Threshold effects in one-dimensional strongly interacting nonequilibrium systems

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In this work we investigate the phenomena associated with the new thresholds in the spectrum of excitations arising when different one-dimensional strongly interacting systems are voltage biased and weakly coupled by tunneling. We develop the perturbation theory with respect to tunneling and derive an asymptotic behavior of physical quantities close to threshold energies. We reproduce earlier results for the electron relaxation at the edge of an integer quantum Hall system and for the nonequilibrium Fermi edge singularity phenomenon. In contrast to previous works, our analysis does not rely on the free-fermionic character of local tunneling; therefore, we are able to extend our theory to a wider class of systems, without well-defined electron excitations, such as spinless Luttinger liquids and chiral quantum Hall edge states at fractional filling factors.

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I. INTRODUCTION

Interactions between particles are an important component for the realistic description of many-body systems. While in a large class of systems, such as electrons in metals, interactions can either be neglected or considered perturbatively, in many systems of reduced dimensionality they manifest themselves in new physical effects and even in new states of matter. Among them, there are models of particular importance such as Luttinger liquids (LLs) [1], quantum dots (QDs) demonstrating the Fermi edge singularity (FES) effect [2], and other systems[3], which are exactly solvable for arbitrary interaction strengths. At equilibrium, these systems have a peculiar behavior. For instance, the tunneling density of states (TDOS) in LLs has a power-law singularity at the Fermi level. In the case of the FES a similar dependence is observed for the transition rate between an impurity and a Fermi sea as a function of the energy of the impurity level.

It seems to be natural to propose a generalization of these models by introducing new energy scales and additional thresholds in the spectrum. For instance, this can be done by accounting for a nonlinearity of the electron spectrum in the LL model [4]. Alternatively, one can inject nonequilibrium electrons into a LL from a metallic reservoir and study their relaxation to a nontrivial stationary state [5,6]. Yet another example is the FES effect, where an impurity that hosts a virtual electronic level couples two electronic reservoirs with different chemical potentials by means of a cotunneling process [7].

It turns out that away from equilibrium these phenomena are deeply related. For instance, the electron relaxation at the edge of an integer quantum Hall (QH) system [5] at filling factor $\nu = 2$ and the nonequilibrium FES effect in a QD embedded in a QH system [8] were solved using the same approach, which is based on the evaluation of the full counting statistics (FCS) [9] of electron tunneling. A different method was used to address the FES problem in Refs. [6,10] and to find the TDOS in LLs in Ref. [6], where the quantities of interest are expressed through a nonequilibrium electron Green's function, represented as Fredholm determinants over single-particle degrees of freedom. Notably, all these methods rely on the free-fermionic character of the injection of electron excitations.

In this paper we present a different approach, which, on the one hand, can reproduce the results mentioned above and, on the other hand, is also applicable to systems without well-defined electronic excitations. Namely, we consider a stationary TDOS at the edge of a fractional QH system and in the bulk of a LL away from equilibrium. In both cases we study the relaxation of the nonequilibrium state, created by injecting electrons via a quantum point contact (QPC) from a reservoir with the chemical potential μ. We assume weak tunneling coupling at the QPC (with tunneling probability $T \ll 1$) and study tunneling perturbatively. The correction to the equilibrium TDOS is then measured by tunneling to a QD at the energy ϵ.

Let us point out that even though the perturbative character of the nonequilibrium tunneling is crucial for our analysis, the obtained results are universal. The quantities of interest (TDOS, transition rates) are typically studied close to the threshold energies [5,6,8,10], where they have a universal power-law behavior $|\epsilon - \epsilon_0|^\kappa$ as a function of the energy ϵ in the vicinity of the energy thresholds $\epsilon_0 = 0$ and $\epsilon_0 = \pm\mu$. High-order tunneling processes at the source QPC smear out the singularities at energies of the order of $T\mu$ at zero temperature [5,6,8,10]. Our main goal, however, is to find universal exponents κ in different physical situations, which justifies our perturbative approach.

The results of our calculations are summarized in Table I for four different physical situations. We study the relaxation of a nonequilibrium state at the edge of a QH system at the filling factor $\nu = 2$. This system has been extensively studied experimentally [11]. The measured quantity is the energy-dependent correction to the TDOS. We consider the nonequilibrium FES phenomenon in a QH-effect-based device. This system was experimentally studied in Ref. [12]. The quantity of interest is the sequential tunneling rate as a function of the energy of the QD level. For both of these

TABLE I. The results for the exponents κ of an asymptotic power-law behavior for different quantities considered in the paper are summarized. In all cases, the physical quantities are weakly perturbed by injecting nonequilibrium electrons from a metallic system with the chemical potential μ and detected at relatively large distances with the help of a QD at energy ϵ. They show an asymptotic behavior $|\epsilon-\epsilon_0|^\kappa$ in the vicinity of different threshold energies $\epsilon_0=-\mu,0,\mu$. The following quantities are presented: (i) The first is the TDOS at the edge of a QH system at a filling factor of 2 as a function of the dimensionless parameter α that characterizes mixing of the edge channels due to the Coulomb interaction. In particular, $\alpha=1/\sqrt{2}$ in the experimentally relevant case of strong interaction (see Sec. II). (ii) The second is the FES in the sequential tunneling rates to (from) a QD embedded in a QH system from (to) the QH edge channel. New FES exponents κ are expressed as a function of the equilibrium FES exponent α_D and of the fraction η_D of the QD charge screened by the edge channel (see Sec. III). (iii) The third is the TDOS in a spinless LL. The exponents κ are expressed as a function of the LL parameter K. (iv) The last is the TDOS at the edge of a chiral fractional QH system at the filling factor $\nu=1/(2n+1)$.

