

Two-dimensional materials in the presence of nonplanar interfacesDanilo T. Alves^{1,2,*} and N. M. R. Peres^{2,3,4,†}¹*Faculdade de Física, Universidade Federal do Pará, 66075-110, Belém, Pará, Brazil*²*Centro de Física, Universidade do Minho, P-4710-057, Braga, Portugal*³*Departamento de Física, Universidade do Minho, P-4710-057, Braga, Portugal*⁴*International Iberian Nanotechnology Laboratory (INL), Av. Mestre José Veiga, 4715-330, Braga, Portugal*

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We consider a planar two-dimensional system between two media with different dielectric constants and in the presence of a third dielectric medium separated by a nonplanar interface. Extending a perturbative method for solving Poisson's equation, developed by Clinton, Esrick, and Sacks [*Phys. Rev. B* **31**, 7540 (1985)], in the presence of nonplanar conducting boundaries to the situation proposed here, we obtain, up to the first order in terms of the function which defines the nonplanar interface, the effective potential, the effective electrostatic field, and the effective dielectric constant for the planar 2D system. We also point to the existence of an effective external field acting in-plane in the 2D system. Implications of the results to properties of 2D systems are discussed. In the limit of planar surfaces, vacuum-dielectric or vacuum-conducting media, our results are in agreement with those found in the literature.

DOI: [10.1103/PhysRevB.99.075437](https://doi.org/10.1103/PhysRevB.99.075437)**I. INTRODUCTION**

Two-dimensional (2D) systems, for example, a 2D electron-gas in a heterostructure or in doped graphene, have properties influenced by the electron-electron interaction [1–6], as well as by the presence of external electric and magnetic fields [7]. In the case of graphene, the electron-electron interactions implies in the renormalization of the Fermi velocity, thus reshaping the Dirac cones [3,8], an effect that was experimentally observed [5].

In the context of quantum field theories applied to the condensed matter, the pseudoquantum electrodynamics (PQED) (sometimes called reduced quantum electrodynamics), an effective and complete description in 2+1 dimensions for electronic systems moving on a plane, was built considering that the static potential of interaction between electrons in the 2D system should be Coulombic, instead of the logarithmic one ($\propto \ln r$) characteristic of quantum electrodynamics in 2+1 dimension [8,9]. On the other hand, the effective interaction between electrons in a two-dimensional system can be changed by the presence of material media. For example, it was recently shown that the logarithmic renormalization of the Fermi velocity in a plane graphene sheet (which, in turn, is related to the Coulombic static potential associated to electrons in the sheet) is inhibited by the presence of a single parallel plate or a cavity formed by conducting plates [10,11], with this inhibition leading to an increase of the optical conductivity.

In addition, the effective interaction between electric charges in a two-dimensional planar system, when it is put in the presence of a planar interface between dielectric

media [7,12,13], has been investigated. This change of the electron-electron interaction due to the presence of boundaries affects the Coulombic drag between graphene single layers [13,14].

The problem of finding the effective interaction between static charges in a two-dimensional system can be viewed as part of a class of problems focusing on a static point charge in the presence of an interface between two media. Essentially, the field of the charge induces an electric polarization on the interface (or a surface charge distribution), which generates an additional electric field, usually named image field.

In the 1970s, the image potential was discussed in the context of several phenomena. For instance, the image-potential states, which are quantum states of electrons localized at surfaces of materials which exhibit negative electron affinity [15]. These electrons cannot escape from the surface due to the image electric potential field and cannot penetrate into the material due to the negative electron affinity [16], as it occurs with electrons in the vicinity of a liquid-helium interface [15–17]. On the other hand, up to 1980, the majority of cases that had been investigated of image-potential effects assumed that the interfaces between the media were planar [18,19]. Motivated by the fact that it is almost impossible to create a perfectly planar surface and interested in determining effects of corrugation on the image potential, Rahman and Maradudin [18] calculated perturbatively the electrostatic image potential for a point charge located near a rough vacuum-isotropic dielectric interface, with the surface of separation described by a random function with mean value equal to zero. The problem of finding the image potential for a point charge in vacuum in the presence of a nonplanar metal surface has been investigated by Clinton *et al.* [15], who, based on a work-energy argument, obtained a general formulation for the image potential for first-order deformations of an arbitrary shape, showed that ions and electrons are always attracted to the elevated part of

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the surface [15]. Clinton *et al.* [20] also presented a formal solution for the electrostatic potential by solving perturbatively Poisson's equation in the presence of a generally modified planar conducting surface, with the solution extendable to any perturbation order in the corrugation function [20].

Nonplanar interfaces occur naturally in graphene-based plasmonic systems [21]. In this class of systems, patterned metallic gratings are positioned at a distance of a single atom from a single graphene sheet, thus leading naturally to the class of problems discussed in this paper. Also, the problem of nanoparticles deposited on graphene [22], and how they change the electron-electron interactions in graphene, is another class of problems that can be solved using the approach we develop ahead. Naturally, the presence of metallic substrates near a 2D material changes the optical conductivity of the material. How this change occurs is also controlled by the nature of the interface near the material and, therefore, incorporating the effect of corrugation in the formalism is a natural application of the problems tackled in this work.

Doped transition-metal dichalcogenides are known to have strong electron-hole interactions (excitonic effects), which can be tuned by the presence of interfaces, being they of dielectric or metallic nature [23]. Again, how the presence of corrugation changes the electron-electron interaction in this class of systems is a highly relevant problem in the field of 2D materials. Finally, if the corrugation [24] occurs in the scale of tens of nanometers, a length scale well in reach of microfabrication techniques, the corrugation plays the role of a scattering potential for the electronic propagation, thus affecting the dc conductivity of the electrons in the 2D material. Since hexagonal boron nitride has allowed an unprecedented control on the distance a 2D system can be positioned near a corrugated interface, the problem discussed in this paper acquires relevance for applications in the field of polaritonics using 2D materials.

In the present paper, we investigate how the presence of nonplanar interfaces changes the effective electrostatic interaction between electrons in a two-dimensional system, producing an effective potential dependent not only on the distance to the source charge (as it occurs for the effective potentials in the case of planar surfaces found in the literature) but also on the position of the charge itself. We also show that nonplanar interfaces generate an effective in-plane external field acting on charges along the two-dimensional system. This effective external field is null for the cases of planar interfaces found in the literature. Specifically, considering a typical configuration [7,13], we investigate a planar two-dimensional system between two media with different dielectric constants, in the presence third dielectric medium separated by a nonplanar interface. Extending the perturbative method for solving Poisson's equation in the presence of nonplanar conducting boundaries, proposed by Clinton, Esrick, and Sacks [20], to the situations discussed here, we obtain the first correction to the effective potential and dielectric constants for the planar two-dimensional system, as well as calculate the coordinate dependent external electric field induced by the nonplanar surface. As an application of our results, we use our results to the case of sinusoidal surfaces. Finally, implications of the results to properties of two-dimensional systems are discussed.

