Editors' Suggestion

Soft modes and nonanalyticities in a clean Dirac metal

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We consider the effects of the electron-electron interaction in a clean Dirac metal, i.e., a Dirac system with the chemical potential not tuned to the Dirac point. We introduce the notion of a Dirac-Fermi liquid (DFL) and discuss the soft two-particle excitations in such systems, which are qualitatively different from the corresponding excitations in a conventional, or Landau-Fermi liquid (LFL). These soft excitations lead to nonanalytic dependencies of observables, including the density of states, the spin susceptibility, and the specific heat, on the temperature, magnetic field, and wave number, which first appear at second order in the interaction. We determine and discuss the nature of these nonanalyticities in a DFL and compare them with the corresponding effects in a LFL. Our results are based on very general arguments regarding the existence and nature of the soft modes, and corroborated by explicit perturbative calculations. We also discuss their consequences for magnetic quantum phase transitions in Dirac metals.

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I. INTRODUCTION

The electronic spectrum of solids in the vicinity of special points in the Brillouin zone where two bands touch has been the subject of investigations since the early days of solid-state physics [1,2]. If one of the touching bands is a conduction band and the other a valence band, and if the chemical potential is tuned to the touching point (sometimes called a diabolical point), one has the situation of a semimetal, a material that in a certain sense is in between a metal and an insulator [3]. Even if the chemical potential lies within the conduction band, the physics that underlies the existence of the touching point can lead to unusual properties. Much interest in recent years has focused on topological properties of such materials, which manifest themselves on the surface, and their relation to topological insulators [4–6]. We will focus on a different aspect, namely, bulk properties such as the density of states and the magnetic susceptibility for which topological considerations are not important, but which nonetheless are strongly influenced by the spin-orbit coupling that also leads to the existence of a touching point.

A special case that is of conceptual interest is a system where the spin-orbit interaction leads to a linear crossing of doubly degenerate bands, such that the effective Hamiltonian in the vicinity of the crossing point is reminiscent of a massless Dirac Hamiltonian [7], although the effects we will discuss persist if the crossing point is gapped out (i.e., if the Dirac system is massive rather than massless). We will consider a situation where the chemical potential lies within the conduction band, so that the material is a true metal with an extended Fermi surface (as opposed to a semimetal with a pointlike Fermi surface) and well-defined quasiparticles. In this sense, we will consider a Fermi liquid. However, the spin-orbit interaction that leads to the band crossing, whether or not the latter is gapped out, modifies the properties of the Fermi liquid in ways that make it qualitatively different from an ordinary Landau-Fermi liquid (LFL) that is realized in the absence of a spin-orbit interaction. This leads us to the notion of a Dirac-Fermi liquid (DFL), which differs from a LFL in the single-particle spectrum, but more importantly in two-particle correlations that are important for the behavior of various thermodynamic properties, as well as the density of states and transport coefficients. As we will see, the spectrum of soft two-particle correlations is qualitatively different in a DFL than in a LFL. Interestingly, this is true even if the spin-orbit interaction is weak and qualitatively affects the single-particle spectrum only asymptotically close to the Dirac point (in gapless systems), or not at all (in gapped ones). By "Dirac metal" we mean a metal whose conduction electrons form a DFL.

It has long been realized that generic soft modes in condensed-matter systems lead to generic scale invariance and nonanalyticities in the thermodynamic and transport properties as functions of the wave number, frequency, temperature, or external fields [8]. Prominent examples include long-time-tail effects in classical fluids [9–11], the long-range nature of the longitudinal susceptibility in the ordered phase of a classical Heisenberg ferromagnet [12,13], and weak-localization effects in disordered electron systems [14,15]. More recently, these effects have been shown to have dramatic consequences for quantum phase transitions. A prime example is the quantum ferromagnetic transition in Landau metals, which is discontinuous as a result of soft modes in a LFL at zero temperature, rather then continuous as the corresponding transition in classical systems [16].

We will focus on effects in a Dirac metal that are universal in the sense that they are determined by the properties of the electron system in the limit of long wavelengths and small frequencies (the "hydrodynamic regime") and hence independent of detailed microscopic properties of the material

other than the underlying Dirac crossing point. Of particular importance are excitations that are soft or massless in this regime, as they can lead to nonanalytic behavior of observables as a function of wave number, frequency, or an external field. We will discuss the existence and nature of such soft modes in a DFL in some detail and compare and contrast them with the analogous soft modes in a LFL. We will then explore the consequences for the density of states, the spin susceptibility, and the specific heat in a Dirac metal.

II. MODEL

A. Single-particle Hamiltonian and symmetries

A linear crossing of two bands can be described by a term $\pm v\boldsymbol{k}\cdot\boldsymbol{\sigma}$ in the single-particle Hamiltonian in momentum space [2], where $\boldsymbol{\sigma}\equiv(\sigma_1,\sigma_2,\sigma_3)$ are the spin Pauli matrices and v is a characteristic velocity whose value reflects the strength of the spin-orbit interaction. The sign of this term reflects the chirality of the electrons: plus and minus describe left-handed and right-handed electrons, respectively. This term is invariant under time reversal (TR), which changes the sign of both the momentum $\boldsymbol{k}$ and the angular momentum $\boldsymbol{\sigma}$. Spatial inversion (SI) changes the sign of $\boldsymbol{k}$ as well as the chirality, but not the sign of $\boldsymbol{\sigma}$, so in a system with spatial inversion symmetry, the relevant part of the Hamiltonian must include both chiralities [17]. If we represent the chirality degree of freedom by a second set of Pauli matrices $\boldsymbol{\pi}=(\pi_1,\pi_2,\pi_3)$, the part of the Hamiltonian that represents the spin-orbit interaction can be written as

$$H_v = v(\pi_3\otimes\boldsymbol{\sigma})\cdot\boldsymbol{k}. \tag{2.1a}$$

We note that a related, but different, class of models contain a term with formally the same structure, but with the Pauli matrices representing a pseudospin degree of freedom [18], as is the case in graphene [19]. The properties of the spin susceptibility are drastically different for these two classes of models, a point we will come back to later.

The most general single-particle Hamiltonian that is invariant under both TR and SI also includes a term that mixes the chiralities in a symmetric way,

$$H_\Delta = \Delta(\pi_1\otimes\sigma_0), \tag{2.1b}$$

with σ_0 the 2×2 unit matrix in spin space, and the usual band Hamiltonian

$$H_\epsilon = (\epsilon_{\boldsymbol{k}}-\mu)(\pi_0\otimes\sigma_0). \tag{2.1c}$$

Here, π_0 is another copy of the 2×2 unit matrix and $\epsilon_{\boldsymbol{k}}$ is the single-particle energy, which for small $\boldsymbol{k}$ is bilinear in the components of $\boldsymbol{k}$. For simplicity, we will assume $\epsilon_{\boldsymbol{k}}$ to be isotropic, $\epsilon_{\boldsymbol{k}}=\boldsymbol{k}^2/2m$, with m an effective mass. μ is the chemical potential. $H_\epsilon+H_v+H_\Delta$ is the most general single-particle Hamiltonian for a system that is invariant under both TR and SI and whose spin-orbit interaction is described by H_v; it is equivalent to the effective Hamiltonian derived in Refs. [7,20] for the Bi_2Se_3 family of topological insulators. In our simple isotropic model, we also have invariance under simultaneous rotations in spin space and real space, which leaves $\boldsymbol{\sigma}\cdot\boldsymbol{k}$ invariant. Coupling the fermions to an external magnetic field $\boldsymbol{h}=(0,0,h)$ via a Zeeman term (which breaks TR, of course), we arrive at a single-particle Hamiltonian

$$H_0 = (\epsilon_{\boldsymbol{k}}-\mu)(\pi_0\otimes\sigma_0)+v(\pi_3\otimes\boldsymbol{\sigma})\cdot\boldsymbol{k}+\Delta(\pi_1\otimes\sigma_0) - h(\pi_0\otimes\sigma_3). \tag{2.2}$$

B. Single-particle spectrum, Green function, and action for a Dirac gas

1. Single-particle spectrum

The single-particle spectrum is readily obtained by finding the eigenvalues of the 4×4 Hamiltonian H. The single-particle energy $E_{\boldsymbol{k}}$ has four branches:

$$E_{\boldsymbol{k}}^{1\pm} = \epsilon_{\boldsymbol{k}}\pm\sqrt{v^2\boldsymbol{k}^2+\Delta^2+h^2-2h\sqrt{v^2k_z^2+\Delta^2}}, \tag{2.3a}$$

$$E_{\boldsymbol{k}}^{2\pm} = \epsilon_{\boldsymbol{k}}\pm\sqrt{v^2\boldsymbol{k}^2+\Delta^2+h^2+2h\sqrt{v^2k_z^2+\Delta^2}}. \tag{2.3b}$$

In order to discuss the various possibilities for different ranges of parameter values, let us introduce an atomic-scale momentum p_0 (on the order of an inverse lattice spacing), velocity $v_0=p_0/2m$, and energy $E_0=p_0^2/2m$. We then measure $E_{\boldsymbol{k}}$, Δ, and h in units of E_0, v in units of v_0, and $\boldsymbol{k}$ in units of k_0.

Let us start with the case of a small spin-orbit coupling. For $v=0$, the spectrum consists of four degenerate parabolic bands, two for each spin projection and two for each chirality. A small spin-orbit coupling v splits the bands, for $\Delta=h=0$, the Fermi surface for a nonzero chemical potential consists of two twofold degenerate sheets, and the split bands display a linear crossing at $k=0$, see Fig. 1(a). $\Delta\neq 0$ gaps out that band crossing, Fig. 1(b), whereas $h\neq 0$ lifts the twofold degeneracy and splits the crossing band pairs vertically, Fig. 1(c). If both Δ and h are nonzero, the band crossings are shifted to nonzero values of k if $h>\Delta$, and they disappear if $\Delta>h$, see Figs. 1(d) and 1(e). At the Fermi surface (for a sizable chemical potential), all of these effects appear as a small splitting of the fourfold degenerate parabolic band in the absence of a spin-orbit coupling. However, as we will discuss, the properties of the resulting Fermi liquid are qualitatively different from the parabolic-band case and the systems can be characterized, in a well-defined sense, as a Dirac [Figs. 1(a) and 1(b)] or Weyl [Figs. 1(c), 1(d), and 1(e)] metal.

For large values of v, when the spin-orbit term H_v dominates over $\epsilon_{\boldsymbol{k}}$ in H_0, the spectrum can be described as follows. For $\Delta=0$ and $h=0$, it consists of two degenerate Dirac cones. The lower cone eventually bends over, but for large v, this happens only for wave vectors large compared to the inverse lattice spacing, so this branch will not contribute to the Fermi surface, see Fig. 2(a). For a chemical potential tuned to the crossing point, $\mu=0$, the system is a Dirac semimetal, and for $\mu>0$, it is natural to call it a Dirac metal. $\Delta>0$ gaps out the crossing point, and the system is a Dirac insulator for $\mu<\Delta$, and a Dirac metal for $\mu>\Delta$, Fig. 2(b). $h>0$ separates the Dirac cones in wave-number space, leading to a Weyl semimtal if $\mu=0$ and a Weyl metal if $\mu>0$, Fig. 2(c). This structure persists for $0<\Delta<h$, Fig. 2(d), whereas for $0<h<\Delta$, the cones are gapped out and one has a Weyl insulator for $\mu=0$ and a Weyl metal for $\mu\gg\Delta$, Fig. 2(e).

Even though the single-particle spectra in Figs. 1 and 2 are related in obvious ways, they appear very different at

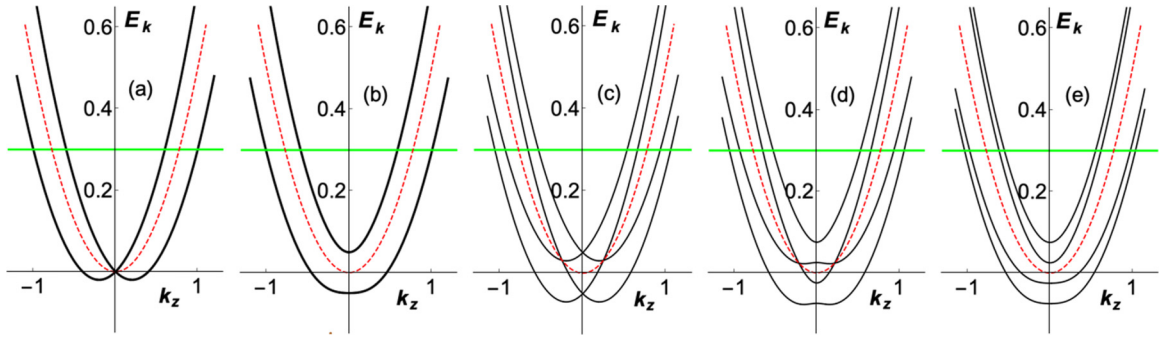

FIG. 1. Single-particle spectra (solid black lines) for $v = 0.2$ and $\Delta = h = 0$ (a), $\Delta = 0.05$, $h = 0$ (b), $\Delta = 0$, $h = 0.05$ (c), $\Delta = 0.025$, $h = 0.05$ (d), and $\Delta = 0.05$, $h = 0.025$ (e), respectively. The bands in (a) and (b) are twofold degenerate and the horizontal green lines denote the chemical potential. The parabolic spectrum for $v = \Delta = h = 0$ is shown for comparison (dashed red line). See the text for further discussion.

wave numbers that are not asymptotically small; this is true in particular at the Fermi surface for generic values of the chemical potential. However, as we will show in Sec. III, the soft-mode spectrum is the same for both small and large values of the spin-orbit coupling v [21], and the systems depicted in Fig. 1 therefore constitute Dirac and Weyl metals in the same sense as those in Fig. 2 for a sufficiently large value of μ. For our purposes, the distinction between the Dirac and Weyl cases is not important, and we will refer to all of the metallic systems depicted in Figs. 1 and 2 as "Diral metals." We also note that in systems that are not invariant under SI, or if TR is broken by effects other than a magnetic field, there are additional possibilities, in addition to the Dirac and Weyl cases, that we will not discuss, see, e.g., Ref. [22].

For $\mu \gg \Delta$ and $h = 0$, the various Fermi surfaces are characterized by Fermi wave numbers that are solutions of the quadratic equations

$$k_{\rm F}^2 = 2m^2(\mu/m + v^2 \pm \sqrt{(\mu/m + v^2)^2 - \mu^2/m^2}). \quad (2.4)$$

Our explicit calculations in Sec. IV will be performed in the limit of a small spin-orbit interaction, $v^2 \ll \mu/m$. v then splits the spherical Fermi surface into two twofold degenerate sheets, see Figs. 1(a) and 1(b), and we have

$$k_{\rm F}^{\pm} = \sqrt{2m\mu} \pm v/2m + O(v^2) \quad (v^2 \ll \mu/m). \quad (2.5a)$$

A small magnetic field lifts the twofold degeneracy, see Figs. 1(c)–1(e). In the limit of a strong spin-orbit interaction, $v^2 \gg \mu/m$, and for $h = 0$, there is only one physical Fermi surface (see Fig. 2) with

$$k_{\rm F} = \mu/v \quad (v^2 \gg \mu/m). \quad (2.5b)$$

The second solution of the quadratic equation, $k_{\rm F} = 2mv$, does not correspond to a physical Fermi surface if it is larger than a reciprocal lattice vector. This case (the Dirac limit for $h = 0$, or Weyl limit for $h > 0$) is commonly considered for problems that are dominated by the cone structure of the spectrum.