ϵ_0	$-\mu+0$	0	$\mu-0$
$\nu=2$	$2(1+\alpha^2)$	-1	$2(1-\alpha^2)$
FES $\Gamma_\pm$	$2-\alpha_D\pm 2(1-\eta_D)$	$-1-\alpha_D$	$2-\alpha_D\mp 2(1-\eta_D)$
LL	$3\frac{K+K^{-1}}{2}+1$	$\frac{K+K^{-1}}{2}-2$	$\frac{K+K^{-1}}{2}-1$
$\nu=\frac{1}{2n+1}$	absent	0	$2n$

systems, we reproduce previously found results, obtained with the nonperturbative methods [5,8]. Finally, we evaluate the TDOS in nonequilibrium LLs and in fractional QH systems [11,13].

The rest of the paper is organized as follows. In Sec. II we focus on the physics of electron relaxation at the edge of an integer QH system. We formulate the problem of finding the nonequilibrium correction to the TDOS, develop the tunneling perturbation theory, and find the asymptotic behavior of the correction at different threshold energies. In this section we also recall the essential elements of the bosonization technique [14]. In Sec. III we concentrate on the nonequilibrium FES and find exponents of singularities in sequential tunneling rates. Finally, in Sec. IV we apply our theory to essentially nonfermionic systems: spinless LLs and chiral fractional QH systems. In the Appendix, we derive the perturbative correction to TDOS at the integer QH edge directly from the Fredholm determinant.

II. QH SYSTEM AT FILLING FACTOR $\nu = 2$

A. Formulation of the problem

To study a stationary state of the strongly interacting QH edge channels at the filling factor $\nu=2$ let us consider the system presented in Fig. 1 that was realized experimentally in [11]. The dynamics in the interacting channels is governed by the Hamiltonian

$$H_0=\pi\int dx\sum_i v_i\rho_i^2(x)+\frac{1}{2}\int dxdy\sum_{ij}\rho_i(x)V_{ij}(x,y)\rho_j(y), \tag{1}$$

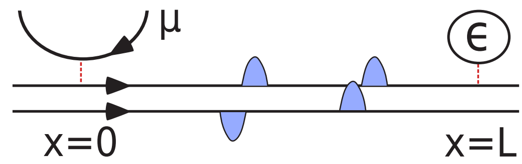

FIG. 1. QH edge states at the filling factor $\nu=2$ are schematically shown. Due to strong interactions at the edge, two free-propagating plasmonic modes arise, dipole and charge mode. At the point $x=0$ nonequilibrium electrons are injected from the source channel biased with the chemical potential μ, and at the point $x=L$ the TDOS $n(\epsilon)$ is measured at the energy ϵ. The measurements can be carried out by attaching a QD with a single level ϵ and monitoring a resonant current through it [11].

where $\rho_i(x)$, $i=U,D$, are the electron densities in the upper and lower edge channel, respectively, and $V_{ij}(x)$ is the Coulomb potential, which is assumed to be screened at distances smaller than the characteristic wavelength of the edge excitations. Thus, it can be written as $V_{ij}(x)=V_{ij}\delta(x)$. Below, this simplification helps us to diagonalize the Hamiltonian (1) using the bosonization technique, which we recall next.

We introduce two bosonic fields $\phi_i(x,t)$, $i=U,D$, corresponding to two edge channels and satisfying the commutation relations

$$[\partial_x\phi_i(x),\phi_j(y)]=2\pi i\delta_{ij}\delta(x-y). \tag{2}$$

Two important identities relate these bosonic fields to edge electrons and charge densities:

$$\psi_i(x)\propto e^{i\phi_i(x)},\quad \rho_i(x)=\frac{1}{2\pi}\partial_x\phi_i(x), \tag{3}$$

where the vertex operator $\psi_i(x)$ annihilates an electron at point x in channel $i=U,D$. Despite interactions, the edge Hamiltonian (1) is quadratic in bosonic fields (quartic in electron operators). Written in terms of bosons, it has the following matrix form:

$$H_0=\int\frac{dx}{8\pi^2}\sum_{ij}\partial_x\phi_i(x)U_{ij}\partial_x\phi_j(x), \tag{4}$$

with

$$U_{ij}=\begin{bmatrix}2\pi v_1+V_{11} & V_{12}\\ V_{12} & 2\pi v_2+V_{22}\end{bmatrix}. \tag{5}$$

This Hamiltonian can be diagonalized by rotating the basis as

$$\phi_c=(\alpha\phi_U+\beta\phi_D),\quad \phi_d=(\beta\phi_U-\alpha\phi_D), \tag{6}$$

where

$$\alpha=\frac{U_{12}}{\sqrt{U_{12}^2+\left[\sqrt{(U_{11}-U_{22})^2+4U_{12}^2}-(U_{11}-U_{22})\right]^2/4}},$$
$$\beta=\sqrt{1-\alpha^2}.$$

The new fields describe fast charge and a slow dipole mode freely propagating at the edge with velocities u_c and u_d. It is important to mention that in the experimentally most relevant regime of strong, long-range interactions, $2\pi v_i\ll V_{ij}$ and $|V_{ii}-V_{12}|\ll V_{12}$, $i=1,2$, the parameters of the rotation

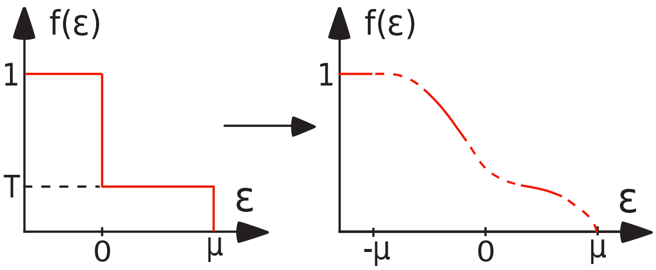

FIG. 2. On the left, the nonequilibrium TDOS is schematically shown directly after the injection from the Fermi sea, biased by the chemical potential μ, to the QH edge at filling factor 2 through a QPC with transparency T. Because of the effectively-free-fermionic character of the local tunneling process, the TDOS acquires a well-known double-step form. At intermediate distances, due to strong interactions, the double-step TDOS relaxes to a stationary state, as schematically shown on the right. In the regions close to the thresholds (shown by dashed lines) the TDOS acquires a singular power-law behavior, which is the subject of our study.

acquire the universal value $\alpha = \beta = 1/\sqrt{2}$. In contrast, in the case of weak interactions these parameters satisfy the following relation: $\alpha \gg \beta$.