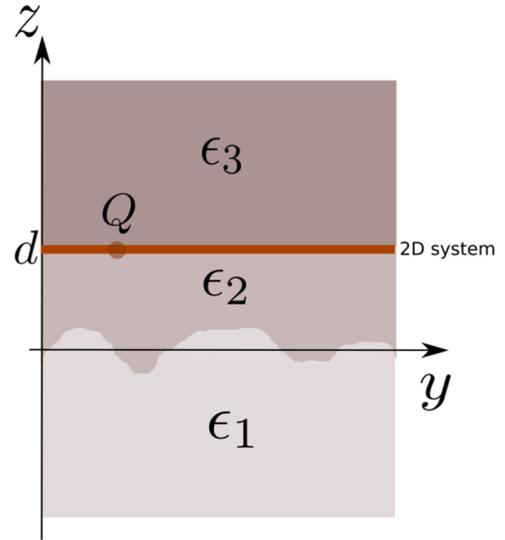


FIG. 1. Illustration of the configuration formed by three different dielectric media (x axis perpendicular to the paper). One can see a planar interface between the regions ϵ_3 and ϵ_2 , where a two-dimensional system is located. All charges of this system, for instance, the charge Q illustrated in the figure, are confined to this plane. The figure also shows a nonplanar interface [described by $z = \lambda h(\mathbf{r}_{\parallel})$] separating the regions ϵ_2 and ϵ_1 .

The paper is organized as follows. In Sec. II, we obtain the total electric potential function for the problem of a point charge between two media with different dielectric constants, and in the presence third dielectric medium separated by a nonplanar surface. We obtain, from our formulas, the particular results for two dielectrics, vacuum-dielectric and vacuum-conducting media, extending and recovering results found in the literature. We obtain the effective potential and dielectric constants for charges living in a 2D planar system put between two dielectric media, also showing the appearance of an effective external field, induced by the nonplanar interface, acting on the charges in this 2D system. In Sec. III, we apply our formulas to the case of sinusoidal surfaces and, using realistic values, obtain estimates for the intensities of effective interaction and external field. In Sec. IV, we present our final comments as well as discuss some implications of our results for two-dimensional systems.

II. POINT-CHARGES CONFINED BETWEEN TWO DIELECTRICS IN THE PRESENCE OF A THIRD DIELECTRIC REGION WITH A NONPLANAR INTERFACE

A. Statement of the problem

We consider a stratified medium containing three different insulators, arranged as as in Fig. 1 (for the purposes of this paper, the first dielectric can also be replaced by a metallic medium). Mathematically, the position of the dielectrics are given by

$$\epsilon(\mathbf{r}) = \begin{cases} \epsilon_3, & z > d \\ \epsilon_2, & \lambda h(\mathbf{r}_{\parallel}) < z < d, \\ \epsilon_1, & z < \lambda h(\mathbf{r}_{\parallel}) \end{cases} \quad (1)$$

where $\mathbf{r}_{\parallel} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, $d > 0$, $z = \lambda h(\mathbf{r}_{\parallel})$ [$h(\mathbf{r}_{\parallel}) < d$] defines a general (nonplanar) surface, and λ ($|\lambda| < 1$) is a dimensionless parameter such that for $\lambda = 0$ one recovers the planar surface case at $z = 0$. We consider $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ unit vectors pointing to the x , y , and z directions, respectively. For practical purposes, we write

$$\begin{aligned} \epsilon(\mathbf{r}) = & \epsilon_3\theta(z-d) + \epsilon_2\theta[z - \lambda h(\mathbf{r}_{\parallel})]\theta(d-z) \\ & + \epsilon_1\theta\{-[z - \lambda h(\mathbf{r}_{\parallel})]\}, \end{aligned} \quad (2)$$

whose expansion in λ leads to

$$\begin{aligned} \epsilon(\mathbf{r}) = & \epsilon_3\theta(z-d) + \epsilon_2\theta(z)\theta(d-z) + \epsilon_1\theta(-z) \\ & + \delta(z)[- \epsilon_2\theta(d-z) + \epsilon_1]\lambda h(\mathbf{r}_{\parallel}) + \mathcal{O}(\lambda^2) + \dots \end{aligned} \quad (3)$$

We consider the problem of a two-dimensional system of point charges confined in the planar interface between the media ϵ_3 and ϵ_2 , as illustrated in Fig. 1. This is achieved, positioning a 2D system between these two dielectrics. From Gauss's law, we have, for a charge Q located at the position $\mathbf{r} = \mathbf{r}'$, with $\mathbf{r}' = \mathbf{r}'_{\parallel} + d\hat{\mathbf{z}}$,

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\phi(\mathbf{r}, \mathbf{r}')] = -4\pi Q\delta(\mathbf{r} - \mathbf{r}'), \quad (4)$$

where the potential ϕ can be written as $\phi(\mathbf{r}, \mathbf{r}') = QG(\mathbf{r}, \mathbf{r}')$. Following Clinton, Esrick, and Sacks [20], we look for a

solution of G as an expansion in powers of λ :

$$G(\mathbf{r}, \mathbf{r}') = G^{(0)}(\mathbf{r}, \mathbf{r}') + \sum_{n=1}^{\infty} \lambda^n G^{(n)}(\mathbf{r}, \mathbf{r}'), \quad (5)$$

where $G^{(0)}(\mathbf{r}, \mathbf{r}')$ is related to the solution of Gauss's equation for planar interfaces. To solve Eq. (4) with $\epsilon(\mathbf{r})$ given by Eq. (2), it is convenient to introduce the Fourier transform in the x , y coordinates,

$$f(\mathbf{r}_{\parallel}, z) = \int \frac{1}{(2\pi)^2} d^2\mathbf{q} f(\mathbf{q}, z) e^{i\mathbf{q}\cdot\mathbf{r}_{\parallel}}, \quad (6)$$

where f can represent any function of \mathbf{r}_{\parallel} considered in the present paper, and $\mathbf{q} = q_x\hat{\mathbf{x}} + q_y\hat{\mathbf{y}}$. We also have

$$f(\mathbf{q}, z) = \int d^2\mathbf{r}_{\parallel} f(\mathbf{r}_{\parallel}, z) e^{-i\mathbf{q}\cdot\mathbf{r}_{\parallel}}, \quad (7)$$

and we are adopting the same nomenclature for a given function of $(\mathbf{r}_{\parallel}, z)$ and for its 2D Fourier transform. Using the representation given in Eq. (6), it can be shown that Eq. (4) can be written as

$$\int \frac{1}{(2\pi)^2} d^2\mathbf{q}' \left\{ (\mathbf{q}' \cdot \mathbf{q}) G(\mathbf{q}', z, \mathbf{r}') \epsilon(\mathbf{q} - \mathbf{q}', z) - \frac{\partial}{\partial z} \left[\frac{\partial G(\mathbf{q}', z, \mathbf{r}')}{\partial z} \epsilon(\mathbf{q} - \mathbf{q}', z) \right] \right\} = 4\pi \delta(z-d) e^{-i\mathbf{q}\cdot\mathbf{r}'_{\parallel}}. \quad (8)$$

The continuity condition for $G(\mathbf{q}, z, \mathbf{r}')$ is required for all values of z . Specifically focusing on the interfaces, we have ($\eta > 0$)

$$\lim_{\eta \rightarrow 0} G(\mathbf{q}, d + \eta, \mathbf{r}') = \lim_{\eta \rightarrow 0} G(\mathbf{q}, d - \eta, \mathbf{r}'), \quad (9)$$

$$\lim_{\eta \rightarrow 0} G(\mathbf{q}, \lambda h(\mathbf{r}_{\parallel}) + \eta, \mathbf{r}') = \lim_{\eta \rightarrow 0} G(\mathbf{q}, \lambda h(\mathbf{r}_{\parallel}) - \eta, \mathbf{r}'). \quad (10)$$

Note that these boundary conditions apply to the full Green's function.