2. *Single-particle Green function*

The single-particle Green function is defined as

$$G_n(\boldsymbol{k}) = [i\omega_n(\pi_0 \otimes \sigma_0) - H_0]^{-1}, \quad (2.6)$$

with ω_n a fermionic Matsubara frequency. It is a 4×4 matrix in chirality-spin space with matrix elements $(G_n(\boldsymbol{k}))^{\alpha\alpha'}_{\sigma\sigma'}$. The complete explicit expression is quite complicated, but we will not need it for our purposes. Rather, we list various special cases and approximations that suffice for our objectives.

For $\Delta = 0$, the Green function is diagonal in the chirality index and one finds

$$G_n^{\Delta=0}(\boldsymbol{k}) = \pi_+ \otimes \mathfrak{g}_n(\boldsymbol{k}, \boldsymbol{h}) + \pi_- \otimes \mathfrak{g}_n(-\boldsymbol{k}, \boldsymbol{h}), \quad (2.7a)$$

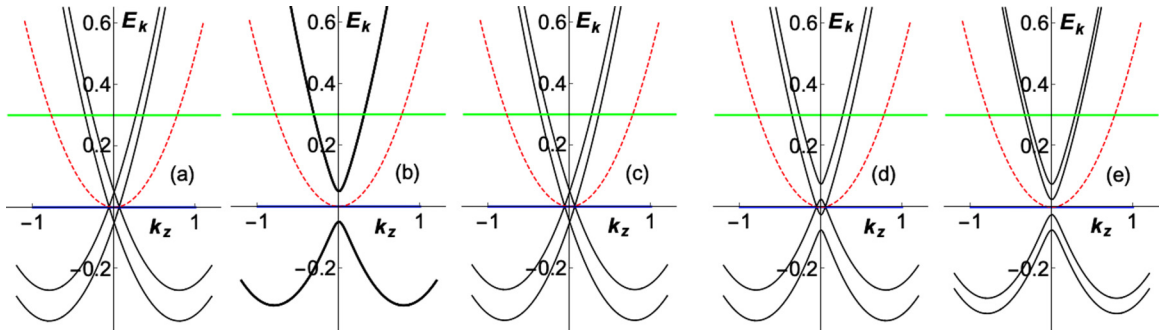

FIG. 2. Same as Fig. 1, but for $v = 0.8$. The horizontal blue line denotes a zero chemical potential, which renders the system a Dirac semimetal (a), Dirac insulator (b), Weyl semimetal [(c) and (d)], and Weyl insulator (e), respectively. For a large chemical potential (horizontal green line) one has a Dirac [(a) and (b)] or Weyl [(c)–(e)] metal. See the text for a discussion of the sense in which the systems shown on Fig. 1 are also properly classified as Dirac and Weyl metals.

where $\pi_\pm = (\pi_0 \pm \pi_3)/2$ and

$$\mathfrak{g}_n(\boldsymbol{k},\boldsymbol{h}) = \frac{(i\omega_n - \xi_{\boldsymbol{k}})\sigma_0 + (v\boldsymbol{k}-\boldsymbol{h})\cdot\boldsymbol{\sigma}}{(i\omega_n-\xi_{\boldsymbol{k}})^2 - (v\boldsymbol{k}-\boldsymbol{h})^2} \tag{2.7b}$$

or, equivalently,

$$\left(G_n^{\Delta=0}(\boldsymbol{k})\right)^{\alpha\alpha'} = \delta_{\alpha\alpha'}\,\frac{(i\omega_n - \xi_{\boldsymbol{k}})\sigma_0 + (\alpha v\boldsymbol{k}-\boldsymbol{h})\cdot\boldsymbol{\sigma}}{(i\omega_n-\xi_{\boldsymbol{k}})^2 - (\alpha v\boldsymbol{k}-\boldsymbol{h})^2}. \tag{2.7c}$$

Here we have defined $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$. A similar simplification occurs in zero field, where one finds

$$G_n^{h=0}(\boldsymbol{k}) = \frac{1}{(i\omega_n-\xi_{\boldsymbol{k}})^2 - v^2\boldsymbol{k}^2 - \Delta^2}\,[(\pi_0\otimes\sigma_0)(i\omega_n-\xi_{\boldsymbol{k}}) + (\pi_3\otimes\boldsymbol{\sigma})\cdot v\,\boldsymbol{k} + (\pi_1\otimes\sigma_0)\Delta]. \tag{2.8}$$

A nonzero field appears in both the numerator and the denominator of the Green function, as is illustrated by the special case $\Delta = 0$ in Eqs. (2.7). For reasons that will become clear in Sec. IV [see the remark before Eq. (4.7)], we do not need to retain the occurrences of h in the numerator for our purposes. In an approximation that suffices for determining the leading nonanalytic behavior of the density of states and the spin susceptibility, we thus keep h only in the denominator and find

$$G_n(\boldsymbol{k}) \approx \frac{[(i\omega_n-\xi_{\boldsymbol{k}})^2 - (v^2\boldsymbol{k}^2+\Delta^2)][(\pi_0\otimes\sigma_0)(i\omega_n-\xi_{\boldsymbol{k}}) + (\pi_3\otimes\boldsymbol{\sigma})\cdot v\,\boldsymbol{k} + (\pi_1\otimes\sigma_0)\Delta]}{\left[(i\omega_n-\xi_{\boldsymbol{k}})^2 - \left(v^2\boldsymbol{k}^2+\Delta^2+h^2-2h\sqrt{v^2k_z^2+\Delta^2}\right)\right]\left[(i\omega_n-\xi_{\boldsymbol{k}})^2 - \left(v^2\boldsymbol{k}^2+\Delta^2+h^2+2h\sqrt{v^2k_z^2+\Delta^2}\right)\right]}. \tag{2.9}$$

This reduces to Eqs. (2.7) in the same approximation if the gap Δ is zero, or if it is is negligible compared to the additive terms in Eq. (2.9). Anticipating that the wave vector $\boldsymbol{k}$ will be close to the Fermi surface, we see that the relevant criterion is $\Delta \ll vk_{\rm F}$, with $k_{\rm F}$ the Fermi wave number for the Fermi surface under consideration. This is the limit we will investigate in Sec. IV. As we will discuss in Sec. V A the presence of a small Δ does not qualitatively change the results.

For $\Delta = 0$, the Green function is diagonal in the chirality index, see Eq. (2.7c). In the same approximation as in Eq. (2.9), i.e., ignoring the $\boldsymbol{h}$ in the numerator, it can be written as

$$\left(G_n^{\Delta=0}\right)^{\alpha\alpha'}(\boldsymbol{k}) \approx \delta_{\alpha\alpha'}\,G_n^{\alpha}(\boldsymbol{k}), \tag{2.10a}$$

where

$$G_n^{\alpha}(\boldsymbol{k}) = \tfrac{1}{2}\left(F_n^{\alpha-}(\boldsymbol{k})M(-\alpha\hat{\boldsymbol{k}}) + F_n^{\alpha+}(\boldsymbol{k})M(\alpha\hat{\boldsymbol{k}})\right) \tag{2.10b}$$

with

$$F_n^{\alpha\pm}(\boldsymbol{k}) = \frac{1}{i\omega_n - \xi_{\boldsymbol{k}} \pm |\alpha v\boldsymbol{k}-\boldsymbol{h}|} \tag{2.10c}$$

and

$$M(\hat{\boldsymbol{k}}) = \sigma_0 - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}. \tag{2.10d}$$

Here, $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ is the unit vector in $\boldsymbol{k}$ direction. In writing the Green function in the form of Eq. (2.10b), we have performed a partial fraction decomposition of the quadratic denominator. The partial Green function $F_n^{\alpha\pm}$ therefore has two indices: α is the chirality index and the additional $\pm$ represents the two branches of the quadratic denominator. Since spin is not a good quantum number in the presence of a spin-orbit interaction, these two branches cannot be interpreted as representing two different spin projections.

Equations (2.10) no longer allow for the Landau limit of zero spin-orbit coupling to be taken, because of the approximation involved. However, performing a partial-fraction decomposition on Eqs. (2.7), which are still exact, shows that the exact Landau Green function can be written in a form analogous to Eqs. (2.10), viz.,

$$\left(G_n^{v=\Delta=0}\right)^{\alpha\alpha'}(\boldsymbol{k}) = \delta_{\alpha\alpha'}\,G_n(\boldsymbol{k}), \tag{2.11a}$$

where

$$G_n(\boldsymbol{k}) = \tfrac{1}{2}(G_{n\downarrow}(\boldsymbol{k})M(\hat{\boldsymbol{h}}) + G_{n\uparrow}(\boldsymbol{k})M(-\hat{\boldsymbol{h}})). \tag{2.11b}$$

Here,

$$G_{n\uparrow\downarrow}(\boldsymbol{k}) = F_n^{\alpha\pm}(\boldsymbol{k})\big|_{v=0} = \frac{1}{i\omega_n - \xi_{\boldsymbol{k}} \pm |\boldsymbol{h}|} \tag{2.11c}$$

are the Green functions for up-spin ($\uparrow \equiv +$) and down-spin ($\downarrow \equiv -$) electrons, respectively.

For later reference, we list five properties of the matrix M, which can easily be checked by direct calculation:

$$M(\hat{\boldsymbol{k}})M(\hat{\boldsymbol{k}}) = 2M(\hat{\boldsymbol{k}}), \tag{2.12a}$$

$$M(\hat{\boldsymbol{k}})\boldsymbol{\sigma}M(\hat{\boldsymbol{k}}) = -2\hat{\boldsymbol{k}}M(\hat{\boldsymbol{k}}), \tag{2.12b}$$

$$M(\hat{\boldsymbol{k}})M(-\hat{\boldsymbol{k}}) = 0, \tag{2.12c}$$

$$M(\hat{\boldsymbol{k}})\boldsymbol{\sigma}M(-\hat{\boldsymbol{k}}) = 2[\boldsymbol{\sigma} + i\hat{\boldsymbol{k}}\times\boldsymbol{\sigma} - \hat{\boldsymbol{k}}(\boldsymbol{k}\cdot\sigma)], \tag{2.12d}$$

$$\mathrm{tr}\,(M(\hat{\boldsymbol{k}})M(\hat{\boldsymbol{p}})) = 2(1+\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}). \tag{2.12e}$$

3. *Action for a Dirac-Fermi gas*

The action for a noninteracting Fermi system governed by the Hamiltonian H_0, or a Dirac-Fermi gas, is given in terms of the inverse Green function,

$$S_0 = \sum_k \sum_{\sigma\sigma'} \sum_{\alpha\alpha'} \bar{\psi}_\sigma^\alpha(k)\left[i\omega_n\delta_{\sigma\sigma'}\delta_{\alpha\alpha'} - (H_0)_{\sigma\sigma'}^{\alpha\alpha'}\right]\psi_{\sigma'}^{\alpha'}(k). \tag{2.13}$$

Here, $\bar{\psi}$ and ψ are fermionic fields with spin index $\sigma = (+,-) \equiv (\uparrow,\downarrow)$ and chirality index $\alpha = (+,-)$, and $k = (\boldsymbol{k},\omega_n)$ is a 4-vector that comprises the wave vector $\boldsymbol{k}$ and the Matsubara frequency ω_n. Introducing spinors $\psi^\alpha = (\psi_\uparrow^\alpha, \psi_\downarrow^\alpha)$ and a scalar product $(\bar{\psi},\psi) = \sum_\sigma \bar{\psi}_\sigma\psi_\sigma$, we can write S_0

more explicitly in the form

$$S_0 = \sum_k \sum_\alpha (\bar\psi^\alpha(k), ((i\omega_n - \xi_{\boldsymbol{k}})\sigma_0 + h\sigma_3 - \alpha\, v\, \boldsymbol{\sigma}\cdot\boldsymbol{k})\psi^\alpha(k)) + \sum_k \sum_{\alpha\neq\alpha'} (\bar\psi^\alpha(k), \Delta\,\sigma_0\,\psi^{\alpha'}(k)). \tag{2.14}$$

To summarize the symmetry properties, this action is invariant under SI, TR (for $h = 0$), and simultaneous rotations in spin space and real space.

C. Electron-electron interaction, and action for a Dirac-Fermi liquid

In order to study correlation effects, we need to add a four-fermion interaction $S_{\rm int}$ to the noninteracting electron system described by S_0. Underlying all correlations is the Coulomb interaction, but the effective interaction in a fully renormalized theory will contain all contributions that are allowed by symmetry. In our case, the noninteracting action is invariant under simultaneous rotations in spin space and momentum space [23], as well as under SI; this also implies invariance under TR. A four-fermion interaction that is consistent with all of these requirements can be written as

$$\begin{aligned} S_{\rm int} = \frac{-T}{2V} {\sum_q}' \sum_{k,p} \Bigg[& \Gamma_{s,1} \sum_\alpha (\bar\psi^\alpha(k)\sigma_0\psi^\alpha(k-q))(\bar\psi^\alpha(p-q)\sigma_0\psi^\alpha(p)) + \Gamma_{s,2} \sum_{\alpha\neq\alpha'} (\bar\psi^\alpha(k)\sigma_0\psi^\alpha(k-q))(\bar\psi^{\alpha'}(p-q)\sigma_0\psi^{\alpha'}(p)) \\ & + \Gamma_{s,3} \sum_{\alpha\neq\alpha'} (\bar\psi^\alpha(k)\sigma_0\psi^{\alpha'}(k-q))(\bar\psi^{\alpha'}(p-q)\sigma_0\psi^\alpha(p)) - \Gamma_{t,1} \sum_\alpha (\bar\psi^\alpha(k)\boldsymbol{\sigma}\psi^\alpha(k-q))\,(\bar\psi^\alpha(p-q)\boldsymbol{\sigma}\psi^\alpha(p)) \\ & - \Gamma_{t,2} \sum_{\alpha\neq\alpha'} (\bar\psi^\alpha(k)\boldsymbol{\sigma}\psi^\alpha(k-q))\,(\bar\psi^{\alpha'}(p-q)\boldsymbol{\sigma}\psi^{\alpha'}(p)) - \Gamma_{t,3} \sum_{\alpha\neq\alpha'} (\bar\psi^\alpha(k)\boldsymbol{\sigma}\psi^{\alpha'}(k-q))\,(\bar\psi^{\alpha'}(p-q)\boldsymbol{\sigma}\psi^\alpha(p)) \Bigg]. \end{aligned} \tag{2.15}$$

Here the prime on the sum over q indicates a restriction $|\boldsymbol{q}| < \Lambda$, with $\Lambda \ll k_{\rm F}$ a momentum cutoff. This is necessary in order to avoid overcounting, as explained in Appendix A. The leading nonanalyticities we are interested in will not depend on the cutoff.