We consider the situation where a nonequilibrium state is created by tunneling processes at the point $x = 0$, described by the Hamiltonian

$$H_T = \tau\psi_\mu(0)^\dagger\psi_U(0) + \text{H.c.},$$

where ψ_μ and $\psi_\mu^\dagger$ are the operators for electrons in the biased source channel (see Fig. 1), with μ denoting the applied bias. For our purposes, it is not necessary to introduce a particular Hamiltonian for electrons in this channel since the only object we need below is the local correlation function, which we choose to have the free-fermion form

$$\langle\psi_\mu^\dagger(0,t)\psi_\mu(0,0)\rangle \sim e^{i\mu t}/(it+0). \tag{7}$$

This is the case for metallic systems at low energies as well as for chiral QH edge channels at integer filling factors [15].

At intermediate distances $x = L$ the TDOS $n(\epsilon)$ at the edge reaches a stationary nonequilibrium form (see Fig. 2) [5]. It can be measured by attaching a QD to the upper edge channel and studying the resonant tunneling current [11]. We are interested in the deviation of the TDOS from its equilibrium value $n_{eq}(\epsilon)$. It can be presented as follows:

$$\delta n(\epsilon) \equiv n(\epsilon) - n_{eq}(\epsilon) = \int_{-\infty}^{\infty} dt e^{-i\epsilon t}\delta N(L,t), \tag{8}$$

$$\delta N(L,t) = \langle\psi_U^\dagger(L,t)\psi_U(L,0)\rangle_{n-eq} - \langle\psi_U^\dagger(L,t)\psi_U(L,0)\rangle_{eq}, \tag{9}$$

where the nonequilibrium correlation function is evaluated with respect to the state excited by the source. The electron operators (3) can be expressed in terms of bosonic eigenmodes by solving equations of motion generated by the Hamiltonian (1),

$$\psi_U(x,t) = \exp\left[i\alpha\phi_c(t - x/u_c) + i\beta\phi_d(t - x/u_d)\right], \tag{10}$$

where the time dependence reflects the free propagation of eigenmodes with different velocities. The first important result that can be easily derived from the bosonic representation is that the local equilibrium TDOS takes the free-fermion form [15] in spite of the strong interactions. At zero temperature, this gives $n_{eq}(\epsilon) = \theta(-\epsilon)$ and allows one to express $\delta n(\epsilon)$ in terms of the FCS of the free-electron transport [9].

In what follows, we rely on several simplifications for the calculation of the correlation function (9). First, due to the separation of the spectrum on the charged and dipole mode, propagating with different speeds, one can neglect their correlations at distances of $\sim L$, where the stationary state is formed. Indeed, at such distances their contributions to the electron correlation function originate from different tunneling events at $x = 0$. Second, we can ignore the correlations of electrons that are separated by a distance of the order of L. As a consequence, only tunneling events that happen at times $\sim -L/u_i$, $i = c, d$, at the point $x = 0$ contribute to the correlator in Eq. (9). Finally, we concentrate on the asymptotic forms of the TDOS in order to study its scaling behavior close to Fermi levels. Even though in Secs. III and IV we consider different systems, the analysis there can also be reduced to finding the bosonic correlators in a nonequilibrium state. In the next section we show how such quantities can be evaluated.

B. Perturbation theory

In order to expand the correlator (9) in powers of the tunneling Hamiltonian H_T, we rewrite it in the interaction representation

$$\delta N(L,t) = \langle U^\dagger(-\infty,t)\psi_U^\dagger(L,t)U(t,0)\psi_U(L,0) \times U(-\infty,0)\rangle_{eq} - \langle\psi_U^\dagger(L,t)\psi_U(L,0)\rangle_{eq}, \tag{11}$$

where $U(t_1,t_2) = \hat{T}\exp[-i\int_{t_2}^{t_1} dt' H_T(t')]$ is the time-ordered evolution operator. Expanding the evolution operators up to the second order in H_T generates $3! = 6$ terms. However, the number of terms can be halved. Indeed, for large L tunneling at the point $x = 0$ taking place between times 0 and t cannot affect the results of the measurement at the point $x = L$. Therefore, the evolution operator $U(t,0)$ can be dropped in the above expression. On the physics level, this amounts to neglecting exchange effects in tunneling events at $x = 0$ and $x = L$; that is, terms like $\langle\psi_U^\dagger(L,t)\psi_U(0,0)\rangle_{eq}$ are neglected.

Moreover, we can safely extend the time domains of the remaining evolution operators to infinity without affecting the correlator (11). Indeed, although by doing so we add extra tunneling events at point $x = 0$, they do not affect the measurements at point $x = L$ since wave packets do not reach this point. Consequently, Eq. (11) can be rewritten as

$$\delta N(L,t) \approx \langle U^\dagger(-\infty,\infty)\psi_U^\dagger(L,t)\psi_U(L,0)U(-\infty,\infty)\rangle_{eq} - \langle\psi_U^\dagger(L,t)\psi_U(L,0)\rangle_{eq}. \tag{12}$$

We note that this approximation is valid only for relatively large energies μ and ϵ. The corrections to Eq. (12) scale as powers of $u_c/[L\min(\mu,\epsilon)]$ with the exponents of the order of 1 (and exactly 1 for free fermions).