B. Method of solution

The central point of the present calculation is to substitute (3) and (5) into (8), and requiring that the coefficients of λ^n vanish. This yields (up to first order in λ) the equation for $G^{(0)}$,

$$\nabla \cdot \{[\epsilon_3\theta(z-d) + \epsilon_2\theta(z)\theta(d-z) + \epsilon_1\theta(-z)]\nabla[G^{(0)}(\mathbf{r}, \mathbf{r}')]\} = -4\pi\delta(\mathbf{r} - \mathbf{r}'), \quad (11)$$

and the equation for $G^{(1)}$,

$$\nabla \cdot \{\epsilon_3\theta(z-d) + \epsilon_2\theta[z]\theta(d-z) + \epsilon_1\theta(-z)\nabla[G^{(1)}(\mathbf{r}, \mathbf{r}')]\} = \epsilon_{21}^{-1} \nabla \cdot \{\delta(z)h(\mathbf{r}_{\parallel})\nabla[G^{(0)}(\mathbf{r}, \mathbf{r}')]\}, \quad (12)$$

where hereafter we consider

$$\epsilon_{ij}^{\pm} = \epsilon_i \pm \epsilon_j. \quad (13)$$

These equations in Fourier space are given, respectively, by

$$q^2 \epsilon(z) G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) - \frac{\partial}{\partial z} \left[\epsilon(z) \frac{\partial G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d)}{\partial z} \right] = 4\pi \delta(z-d) e^{-i\mathbf{q}\cdot\mathbf{r}'_{\parallel}} \quad (14)$$

and

$$\begin{aligned} & q^2 G^{(1)}(\mathbf{q}, z, \mathbf{r}') [\epsilon_3\theta(z-d) + \epsilon_2\theta(z)\theta(d-z) + \epsilon_1\theta(-z)] - \frac{\partial}{\partial z} \left\{ \frac{\partial G^{(1)}(\mathbf{q}, z, \mathbf{r}')}{\partial z} [\epsilon_3\theta(z-d) + \epsilon_2\theta(z)\theta(d-z) + \epsilon_1\theta(-z)] \right\} \\ & = \epsilon_{21}^{-1} \int \frac{1}{(2\pi)^2} d^2\mathbf{q}' \left\{ (\mathbf{q}' \cdot \mathbf{q}) G^{(0)}(\mathbf{q}', z, \mathbf{r}') \delta(z) h(\mathbf{q} - \mathbf{q}') - \frac{\partial}{\partial z} \left[\frac{\partial G^{(0)}(\mathbf{q}', z, \mathbf{r}')}{\partial z} \delta(z) h(\mathbf{q} - \mathbf{q}') \right] \right\}. \end{aligned} \quad (15)$$

The solution for (14), taking into account that $G^{(0)}$ is continuous through the interfaces, is known and given by [7,12,13]

$$G^{(0)}(\mathbf{q}, z, \mathbf{r}') = \begin{cases} G_{\text{III}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & z \geq d \\ G_{\text{II}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & 0 \leq z \leq d. \\ G_{\text{I}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & z \leq 0 \end{cases} \quad (16)$$

The functions $G_{\text{I}}^{(0)}$, $G_{\text{II}}^{(0)}$, and $G_{\text{III}}^{(0)}$ are explicitly exhibited in Appendix A.

To solve Eq. (15), we take into account a set of four equations describing the boundary conditions for $G^{(1)}$. A first pair of equations is given by (see Appendix B)

$$G^{(1)}(\mathbf{q}, d^+, \mathbf{r}'_{\parallel}, d) = G^{(1)}(\mathbf{q}, d^-, \mathbf{r}'_{\parallel}, d), \quad (17)$$

$$G^{(1)}(\mathbf{q}, 0^+, \mathbf{r}') - G^{(1)}(\mathbf{q}, 0^-, \mathbf{r}') = \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' h(\mathbf{q} - \mathbf{q}') \left[- \left(\frac{\partial G^{(0)}}{\partial z} \right) (\mathbf{q}', 0^+, \mathbf{r}') + \left(\frac{\partial G^{(0)}}{\partial z} \right) (\mathbf{q}', 0^-, \mathbf{r}') \right]. \quad (18)$$

Then, the initial problem of finding G via Eq. (8) with the boundary conditions (9) and (10) (both requiring the continuity of G , with the former taken on a planar and the latter taken on a nonplanar surface), is now effectively replaced (up to first order in λ) by the problem of finding $G^{(1)}$ via Eq. (15) with the boundary conditions (17) and (18), which are both taken on planar surfaces, but with the latter showing a discontinuity of $G^{(1)}$ when it passes through $z = 0$.

Looking for a boundary condition for the z derivative of $G^{(1)}$ across $z = 0$, we integrate Eq. (15) in z in the regions $(-\eta, +\eta)$, sending $\eta \rightarrow 0$, obtaining

$$-\epsilon_2 \left[\frac{\partial}{\partial z} G^{(1)}(\mathbf{q}, z, \mathbf{r}') \right]_{z=0^+} + \epsilon_1 \left[\frac{\partial}{\partial z} G^{(1)}(\mathbf{q}, z, \mathbf{r}') \right]_{z=0^-} = \epsilon_{21} \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' h(\mathbf{q} - \mathbf{q}') (\mathbf{q}' \cdot \mathbf{q}) G^{(0)}(\mathbf{q}', 0, \mathbf{r}'). \quad (19)$$

Repeating the procedure for the region $(d - \eta, d + \eta)$, we get

$$\epsilon_3 \frac{\partial}{\partial z} G^{(1)}(\mathbf{q}, z, \mathbf{r}')_{d+\eta} - \epsilon_2 \frac{\partial}{\partial z} G^{(1)}(\mathbf{q}, z, \mathbf{r}')_{d-\eta} = 0. \quad (20)$$