The six terms in Eq. (2.15) are separately invariant under the required symmetries, and come with six interaction amplitudes. They are graphically represented in Fig. 3. As we will see in Sec. III, interactions that mix chiralities ($\Gamma_{s,2}$, $\Gamma_{s,3}$, $\Gamma_{t,2}$, $\Gamma_{t,3}$) play a qualitatively different role that those that do not ($\Gamma_{s,1}$, $\Gamma_{t,1}$). A sketch of how this effective interaction is generated from a Coulomb interaction between density fluctuations is provided in Appendix A.

The full action for our model of interacting Dirac fermions, i.e., a Dirac-Fermi liquid, is given by

$$S_{\rm DL} = S_0 + S_{\rm int}. \tag{2.16}$$

It still contains the Landau-Fermi liquid as a limiting case if one puts the parameters v and Δ in H_0 equal to zero.

III. SOFT MODES IN A DIRAC METAL

The nonanalytic behavior of various observables we are interested in is the result of four-fermion correlation functions that are soft or massless at zero temperature and the dependence of these soft modes on the fields conjugate to the respective observables. Prior to performing explicit calculations it is helpful to discuss the existence, nature, and origin of these soft modes. We start by discussing the soft modes in an ordinary LFL and then generalize to the case of a DFL.

A. Ward identity for a Landau-Fermi liquid

Consider the LFL Green function as given by Eqs. (2.11). It is easy to check that the following identity holds:

$$G_{n_1\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2)\, G_{n_2\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2) = -\frac{G_{n_1\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2) - G_{n_2\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2)}{i\Omega_{n_1-n_2} - \boldsymbol{k}\cdot\boldsymbol{q}/m + (\sigma_1-\sigma_2)h}, \tag{3.1}$$

where $\Omega_{n_1-n_2} = \omega_{n_1} - \omega_{n_2}$ is a bosonic Matsubara frequency, and $h = |\boldsymbol{h}|$. This is a generalization of Velicky's Ward identity [24] for noninteracting electrons to a nonzero magnetic field. Notice that the denominator on the right-hand side has the structure of the differential operator of a Boltzmann equation in the absence of the collision operator (i.e., a time derivative plus a streaming term). It also holds, with the ballistic dynamics replaced by diffusive ones, for electron systems with quenched disorder if one takes G to be the unaveraged Green function and takes a disorder average on either side of Eq. (3.1) [24–26]. It further still holds, with a slightly more complicated frequency structure and m replaced by an appropriately renormalized effective mass, for interacting electrons at zero temperature, with or without quenched disorder, if the left-hand side is replaced by the appropriate four-fermion correlation function that factorizes into a product of two Green functions in the noninteracting limit [27,28]. This is a consequence of the adiabatic correspondence between single-quasiparticle states in the noninteracting and interacting Fermi systems, respectively, that is inherent in Landau-Fermi-liquid theory. In an interacting system at nonzero temperature, a dephasing rate appears in the denominator of the right-hand side of the Ward identity and its generalization discussed below. A nonzero temperature thus gives a mass to all of

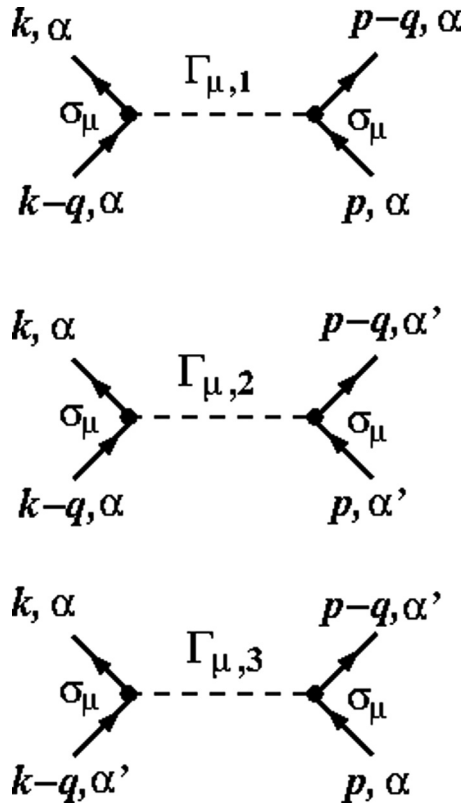

FIG. 3. Graphical representation of the electron-electron interaction. $\mu = 0$ and $\mu = 1, 2, 3$ represent spin-singlet and spin-triplet interactions, respectively. The spin-triplet amplitudes $\Gamma_{(\mu=1,2,3),i} \equiv \Gamma_{t,i}$ are identical due to spin rotational invariance. The chirality index is denoted by α and $\alpha' \neq \alpha$. $\Gamma_{\mu,1}$ describes intrachirality interactions and $\Gamma_{\mu,2}$ and $\Gamma_{\mu,3}$ describe interchirality interactions.

the soft modes we will discuss. However, for our purposes there is a stronger effect of a nonzero temperature that is most easily seen by considering convolutions of Green functions, see Eq. (3.4c) below.

In order to see the significance of the Ward identity, analytically continue the Green function in Eq. (2.11c) to complex frequencies z and consider its analytic structure. The function $G_\pm(\boldsymbol{k}, z) = 1/(z - \xi_{\boldsymbol{k}} \pm |\boldsymbol{h}|)$ has a cut on the real axis, and close to the real axis it is given by

$$G_\pm(\boldsymbol{k}, \omega \pm i0) = \frac{1}{\omega - \xi_{\boldsymbol{k}} \pm |h|} \mp i\pi\delta(\omega - \xi_{\boldsymbol{k}} \pm |\boldsymbol{h}|). \quad (3.2)$$

We see that the behavior of the right-hand side of Eq. (3.1) is drastically different depending on whether the two fermionic Matsubara frequencies ω_{n_1} and ω_{n_2} have the same sign, or opposite signs. In latter case, the numerator on the right-hand side is nonzero in the limit $\Omega_{n_1-n_2} \to 0$, $\boldsymbol{q} \to 0$ for all frequencies or energies where the spectrum of the Green function is nonzero. The denominator, on the other hand, is small of $O(\epsilon)$, where ϵ stands for Ω, or $q = |\boldsymbol{q}|$, or h, all of which scale the same way. In particular, it vanishes in the limit of zero frequency Ω and wave number q if the two spin projections are the same ($\sigma_1 = \sigma_2$). The four-fermion correlation on the left-hand side is thus a soft mode of $O(1/\epsilon)$. Whereas, if n_1 and n_2 have the same sign, then both the numerator and the denominator are small of $O(\epsilon)$, and the four-fermion correlation is of $O(1)$. Similarly, products of N Green functions $G_{n_1} G_{n_2} \ldots G_{n_N}$ are of $O(1/\epsilon^{N-1})$ if the N frequency indices n_i do not all have the same sign, but of $O(1)$ if they do.

It is illustrative to derive the same result as a Ward identity that results from a unitary transformation of the fermionic fields that represents a rotation in frequency space [25], see Appendix B. This shows that the soft four-fermion excitations can be interpreted as the Goldstone modes of a spontaneously broken continuous symmetry, viz., the symmetry between retarded (positive Matsubara frequency) and advanced (negative Matsubara frequency) degrees of freedom. The relevant order parameter is the spectrum of the Green function.

B. Ward identity for a Dirac-Fermi liquid

We now consider the analogous identity involving the Green function $F_n^{\alpha\beta}(\boldsymbol{k})$ given in Eq. (2.10c). We find

$$F_{n_1}^{\alpha_1\beta_1}(\boldsymbol{k}+\boldsymbol{q}/2)F_{n_2}^{\alpha_2\beta_2}(\boldsymbol{k}-\boldsymbol{q}/2) = -\frac{F_{n_1}^{\alpha_1\beta_1}(\boldsymbol{k}+\boldsymbol{q}/2) - F_{n_2}^{\alpha_2\beta_2}(\boldsymbol{k}-\boldsymbol{q}/2)}{i\Omega_{n_1-n_2} - \boldsymbol{k}\cdot\boldsymbol{q}/m + \beta_1|\alpha_1 v(\boldsymbol{k}+\boldsymbol{q}/2) - \boldsymbol{h}| - \beta_2|\alpha_2 v(\boldsymbol{k}-\boldsymbol{q}/2) - \boldsymbol{h}|}. \quad (3.3)$$

For the numerator, the same discussion as in the LFL case applies, but the denominator now allows for more possibilities. Anticipating that in perturbation theory with respect to the electron-electron interaction we will encounter convolutions of Green functions, we define a correlation function

$$\varphi_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\boldsymbol{q}, i\Omega_n; i\omega_m) = \frac{1}{V}\sum_{\boldsymbol{k}} F_k^{\alpha_1\beta_1} F_{k+q}^{\alpha_2\beta_2}, \quad (3.4a)$$

where $k = (\boldsymbol{k}, i\omega_m)$, $q = (\boldsymbol{q}, i\Omega_n)$, and $F_k^{\alpha\beta} = F_n^{\alpha\beta}(\boldsymbol{k})$. Rather than performing the integral exactly, we note that the singular structure at small momenta and wave numbers we are interested in is preserved in the well-known approximation that performs the radial part of the momentum integral by means of a contour integration over the interval $-\infty < \xi_{\boldsymbol{k}} < \infty$ [29]. Within this scheme, which we refer to as the AGD approximation, we are left with only an angular integral,

$$\varphi_{\beta_1\beta_2}^{\alpha_1\alpha_2}(\boldsymbol{q}, i\Omega_n; i\omega_m) = N_F \int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{-2\pi i\,\mathrm{sgn}\,(\omega_m)\,\Theta(-\omega_m(\omega_m+\Omega_n))}{i\Omega_n - v_F\hat{\boldsymbol{k}}\cdot\boldsymbol{q} + \beta_2|\alpha_2 v k_F\hat{\boldsymbol{k}} + \alpha_2 v\boldsymbol{q} - \boldsymbol{h}| - \beta_1|\alpha_1 v k_F\hat{\boldsymbol{k}} - \boldsymbol{h}|}. \quad (3.4b)$$

Here, $\Omega_{\boldsymbol{k}}$ is the solid angle with respect to the wave vector $\boldsymbol{k}$ (not to be confused with the bosonic Matsubara frequency Ω_n), $v_{\rm F} = k_{\rm F}/m$ and $N_{\rm F} = k_{\rm F} m/2\pi^2$ are the Fermi velocity and the density of states per spin and chirality, respectively, at the Fermi surface under consideration [30], and the expression is valid for the leading singular behavior only.

Depending on the relative values of $\alpha_{1,2}$ and $\beta_{1,2}$ there may or may not be a mass in the denominator of Eq. (3.4b). Before we discuss the various cases, we consider the dependence on the Matsubara frequency Ω_n and its consequences for the soft-mode structure at nonzero temperature. To this end, it is useful to perform the sum over the fermionic Matsubara frequency. We define

$$\varphi^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\boldsymbol{q}, i\Omega) = T\sum_{i\omega_m} \varphi^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\boldsymbol{q}, i\Omega_n; i\omega_m) = N_{\rm F}\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{i\Omega_n}{i\Omega_n - v_{\rm F}\hat{\boldsymbol{k}}\cdot\boldsymbol{q} + \beta_2|\alpha_2 v k_{\rm F}\hat{\boldsymbol{k}} + \alpha_2 v\boldsymbol{q} - \boldsymbol{h}| - \beta_1|\alpha_1 v k_{\rm F}\hat{\boldsymbol{k}} - \boldsymbol{h}|}. \tag{3.4c}$$

The factor of $i\Omega_n$ in the numerator is a consequence of the frequency-mixing requirement, which in Eq. (3.4b) was reflected by the Θ function; only products of retarded and advanced Green functions can produce convolutions that are potentially soft, which makes the frequency summation trivial. In any frequency sum over products of convolutions, the term with $n = 0$ therefore comes with zero weight, and the smallest frequency index that contributes is $n = 1$. The $i\Omega_n$ in the denominator at nonzero temperature thus effectively acts as a mass that is linear in T. This is a stronger effect than the dephasing rate mentioned after Eq. (3.1), which vanishes faster than T for $T \to 0$, and it is responsible for the temperature scaling as the wave number and the magnetic field.

C. Soft modes in Landau- and Dirac-Fermi liquids

We now discuss two limiting cases.

First case: $h \gg v k_F$. To zeroth approximation we can put $v = 0$ and recover the LFL case. φ is then independent of the chirality index, and $\beta = \pm \equiv \uparrow\downarrow$ labels the spin projection. Consistent with Eq. (3.1), we have two types of singular behavior. The first type is realized by the $\uparrow\downarrow$ and $\downarrow\uparrow$ combinations:

$$\varphi_{\beta,-\beta}(\boldsymbol{q}, i\Omega_n) = N_{\rm F}\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{i\Omega_n}{i\Omega_n - v_{\rm F}\hat{\boldsymbol{k}}\cdot\boldsymbol{q} - 2\beta h}. \tag{3.5a}$$

It is characterized by singular behavior for $\Omega, |\boldsymbol{q}| \to 0$ that is cut off by a magnetic field h (which scales as Ω and $|\boldsymbol{q}|$). We will refer to soft modes of this type as "soft modes of the first kind" (with respect to h, see Sec. III D below). The second kind is realized by the $\uparrow\uparrow$ and $\downarrow\downarrow$ combinations,

$$\varphi_{\beta\beta}(\boldsymbol{q}, i\Omega_n) = N_{\rm F}\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{i\Omega_n}{i\Omega_n - v_{\rm F}\hat{\boldsymbol{k}}\cdot\boldsymbol{q}}, \tag{3.5b}$$

which we recognize as the hydrodynamic part of the Lindhard function. Here the hydrodynamic singularity is *not* cut off by a magnetic field. We will refer to this type as "soft modes of the second kind".