After expanding the evolution operator up to the second order in H_T and expressing the tunneling Hamiltonian in terms of plasmonic eigenmodes,

$$H_T = \tau \psi_\mu^\dagger \exp[i(\alpha\phi_c + \beta\phi_d)] + \text{H.c.}, \quad (13)$$

we evaluate the average in Eq. (12) with respect to the equilibrium state. The result can be expressed in terms of the four-point correlation functions of the following form:

$$\langle e^{-i\xi\phi_i(t_1)} e^{i\xi\phi_i(t_2)} e^{-i\lambda\phi_i(t-L/u_i)} e^{i\lambda\phi_i(-L/u_i)}\rangle_{eq}$$
$$= K_{\xi^2}(t_1,t_2) K_{\lambda^2}(t,0) \frac{K_{\xi\lambda}(t_1,-L/u_i) K_{\xi\lambda}(t_2, t-L/u_i)}{K_{\xi\lambda}(t_1, t-L/u_i) K_{\xi\lambda}(t_2,-L/u_i)}, \quad (14)$$

where the two-point correlator

$$K_\gamma(t_1,t_2) = \langle e^{-i\sqrt{\gamma}\phi_i(t_1)} e^{i\sqrt{\gamma}\phi_i(t_2)}\rangle \propto [i(t_1-t_2)+0]^{-\gamma} \quad (15)$$

takes the same form for the two eigenmodes $i = c, d$.

The correlation function (14) has the important property [16] that it acquires a nontrivial form only at t_1 and t_2 close to the flight time of one of the eigenmodes, $\sim -L/u_i$. Therefore, one can split the function (12) into two contributions, $\delta N(L,t) = \delta N_c(L,t) + \delta N_d(L,t)$ from the charged and dipole modes. The contribution of the charged mode reads

$$\delta N_c(L,t) \propto K_1(t,0) \iint dt_1 dt_2 e^{i\mu(t_1-t_2)}$$
$$\times \left(K_2(t_1,t_2) \frac{K_{\alpha^2}(t_1,t)}{K_{\alpha^2}(t_1,0)} - \text{c.c.}\right)\left(\frac{K_{\alpha^2}(t_2,0)}{K_{\alpha^2}(t_2,t)} - \text{c.c.}\right). \quad (16)$$

The contribution of the dipole mode can be obtained by replacing $\alpha \to \beta$.

We have arrived at expression (16) by applying the perturbation expansion directly to the correlation function (9). Alternatively, one can apply an expansion in tunneling amplitude to the nonperturbative expression for the TDOS in the form of a Fredholm determinant. This method, presented in the Appendix, is based on the free-fermion character of the local tunneling transport. The advantage of the approach presented in this section is that it can also be used for tunneling to non-Fermi-liquid states, as discussed in Sec. IV.

C. Asymptotic behavior of TDOS

In this section we evaluate the TDOS (8) asymptotically close to the threshold energies $\epsilon_0 = 0$ and $|\epsilon_0| = \mu$ (see Fig. 2). Starting with $\mu, \epsilon > 0$, the contribution of the charged mode $\delta n_c(\epsilon) = \int dt e^{-i\epsilon t} \delta N_c(L,t)$ can be written as

$$\delta n_c(\epsilon) \propto \int_{-\infty}^{\infty} dt_- \int_0^\infty dt_+ \int_0^\infty dt \frac{e^{-\epsilon(t+t_+)} e^{i(\mu-\epsilon)t_-}}{(it_- + t_+ + t)(t_- - i0)^2}$$
$$\times \frac{(-t_- + it)^{\alpha^2}(t_- - it_+)^{\alpha^2}}{(-it_+)^{\alpha^2}(it)^{\alpha^2}}, \quad (17)$$

where we changed the variables in the integral (16) to $t_- = t_2 - t_1$, $t_+ = (t_1+t_2)/2$. For $\epsilon \ll \mu$, this expression takes the form

$$\delta n_c(\epsilon) \propto \int dt_- \int_0^\infty dt \int_0^\infty dt_+ \frac{e^{i\mu t_-} e^{-\epsilon(t+t_+)}}{(t_- - i0)^2 (t_+ + t)} \propto \frac{\mu}{\epsilon}, \quad (18)$$

where we dropped a small prefactor of tunneling probability $T \ll 1$ since we are interested in only a power-law scaling.

TABLE II. For electron tunneling to the edge of an integer QH system at filling factor $\nu = 2$ the correction to the equilibrium TDOS acquires the general asymptotic form $\delta n(\epsilon) \propto |\epsilon - \epsilon_0|^\kappa$. The exponents κ of this asymptotic behavior in the vicinity of different thresholds ϵ_0 are shown, where α is an interaction constant.

ϵ_0	$-\mu + 0$	0	$\mu - 0$
κ	$2(1+\alpha^2)$	-1	$2(1-\alpha^2)$

This results agrees with the findings of Ref. [5]. The dipole contribution $\delta n_d(\epsilon)$ scales in the same way.