Requiring that $\lim_{z \rightarrow \pm\infty} G^{(1)} = 0$, we write the solution for (15), (17), (18), (19) and (20) as

$$G^{(1)}(\mathbf{q}, z, \mathbf{r}') = \begin{cases} G_{\text{III}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & z \geq d \\ G_{\text{II}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & 0 < z \leq d, \\ G_{\text{I}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d), & z < 0 \end{cases} \quad (21)$$

with $G_{\text{I}}^{(1)}$, $G_{\text{II}}^{(1)}$, and $G_{\text{III}}^{(1)}$ shown in Appendix A. The solution for $G^{(1)}$ in terms of x and y is given by (see Appendix C)

$$G_{\text{III}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{1}{4\pi} \epsilon_{21}^- \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) [\mathcal{G}_1(\mathbf{r}_{\parallel}, d - z, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) + \frac{\epsilon_1}{\epsilon_2} \mathcal{G}_2(\mathbf{r}_{\parallel}, d - z, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel})], \quad (22)$$

$$G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{1}{8\pi} \frac{\epsilon_{21}^-}{\epsilon_2} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ \epsilon_{23}^- \left[\mathcal{G}_1(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) + \frac{\epsilon_1}{\epsilon_2} \mathcal{G}_2(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) \right] \right. \\ \left. + \epsilon_{23}^+ \left[\mathcal{G}_1(\mathbf{r}_{\parallel}, d - z, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) + \frac{\epsilon_1}{\epsilon_2} \mathcal{G}_2(\mathbf{r}_{\parallel}, d - z, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) \right] \right\}, \quad (23)$$

$$G_{\text{I}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{1}{8\pi} \frac{\epsilon_{21}^-}{\epsilon_2} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \{ \epsilon_{23}^- [\mathcal{G}_1(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) + \mathcal{G}_2(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel})] \\ + \epsilon_{23}^+ [\mathcal{G}_1(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) - \mathcal{G}_2(\mathbf{r}_{\parallel}, z - d, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel})] \}, \quad (24)$$

where

$$\mathcal{G}_1(\mathbf{r}_{\parallel}, \zeta, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) = \nabla_{\parallel} G_{\text{I}}^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \zeta, \mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\tilde{\mathbf{r}}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d), \quad (25)$$

$$\mathcal{G}_2(\mathbf{r}_{\parallel}, \zeta, \mathbf{r}'_{\parallel}, d, \tilde{\mathbf{r}}_{\parallel}) = \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=\zeta} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0}. \quad (26)$$

We have, therefore, concluded the solution of the problem in its most general form.

C. Particular results

If we consider the vacuum-dielectric case ($\epsilon_3 = \epsilon_2 = 1$), we get $G_{\text{III}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d)$, so that

$$G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{1}{4\pi} (1 - \epsilon_1) \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ \nabla_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, d - z, \mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d) \right. \\ \left. + \epsilon_1 \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=d-z} \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right\}, \quad (27)$$

$$G_1^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{1}{4\pi} (1 - \epsilon_1) \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ \nabla_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, z + d, \mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') \right. \\ \left. - \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}_{\parallel}, z') \right]_{\tilde{z}=z+d} \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right\}, \quad (28)$$

where for this case,

$$G_1^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') = \frac{2}{(\epsilon_1 + 1)} \frac{1}{[|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|^2 + (z - d)^2]^{1/2}}. \quad (29)$$

The particular result given by Eqs. (27) and (28) also generalize that found in the literature [20]. Finally, if we consider the vacuum-conducting case ($\epsilon_3 = \epsilon_2 = 1$ and $\epsilon_1 \rightarrow \infty$), we get

$$G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = -\frac{1}{4\pi} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=d-z} \\ \times \left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0}, \quad (30)$$

$$G_1^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = 0, \quad (31)$$

where for this case,

$$G_1^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \frac{2}{[|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|^2 + (z - d)^2]^{1/2}}. \quad (32)$$

This result recovers the result found in the literature [20] and is formally identical to Hadamard's theorem for Green's functions [20,25], which gives the solution (up to first order in λ) of

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}'), \quad (33)$$

with the boundary condition

$$G(\mathbf{r} - \mathbf{r}')|_{z=\lambda h(\mathbf{r}_{\parallel})} = 0. \quad (34)$$

D. Interaction between a charge and the surrounding polarized matter induced by it

When we bring a charge Q to the position \mathbf{r}' , we produce a state of polarization in the dielectric media. The energy of interaction W between the charge Q and the polarized dielectrics (we are considering a linear behavior for the dielectrics) is given by (see Appendix D)

$$W = \frac{1}{2} Q \phi_{\text{ind}}(\mathbf{r}'), \quad (35)$$

where ϕ_{ind} is the induced (or image) potential function, which, taken at \mathbf{r}' , is given by

$$\phi_{\text{ind}}(\mathbf{r}') \approx Q [G_{\text{ind}}^{(0)}(\mathbf{r}, \mathbf{r}')_{\mathbf{r}=\mathbf{r}'} + \lambda G^{(1)}(\mathbf{r}, \mathbf{r}')_{\mathbf{r}=\mathbf{r}'}]. \quad (36)$$

Then, we have

$$W \approx W^{(0)} + \lambda W^{(1)}, \quad (37)$$

where

$$W^{(0)} = \frac{1}{2} Q^2 G_{\text{ind}}^{(0)}(\mathbf{r}, \mathbf{r}')_{\mathbf{r}=\mathbf{r}'}, \\ W^{(1)} = \frac{1}{2} Q^2 G^{(1)}(\mathbf{r}, \mathbf{r}')_{\mathbf{r}=\mathbf{r}'}, \quad (38)$$

with

$$G_{\text{ind}}^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = -\int \frac{1}{(2\pi)^2} d^2 \mathbf{q} e^{-qz} \frac{1}{q} e^{i\mathbf{q} \cdot \mathbf{r}_{\parallel}} e^{-i\mathbf{q} \cdot \mathbf{r}'_{\parallel}} \\ \times \frac{1}{\epsilon_{23}^+ [\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}. \quad (39)$$

Considering the solution for $G^{(1)}$ for $z = d$, we have

$$W = -Q^2 \frac{1}{\pi} \frac{\epsilon_{21}^- \epsilon_{23}^-}{\epsilon_{23}^+} \int d^2 \mathbf{q} \frac{1}{q} \frac{e^{-qd}}{[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]} \\ + \lambda Q^2 \epsilon_{21}^- \frac{1}{8\pi} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ |\nabla'_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d)|^2 \right. \\ \left. + \frac{\epsilon_1}{\epsilon_2} \left(\left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right)^2 \right\}. \quad (40)$$

If we consider the case vacuum-dielectric ($\epsilon_3 = \epsilon_2 = 1$) in Eq. (40), we obtain

$$W = -Q^2 \frac{1}{4d} \frac{(\epsilon_1 - 1)}{(\epsilon_1 + 1)} + \lambda Q^2 \frac{(1 - \epsilon_1)}{8\pi} \\ \times \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ |\nabla'_{\parallel} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d)|^2 \right. \\ \left. + \epsilon_1 \left(\left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right)^2 \right\}, \quad (41)$$

where $G_1^{(0)}$ for this case is obtained considering $\epsilon_3 = \epsilon_2 = 1$ in Eqs. (A8) and (A9), and using (6). The result shown in Eq. (41) is agreement with that found in the literature [20]. For the vacuum-conducting case ($\epsilon_3 = \epsilon_2 = 1$ and $\epsilon_1 \rightarrow -\infty$),

we obtain from Eq. (40) the result

$$W = -Q^2 \frac{1}{4d} - \lambda Q^2 \frac{1}{8\pi} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \left\{ \left(\left[\frac{\partial}{\partial \tilde{z}} G_1^{(0)}(\tilde{\mathbf{r}}_{\parallel}, \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right)^2 \right\}, \quad (42)$$