Second case: $h \ll v k_F$. In this case, we can ignore h compared to $v k_{\rm F}$ in Eq. (3.4b). The correlations with $\beta_1 = -\beta_2$ then become independent of the chirality index, and we have

$$\varphi^{\alpha_1\alpha_2}_{\beta,-\beta}(\boldsymbol{q}, i\Omega_n) = i\Omega_n N_{\rm F}\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{1}{i\Omega_n - v_{\rm F}\hat{\boldsymbol{k}}\cdot\boldsymbol{q} - 2\beta v k_{\rm F}}. \tag{3.6a}$$

The spin-orbit interaction thus gives a mass to the modes that were soft of the first kind in the LFL case, see Eq. (3.5a). The correlations with $\beta_1 = \beta_2$, on the other hand, remain soft:

$$\varphi^{\alpha_1\alpha_2}_{\beta\beta}(\boldsymbol{q}, i\Omega_n) = i\Omega_n N_{\rm F}\int \frac{d\Omega_{\boldsymbol{k}}}{4\pi} \frac{1}{i\Omega_n - (v_{\rm F} - \beta v)\hat{\boldsymbol{k}}\cdot\boldsymbol{q} + \beta(\alpha_1 - \alpha_2)\hat{\boldsymbol{k}}\cdot\boldsymbol{h}}. \tag{3.6b}$$

They are soft of the first kind for correlations that mix the chiralities ($\alpha_1 \neq \alpha_2$), and soft of the second kind for correlations within a given chirality ($\alpha_1 = \alpha_2$). We note that the soft modes of the first kind have a chirality structure that is different from the one necessary for forming a density of a spin density, which is diagonal in α:

$$\boldsymbol{n}_{\rm s}(q) = \sum_{k,\alpha} (\bar{\psi}^{\alpha}(k), \boldsymbol{\sigma}\psi^{\alpha}(k-q)). \tag{3.7}$$

These are nevertheless the modes that produce a nonanalyticity in the spin susceptibility, as we will discuss next.

We also note that models of the graphene-type mentioned after Eq. (2.1a), while allowing for a formally very similar discussion, lead to a drastically different physical interpretation. If the Pauli matrix in the $\boldsymbol{\sigma}\cdot\boldsymbol{k}$ term is a pseudospin, then the coupling constant v does not give a mass to the modes that were soft of the first kind in a LFL. This has important consequences for the effects of the soft modes on the spin susceptibility, as we will discuss in Sec. V.

D. Physical consequences of soft modes: nonanalytic behavior of observables

In order to determine the effects of the soft modes on observables, we need to generalize the notion of soft modes of the first and second kinds from the last subsection. Consider an observable $\mathcal{O}$ and its conjugate field $\mathfrak{h}$. Each soft mode can then be classified as being of the first or second kind with respect to $\mathfrak{h}$, depending on whether or not $\mathfrak{h}$ cuts off the singularity. In Sec. III C, $\mathfrak{h}$ was the magnetic field h, and $\mathcal{O}$ was the spin density. As illustrated by Eq. (3.6b), the correlations we denoted by $\varphi^{\alpha\alpha'}_{\beta\beta}$ are of the first kind with respect to h if $\alpha \neq \alpha'$, and of the second kind if $\alpha = \alpha'$. However, if we take $\mathfrak{h}$ to be the chemical potential μ, which makes $\mathcal{O}$ the number density, then all of the soft modes are of the second kind with respect to μ. Now consider the partition function Z as a generating functional $Z[\mathfrak{h}]$ that depends on $\mathfrak{h}$. Then the free energy is given by $f[\mathfrak{h}] = -(T/V)\ln Z[\mathfrak{h}]$, the observable by $\mathcal{O} = \partial f/\partial\mathfrak{h}$, and the corresponding susceptibility by $\chi_{\mathcal{O}} = \partial^2 f/\partial\mathfrak{h}^2$. Now f, and all of its derivatives, will be nonanalytic functions of $\mathfrak{h}$ if some mode is soft of the first kind with respect to $\mathfrak{h}$. Generically, the field will

scale linearly with the frequency or temperature, and with the external wave number, and scaling implies that $\chi_{\mathcal{O}}$ will also be a nonanalytic function of those. Necessary conditions for these nonanalyticities to be realized are of course the mixing of positive and negative frequencies, and a coupling to the relevant soft modes of the first kind. The first condition requires an interacting fermion system, and the second one implies that only interaction amplitudes in certain channels will lead to nonanalyticities.

We will now apply these considerations, which were first discussed in Ref. [31], to various specific observables. We will first show that they correctly reproduce the known nonanalyticities in a LFL, and then use them to predict the corresponding results in a DFL.

1. Nonanalyticities in a Landau-Fermi liquid revisited

Consider the static spin susceptibility $\chi_s = \partial^2 f/\partial h^2$ in a clean LFL. In the absence of an interaction there is no frequency mixing, and hence χ_s in a Fermi gas, i.e., the Lindhard function, is analytic at zero wave number and zero field. It is easy to see in perturbation theory that there still is no frequency mixing to first order in the interaction amplitudes, but starting at second order, an electron-electron interaction mixes retarded and advanced degrees of freedom, and the relevant soft modes of the first kind are given by Eq. (3.5a). They involve a mixing of the spin projections, and therefore at least one spin-triplet interaction amplitude Γ_t (more precisely, one of the two components of the spin triplet that are transverse with respect to the field) is required in order to produce a nonanalytic behavior of χ_s. This can be seen from Eq. (2.15): only the terms that involve the Pauli matrices σ^1 and σ^2 have the requisite $\uparrow\downarrow$ spin structure. We thus expect χ_s in a LFL to be a nonanalytic function of the wave number Q, the magnetic field h, and also the temperature (which scales the same as the frequency) with a prefactor of the nonanalyticity that for weak interactions is quadratic in the Γ and contains at least one Γ_t. Simple scaling arguments [32] show that the leading nonanalyticity takes the form $\delta\chi_s = \chi_s(Q) - \chi_s(Q = 0) \propto Q^{d-1}$ in generic spatial dimensions d, and $\delta\chi_s \propto Q^2 \ln Q$ in $d = 3$. Using only the soft-mode structure of the LFL, we thus conclude that the spin susceptibility as a function of either Q, or h, of T, with the other two variables equal to zero, has a nonanalytic behavior

$$\delta\chi_s \propto \begin{cases} Q^{d-1} & \text{for} \quad h = T = 0 \\ h^{d-1} & \text{for} \quad Q = T = 0\,, \\ T^{d-1} & \text{for} \quad Q = h = 0 \end{cases} \tag{3.8}$$

in generic dimensions, and $\delta\chi_s \propto Q^2 \ln Q$, etc. in $d = 3$.

This is indeed the case, as is well known from explicit calculations [33,34], with one exception; in 3D systems, the leading temperature nonanalyticity at $O(\Gamma_t^2)$ has a zero prefactor [35]. This zero is accidental, however, and has no bearing on the general argument. Analogous considerations and results hold for LFLs in the presence of quenched disorder [15,36], only the soft modes are diffusive rather than ballistic. As a result, the singularity is stronger, $\delta\chi_s \propto Q^{d-2}$, etc., and the prefactors are linear in Γ_t for weak interactions.

It is also illustrative to note that none of the soft modes are cut off by the chemical potential, which is the field conjugate to the electron number density. The density susceptibility $\chi_n = \partial n/\partial\mu = \partial^2 f/\partial\mu^2$ therefore can *not* be a nonanalytic function of the wave number or the temperature. The same is true for the susceptibilities of the density current, as well as the related higher-rank tensor correlation functions. For instance, a source term $(\boldsymbol{J}\cdot\boldsymbol{k})\,\bar\psi(k)\psi(k)$ in the action generates the number-current density. Such a term can be absorbed into the single-particle energy $\epsilon_{\boldsymbol{k}}$ by completing the square, and hence does not cut off any of the soft modes. This is also confirmed by explicit calculations [33], and we will discuss the consequences in Sec. V.

The extreme opposite case is represented by a nonzero temperature, which cuts off *all* of the soft modes, see the discussion after Eq. (3.4b) above. Focusing on $d = 3$ from here on, we thus expect the specific-heat coefficient $\gamma_V = \partial^2 f/\partial T^2$ in a clean 3D LFL to have a $T^2 \ln T$ nonanalyticity to which both spin-singlet and spin-triplet soft modes contribute. This is indeed the case [37], and an analogous result holds for disordered electrons [36]. In a magnetic field, two of the three spin-triplet modes become massive, and the corresponding $T^2 \ln T$ contributions turn into $h^2 \ln h$ terms.

We finally consider the density of states, which is given in terms of the spectrum of the Green function. It is not a thermodynamic quantity but can be generated if we add to the action a source term that has the structure of a frequency-dependent chemical potential:

$$S_{\delta\mu} = \sum_n \delta\mu(\omega_n)\frac{1}{V}\sum_{\boldsymbol{k}}\sum_\sigma\sum_\alpha \left(\bar\psi^\alpha_\sigma(\boldsymbol{k},\omega_n)\psi^\alpha_\sigma(\boldsymbol{k},\omega_n) - \bar\psi^\alpha_\sigma(\boldsymbol{k},-\omega_n)\psi^\alpha_\sigma(\boldsymbol{k},-\omega_n)\right) \tag{3.9}$$

with $\delta\mu(\omega_n) = -\delta\mu(-\omega_n)$ a source field that is an odd function of the Matsubara frequency and discontinuous at $\omega_n = 0$, $\delta\mu(0+) = -\delta\mu(0-) \neq 0$. It is not a physically realizable field, but generates the density of states N at the Fermi level via

$$N = \frac{i}{2\pi}\lim_{\omega_n\to 0+}\left.\frac{\partial}{\partial\,\delta\mu(\omega_n)}\right|_{\delta\mu=0}\ln Z[\delta\mu]. \tag{3.10}$$

This source field gives all of the soft modes a mass, just as a nonzero temperature does; in the denominator on the right-hand side of the Ward identity, Eqs. (3.1) or (3.3), a term $\delta\mu(\omega_{n_1}) - \delta\mu(\omega_{n_2})$ appears, which for $n_1 \to 0+$, $n_2 \to 0-$ becomes $2\delta\mu(0+)$.

We thus expect the density of states, as a function of the frequency or energy ω at zero temperature, to have a leading nonanalytic contribution that is proportional to $\omega^2 \ln\omega$, or to $T^2 \ln T$ for the density of states on the Fermi surface as a function of the temperature, and with both the spin-singlet and the spin-triplet modes contributing. As in the case of the specific-heat coefficient, a magnetic field will give two of the three spin-triplet modes a mass and turn the corresponding $T^2 \ln T$ or $\omega^2 \ln\omega$ into $h^2 \ln h$. In generic dimensions $1 < d < 3$, all of these nonanalyticities turn into power laws with an exponent $d - 1$.

2. Nonanalyticities in a Dirac-Fermi liquid

In the preceding section, we saw that one can deduce the known results for nonanalyticities in a LFL from the structure

of the soft modes alone. We will now employ the same line of reasoning to deduce the nature of the nonanalyticities in a DFL. For simplicity, we will restrict ourselves to $d = 3$. It is obvious from power counting what the corresponding results are for $1 < d < 3$. In Sec. IV, we will demonstrate that explicit calculations of the spin and density susceptibilities, as well as the density of states, confirm these results.

For the density susceptibility, we obtain the same null result as in the LFL case; the chemical potential μ, which is the field conjugate to the particle number, gives none of the soft modes a mass. The free energy therefore is an analytic function of μ, and the density susceptibility $\partial n/\partial\mu$ has no nonanalyticities.

The fields conjugate to the density of states and the specific-heat coefficient, viz., the frequency-dependent chemical potential and the temperature, respectively, give all of the soft modes a mass. The density of states will thus have a nonanalytic contribution:

$$\delta N(\omega, T = 0, h = 0) \propto N_{\rm F}(\omega/\epsilon_{\rm F})^2 \ln(|\omega|/\epsilon_{\rm F}), \quad (3.11a)$$

$$\delta N(\omega = 0, T, h = 0) \propto N_{\rm F}(T/\epsilon_{\rm F})^2 \ln(T/\epsilon_{\rm F}), \quad (3.11b)$$

where ω is the energy measured from the Fermi surface. All of the coupling constants defined in Sec. II C will contribute to the prefactors. A magnetic field gives modes that couple different chiralities a mass, see Eq. (3.6b), and we expect for the corresponding contributions

$$\delta N(\omega = 0, T = 0, h) \propto N_{\rm F}(h/\epsilon_{\rm F})^2 \ln(h/\epsilon_{\rm F}). \quad (3.11c)$$

Contributions that do not mix chiralities will remain unaffected by the field. By the same argument, we obtain for the specific-heat coefficient:

$$\delta\gamma_V(T, h = 0) \propto N_{\rm F}(T/\epsilon_{\rm F})^2 \ln(T/\epsilon_{\rm F}), \quad (3.12a)$$

$$\delta\gamma_V(T = 0, h) \propto N_{\rm F}(h/\epsilon_{\rm F})^2 \ln(h/\epsilon_{\rm F}), \quad (3.12b)$$

with all interaction amplitudes contributing to the former result, but only the $\Gamma_{\mu,(2,3)}$ to the latter.

For the spin susceptibility, the relevant soft modes are the ones that are made massive by a nonzero magnetic field. From Eq. (3.6b), we see that the only soft modes that contribute are those that mix chiralities. We thus expect

$$\chi_s(Q, T = 0, h = 0) \propto N_{\rm F}\,(Q/k_{\rm F})^2 \ln(q/k_{\rm F}), \quad (3.13a)$$

$$\chi_s(Q = 0, T, h = 0) \propto N_{\rm F}\,(T/\epsilon_{\rm F})^2 \ln(T/\epsilon_{\rm F}), \quad (3.13b)$$

$$\chi_s(Q = 0, T = 0, h) \propto N_{\rm F}\,(h/\epsilon_{\rm F})^2 \ln(h/\epsilon_{\rm F}). \quad (3.13c)$$

In all three cases, the prefactors will depend only on the interchirality interaction amplitudes. Note that the modes that caused the nonanalyticities in χ_s in a LFL are massive in a DFL, see Eq. (3.6a). As a result, an interaction that couples different chiralities is necessary in order to obtain any nonanalytic behavior of χ_s in a DFL.

IV. NONANALYTIC BEHAVIOR OF OBSERVABLES: EXPLICIT CALCULATIONS

We now demonstrate that the results predicted by the general arguments in Sec. III D 2 are confirmed by explicit perturbative calculations. We perform the calculations for the case of a spin-orbit interaction that is weak in the sense $v \ll \sqrt{\mu/m}$. In particular, we ignore the differences between the various values of $k_{\rm F}$, $v_{\rm F}$, and $N_{\rm F}$. We will ignore all contributions from modes that are massive due to $v > 0$. This implies that our results are observable for wave numbers $q/k_{\rm F} \lesssim v/v_{\rm F}$, see Eq. (3.6a). In particular, this implies that we will not be able to recover the LFL limit by taking $v \to 0$. We will further work for $\Delta = 0$; as we will show in Sec. V A, the results remain qualitatively valid for $\Delta \neq 0$ as long as $\Delta \ll v k_{\rm F}$ [38]. We start with the spin susceptibility of a DFL.