We now concentrate on the behavior close to the second threshold in the TDOS: $\mu - \epsilon \ll \epsilon$. We stress that this threshold arises in the weak tunneling limit, to leading order in tunneling at the source QPC, because the maximum energy that can be injected with one electron from the source is equal to μ. Consequently, to leading order in tunneling $\delta n(\varepsilon) = 0$ for $\varepsilon > \mu$. High-order tunneling processes smear out the singularity. Close to the threshold the charged-mode contribution reads

$$\delta n_c(\epsilon) \propto \int dt_- \int_0^\infty dt \int_0^\infty dt_+ \frac{e^{i\mu t_-} e^{-\epsilon(t+t_+)}}{i(t_- - i0)^{3-2\alpha^2}(it)^{\alpha^2}(-it_+)^{\alpha^2}}$$
$$\propto \left(\frac{\mu - \epsilon}{\mu}\right)^{2(1-\alpha^2)}, \quad (19)$$

and a similar expression is obtained for the dipole mode by replacing $\alpha \to \beta$. In the case of strong, long-range interactions the charge of the tunneling electron equally splits between charged and dipole modes, $\alpha = \beta = 1/\sqrt{2}$, which leads to the linear dependence: $\delta n(\epsilon) \propto (\mu - \epsilon)/\mu$.

In the next step we analyze the hole part of the TDOS, $\epsilon < 0$. Since the details of the evaluation of the TDOS (8) are the same, we present the results without the derivation (see Table II). Finally, we note that for $\mu < 0$ the TDOS is immediately obtained by exchanging electrons and holes, and thus, the following identity holds:

$$\delta n(-\epsilon)|_{\mu \to -\mu} = -\delta n(\epsilon), \quad (20)$$

which can be derived directly from Eq. (16) and is intuitive from the physics perspective.

III. FERMI EDGE SINGULARITY

In this section we apply the technique developed above to the problem of nonequilibrium FES. Motivated by the recent experiment [12], we study this effect in the QH setup, shown in Fig. 3. In this system, FES appears as a universal energy dependence of the transition rate between the edge channel and the QD level. We follow the bosonization approach of [8] and present the Hamiltonian of the system in the form

$$H = H_0 + H_{\text{int}} + H_T + H_T', \quad (21)$$

where H_0 is the free-fermion part,

$$H_0 = \int \frac{dx}{4\pi^2} \sum_i v_i [\partial_x \phi_i(x)]^2 + \epsilon\, d^\dagger d, \quad (22)$$

describing excitations in the QH channels and in the QD, respectively. The summation in the first term runs over four

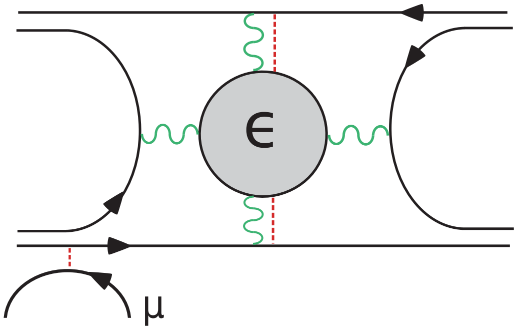

FIG. 3. The QH system at integer filling factor with the embedded QD, perturbed by tunneling at the voltage-biased QPC, is schematically shown (for details, see the experiment [12]). The QD strongly interacts with surrounding edge channels (shown by arrows), which partially or completely screen an electron added to the QD energy level ϵ. In equilibrium, this leads to the well-known FES phenomenon: sequential tunneling rates (shown by dashed lines) to and from the energy level ε [shifted by the interactions, see Eq. (26)], acquire the universal low-energy behavior $\Gamma_\pm(\varepsilon) \propto 1/|\varepsilon|^{\alpha_D}$, where the exponent α_D depends only on the charges induced in surrounding channels. Weak tunneling at the upstream QPC, biased with the chemical potential μ, creates new thresholds and modifies FES exponents. They are presented in Table III. The tunneling rates $\Gamma_\pm(\varepsilon)$ can be found from the measurements of the current in an upper or lower channel.

channels surrounding the QD, $i = U, D, L, R$, where $\phi_i(x)$ are bosonic operators introduced in Sec. II. The QD is tuned to the resonant tunneling regime via the energy level ϵ, and operators $d^\dagger$ and d create and annihilate an electron at this level. The case of a large QD hosting more than one energy level was considered in Ref. [17].

The key ingredient of the FES is Coulomb interactions between the charge localized on the QD and the density accumulated in the channels. It is described by the Hamiltonian

$$H_{\text{int}} = \frac{1}{2\pi} d^\dagger d \int dx \sum_i U_i(x) \partial_x \phi_i(x), \tag{23}$$

where $U_i(x)$ are the Coulomb potentials and the sum runs over the surrounding channels, $i = U, D, L, R$. While the general universal solution of the problem can be found in [17], we replace potentials with $U_i(x) = U_i \delta(x)$ for simplicity since we are interested in the low-energy physics, where the length of edge excitations is larger than the range of potentials [18].

The upper and lower channels are coupled to the QD through the tunneling Hamiltonian

$$H_T = d^\dagger \sum_{i=U,D} \tau_i e^{i\phi_i(L)} + \text{H.c.}, \tag{24}$$

where $x = L$ is the point at the edges, where tunneling takes place. A nonequilibrium state in the lower channel is created by electron tunneling at point $x = 0$ from the source channel, biased with the chemical potential μ. The Hamiltonian that accounts for this process is given by

$$H_T' = \tau \psi_\mu^\dagger(0) e^{i\phi_D(0)} + \text{H.c.}, \tag{25}$$

TABLE III. The exponents κ of the asymptotic behavior $\Gamma_\pm(\varepsilon) \propto |\varepsilon - \varepsilon_0|^\kappa$ of the transition rates in the vicinity of different thresholds ε_0 are shown.

ε_0	$-\mu + 0$	0	$\mu - 0$
κ	$2 - \alpha_D \pm 2(1 - \eta_D)$	$-1 - \alpha_D$	$2 - \alpha_D \mp 2(1 - \eta_D)$

where operators ψ_μ and $\psi_\mu^\dagger$ describe electrons in the biased channel.