We can also rewrite Eq. (40) in the following manner:

$$W = -Q^2 \frac{\epsilon_2 \epsilon_{12}^-}{\epsilon_{23}^+} \frac{1}{d} \mathcal{F}_1(\epsilon_{23}^- \epsilon_{12}^-, \epsilon_{23}^+ \epsilon_{12}^+) + \lambda Q^2 \epsilon_{21}^- \frac{1}{\pi^2} \int d^2 \tilde{\mathbf{q}} h(\tilde{\mathbf{q}}) e^{i\tilde{\mathbf{q}} \cdot \mathbf{r}'_{\parallel}} \int_0^{\infty} dR R J_0(\tilde{q}R) \int_0^{\infty} dq \times \int_0^{\infty} dq' \mathcal{F}_2(q, \epsilon_{23}^- \epsilon_{12}^-, \epsilon_{23}^+ \epsilon_{12}^+, d) \mathcal{F}_2(q', \epsilon_{23}^- \epsilon_{12}^-, \epsilon_{23}^+ \epsilon_{12}^+, d) [\epsilon_2^2 J_1(qR) J_1(q'R) + \epsilon_1 \epsilon_2 J_0(qR) J_0(q'R)], \quad (43)$$

where

$$\mathcal{F}_1(a, b) = \begin{cases} [\ln(a+b) - \ln(b)]/a, & a \neq 0 \\ 1/b, & a = 0 \end{cases}, \quad (44)$$

$$\mathcal{F}_2(q, a, b, d) = q/(ae^{-qd} + be^{qd}), \quad (45)$$

and J_0 and J_1 are Bessel functions of the first kind of zeroth and first order, respectively. The form for W given in Eq. (43) is very convenient for numerical calculations. Next, we take some limits of the presented formulas and recover some results found in the literature.

For the case with $\epsilon_3 = \epsilon_2$, we have the problem involving two dielectrics ϵ_2 and ϵ_1 , for which we get

$$W = Q^2 \frac{\epsilon_{21}^-}{\epsilon_{12}^+} \frac{1}{4d} \frac{1}{\epsilon_2} + \lambda Q^2 \frac{\epsilon_{21}^-}{\epsilon_2 (\epsilon_{12}^+)^2} \frac{1}{4\pi^2} \int d^2 \tilde{\mathbf{q}} h(\tilde{\mathbf{q}}) e^{i\tilde{\mathbf{q}} \cdot \mathbf{r}'_{\parallel}} \times \int_0^{\infty} dR R J_0(\tilde{q}R) \left[\frac{\epsilon_2 R^2 + \epsilon_1 d^2}{(R^2 + d^2)^3} \right]. \quad (46)$$

The vacuum-dielectric case can be recovered by doing $\epsilon_2 = 1$ in Eq. (46). The vacuum-conducting case is recovered taking $\epsilon_1 \rightarrow -\infty$. Both results for these limit cases coincide with those found in the literature [15,20].

The presence of a nonplanar interface between ϵ_2 and ϵ_1 media also induces an external field, so that on each charge Q in the two-dimensional system acts an effective force parallel to the plane $z = d$ given by

$$\mathbf{F}_{\parallel}^{(\text{ext})} \approx -\frac{1}{2} \lambda Q^2 \nabla_{\parallel} G^{(1)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}'_{\parallel}, d). \quad (47)$$

This force depends on the magnitude of the charge (specifically, on Q^2) and it can point to the next valley or peak of the nonplanar interface, depending on the sign of $\epsilon_2 - \epsilon_1$. This generalizes the result found in the literature for the vacuum-conductor case [15], where the correspondent force always points to the next peak of the nonplanar interface.

We also have a perpendicular force acting on Q , given by

$$\mathbf{F}_{\perp}^{(\text{ext})} \approx -\frac{1}{2} Q^2 \frac{\partial}{\partial d} [G_{\text{ind}}^{(0)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}'_{\parallel}, d) + \lambda G^{(1)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}'_{\parallel}, d)]. \quad (48)$$

Part of this force (proportional to λ) can be quite relevant for suspended graphene, since the force induces a deformation of the material, which, in turn, affects its optical and dc transport properties.

E. Effective charge-charge interaction in the 2D material

Let us now consider the effective electron-electron interaction, which alters its usual form due to the presence of corrugation. The effective electric potential $\phi^{(\text{eff})}$ associated to a point charge Q in the position \mathbf{r}'_{\parallel} is

$$\phi^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) = \phi(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = QG(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d). \quad (49)$$

Up to the first order, we have

$$\phi^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) \approx \phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) + \lambda \phi^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d). \quad (50)$$

Using Eqs. (16) and (21), we get

$$\phi^{(\text{eff})}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) \approx \frac{2\pi Q}{q} e^{-i\mathbf{q} \cdot \mathbf{r}'_{\parallel}} \left\{ \frac{2[\epsilon_{21}^- e^{-2qd} + \epsilon_{12}^+]}{[\epsilon_{23}^- \epsilon_{12}^- e^{-2qd} + \epsilon_{23}^+ \epsilon_{12}^+]} + \lambda \frac{\epsilon_2 (\tilde{T}_2 \epsilon_{21}^- + \tilde{T}_1 \epsilon_1 q) e^{qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]} \right\}, \quad (51)$$

where

$$\tilde{T}_1 = \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' \frac{4qh(\mathbf{q} - \mathbf{q}') \epsilon_{21}^- e^{i\mathbf{r}'_{\parallel} \cdot (\mathbf{q} - \mathbf{q}')}}{[\epsilon_{23}^- \epsilon_{12}^- e^{-q'd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{q'd}]}, \quad (52)$$

$$\tilde{T}_2 = \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' \frac{4qh(\mathbf{q} - \mathbf{q}') (\mathbf{q}' \cdot \mathbf{q}) \epsilon_2 e^{i\mathbf{r}'_{\parallel} \cdot (\mathbf{q} - \mathbf{q}')}}{q' [\epsilon_{23}^- \epsilon_{12}^- e^{-q'd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{q'd}]}. \quad (53)$$

From Eq. (51), we get the effective dielectric constant ϵ_{eff} :

$$\frac{1}{\epsilon_{\text{eff}}} = \frac{2[\epsilon_{21}^- e^{-2qd} + \epsilon_{12}^+]}{[\epsilon_{23}^- \epsilon_{12}^- e^{-2qd} + \epsilon_{23}^+ \epsilon_{12}^+]} + \lambda \frac{\epsilon_2 (\tilde{T}_2 \epsilon_{21}^- + \tilde{T}_1 \epsilon_1 q) e^{qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}. \quad (54)$$

Notice that ϵ_{eff} depends on \mathbf{r}'_{\parallel} . When $\lambda = 0$, the result given in Eq. (54) recovers that found in the literature [7,12,13]. The functions $\phi^{(0)}$ and $\phi^{(1)}$ in Eq. (50) are given explicitly in Appendix E and exhibit the symmetry properties

$$\phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = \phi^{(0)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}_{\parallel}, d), \quad (55)$$

$$\phi^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = \phi^{(1)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}_{\parallel}, d), \quad (56)$$

from which the energy interaction W_{12} between two charges Q_1 and Q_2 located at $\mathbf{r}_{1\parallel}$ and $\mathbf{r}_{2\parallel}$, respectively, is

$$W_{12} \approx Q_2 \phi_1^{(\text{eff})}(\mathbf{r}_{2\parallel}, \mathbf{r}_{1\parallel}, d) = W_{21} \approx Q_1 \phi_2^{(\text{eff})}(\mathbf{r}_{1\parallel}, \mathbf{r}_{2\parallel}, d), \quad (57)$$

as expected, with $\phi_1^{(\text{eff})}$ and $\phi_2^{(\text{eff})}$ the potential functions associated to the charges 1 and 2.