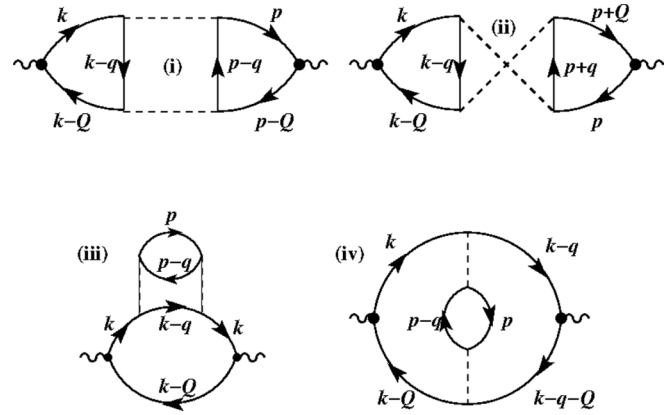

FIG. 4. Contributions to the spin susceptibility at second order in the interaction amplitudes. Directed solid lines represent Green functions, dashed lines represent interaction amplitudes. The external vertex (heavy dot) represents a Pauli matrix σ_3. The internal vertices carry Pauli matrices that depend on the spin channel represented by the corresponding dashed line. Diagram (iii) carries a multiplication factor of 2.

A. Spin susceptibility

The spin susceptibility is given by the two-point correlation function of the spin density, Eq. (3.7). We consider the static longitudinal susceptibility

$$\chi_{\rm s}^{\rm L}(Q) = \left\langle n_{\rm s}^3(Q)\, n_{\rm s}^3(-Q)\right\rangle \quad (4.1)$$

with $Q = (\boldsymbol{Q}, i0)$ the external wave vector/frequency. We are interested in the dependence of $\chi_{\rm s}^{\rm L}$ on the external wave number $|\boldsymbol{Q}|$, the magnetic field h, and the temperature T. Any nonanalytic dependence on these parameters requires that one goes to at least second order in perturbation theory with respect to the electron-electron interaction, and we denote this contribution to the spin susceptibility by $\delta\chi_{\rm s}^{\rm L}$. With the interaction given by Eq. (2.15), and to lowest order in a loop expansion, there are four structurally different diagrams that contribute to $\delta\chi_{\rm s}^{\rm L}$, which are shown in Fig. 4 [39].

As mentioned above, we keep only contributions that are not made massive by the spin-orbit coupling v via Eq. (3.6a). That is, we keep only Green function convolutions where all of the Green functions $F^{\alpha\beta}$ share the same branch index β.

1. Wave-number dependence

We first consider the dependence on the external wave number Q for $h = T = 0$. Diagrams (i) and (ii) in Fig. 4 lead to products of convolutions of three Green functions each.

They can be realized with the interaction amplitudes $\Gamma_{s,1}$ and $\Gamma_{t,1}$, or with $\Gamma_{s,2}$ and $\Gamma_{t,2}$ (the $\Gamma_{\mu,1}$ and $\Gamma_{\mu,2}$ do not mix). From the discussion in Sec. III D, we do not expect these processes to contribute, since the corresponding soft modes are of the first kind with respect to h. Indeed, the contributions from the individual diagrams cancel due to the symmetry properties of convolutions of Green functions listed in Appendix C: diagrams (i) and (ii), if realized with $\Gamma_{\mu,1}$ and $\Gamma_{\mu,2}$, cancel due to Eq. (C1b), and diagrams (iii) and (iv) cancel due to Eq. (C3c).

This leaves $\Gamma_{\mu,3}$ as the only candidates for producing singular behavior, as expected from the general arguments in Sec. III D. An analysis of the spin structure of the diagrams alone shows that any realization that contains a $\Gamma_{s,3}$ cannot contribute to the leading singularities since the leading terms contain spin-matrix products of the type shown in Eq. (2.12c), which vanish. Therefore the only terms that require a full calculation are ones proportional to $(\Gamma_{t,3})^2$.

Realizing the diagrams in Fig. 4 by means of $\Gamma_{t,3}$ leads to the following expression for $\delta\chi_s^L$ in terms of integrals:

$$\begin{aligned}\delta\chi_s^L(\boldsymbol{Q}) = \frac{1}{4}(\Gamma_{t,3})^2 {\sum_q}' \sum_{k,p} \Bigg\{ & \hat{k}_z\hat{p}_z[1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta \beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \beta' \left(F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} - F_p^{-\beta'} F_{p+q}^{+\beta'} F_{p+Q}^{-\beta'}\right) \\ & + 2\hat{k}_z\hat{p}_z(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \left(F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} - F_p^{-\beta'} F_{p+q}^{+\beta'} F_{p+Q}^{-\beta'}\right) \\ & + \hat{k}_z^2[1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta \left(2F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} - F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} F_{k-q-Q}^{-\beta}\right) \sum_{\beta'} F_p^{+\beta'} F_{p-q}^{-\beta'} \\ & + 2\hat{k}_z^2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta \beta \left(2F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} - F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} F_{k-q-Q}^{-\beta}\right) \sum_{\beta'} \beta' F_p^{+\beta'} F_{p-q}^{-\beta'} \Bigg\} \\ & + (\text{the same terms with } \boldsymbol{Q} \to -\boldsymbol{Q}),\end{aligned} \tag{4.2a}$$

where $F_k^{\alpha\beta} \equiv F_n^{\alpha\beta}(\boldsymbol{k})$ is given by Eq. (2.10c). This expression is valid only for the leading singular contributions to $\delta\chi_s^L$. The first two terms come from diagrams (i) and (ii), and the last two from (iii) and (iv). By using the symmetry relations expressed in Eqs. (C2b) and (C4c), we can rewrite this result as follows:

$$\begin{aligned}\delta\chi_s^L(\boldsymbol{Q}) = \frac{1}{2}(\Gamma_{t,3})^2 {\sum_q}' \sum_{k,p} \Bigg\{ & \hat{k}_z\hat{p}_z[1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta \beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \beta' F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} \\ & + 2\hat{k}_z\hat{p}_z(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} + 2\hat{k}_z^2[1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} F_p^{+\beta'} F_{p-q}^{-\beta'} \\ & + 4\hat{k}_z^2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta \beta F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \beta' F_p^{+\beta'} F_{p-q}^{-\beta'} \Bigg\} + (\text{the same terms with } \boldsymbol{Q} \to -\boldsymbol{Q}).\end{aligned} \tag{4.2b}$$

Power counting shows that all of the integrals in Eq. (4.2b) scale as a constant plus $|\boldsymbol{Q}|^{d-1}$ for spatial dimensions $d < 3$, and as $\boldsymbol{Q}^2 \ln|\boldsymbol{Q}|$ for $d = 3$, which is the expected nonanalytic behavior. Determining the prefactor for $d < 3$ would be hard. However, for $d = 3$, and to leading logarithmic accuracy, we can expand the integrands in powers of $\boldsymbol{Q}$ to $O(\boldsymbol{Q}^2)$, and replace the resulting logarithmic divergence with $\ln|\boldsymbol{Q}|$. After lengthy calculations, we obtain, for $h = T = 0$ and $v \ll v_F$,

$$\delta\chi_s^L(\boldsymbol{Q}) = \frac{1}{30}(N_F\Gamma_{t,3})^2 (v_F\boldsymbol{Q})^2 \frac{1}{V} \sum_{\boldsymbol{q}} \frac{1}{(v_F|\boldsymbol{q}|)^3} = N_F (N_F\Gamma_{t,3})^2 \frac{1}{30} (\boldsymbol{Q}/k_F)^2 \ln(k_F/|\boldsymbol{Q}|), \tag{4.3}$$

where we have omitted the constant contribution.

While this result has the same functional form as the corresponding one in a LFL [33], we stress that the underlying soft-mode structure is very different; the LFL soft modes that were responsible for the nonanalytic behavior in the absence of a spin-orbit interaction have become massive, see Eq. (3.6a), and the soft modes that are responsible for the present result mix chiralities, which is why only the interaction amplitude $\Gamma_{t,3}$ contributes. We also note that our result is independent of v and nonzero in the limit $v \ll v_F$. This does *not* imply that it remains unchanged in the true $v \to 0$ limit; we have neglected terms that have become massive due to $v \neq 0$, so we can no longer take the $v \to 0$ limit. If one keeps the terms that are soft for $v = 0$, but massive for $v > 0$, then one finds that in the limit $v \to 0$, they cancel the contribution shown in Eq. (4.3), leaving behind the known result for a LFL [33]. We will come back to these points in Sec. V.

2. Temperature dependence

We now turn to the temperature dependence of $\delta\chi_s^L$ at $\boldsymbol{Q} = h = 0$. For $\boldsymbol{Q} = 0$, it is possible to identically rewrite all of the terms in Eq. (4.2b) as products of convolutions of three Green functions. This can be achieved by means of partial integrations with respect to the z component of the hydrodynamic wave vector $\boldsymbol{q}$. We find

$$\delta\chi_s^L(T) = -2(\Gamma_{t,3})^2 \frac{v_F^2}{v_F^2 - v^2} \sum_q{}' \sum_{k,p} \hat{k}_z \hat{p}_z (1 - \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}})^2 (F_k^{++})^2 F_{k-q}^{-+} (F_p^{+-})^2 F_{p-q}^{--}$$

$$= -2(\Gamma_{t,3})^2 \frac{v_F^2}{v_F^2 - v^2} \sum_q{}' \left[-2 \sum_{i=1}^{3} \mathcal{F}_{zi}^{(2)+}(q) \mathcal{F}_{zi}^{(2)-}(q) + \sum_{i,j=1}^{3} \mathcal{F}_{zij}^{(3)+}(q) \mathcal{F}_{zij}^{(3)-}(q) \right], \tag{4.4a}$$

where

$$\mathcal{F}_z^{(1)\beta}(q) = \sum_k \hat{k}_z \left(F_k^{+\beta}\right)^2 F_{k-q}^{-\beta}, \quad \mathcal{F}_{zi}^{(2)\beta}(q) = \sum_k \hat{k}_z \hat{k}_i \left(F_k^{+\beta}\right)^2 F_{k-q}^{-\beta}, \quad \mathcal{F}_{zij}^{(3)\beta}(q) = \sum_k \hat{k}_z \hat{k}_i \hat{k}_j \left(F_k^{+\beta}\right)^2 F_{k-q}^{-\beta}. \tag{4.4b}$$

We now use a spectral representation for the $\mathcal{F}$ to perform the summation over the hydrodynamic frequency. This yields

$$\delta\chi_s^L(T) = 2(\Gamma_{t,3})^2 \frac{v_F^2}{v_F^2 - v^2} \frac{1}{V} \sum_{\boldsymbol{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \frac{dy}{\pi} \frac{n(x/T) - n(y/T)}{x - y} \left[\mathcal{F}_z^{(1)+\prime\prime}(q) \mathcal{F}_z^{(1)-\prime\prime}(q) \right.$$

$$\left. - 2 \sum_{i=1}^{3} \mathcal{F}_{zi}^{(2)+\prime\prime}(q) \mathcal{F}_{zi}^{(2)-\prime\prime}(q) + \sum_{i,j=1}^{3} \mathcal{F}_{zij}^{(3)+\prime\prime}(q) \mathcal{F}_{zij}^{(3)-\prime\prime}(q) \right]. \tag{4.5}$$

Here, the $\mathcal{F}''$ are the spectra of the $\mathcal{F}$, which contain a continuous part as well as delta-function contributions, and $n(x) = 1/(e^x - 1)$ is the Bose distribution function. The low-T behavior can now be determined by standard asymptotic analysis. Power counting shows that the leading singularity in $d < 3$ spatial dimensions is T^{d-1}, and $T^2 \ln T$ in $d = 3$, as expected. Focusing on $d = 3$ again, structural considerations show that $\mathcal{F}^{(2)}$ does not contribute to the singular behavior, while $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(3)}$ do. The contributions from $\mathcal{F}^{(1)}$ cancel internally, and those from $\mathcal{F}^{(3)}$ yield the final result

$$\delta\chi_s^L(T) = N_F (N_F \Gamma_{t,3})^2 \frac{\pi^4}{48} (T/\epsilon_F)^2 \ln(\epsilon_F/T). \tag{4.6}$$

As expected from the general arguments in Sec. III, there is a nonanalytic temperature dependence whose functional form and sign are the same as for the $\boldsymbol{Q}$ dependence at $T = 0$. While the result is formally independent of v, the comments made after Eq. (4.3) apply here as well: we have neglected modes that are massive for $v > 0$. If kept, they will cancel the term shown in the limit $v \to 0$. In contrast to the case of the Q dependence, here the cancellation is exact, and the prefactor of the $T^2 \ln T$ term in a LFL is zero, at least to second order in the interaction [33,35]. The nonzero result in a DFL underscores the point that the absence of a $T^2 \ln T$ term in a in a LFL is not of structural significance, but due to a prefactor that is accidentally zero in $d = 3$, see the discussion in Sec. V.

3. Magnetic-field dependence

Finally, we consider the dependence on the magnetic field h at $\boldsymbol{Q} = T = 0$. Since $\boldsymbol{Q} = 0$, we can write $\delta\chi_s^L$ in the form of Eqs. (4.4). In order to extract the leading nonanalyticity in $d = 3$, we expand the integrand to $O(h^2)$ and realize that the resulting logarithmic divergence is cut off by h. The leading singular behavior is obtained from terms that maximize the number of Green functions in the integrands. This justifies the approximation we made in Sec. II B 2, where we neglected all occurrences of h in the numerator of the Green function. To leading logarithmic accuracy, we obtain

$$\delta\chi_s^L(h) = \frac{8}{5} (N_F \Gamma_{t,3})^2 h^2 \frac{1}{V} \sum_{\boldsymbol{q}} \frac{1}{(v_F |\boldsymbol{q}|)^3}$$

$$= N_F (N_F \Gamma_{t,3})^2 \frac{2}{5} (h/\epsilon_F)^2 \ln(\epsilon_F/h). \tag{4.7}$$

Again, this is consistent with the prediction based on the general arguments in Sec. III. As explained there, this is, in a well defined sense, the most fundamental nonanalyticity, since h is the field conjugate to the observable whose susceptibility we are calculating, viz., the magnetization. The nonanalytic dependencies on $\boldsymbol{Q}$ and T follow from the one on h due to scaling.

