k_F^4 \ell}{\rho_i}, \quad (\text{B15})$$

so that

$$\frac{1}{\tau_{\text{ph}}^{(l)}} = \frac{1}{\tau_{\text{ph}}^{(1,l)}} + \frac{1}{\tau_{\text{ph}}^{(2)}} = \frac{2q_\parallel^2}{45\pi^2} \frac{k_F^4 \ell}{\rho_i}. \quad (\text{B16})$$

Correspondingly, the out-cooling rate due to longitudinal phonons is

$$J_l(T) = \frac{8\pi^4}{945} \frac{1}{[v_s^{(l)}]^5} (k_F \ell) n_e T^6. \quad (\text{B17})$$

Thus the ratio of the total contribution of the transverse $[2J_t(T)]$ and the longitudinal $[J_l(T)]$ phonons to the cooling rate is $(3/2)[v_s^{(l)}/v_s^{(t)}]^5$, in agreement with earlier results [see, e.g., Eq. (31) of Ref. [18] of the paper].

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