The effective electric field $\mathbf{E}_{\parallel}^{(\text{eff})}$ produced by a charge Q is

$$\mathbf{E}_{\parallel}^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) = -\nabla_{\parallel} \phi^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d), \quad (58)$$

which can be written as

$$\mathbf{E}_{\parallel}^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) \approx \mathbf{E}_{\parallel}^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) + \lambda \mathbf{E}_{\parallel}^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d), \quad (59)$$

with

$$\mathbf{E}_{\parallel}^{(0)}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) = -\nabla_{\parallel} \phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d), \quad (60)$$

$$\mathbf{E}_{\parallel}^{(1)}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) = -\nabla_{\parallel} \phi^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d), \quad (61)$$

and $\mathbf{E}_{\parallel}^{(0)}$ and $\mathbf{E}_{\parallel}^{(1)}$ given explicitly in Appendix E.

For $\mathbf{E}_{\parallel}^{(0)}$, we have the usual symmetry

$$\mathbf{E}_{\parallel}^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = -\mathbf{E}_{\parallel}^{(0)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}_{\parallel}, d), \quad (62)$$

but for $\mathbf{E}_{\parallel}^{(1)}$, in general, we find

$$\mathbf{E}_{\parallel}^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) \neq -\mathbf{E}_{\parallel}^{(1)}(\mathbf{r}'_{\parallel}, d, \mathbf{r}_{\parallel}, d). \quad (63)$$

Then, the effective 2D electrical field is such that

$$\mathbf{E}_{\parallel}^{(\text{eff})}(\mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}, d) \neq -\mathbf{E}_{\parallel}^{(\text{eff})}(\mathbf{r}'_{\parallel}, \mathbf{r}_{\parallel}, d), \quad (64)$$

so that the effective forces between two charges do not point along the line from one charge to the other.

The above result can be understood as follows. Under the external field associated to a charge Q , the atoms of the dielectric media become polarized, or with permanent dipoles aligned with the field [26]. Let us consider that these dipole moments contribute to the averaged charge density of the dielectric media [26],

$$\langle \rho'(\mathbf{r}) \rangle = \frac{1}{\Delta V} \int_{\Delta V} \rho'(\mathbf{r} + \boldsymbol{\xi}) d^3 \boldsymbol{\xi}, \quad (65)$$

where ρ' is the exact position (in a certain instant of time) of the charges in motion (thermal or zero point effects) in the dielectric media, ΔV is a macroscopically small volume, $\boldsymbol{\xi}$ ranges over the this small volume, and $\langle \cdot \rangle$ means the average value [26]. For the situation where all interfaces are flat, $\langle \rho'(\mathbf{r}) \rangle$, now relabeled as $\langle \rho'^{(0)}(\mathbf{r}) \rangle$, is (by symmetry arguments) a function of z and of the distance $|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|$. Then, the effective potential $\phi^{(\text{eff})}$, for this case, is $\phi^{(\text{eff})} = \phi^{(0)}$, which is associated to the distribution of charges $Q\delta(\mathbf{r} - \mathbf{r}') + \langle \rho'^{(0)}(\mathbf{r}) \rangle$. From Eq. (E1), we can see that $\phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d)$ depends on d and on the distance $|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|$. This means that the equipotential lines are circular lines with the charge Q at the center of the circle. For this case, the effective electric field $\mathbf{E}_{\parallel}^{(\text{eff})}$ is $\mathbf{E}_{\parallel}^{(\text{eff})} = \mathbf{E}_{\parallel}^{(0)}$, with $\mathbf{E}_{\parallel}^{(0)}$ [Eq. (E5)] proportional to the vector $\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}$, depending on d , on the distance $|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|$, and exhibiting the symmetry shown in Eq. (62). On the other hand, if we consider a nonplanar surface, for example, the surface defined by the 2D gaussian function,

$$z = \lambda h(\mathbf{r}_{\parallel}) = \lambda d e^{-k^2(x^2+y^2)}, \quad (66)$$

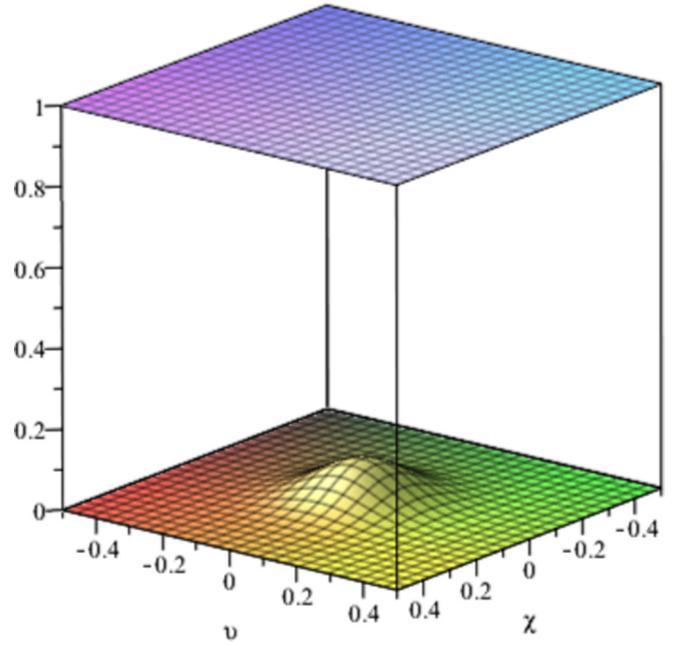


FIG. 2. Illustration of the planar two-dimensional system in the presence of a nonplanar surface described by Eq. (66), with $d = 300$ nm, $k = 2\pi/d$, and $\lambda = 1/10$. The vertical axis exhibits z/d , whereas the other axes represent $\chi = x/d$ and $v = y/d$.

as illustrated in Fig. 2, the mean charge density $\langle \rho'(\mathbf{r}) \rangle$ can be written as

$$\langle \rho'(\mathbf{r}) \rangle \approx \langle \rho'^{(0)}(\mathbf{r}) \rangle + \lambda \langle \rho'^{(1)}(\mathbf{r}) \rangle, \quad (67)$$