B. Density susceptibility and related correlation functions

We next ascertain that the density susceptibility χ_n has no nonanalytic behavior, as predicted by the general arguments of Sec. III. To second order in the interaction the diagrams are the same as in Fig. 4, but the external vertices now carry a unit matrix in spin space rather than a σ_3, which changes the result of performing the spin traces. The realizations of the diagrams in terms of $\Gamma_{\mu,1}$ and $\Gamma_{\mu,2}$ vanish for the same symmetry reasons as in the case of χ_s^L. For the realizations in

terms of $\Gamma_{\mu,3}$, we find

$$\delta\chi_{\rm n}(\boldsymbol{Q}) = \frac{1}{4}(\Gamma_{t,3})^2 {\sum_q}' \sum_{k,p} \Big\{ [1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \big(F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} + F_p^{-\beta'} F_{p+q}^{+\beta'} F_{p+Q}^{-\beta'} \big)$$
$$+ 2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta \beta F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} \sum_{\beta'} \beta' \big(F_p^{+\beta'} F_{p-q}^{-\beta'} F_{p-Q}^{+\beta'} + F_p^{-\beta'} F_{p+q}^{+\beta'} F_{p+Q}^{-\beta'} \big)$$
$$+ [1+(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})^2] \sum_\beta \big(2F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} + F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} F_{k-q-Q}^{-\beta} \big) \sum_{\beta'} F_p^{+\beta'} F_{p-q}^{-\beta'}$$
$$+ 2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) \sum_\beta \beta \big(2F_k^{+\beta} F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} + F_k^{+\beta} F_{k-q}^{-\beta} F_{k-Q}^{+\beta} F_{k-q-Q}^{-\beta} \big) \sum_{\beta'} \beta' F_p^{+\beta'} F_{p-q}^{-\beta'} \Big\}$$
$$+ (\text{the same terms with } \boldsymbol{Q} \to -\boldsymbol{Q}), \tag{4.8a}$$
$$= 0. \tag{4.8b}$$

As in the case of $\delta\chi_{\rm s}^{\rm L}$, the expression in terms of integrals, Eq. (4.8a), is valid only for the leading nonanalytic contributions to $\delta\chi_{\rm n}$. We see that Eq. (4.8a) is very similar to the corresponding result for $\delta\chi_{\rm s}^{\rm L}$, Eq. (4.2a). However, the symmetry relations from Eqs. (C2b) and (C4c) now lead to terms being equal and opposite that were equal in the case of $\chi_{\rm s}^{\rm L}$. As a result, the contributions proportional to $\Gamma_{\mu,3}^2$ cancel as well, as they must according to the general arguments. We conclude that the density susceptibility $\chi_{\rm n}$ is an analytic function of the wave number, the temperature, and the magnetic field. Consistent with the very general reasons underlying this result, the cancellation occurs at a structural level, and it is not necessary to evaluate any of the integrals in order to see it.

The symmetry relations that lead to this null result are independent of the angular factors encoded in the function $f(\hat{\boldsymbol{k}})$ in Eqs. (3.1)–(3.4). As a result, the absence of nonanalyticities also applies to the density current susceptibility and to all high-rank tensor susceptibilities related to the density. This is important for the stability of Fermi liquids, as we explain in Sec. V.

C. Density of states

The density of states (DOS) $N(\omega)$, with ω the difference in energy space from the Fermi surface, is given by the trace of the Green function via

$$N(\omega) = \mathrm{Re}\, Q(i\omega \to \omega \to i0), \tag{4.9a}$$

where

$$Q(i\omega_n) = \frac{i}{\pi}\frac{1}{V}\sum_{\boldsymbol{k}} \mathrm{tr}\; G_n(\boldsymbol{k}). \tag{4.9b}$$

For noninteracting electrons, $N(\omega \to 0) = 4N_{\rm F}$. We are interested in corrections $\delta N(\omega)$ that are nonanalytic functions of the frequency ω due to the effects of the soft modes. To second order in the interaction, there is only one diagram that is of one-loop type in the sense of Ref. [39], which is shown in Fig. 5. Realizing this diagram in all possible ways with the interaction amplitudes defined in Eq. (2.15), and keeping only terms with hydrodynamic content, we find seven different contributions. In terms of convolutions of the Green function F defined in Eq. (2.10c), these are

$$\delta Q_{{\rm ss},1}(i\omega) = \frac{-i}{\pi}(\Gamma_{{\rm s},1})^2 \sum_\alpha {\sum_q}' \frac{1}{V} \sum_{\boldsymbol{k}} \sum_\beta F_k^{\alpha\beta} F_k^{\alpha\beta} F_{k-q}^{\alpha\beta} \sum_p \sum_{\beta'} F_p^{\alpha\beta'} F_{p-q}^{\alpha\beta'}, \tag{4.10a}$$

$$\delta Q_{{\rm st},1}(i\omega) = \frac{-2i}{\pi}\Gamma_{{\rm s},1}\Gamma_{{\rm t},1} \sum_\alpha {\sum_q}' \frac{1}{V} \sum_{\boldsymbol{k}} (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}) \sum_\beta \beta F_k^{\alpha\beta} F_k^{\alpha\beta} F_{k-q}^{\alpha\beta} \sum_p (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{p}}) \sum_{\beta'} \beta' F_p^{\alpha\beta'} F_{p-q}^{\alpha\beta'}, \tag{4.10b}$$

$$\delta Q_{{\rm tt},1}(i\omega) = \frac{-i}{\pi}(\Gamma_{{\rm t},1})^2 \sum_\alpha {\sum_q}' \frac{1}{V} \sum_{\boldsymbol{k}} (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}})^2 \sum_\beta F_k^{\alpha\beta} F_k^{\alpha\beta} F_{k-q}^{\alpha\beta} \sum_p (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{p}})^2 \sum_{\beta'} F_p^{\alpha\beta'} F_{p-q}^{\alpha\beta'}, \tag{4.10c}$$

$$\delta Q_{{\rm ss},2}(i\omega) = \frac{-i}{\pi}(\Gamma_{{\rm s},2})^2 \sum_{\alpha\neq\alpha'} {\sum_q}' \frac{1}{V} \sum_{\boldsymbol{k}} \sum_\beta F_k^{\alpha\beta} F_k^{\alpha\beta} F_{k-q}^{\alpha\beta} \sum_p \sum_{\beta'} F_p^{\alpha'\beta'} F_{p-q}^{\alpha'\beta'}, \tag{4.10d}$$

$$\delta Q_{{\rm st},2}(i\omega) = \frac{-2i}{\pi}\Gamma_{{\rm s},2}\Gamma_{{\rm t},2} \sum_{\alpha\neq\alpha'} {\sum_q}' \frac{1}{V} \sum_{\boldsymbol{k}} (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}) \sum_\beta \beta F_k^{\alpha\beta} F_k^{\alpha\beta} F_{k-q}^{\alpha\beta} \sum_p (\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{p}}) \sum_{\beta'} \beta' F_p^{\alpha'\beta'} F_{p-q}^{\alpha'\beta'}, \tag{4.10e}$$

$$\delta Q_{\mathrm{tt},2}(i\omega) = \frac{-i}{\pi}(\Gamma_{\mathrm{t},2})^2 \sum_{\alpha\neq\alpha'}\sum_{q}{}' \frac{1}{V}\sum_{\boldsymbol{k}}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}})^2 \sum_{\beta} F_k^{\alpha\beta}F_k^{\alpha\beta}F_{k-q}^{\alpha\beta}\sum_{p}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{p}})^2\sum_{\beta'}F_p^{\alpha'\beta'}F_{p-q}^{\alpha'\beta'}, \tag{4.10f}$$

$$\begin{aligned}\delta Q_{\mathrm{tt},3}(i\omega) = \frac{-i}{\pi}(\Gamma_{\mathrm{t},3})^2 \sum_{\alpha\neq\alpha'}\sum_{q}{}' \Bigg[& \frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\beta} F_k^{\alpha\beta}F_k^{\alpha\beta}F_{k-q}^{\alpha'\beta}\sum_{p}\sum_{\beta'}F_p^{\alpha\beta'}F_{p-q}^{\alpha'\beta'} \\ & + \frac{2}{V}\sum_{\boldsymbol{k}}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}})\sum_{\beta} F_k^{\alpha\beta}F_k^{\alpha\beta}F_{k-q}^{\alpha'\beta}\sum_{p}\sum_{\beta'}F_p^{\alpha\beta'}F_{p-q}^{\alpha'\beta'} \\ & + \frac{1}{V}\sum_{\boldsymbol{k}}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}})^2\sum_{\beta} F_k^{\alpha\beta}F_k^{\alpha\beta}F_{k-q}^{\alpha'\beta}\sum_{p}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{p}})^2\sum_{\beta'}F_p^{\alpha\beta'}F_{p-q}^{\alpha'\beta'}\Bigg].\end{aligned} \tag{4.10g}$$

Power counting shows that these integrals scale as a constant plus $|\omega|^{d-1}$ for spatial dimensions $1 < d < 3$, and as $\omega^2\ln|\omega|$ in $d = 3$, which is the behavior expected from the general arguments in Sec. III D 1. An explicit calculation shows that the nonanalyticity is absent for $d = 3$ (but *not* for $d < 3$) in $\delta Q_{\mathrm{tt},1}$, $\delta Q_{\mathrm{tt},2}$, and the third contribution to $\delta Q_{\mathrm{tt},3}$ [40]. We obtain

$$\delta N(\omega) = 4N_{\mathrm{F}}\{[(\gamma_{\mathrm{s},1})^2+(\gamma_{\mathrm{s},2})^2+(\gamma_{\mathrm{s},3})^2]f_0(v/v_{\mathrm{F}}) + [\gamma_{\mathrm{s},1}\gamma_{\mathrm{t},1}+\gamma_{\mathrm{s},2}\gamma_{\mathrm{t},2}+(\gamma_{\mathrm{t},3})^2]f_1(v/v_{\mathrm{F}})\}\frac{1}{32}\left(\frac{\omega}{\epsilon_{\mathrm{F}}}\right)^2\ln(|\omega|/\epsilon_{\mathrm{F}}) + O(\omega^2), \tag{4.11a}$$

where $\gamma_a = 4N_{\mathrm{F}}\Gamma_a$ ($a = \mathrm{s}, 1$, etc.), and

$$f_0(x) = (1+x^2)/(1-x^2)^3, \quad f_1(x) = -2x^2/(1-x^2)^3. \tag{4.11b}$$

On the Fermi surface, $\omega = 0$, we find an analogous result for $\delta N(T)$ as a function of the temperature; the result has the same form as Eq. (4.11a) with $\omega^2\ln|\omega|$ replaced by $T^2\ln T$, only the numerical prefactor is slightly different.

A nonzero magnetic field affects only $\delta Q_{\mathrm{tt},3}$, which is the only contribution containing convolutions that mix chiralities. Accordingly, the term proportional to $(\gamma_{\mathrm{t},3})^2\omega^2\ln|\omega|$ gets replaced, for $h > \omega, T$, by $(\gamma_{\mathrm{t},3})^2h^2\ln h$, again with a different numerical prefactor. The other contributions remain unaffected.

V. DISCUSSION

We conclude by discussing various aspects of our results, and by elaborating on some points we have mentioned in passing only.

A. Different types of Fermi liquids

1. Landau-Fermi liquids versus Dirac-Fermi liquids

A Fermi liquid in general is characterized by the existence of well-defined quasiparticles that are continuously related to the excitations in a Fermi gas. As we have mentioned in the Introduction, the action given by Eq. (2.16), with or without the spin-orbit coupling v, does indeed describe a Fermi liquid. This can be seen explicitly by considering the electron self energy, which is given by the diagram in Fig. 5 without the two external Green functions. If we average over the Fermi surface, we obtain the expressions in Eqs. (4.10) with the factor $F_k^{\alpha\beta}F_k^{\alpha\beta}$ common to all of them replaced by a delta function $\delta(\xi_{\boldsymbol{k}})$ that pins $\boldsymbol{k}$ to the Fermi surface. Given that the DOS scales as ω^{d-1}, it is obvious that the self energy scales as ω^d, i.e., well-defined quasiparticles exist for all dimensions $d > 1$.

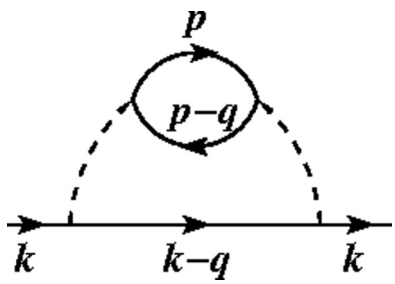

FIG. 5. Contribution to the density of states at second order in the interaction amplitudes. The notation is the same as in Fig. 4.

However, the nature of the Fermi liquid is drastically different depending on whether or not the spin-orbit coupling v is present. As illustrated in Eqs. (3.5) and (3.6), the modes that are soft of the first kind with respect to the magnetic field h for $v = 0$ are no longer soft for $v > 0$. This is related to the fact that the spin-orbit coupling destroys spin conservation. The modes that are soft instead are excitations that mix chiralities, see Eq. (3.6b), a concept that disappears in the limit $v \to 0$ when the two chirality channels become degenerate. For our purposes, this is the crucial difference between a Landau-Fermi liquid at $v = 0$ and a Dirac-Fermi liquid at $v > 0$. As noted in connection with Eq. (3.7), the soft excitations in a DFL have a structure that is qualitatively different from that of density fluctuations. In this sense, they are as different from the soft modes in a LFL as particle-particle or Cooper-channel excitations are from particle-hole excitations.

2. Effects of the gap Δ and the limit of large v

We still need to show that our results are not qualitatively altered if one considers the general model with $\Delta \neq 0$. As can be seen from Eq. (2.9), the gap affects the Green function in two distinct ways. A trivial effect is that the expression $(\alpha v\boldsymbol{k} - \boldsymbol{h})^2$ in the denominator of the Green function gets

modified according to

$$(\alpha v\boldsymbol{k}-\boldsymbol{h})^2 = v^2\boldsymbol{k}^2-\alpha 2vk_z h+h^2 \rightarrow v^2\boldsymbol{k}^2+\Delta^2-\alpha 2h\sqrt{v^2k_z^2+\Delta^2}+h^2, \quad (5.1)$$

which modifies $F_k^{\alpha\beta}$, Eq. (2.10c). A less trivial effect is that the Green function is no longer diagonal in the chirality index due the term $(\pi_1\otimes\sigma_0)\Delta$ in the numerator of Eq. (2.9). This allows for realizations of the diagrams in Fig. 4 that are not possible if $\Delta=0$. However, in order for convolutions of the F to be soft of the first kind with respect to the magnetic field they still need to mix chiralities. This can now be achieved by means of the Green functions rather than the interaction amplitudes, but the soft-mode structure remains unchanged and the integrals are still the same as in Eqs. (4.2) (except for the modified F). The only effect of the gap is therefore to produce small corrections of $O((\Delta/vk_F)^2)$ to the results we obtained in Sec. IV.