According to [8], the bosonization technique allows us to treat the interaction term H_{int} exactly. One can perform a unitary transformation that removes this term at the cost of the modification of the energy of the QD level $\epsilon \to \varepsilon$ and of the transformation of the tunneling Hamiltonian $H_T \to \tilde{H}_T$:

$$\varepsilon = \epsilon + \sum_i \eta_i U_i, \tag{26}$$

$$\tilde{H}_T = d^\dagger \sum_i \tau_i e^{i\phi_i(L) - \sum_j \eta_j \phi_j(L)} + \text{H.c.}, \tag{27}$$

where the dimensionless numbers $0 \leqslant \eta_i \leqslant 1$ are the charges accumulated in surrounding channels in response to adding an electron to the QD. For the QD screened solely by these channels, $\sum_i \eta_i = 1$. Since the microscopic details of the setup are reduced solely to the set of parameters η_i, the discussed approach can be naturally generalized beyond the setup in Fig. 3, i.e., to any number of transport channels with any strength of the interaction with the QD.

The transition rate from the lower channel to the QD Γ_+ and the rate for the reversed process Γ_- can be found perturbatively by applying the Fermi golden rule with respect to the modified tunneling Hamiltonian $\tilde{H}_T$,

$$\Gamma_\pm(\varepsilon) \propto \int dt e^{-i\varepsilon t} \chi_D(t, \pm(1 - \eta_D)) \prod_{i \neq D} \chi_i(t, \mp\eta_i), \tag{28}$$

where the correlation functions

$$\chi_i(t, \lambda) = \langle e^{-i\lambda\phi_i(t)} e^{i\lambda\phi_i(0)} \rangle_{n-eq} \tag{29}$$

for $i = U, D, L, R$ are evaluated over a nonequilibrium state created by tunneling from the source channel, described by the Hamiltonian (25). Since at low energies H_T' perturbs only the lower channel [18], for other channels, $i \neq D$, the averaging is performed over the equilibrium state, which gives $\chi_i(t, \lambda) = K_{\lambda^2}(t, 0)$ [see Eq. (15)]. For the lower channel, averaging has to be evaluated over the nonequilibrium state created by tunneling from the source. We therefore apply the perturbation theory, introduced in Sec. II.

We skip the details of the calculations, outlined in the Sec. II, and present the results for the asymptotic behavior of the transition rates for $\mu > 0$ in Table III. The transition rates for the negative bias follow from the electron-hole symmetry, $\Gamma_\pm(-\varepsilon)|_{\mu \to -\mu} = \Gamma_\mp(\varepsilon)$. These results have to be compared to the well-known FES exponents in equilibrium: $\Gamma_\pm(\varepsilon) \propto 1/|\varepsilon|^{\alpha_D}$, where $\alpha_D = 2\eta_D - \sum_i \eta_i^2$. Note that the equilibrium values of the rates can be immediately obtained from Eq. (28) by substituting equilibrium correlation functions. Our findings are consistent with those of Refs. [6,8], where the transition rates are evaluated for arbitrary tunneling, so that the singular

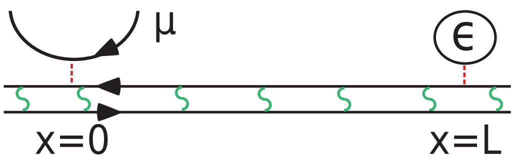

FIG. 4. A spinless LL containing left- and right-moving electrons (presented by arrows) with an attached QPC and QD is schematically shown. Wavy lines indicate interactions between electron channels. At the point $x = 0$ nonequilibrium electrons are injected from a free-fermionic reservoir with the chemical potential μ. At the point $x = L$ the TDOS $n(\epsilon)$ is measured with the help of a QD.

behavior at the thresholds acquires the natural cutoff at energies of the order of μ times the small transparency of the source QPC.

IV. TUNNELING TO NON-FERMI LIQUIDS

The systems considered in the previous sections can be investigated using a nonperturbative method, as discussed in the Appendix, which is based on the free-fermion character of the local tunneling process. The goal of this section is to present examples of systems in which the application of the perturbation theory approach developed in Sec. II B cannot be avoided. Namely, we investigate tunneling transport and a stationary nonequilibrium state in a spinless LL and at the edge of a fractional QH system.

A. Luttinger liquid

The interaction-induced relaxation in spinless LLs was studied in Ref. [6]. However, the analysis in that paper is restricted to the case where a LL is coupled to free-fermion reservoirs away from the interaction region. This allows one to reduce the problem to the calculation of a Fredholm determinant of a single-particle operator. Although the results of that paper can be reproduced with our approach, we go beyond this restriction and consider a LL system, shown in Fig. 4, in which a nonequilibrium state is created inside a LL and interactions cannot be neglected. In the case of a LL, the tunneling contacts introduce the backscattering terms starting from the second order. These terms are relevant in the renormalization group sense and can drastically modify the ground state of the LL. However, in experiment one can always pinch the QPCs off and bring them into the regime of weak tunneling. In this section we assume that this is the case and the effect of the tunneling contacts is perturbative.

As in the Sec. II [see Eqs. (8) and (9)], we consider the nonequilibrium correction to the TDOS, which can be measured using resonant tunneling through a QD:

$$\delta n(\epsilon) = \int dt e^{-i\epsilon t} \langle \psi^\dagger(L,t)\psi(L,0)\rangle_{n-eq} - n_{eq}(\epsilon), \quad (30)$$

where $\psi^\dagger$ and ψ are the creation and annihilation electron operators in the LL. Note that in the case of tunneling to a LL even the TDOS $n_{eq}(\epsilon) = \int dt e^{-i\epsilon t}\langle \psi^\dagger(L,t)\psi(L,0)\rangle_{eq}$ has a nontrivial energy dependence.