where the term $\lambda \langle \rho'^{(1)}(\mathbf{r}) \rangle$ is related to the presence of the surface $z = \lambda h(\mathbf{r}_{\parallel})$. If a charge Q , in the 2D system ($z = d$), is put exactly over the center (peak) of the gaussian, the term $\lambda \langle \rho'^{(1)}(\mathbf{r}) \rangle$ in Eq. (67) is, by the symmetry of this situation, a function of z and of the distance $|\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}|$. For this specific position of Q , the effective electric field $\mathbf{E}_{\parallel}^{(\text{eff})} \approx \mathbf{E}_{\parallel}^{(0)} + \lambda \mathbf{E}_{\parallel}^{(1)}$ is along the line from the charge Q to any other point of the plane $z = d$, since both parts, $\mathbf{E}_{\parallel}^{(0)}$ and $\lambda \mathbf{E}_{\parallel}^{(1)}$, are proportional to $\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}$. However, this is a particular situation. If Q is not over the peak, but displaced along the x axis, as shown in Fig. 3, its expected that external field associated to the charge Q contributes to different average charge densities in the dielectric media for the left and right sides of Q , in the sense that, in general, $\langle \rho'^{(1)}(\mathbf{r}'_{\parallel} - \delta_x \hat{\mathbf{x}} + \delta_y \hat{\mathbf{y}} + z \hat{\mathbf{z}}) \rangle \neq \langle \rho'^{(1)}(\mathbf{r}'_{\parallel} + \delta_x \hat{\mathbf{x}} + \delta_y \hat{\mathbf{y}} + z \hat{\mathbf{z}}) \rangle$. This means that for the potential $\phi^{(\text{eff})} \approx \phi^{(0)} + \lambda \phi^{(1)}$, although for the term $\phi^{(0)}$ the equipotential lines are circular lines with Q at the center, those associated with $\phi^{(1)}$ are not circular lines, so that $\lambda \mathbf{E}_{\parallel}^{(1)}$ (and, as a consequence, $\mathbf{E}_{\parallel}^{(\text{eff})}$) is not proportional to the vector $\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}$, and consequently is not along the line from the charge Q (point A) to the point B in Fig. 3. Inversely, putting the charge Q at the point B, by analogous arguments we expect the behavior of $\mathbf{E}_{\parallel}^{(\text{eff})}$ is as shown in Fig. 4. Comparing $\mathbf{E}_{\parallel}^{(\text{eff})}$ in Figs. 3 and 4, one can visualize the inequality given in Eq. (64). Comparing the behavior of $\mathbf{E}_{\parallel}^{(0)}$ and $\mathbf{E}_{\parallel}^{(1)}$, one can also visualize Eqs. (62) and (63).

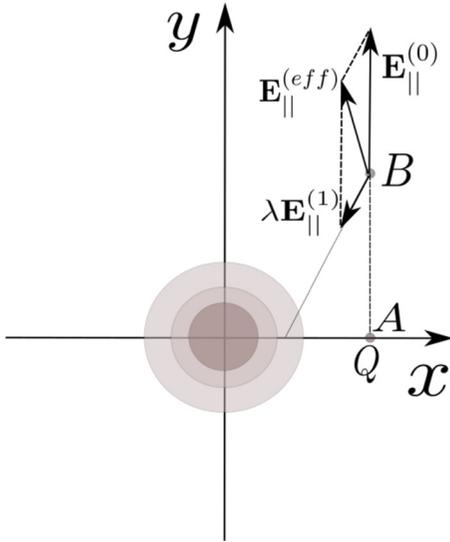


FIG. 3. Front view of the 2D system located on the plane $z = d$ (z axis is perpendicular to the paper). The circular lines are projections on the plane $z = d$ of some contour lines indicating points of equal altitude of the gaussian surface shown in Fig. 2. A charge $Q > 0$ is located at the point A , whereas the electric field is shown at the point B .

III. SOME APPLICATIONS

For simplicity, let us consider the situation with $\epsilon_3 = \epsilon_2$, for which can use Eq. (46), which we write as $W = W^{(0)} + \lambda W^{(1)}$, where $W^{(0)}$ is obtained making $\lambda = 0$ in Eq. (46). For this case, and when $\lambda = 0$ (the interface between ϵ_2 and ϵ_1 is plane), a perpendicular force $\mathbf{F}_{\perp}^{(0)} = -\frac{\partial}{\partial d} W^{(0)} \hat{\mathbf{z}}$ acts on a point charge Q in the two-dimensional system, which will be used as reference in comparison to the external parallel force given in Eq. (47).

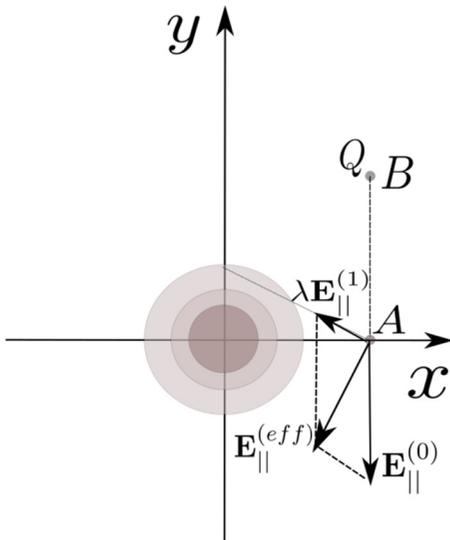


FIG. 4. Front view of the 2D system located on the plane $z = d$ (z axis is perpendicular to the paper). The circular lines are projections on the plane $z = d$ of some contour lines indicating points of equal altitude of the gaussian surface shown in Fig. 2. A charge $Q > 0$ is located at the point B , whereas the electric field is shown at the point A .

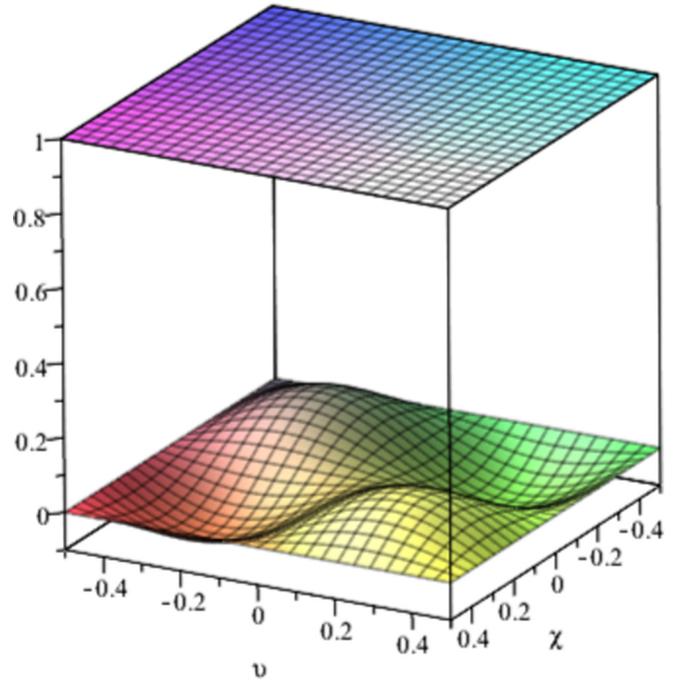


FIG. 5. Illustration of the planar two-dimensional system in the presence of a nonplanar surface described by Eq. (68), with $L_x = L_y = d = 300$ nm and $\lambda = 1/10$. The vertical axis exhibits z/d , whereas the other axes represent $\chi = x/d$ and $\nu = y/d$.