Our explicit results have been obtained in the limit $v \ll v_0$, with v_0 the atomic-scale velocity. An obvious question is what happens in the limit $v/v_0 \lesssim 1$. Consider the spin susceptibility χ_s as an example. The general considerations in Sec. III show that the soft modes of the first kind with respect to the magnetic field are given by convolutions with all branch indices β the same, while the chirality indices must not be all the same. Furthermore, in order for the magnetic field dependence to not disappear upon a simple shift of the wave number, the two soft modes in any one-loop contribution to χ_s must carry different branch indices. This is consistent with the structure of the branch and chirality indices in, e.g., Eq. (4.4a). As a result, the interaction couples a branch $E^{1\beta}$ of the single-particle specrum, Eqs. (2.3), with a branch $E^{2\beta}$. However, one of these two branches will not contribute to the Fermi surface in the limit of large v, as can be seen from Fig. 2. This suggests that in this limit the spin susceptibility does not have a nonanalyticity in systems with $\Delta=0$. $\Delta\neq 0$, on the other hand, will eliminate the condition that the two soft modes must carry different branch indices and will likely restore the nonanalyticity, albeit with a prefactor that is proportional to Δ. These remarks apply to the spin susceptibility, other observables will behave differently. For instance, for the density of states, all of the soft modes are of the first kind. The restrictions that are in place for χ_s therefore do not apply, and we expect the behavior for large v to be qualitatively the same as for small v. These qualitative considerations suggest that the behavior in the limit of large v is quite complex, and different in various aspects from that for small v. This limit requires a separate detailed investigation.

3. Different types of Dirac-Fermi liquids

A linear spectrum at small momenta is not sufficient to uniquely define a Dirac-Fermi liquid. This can be seen by comparing the model for a Dirac metal we have discussed, where the Pauli matrices in the $v\boldsymbol{\sigma}\cdot\boldsymbol{k}$ term represents the physical spin degree of freedom, with the graphene-type models mentioned after Eq. (2.1a), where they represent a pseudospin. In the former case, the discussion in Sec. V A 1 applies. In a graphene-type model, on the other hand, the situation is different. Since the Pauli matrices in the $v\boldsymbol{\sigma}\cdot\boldsymbol{k}$ term and in the Zeeman term represent different degrees of freedom, the soft modes of the first kind with respect to the magnetic field are unaffected by the coupling constant v, and the physics behind the nonanalytic spin susceptibility the same as in a LFL. We note, however, that these remarks do not pertain to, e.g., the density of states. Since the nonanalyticity of the latter relies on modes that are soft with respect to the frequency-dependent chemical potential introduced in Eq. (3.9), rather than the magnetic field, the physical difference between the two models does not apply.

We also note that other types of spin-orbit interactions exist, which leads to different types of single-particle spectra. For instance, the touching point may be quadratic rather than linear [2]. For some of these systems, the behavior in the metallic regime will likely be the same as for the Dirac metals discussed here. For a two-dimensional electron system with a Rashba spin-orbit coupling, the analog of Landau Fermi-liquid theory has been developed in Ref. [41], which also calculated the effective mass and various thermodynamic quantities in terms of Fermi-liquid parameters. A similar study for the two-dimensional helical Fermi liquid on the surface of certain bulk topological insulators was carried out in Ref. [42]. The spin susceptibility for a two-dimensional electron system with a spin-orbit interaction of Rashba type was considered in Ref. [43], and the consequences for nuclear spin polarization were discussed in Ref. [44]. Our result that the chirality-independent modes, Eq. (3.6a), become massive as a result of the spin-orbit interaction, are consistent with the results of these papers. However, we find that as a result of the chirality degrees of freedom there are different soft modes that remain soft for $v>0$, see Eq. (3.6b). It is the existence of the latter that leads to a DFL having nonanalyticities very similar to those in a LFL, although as a result of a different set of soft modes.

B. Nature of the soft modes

As we have discussed in Sec. III, the soft modes that underly all of the nonanalyticities we have derived are phase-space four-fermion correlation functions that are local in frequency space and of the general form

$$\left\langle \bar{\psi}_{n_1+m}(\boldsymbol{k}+\boldsymbol{q}/2)\,\psi_{n_1}(\boldsymbol{k}-\boldsymbol{q}/2) \times \bar{\psi}_{n_2-m}(\boldsymbol{p}-\boldsymbol{q}/2)\,\psi_{n_2}(\boldsymbol{p}+\boldsymbol{q}/2)\right\rangle. \quad (5.2)$$

For noninteracting electrons, the correlation factorizes into a product of two Green functions and we obtain, in the LFL case, Eq. (3.1), but at $T=0$ it remains soft even in the case of interactions [28]. Their physical interpretation is in terms of the Goldstone modes of a spontaneously broken rotational symmetry in frequency space, see Appendix B. In calculations that are perturbative with respect to the interaction they enter via convolutions of noninteracting Green functions, the most basic one for a DFL is given in Eqs. (3.4). We note that these Goldstone modes do not describe density fluctuations: the latter are local in time space, whereas the Goldstone modes are local in frequency space. This distinction gets blurred in the case of noninteracting electrons [e.g., Eq. (3.5b) is just

the hydrodynamic part of the Lindhard function, which does describe density fluctuations], but it is physically important in general. For instance, the physical density susceptibility in an interacting system has no dephasing rate, and remains soft, as a result of particle number conservation. This is not true for the Goldstone modes, which do acquire a mass at nonzero temperature. In a DFL, the chirality structure invalidates the interpretation as density fluctuations of even the noninteracting reference system, see the discussion after Eq. (3.6b)

As the discussion in Sec. III D has shown, the properties of the soft modes in the presence of the external field conjugate to the observable under consideration are crucial. Specifically, only modes that are soft of the first kind with respect to the conjugate field, i.e., acquire a small mass upon the application of a small field, can lead to nonanalyticities of the corresponding variable. This observation also underscores the fact that the fundamental nonlinear behavior of any susceptibility is the one with respect to the relevant conjugate field. The nonanalytic dependencies on anything else (e.g., wave number, frequency, or any fields other than the conjugate one) are derivative and follow from how these parameters scale with the conjugate field. As an example, the nonanalytic dependence of the spin susceptibility on the magnetic field h expressed in Eq. (4.7) is the fundamental one for this observable. The nonanalyticities as a function of the temperature, Eq. (4.6), and the wave number, Eq. (4.3), are consequences of the fact that both the temperature and the wave number scale linearly with h.

Conversely, without a mode that is soft of the first kind with respect to the respective conjugate field there can be no nonanalytic dependence on any parameter. An example is the absence of a nonanalyticity in the density susceptibility and related correlation functions that is predicted by the arguments in Sec. III D and confirmed by the calculation in Sec. IV B. This null result is of great physical importance; since the diamagnetic susceptibility is related to the coefficient of the $\boldsymbol{Q}^2$ term in the wave-number dependent number-density current suceptibility [45], a nonanalyticity in the density channel analogous to the one in the spin-density channel would imply ideal diamagnetism.

Another consequence of the structure of the soft modes, and their dependence on various parameters, is the fact that the results for the nonanalyticity in the spin susceptibility, Eqs. (4.3), (4.6), and (4.7) are nonzero in the limit $v \to 0$. This does not contradict the fact that the spin-orbit coupling is necessary in order to obtain these terms in the first place. The point is that other modes have acquired a mass due to the spin-orbit interaction and therefore do not contribute to any nonanalytic behavior. If one kept all contributions to the susceptibility, from massive modes as well as from massless ones, then some of the terms that are made massive by the spin-orbit interaction would, in the limit $v \to 0$, cancel the terms we have calculated, leaving behind the result for a LFL. As written, our results for the spin susceptibility are valid for $\Delta, h \ll v k_F \ll \epsilon_F$. Corrections of order $\Delta/v k_F$ and v/v_F can be calculated if desired, they just make the prefactors of the nonanalyticities more complicated. In our calculation of the density of states, Sec. IV C, we have relaxed the condition $v k_F \ll \epsilon_F$ since in this case some contributions vanish as $v \to 0$.

C. Importance of the soft-mode structure

An important conceptual point is that the general considerations in Sec. III suffice for determining the universal nonanalytic behavior of any observable. No explicit calculations are necessary unless one wants to determine the prefactor of the nonanalyticity, which is model dependent and hence nonuniversal in any case. The calculations in Sec. IV merely demonstrate that the results of explicit calculations are consistent with the general arguments, as they must be. It is also worth noting that the explicit calculations for the DFL are very involved, and guidance from the general structural arguments is very helpful for the asymptotic analysis needed to extract the leading nonanalytic behavior.

The structural arguments can be taken to a higher level by employing field-theoretic and renormalization-group (RG) techniques. In fact, some elements of such an analysis we already employed in Sec. IV by restricting the calculation to diagrams that correspond to one-loop terms in a field-theoretic description. More generally, a LFL is associated with a stable renormalization-group (RG) fixed point that characterizes the entire LFL phase (as opposed to a critical RG fixed point that characterizes a phase transition, see Ref. [46] for a discussion of this important distinction). Such a fixed point has been identified in both a fermionic field theory [47] and an effective bosonic one [28]. Our soft-mode analysis in Sec. III implies that the spin-orbit interaction is a relevant operator with respect to this fixed point and hence makes it unstable. In addition, there must be another unstable fixed point, one describing a Dirac semimetal, with respect to which the chemical potential is a relevant operator. In the presence of both the spin-orbit interaction and a nonzero chemical potential, the RG flow must lead from either unstable fixed point to a stable one that describes a DFL. In such a description, the leading nonanalyticities we have derived will be associated with the least irrelevant operators at the stable fixed point. Such a description will do much more than just confirm the perturbative results. Since the nature the nonanalyticities is tied to the scale dimensions of the least irrelevant operators, it can provide a proof of the hypothesis that the perturbative results are exact as far as the exponents of the nonanalyticities are concerned. Higher orders in perturbation theory and/or a loop expansion will change the prefactor, but cannot change the exponents. This is impossible to ascertain within perturbation theory. It has been shown to be true for a LFL [32], and it is important to establish the corresponding result for a DFL. Such an analysis will also underscore the point that the DFL is qualitatively different from a LFL, the similarity of the resulting nonanalyticities notwithstanding; the similarities just reflects the fact that the least irrelevant operators have the same scale dimensions in both cases, but they obscure the fact that they belong to different fixed points.

In $d = 3$ spatial dimensions, the least irrelevant operators with respect to the DLF fixed point must have a scale dimension of 2, which leads to the logarithmic behavior we have found. In this sense $d = 3$, represents an upper critical dimension that marks the boundary between dimensions $d < 3$ where the least irrelevant operators represent the leading corrections to the fixed-point behavior, and dimensions $d > 3$ where the leading corrections are analytic. As we have seen in Sec. IV, the prefactors of some of the $T^2 \ln T$ and analogous

terms vanish in perturbation theory. There is no structural reason for this that we are aware of. These zeros are likely accidental and an artifact of the low order in the perturbation theory and/or the loop expansion. One example of such an accidental zero is the absence of a $T^2 \ln T$ contribution to the spin susceptibility of a LFL to second order in perturbation theory in $d = 3$ [33,35]. By contrast, the specific-heat coefficient in $d = 3$ does have a $T^2 \ln T$ contribution [37], and an analysis of the spin susceptibility for generic dimensions yields a T^{d-1} nonanalyticity with a d-dependent prefactor that changes sign and is zero for $d = 3$ [48].

D. Consequences for quantum phase transitions in Dirac metals

We finally mention that our results for the spin susceptibility of a DFL have important implications for various magnetic quantum phase transitions in Dirac metals. It is well known that a nonanalytic dependence of the spin susceptibility on the magnetic field h leads to a corresponding nonanalyticity in the magnetic equation of state [16,33,49]. Equation (4.7) implies, at $T = 0$ and in a mean-field approximation for the dimensionless magnetization m, an equation of state of the form

$$t\, m - w\, m^3 \ln m + u\, m^3 = h. \tag{5.3}$$

Here, t and $u > 0$ are Landau parameters, and $w > 0$ due to the sign of the prefactor in Eq. (4.7) [50]. This implies that any quantum phase transition from a paramagnetic Dirac metal to a homogeneous ferromagnetic one is necessarily first order. (This applies only at zero temperature, and in the absence of quenched disorder.) There are, however, other possibilities, such as a transition to a state with modulated magnetic order, with the wave-number scale of the modulation given by the maximum in the wave-number dependent spin susceptibility. Analogous conclusions apply to spin-nematic quantum transitions in Dirac metals. Also of interest are transitions to helimagnetic states with a long pitch wavelength, but our the SI-invariant model studied in the present paper does not directly apply to those. These implications will be pursued in a separate project.

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APPENDIX A: ELECTRON-ELECTRON INTERACTION IN A DIRAC METAL

Consider a Coulomb interaction between fluctuations of the particle number density

$$n(q) = \sum_\alpha n^\alpha(q) = \sum_{k,\alpha} (\bar\psi^\alpha(k), \sigma_0 \psi^\alpha(k-q)). \tag{A1}$$

Here and in what follows, we use an obvious spinor notation: $\psi^\alpha(k) = (\psi^\alpha_\uparrow, \psi^\alpha_\downarrow)$, and $\psi(k) = (\psi^+_\uparrow, \psi^+_\downarrow, \psi^-_\uparrow, \psi^-_\downarrow)$, and the same four-vector notation $k = (\boldsymbol{k}, i\omega_n)$, $q = (\boldsymbol{q}, i\Omega_n)$ as in Sec. III. With $v(\boldsymbol{q})$ the interaction potential, the interaction part of the action can be written

$$\begin{aligned} S^{\rm c}_{\rm int} &= \frac{-T}{2V} \sum_q v(\boldsymbol{q}) \sum_{k,p} \sum_{\sigma,\sigma'} \sum_{\alpha\alpha'} \bar\psi^\alpha_\sigma(k) \psi^\alpha_\sigma(k-q)\, \bar\psi^{\alpha'}_{\sigma'}(p-q) \psi^{\alpha'}_{\sigma'}(p) \\ &= \frac{T}{2V} \sum_q v(\boldsymbol{k}-\boldsymbol{p}) \sum_{k,p} \sum_{\sigma,\sigma'} \sum_{\alpha\alpha'} \bar\psi^\alpha_\sigma(k) \psi^{\alpha'}_{\sigma'}(k-q)\, \bar\psi^{\alpha'}_{\sigma'}(p-q) \psi^\alpha_\sigma(p) \\ &= \frac{-T}{2V} \sum_q v(\boldsymbol{k}+\boldsymbol{p}) \sum_{k,p} \sum_{\sigma,\sigma'} \sum_{\alpha\alpha'} \bar\psi^\alpha_\sigma(-k) \bar\psi^{\alpha'}_{\sigma'}(k-q)\, \psi^{\alpha'}_{\sigma'}(-p-q) \psi^\alpha_\sigma(p). \end{aligned} \tag{A2}$$

These are three identical ways to write the interaction. The first two represent the small-angle and large-angle scattering processes in the particle-hole channel, respectively, from Ref. [33]. The third one represents scattering in the particle-particle channel, i.e., scattering across the Fermi surface. The latter processes do not contribute to the leading singularities we are interested in, and we neglect them. Since we are interested in metals (as defined in Sec. I) we can consider the interaction potential $v(\boldsymbol{q})$ to be screened and hence short ranged.