TABLE IV. The exponents κ of the asymptotic behavior $\delta n(\epsilon) \propto |\epsilon - \epsilon_0|^\kappa$ in the vicinity of different thresholds ϵ_0 for tunneling to a LL are expressed in terms of the LL interaction parameter K. The results for different processes are listed, i.e., when a left or right mover is injected and a left or right mover is detected. The process R to L gives the same contribution as L to R.

ϵ_0	$-\mu + 0$	0	$\mu - 0$
R to R	$3\frac{K+K^{-1}}{2} + 1$	$\frac{K+K^{-1}}{2} - 2$	$\frac{K+K^{-1}}{2} - 1$
L to L	$3\frac{K+K^{-1}}{2} - 1$	$\frac{K+K^{-1}}{2} - 2$	$\frac{K+K^{-1}}{2} + 1$
L to R	$3\frac{K+K^{-1}}{2}$	$\frac{K+K^{-1}}{2} - 2$	$\frac{K+K^{-1}}{2}$

In a spinless LL, the fermion creation operator $\psi(x,t)$ has two components, $\psi(x,t) = \psi_R(x,t)e^{ik_F x} + \psi_L(x,t)e^{-ik_F x}$, that correspond to the right- and left-moving fermions, where k_F denotes the Fermi wave vector. Right and left movers can be expressed in terms of eigenmodes ϕ_R and ϕ_L of the LL Hamiltonian, which describe right- and left-moving bosons, respectively [14],

$$\psi_R \propto e^{i(\phi_R \cosh\theta + \phi_L \sinh\theta)},$$
$$\psi_L \propto e^{i(\phi_R \sinh\theta + \phi_L \cosh\theta)},$$

where the mixing angle $\theta = \frac{1}{2}\ln K$ is determined by the LL interaction parameter K.

A nonequilibrium state in the LL is created by a voltage-biased QPC and is described by the tunneling Hamiltonian H_T, acting at the point $x = 0$:

$$H_T = \tau\psi_\mu^\dagger(0)\psi(0) + \text{H.c.}, \quad (31)$$

where ψ_μ is an electron operator in the biased channel. In order to focus our analysis on the nonequilibrium LL effects, we consider these electrons to be effectively free, with the local correlation function (7).

Four different terms contribute to the correction (30). One can inject either a right- or a left-moving electron and collect either a right or a left mover [19]. The asymptotic behavior of these contributions is summarized in Table IV for a positive bias $\mu > 0$ [20]. For negative biases, $\mu < 0$, one can use the electron-hole symmetry discussed in Sec. II C [see Eq. (20)].

B. Fractional quantum Hall edge states

Another interesting problem that cannot be solved by evaluating the Fredholm determinant is the problem of the relaxation of a nonequilibrium stationary state at the edge of a fractional QH system. It is well known that at filling factors of the form $\nu = (2n+1)^{-1}$, $n \in \mathbb{N}$, there exists a single channel of the freely propagating bosonic field ϕ at the edge [21]. This field is related to the electron operator by the identity

$$\psi \propto e^{i\sqrt{2n+1}\phi}. \quad (32)$$

We consider the system shown in Fig. 5, where electrons tunnel between two fractional QH edges at the point $x = 0$, and the relaxed stationary state is studied downstream at the point $x = L$. In this case, the tunneling Hamiltonian is given by

$$H_T = \tau\psi_\mu^\dagger\psi + \text{H.c.}, \quad (33)$$

FIG. 5. A QH edge state at the filling factor $\nu = (2n+1)^{-1}$, $n \in \mathbb{N}$, is schematically shown. Electrons are injected at the point $x = 0$ via the voltage-biased QPC and detected at the point $x = L$ at energies ϵ with the help of a QD. A measurement setup analogous to the one for integer QH (see Fig. 1) can be utilized.

where ψ_μ is the fermion operator in the biased edge channel and ψ describes electrons in the edge channel, where we study the correction $\delta n(\epsilon)$ to the stationary TDOS given by Eq. (30). Note that for the fractional QH system considered here the correlator of electron fields in the biased channel takes the following form: $\langle \psi_\mu^\dagger(t)\psi_\mu(0)\rangle_{eq} \propto e^{i\mu t}/(it+0)^{2n+1}$, which generalizes Eq. (7).

The TDOS at the point $x = L$ is measured by a resonant tunneling through a QD level ϵ. The results for $\mu > 0$ are summarized in Table V. The result for injecting holes, i.e., for $\mu < 0$, can be obtained by using the symmetry $\delta n(-\epsilon)|_{\mu\to-\mu} = -\delta n(\epsilon)$. Note that electronlike excitations $\mu > 0$ do not affect the TDOS at negative energies ($\epsilon < 0$). Technically, this follows from the fact that the expression under the integral in Eq. (17) becomes analytical; that is, instead of integrals along the branch cuts, one needs to compute residues of the poles. Another curious result is that at the threshold $\epsilon_0 = 0$ the exponent κ vanishes. This implies that at these energies the correction may dominate over the background equilibrium TDOS, which vanishes at this point as $|\epsilon|^{2n}$ ($\epsilon < 0$).

Finally, we would like to mention that various combinations of the electron and quasiparticle tunneling at the source and detector, as well as various other filling fractions, can be experimentally relevant. They will be investigated elsewhere.

V. CONCLUSION

Exactly solvable strongly interacting systems provide an important platform for studying the interplay between strong interaction and nonequilibrium physics. This is because it is possible to extend analytical results even beyond an equilibrium regime. With analytical predictions, one can test experimentally the current theoretical understanding both of interaction effects and of the nonequilibrium physics. In this paper we developed a theoretical method that allows one to analyze strongly interacting systems out of equilibrium by studying an asymptotic universal behavior of physical quantities in the vicinity of the thresholds in the spectrum of excitations. Our approach is based on perturbation theory with respect to a small parameter, the number of nonequilibrium excitations, which is controlled by weak tunneling. We extended the results of previous works that used the Fredholm determinant technique to a class of systems without well-defined electron excitations. Namely, in this paper along with conventional systems we studied the relaxation of nonequilibrium electrons in spinless LLs and at the edge of chiral fractional QH systems, where previously introduced methods cannot be applied. The universal exponents that we found depend only on a small set of parameters that encode all effects of interaction. Moreover, a linear combination of these exponents reduces the result to a number. For instance, a sum of κ at $-\mu$ and μ in the case of integer QH (Table I) is 4 for any interaction strength. This universality can facilitate an experimental test of our predictions.

TABLE V. The exponents κ of the asymptotic behavior $\delta n(\epsilon) \propto |\epsilon - \epsilon_0|^\kappa$ in the vicinity of different thresholds ϵ_0 for electron tunneling to a fractional QH edge at the filling factor $\nu = (2n+1)^{-1}$, $n \in \mathbb{N}$ are listed. Note that the correction vanishes at negative energies $\epsilon < 0$.

ϵ_0	<0	$+0$	$\mu - 0$
κ	absent	0	$2n$

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APPENDIX: PERTURBATIVE DERIVATION OF THE ELECTRON CORRELATION FUNCTION FROM THE FREDHOLM DETERMINANT

In this Appendix we derive Eq. (16) from the Fredholm determinant representation of the electron correlation functions in a case when the local tunneling process is effectively free fermionic. We use the fact that expression (9) can be reduced to a determinant of a single-particle operator by means of the nonequilibrium bosonization technique [5,15]. One of the key steps in this approach is to relate the bosonic fields at point $x = L$ to the transferred charge into the edge channel through the QPC at the point $x = 0$.

Since the eigenmodes propagate with constant speeds, one can write [5]

$$\phi_U(L,t) = \alpha^2\phi_U(0, t - L/u_c) + \beta^2\phi_U(0, t - L/u_d) + \alpha\beta\phi_D(0, t - L/u_d) - \alpha\beta\phi_D(t - L/u_c), \quad \text{(A1)}$$

which allows one to present the electron correlator in terms of nonequilibrium correlators of bosonic field (29),

$$\langle \psi_U^\dagger(L,t)\psi_U(L,0)\rangle_{n-eq} = \chi_U(t,\alpha^2)\chi_U(t,\beta^2)\chi_D(t,\alpha\beta)\chi_D(t,-\alpha\beta), \quad \text{(A2)}$$

where we used relations (3) and (6) and the simplification arising from the fact that dipole and charged wave packets are well separated in space in the $L \to \infty$ limit. Given the relation of the bosonic fields to the charge in the channel (3), one can express $\chi_i(t,\lambda)$ in terms of the FCS of the charge $Q_i(t)$, $i = U, D$, transferred through the junction over time t:

$$\chi_i(t,\lambda) = \langle e^{-2\pi i\lambda Q_i(t)} e^{2\pi i\lambda Q_i(0)}\rangle_{n-eq}. \quad \text{(A3)}$$

In the case where a local electron tunneling is effectively free fermionic, the evaluation of the correlator (A3) amounts to solving the scattering problem at the source QPC and expressing the FCS generator in terms of the Fredholm determinant [9] (we use $\ln\det = \mathrm{Tr}\ln$):

$$\ln \chi_U(t,\lambda) = \mathrm{Tr}\ln\left(1 - F + U_\lambda F\right), \tag{A4}$$

where F is the diagonal in the energy basis matrix, with the elements being the Fermi distribution functions in the incoming scattering channels. The matrix U_λ is obtained from the scattering matrix (see Ref. [9] for details) and is given by

$$U_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1_{0,t}\left(e^{i\lambda} - 1\right)\begin{bmatrix} T & rt^* \\ r^*t & (1-T) \end{bmatrix}, \tag{A5}$$

where r and t are reflection and transmission amplitudes, respectively, and $T = |t|^2$ is the tunneling probability. The role of this matrix is to "count" electrons, which end up in the channel of interest after scattering.

We are interested in the limit of weak tunneling; therefore, we can expand Eq. (A4) in small T. The zeroth-order term $\ln \chi_U^{(0)}(t,\lambda)$ represents the equilibrium charge fluctuations. The linear in T contribution is given by

$$\ln \chi_U^{(1)}(t,\lambda) \propto T \int_0^t\int_0^t dt_1 dt_2 \left(\frac{t_1}{t-t_1}\right)^\lambda \left(\frac{t-t_2}{t_2}\right)^\lambda \times e^{i\mu(t_1-t_2)}\left\{\frac{1}{(t_1-t_2-i0)^2} - \frac{e^{-2\pi i\lambda}}{(t_2-t_1-i0)^2}\right\}. \tag{A6}$$

There are two contributions in Eq. (A2) that contain this term: one from the charged mode, $\chi_U(t,\alpha^2)$, and another one from the dipole mode, $\chi_U(t,\beta^2)$. These are exactly the two contributions that we obtained in Sec. II B. The contribution of the charge mode to the correction (8) reads

$$\delta n_c(\epsilon) \propto \int dt \frac{e^{-i\epsilon t}}{it+0} \ln \chi_U^{(1)}(t,\alpha^2). \tag{A7}$$

This is nothing but Eq. (16), written in a different form. The contribution of the dipole mode $\delta n_d(\epsilon)$ is obtained by replacing $\alpha \to \beta$.

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