The singular behavior we are interested in manifests itself at small values of the scattering wave vector $\boldsymbol{q}$ (the "hydrodynamic regime"). If we restrict the sum over this hydrodynamic wave vector to values $|\boldsymbol{q}| < \Lambda$, with Λ a UV cutoff [51], we can add the first two identical expressions for $S_{\rm int}$ in Eq. (A2) to write an effective interaction

$$S_{\rm int} = \frac{-T}{2V} {\sum_q}' \sum_{k,p} \sum_{\sigma,\sigma'} \sum_{\alpha,\alpha'} \big(v(\boldsymbol{q}) \bar\psi^\alpha_\sigma(k) \psi^\alpha_\sigma(k-q)\, \bar\psi^{\alpha'}_{\sigma'}(p-q) \psi^{\alpha'}_{\sigma'}(p) + v(\boldsymbol{k}-\boldsymbol{p}) \bar\psi^\alpha_\sigma(k) \psi^{\alpha'}_{\sigma'}(k-q)\, \bar\psi^{\alpha'}_{\sigma'}(p-q) \psi^\alpha_\sigma(p) \big). \tag{A3}$$

Here the prime on the sum over q indicates the restriction $|\boldsymbol{q}| < \Lambda$.

The two terms in Eq. (A3) can be decomposed into a spin singlet and a spin triplet by using the completeness relation for the Pauli matrices,

$$\boldsymbol{\sigma}_{\sigma_1\sigma_2} \cdot \boldsymbol{\sigma}_{\sigma_3\sigma_4} = 2\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3} - \delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4}, \tag{A4}$$

and the chirality dependence can be written analogously. We obtain

$$\begin{aligned} S_{\rm int} = \frac{-T}{2V} {\sum_q}' \sum_{k,p} \Big[& (v(\boldsymbol{q}) - \frac{1}{2} v(\boldsymbol{k}-\boldsymbol{p}))(\bar\psi(k)(\pi_0 \otimes \sigma_0)\psi(k-q))(\bar\psi(p-q)(\pi_0 \otimes \sigma_0)\psi(p)) \\ & - \frac{1}{8} v(\boldsymbol{k}-\boldsymbol{p}) \sum_{i=1}^{3} (\bar\psi(k)(\pi_0 \otimes \sigma_i)\psi(k-q))(\bar\psi(p-q)(\pi_0 \otimes \sigma_i)\psi(p))) \\ & - \frac{1}{8} v(\boldsymbol{k}-\boldsymbol{p}) \sum_{j=1}^{3} (\bar\psi(k)(\pi_j \otimes \sigma_0)\psi(k-q)(\bar\psi(p-q)(\pi_j \otimes \sigma_0)\psi(p)) \\ & - \frac{1}{8} v(\boldsymbol{k}-\boldsymbol{p}) \sum_{i,j=1}^{3} (\bar\psi(k)(\pi_j \otimes \sigma_i)\psi(k-q))(\bar\psi(p-q)(\pi_j \otimes \sigma_i)\psi(p)) \Big]. \end{aligned} \tag{A5}$$

In the calculation of any observable, the wave vectors $\boldsymbol{k}$ and $\boldsymbol{p}$ will be pinned to the Fermi surface by the poles of the Green functions. One can thus expand $v(\boldsymbol{k}-\boldsymbol{p})$ in a complete set of functions on the Fermi surface. For our isotropic model, this results in an expansion in spherical harmonics, as in Landau-Fermi-liquid theory [29]. For our purposes, the $\ell = 0$ (s-wave) term in this expansion suffices, which amounts to replacing $v(\boldsymbol{k}-\boldsymbol{p})$ by its average over the Fermi surface, and for studying leading the hydrodynamic singularities we can replace $v(\boldsymbol{q})$ by $v(\boldsymbol{q}=0)$. Furthermore, the four terms in Eq. (A5) are individually invariant under rotations in spin space and chirality space, and their respective coupling constants will behave differently under renormalization. We thus obtain Eq. (A5) with four independent number-valued interaction amplitudes for the four different terms:

$$\begin{aligned} S_{\rm int} = \frac{-T}{2V} {\sum_q}' \sum_{k,p} \Big[& \Gamma_{\rm s}^{\rm s} (\bar\psi(k)(\pi_0 \otimes \sigma_0)\psi(k-q))(\bar\psi(p-q)(\pi_0 \otimes \sigma_0)\psi(p)) \\ & - \frac{1}{8} \Gamma_{\rm t}^{\rm s} \sum_{i=1}^{3} (\bar\psi(k)(\pi_0 \otimes \sigma_i)\psi(k-q))(\bar\psi(p-q)(\pi_0 \otimes \sigma_i)\psi(p)) \\ & - \frac{1}{8} \Gamma_{\rm s}^{\rm t} \sum_{j=1}^{3} (\bar\psi(k)(\pi_j \otimes \sigma_0)\psi(k-q))(\bar\psi(p-q)(\pi_j \otimes \sigma_0)\psi(p)) \\ & - \frac{1}{8} \Gamma_{\rm t}^{\rm t} \sum_{i,j=1}^{3} (\bar\psi(k)(\pi_j \otimes \sigma_i)\psi(k-q))(\bar\psi(p-q)(\pi_j \otimes \sigma_i)\psi(p)) \Big]. \end{aligned} \tag{A6}$$

Equation (A6) represents the most general interaction in the s-wave particle-hole channel that is invariant under rotations in both spin and chirality space. In particular, it is the most general interaction in that channel for the model defined in Sec. II in the LFL limit, i.e., for a vanishing spin-orbit interaction, $v = 0$. $v \neq 0$ breaks the rotational invariance in chirality space, and $S_{\rm int}$ as written in Eq. (A6) therefore still has a higher symmetry than the single-particle action. If we require invariance only under spatial inversion, rather than under rotation in chirality space, we see that the four terms in Eq. (A6) give rise to six different structures that are all individually invariant under spatial inversion and simultaneous rotations in spin space and real space and come with six independent interaction amplitudes [52]. These are given in Eq. (2.15).

APPENDIX B: WARD IDENTITIES FOR A LANDAU-FERMI LIQUID FROM ROTATIONS IN FREQUENCY SPACE

Consider noninteracting electrons described by the action S_0 in Eq. (2.13) with $v = 0$. The partition function is given by

$$Z = \int D[\bar\psi, \psi]\, e^{S_0[\bar\psi,\psi]}. \tag{B1}$$

We augment the action by a source term for products $\bar\psi\psi$,

$$\begin{aligned} S_J[\bar\psi, \psi] = \sum_{nm} \int d\boldsymbol{x}\, d\boldsymbol{y} \sum_{\sigma\sigma'} \sum_{\alpha} J^{\alpha}_{nm,\sigma\sigma'}(\boldsymbol{x},\boldsymbol{y}) \\ \times \bar\psi^{\alpha}_{\sigma}(\boldsymbol{x},\omega_n)\, \psi^{\alpha}_{\sigma'}(\boldsymbol{y},\omega_m) \end{aligned} \tag{B2a}$$

and define a generating functional

$$Z[J] = \int D[\bar\psi, \psi]\, e^{S_0[\bar\psi,\psi] + S_J[\bar\psi\psi]}. \tag{B2b}$$

Now consider transformations of the fermionic fields

$$\psi^{\alpha}_{\sigma}(\boldsymbol{x},\omega_n) \to \int d\boldsymbol{y} \sum_{m} \sum_{\sigma'} T^{(i)}_{nm\,\sigma\sigma'}(\boldsymbol{x},\boldsymbol{y})\, \psi^{\alpha}_{\sigma'}(\boldsymbol{y},\omega_m), \tag{B3a}$$

$$\bar\psi^{\alpha}_{\sigma}(\boldsymbol{x},\omega_n) \to \int d\boldsymbol{y} \sum_{m} \sum_{\sigma'} T^{(i)*}_{nm\,\sigma\sigma'}(\boldsymbol{x},\boldsymbol{y})\, \bar\psi^{\alpha}_{\sigma'}(\boldsymbol{y},\omega_m). \tag{B3b}$$

This defines four different transformations for $i = 0, 1, 2, 3$. The $T^{(i)}$ are defined as

$$T^{(i)}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \delta_{nm}\,\delta(\boldsymbol{x}-\boldsymbol{y})\,\delta_{\sigma\sigma'} + t_{nm}[\phi(\boldsymbol{x},\boldsymbol{y}) + \phi(\boldsymbol{y},\boldsymbol{x})](\sigma^i)_{\sigma\sigma'} \tag{B4a}$$

with σ^0 the 2×2 unit matrix and $\sigma^{1,2,3}$ the Pauli matrices and

$$t_{nm} = \delta_{nn_1}\delta_{mn_2} - \delta_{nn_2}\delta_{mn_1}. \tag{B4b}$$

n_1 and $n_2 \neq n_1$ are two fixed Matsubara frequency labels, and $\phi(\boldsymbol{x},\boldsymbol{y})$ is a nonlocal mixing angle. Note that ψ and $\bar{\psi}$ transform via $T^{(i)}$ and its complex conjugate $T^{(i)*}$, respectively.

All four of the transformations are unitary with respect to the scalar product of $\bar{\psi}$ and ψ that is defined by Eq. (B2a) with J replaced by the unit matrix. The functional integration measure is therefore invariant under the transformations, and we have

$$\begin{aligned} Z[J] &= \int D[\bar{\psi},\psi]\, e^{S_0+S_J} \\ &= \int D[\bar{\psi},\psi]\, e^{S_0+\delta S_0+S_J+\delta S_J} \\ &= \int D[\bar{\psi},\psi]\, e^{S_0+S_J}\,[1+\delta S_0+\delta S_J+O(\phi^2)] \end{aligned} \tag{B5a}$$

or

$$0 = \int D[\bar{\psi},\psi]\,(\delta S_0+\delta S_J)e^{S_0+S_J}. \tag{B5b}$$

Here, δS_0 and δS_J are the variations of the action under any of the transformations. If we differentiate Eq. (B5b) with respect to J and put $J=0$, we obtain

$$\begin{aligned} &\left\langle \delta S_0\,\bar{\psi}^\alpha_\sigma(\boldsymbol{x},\omega_n)\,\psi^\alpha_{\sigma'}(\boldsymbol{y},\omega_m)\right\rangle \\ &\quad = -\left\langle \delta\bar{\psi}^\alpha_\sigma(\boldsymbol{x},\omega_n)\,\psi^\alpha_{\sigma'}(\boldsymbol{y},\omega_m)\right\rangle \\ &\qquad -\left\langle \bar{\psi}^\alpha_\sigma(\boldsymbol{x},\omega_n)\,\delta\psi^\alpha_{\sigma'}(\boldsymbol{y},\omega_m)\right\rangle, \end{aligned} \tag{B6}$$

where $\langle\ldots\rangle$ is an average with respect to the action S_0.

Equation (B6) represents four Ward identities, one for each of the four transformations defined above. Calculating the variations, and factorizing the four-fermion correlation on the left-hand side of Eq. (B6) we find that the four identities provide two independent pieces of information: The transformations with $i=0$ or $i=3$ relate products $G_{n_1\sigma}G_{n_2\sigma}$ to the difference of the same Green functions, and those with $i=1$ or $i=2$ do the same for $G_{n_1\sigma}G_{n_2,-\sigma}$. The results are equivalent to Eq (3.1).

This derivation emphasizes that the soft modes are the Goldstone modes of a broken rotational symmetry in frequency space that reflects the difference between retarded and advanced degrees of freedom. The order parameter associated with this broken symmetry is the spectrum of the Green function, which is nonzero everywhere inside the band. This interpretation of the soft modes as Goldstone modes was first discussed by Wegner in the context of disordered electron systems [25].

APPENDIX C: SYMMETRY PROPERTIES OF CORRELATION FUNCTIONS

Here we list symmetry properties of convolutions of Green functions that are useful for simplifying the integrals one encounters in perturbation theory. They can be checked by explicit calculations and are valid only for the leading singular behavior of the convolutions.

Define a convolution

$$J^{(3)\alpha\beta}(q,Q) = \int_k f(\hat{\boldsymbol{k}})F^{\alpha\beta}_k F^{\alpha\beta}_{k-q}F^{\alpha\beta}_{k-Q} \tag{C1a}$$

with $f(\hat{\boldsymbol{k}})$ an arbitrary tensor-valued function of the components of the unit wave vector $\hat{\boldsymbol{k}}$, i.e., $f(\hat{\boldsymbol{k}}) = \text{const.}, \hat{k}_i, \hat{k}_i\hat{k}_j$, etc. $J^{(3)}$ scales as $J^{(3)} \sim 1/q$. The leading singular contributions obey

$$J^{(3)\alpha\beta}(-q,-Q) = -J^{(3)\alpha\beta}(q,Q). \tag{C1b}$$

Similarly the convolutions

$$\tilde{J}^{(3)\beta}_{\pm\mp\pm}(q,Q) = \sum_k f(\hat{\boldsymbol{k}})F^{\pm\beta}_k F^{\mp\beta}_{k-q}F^{\pm\beta}_{k-Q} \tag{C2a}$$

obey

$$\tilde{J}^{(3)\beta}_{+-+}(q,Q) = -\tilde{J}^{(3)\beta}_{-+-}(-q,-Q). \tag{C2b}$$

Useful properties of convolutions of four Green functions involve

$$J^{(4)\alpha\beta}_1(q,Q) = \int_k f(\hat{\boldsymbol{k}})F^{\alpha\beta}_k F^{\alpha\beta}_k F^{\alpha\beta}_{k-q}F^{\alpha\beta}_{k-Q}, \tag{C3a}$$

$$J^{(4)\alpha\beta}_2(q,Q) = \int_k f(\hat{\boldsymbol{k}})F^{\alpha\beta}_k F^{\alpha\beta}_{k-q}F^{\alpha\beta}_{k-Q}F^{\alpha\beta}_{k-q-Q}, \tag{C3b}$$

which scale as $1/q^2$. One finds, for the leading singular behavior,

$$J^{(4)\alpha\beta}_2(q,Q) = -2J^{(4)\alpha\beta}_1(q,Q). \tag{C3c}$$

Finally, the convolutions

$$\tilde{J}^{(4)\beta}_{1,\pm\pm\mp\pm}(q,Q) = \int_k f(\hat{\boldsymbol{k}})F^{\pm\beta}_k F^{\pm\beta}_k F^{\mp\beta}_{k-q}F^{\pm\beta}_{k-Q}, \tag{C4a}$$

$$\tilde{J}^{(4)\beta}_{2,\pm\mp\pm\mp}(q,Q) = \int_k f(\hat{\boldsymbol{k}})F^{\pm\beta}_k F^{\mp\beta}_{k-q}F^{\pm\beta}_{k-Q}F^{\mp\beta}_{k-q-Q} \tag{C4b}$$

obey

$$\tilde{J}^{(4)\beta}_{2,+-+-}(q,Q) = -2\tilde{J}^{(4)\beta}_{1,++-+}(q,Q). \tag{C4c}$$

These symmetry properties are generalizations of analogous properties in the LFL case [33].

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