A. 2D sine grating

For the case of a two-dimensional sine grating (see Fig. 5),

$$z = \lambda h(\mathbf{r}_{\parallel}) = \lambda d \sin(k_x x) \sin(k_y y), \quad (68)$$

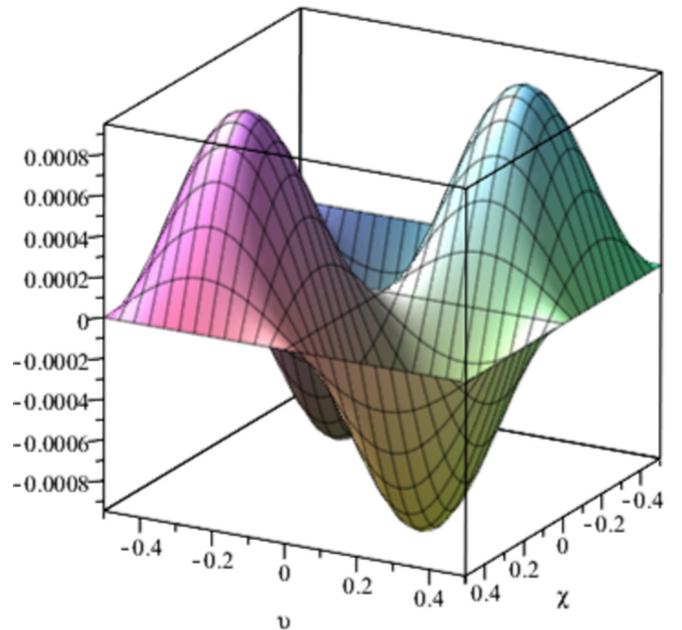


FIG. 6. Ratio $\lambda W^{(1)}/|W^{(0)}|$, for the interface described by Eq. (68), with $L_x = L_y = d = 300$ nm, $\epsilon_1 = 4$, $\epsilon_2 = 1$, and $\lambda = 1/10$. The vertical axis exhibits $\lambda W^{(1)}/|W^{(0)}|$, whereas the other axes represent $\chi = x/d$ and $\nu = y/d$.

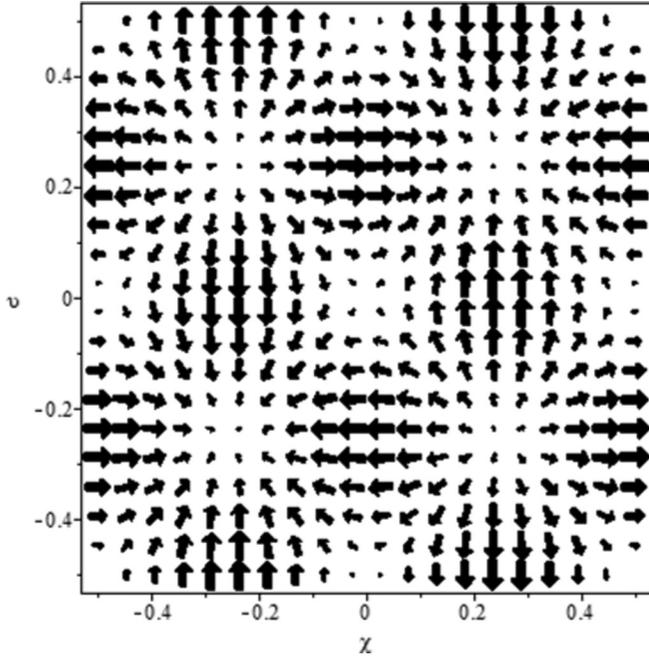


FIG. 7. Effective external force acting on an electron in the two-dimensional system, for the interface described by Eq. (68), with $L_x = L_y = d = 300$ nm, $\epsilon_1 = 4$, $\epsilon_2 = 1$, and $\lambda = 1/10$. The vectors represent exhibits $\mathbf{F}_{\parallel}^{(\text{ext})}/|\mathbf{F}_{\perp}^{(0)}|$, whereas the axes represent $\chi = x/d$ and $\nu = y/d$.

where and hereafter $k_x = 2\pi/L_x$ and $k_y = 2\pi/L_y$, we have

$$W^{(1)} = Q^2 d \frac{\epsilon_{21}^-}{\epsilon_2 (\epsilon_{12}^+)^2} \sin(k_y y') \sin(k_x x') [k_x^2 + k_y^2] \times \int_0^\infty d\tilde{R} \tilde{R} \left\{ \frac{\epsilon_2 \tilde{R}^2 + \epsilon_1 [k_x^2 + k_y^2] d^2}{(\tilde{R}^2 + [k_x^2 + k_y^2] d^2)^3} \right\} J_0(\tilde{R}) \quad (69)$$

(whose behavior can be visualized in Fig. 6), which is related to the charge-polarized matter interaction and with an effective external force parallel to the 2D material, as shown in Fig. 7.

Note that, since $\epsilon_2 < \epsilon_1$, the force points to the next peak of the nonplanar surface.

B. 1D sine grating

For the case of a one-dimensional sine grating (see Fig. 8)

$$z = \lambda h(\mathbf{r}_{\parallel}) = \lambda d \sin(k_y y), \quad (70)$$

with $\epsilon_3 = \epsilon_2$, we have

$$W^{(1)} = Q^2 d \frac{\epsilon_{21}^-}{\epsilon_2 (\epsilon_{12}^+)^2} \sin(k_y y') k_y^2 \quad (71)$$

$$\times \int_0^\infty d\tilde{R} \tilde{R} \left[\frac{\epsilon_2 \tilde{R}^2 + \epsilon_1 k_y^2 d^2}{(\tilde{R}^2 + k_y^2 d^2)^3} \right] J_0(\tilde{R}) \quad (72)$$

(whose behavior can be visualized in Fig. 9), which is related to the interaction charge-polarized matter and the

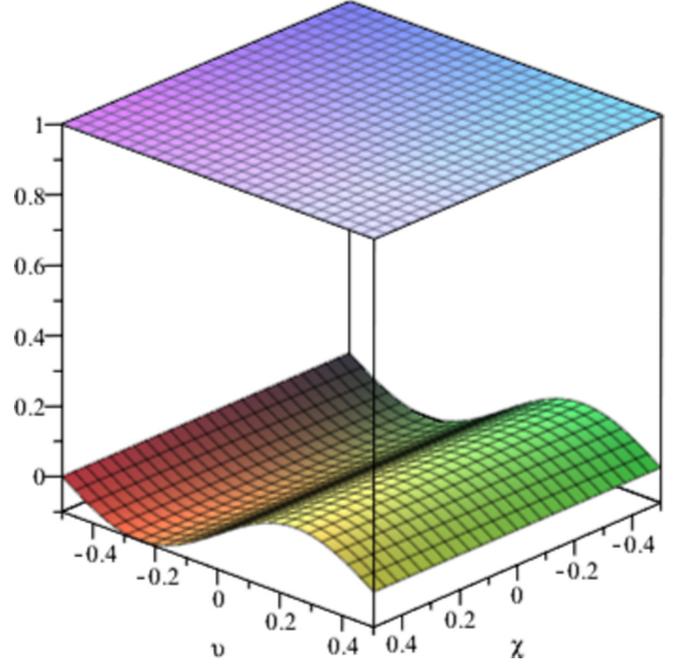


FIG. 8. Illustration of the planar two-dimensional system in the presence of a nonplanar surface described by Eq. (70), with $L_y = d = 300$ nm and $\lambda = 1/10$. The vertical axis exhibits z/d , whereas the other axis represent $\chi = x/d$ and $\nu = y/d$.

effective external force shown in Fig. 10. Note that since $\epsilon_2 > \epsilon_1$, the force points to the next valley of the nonplanar surface.

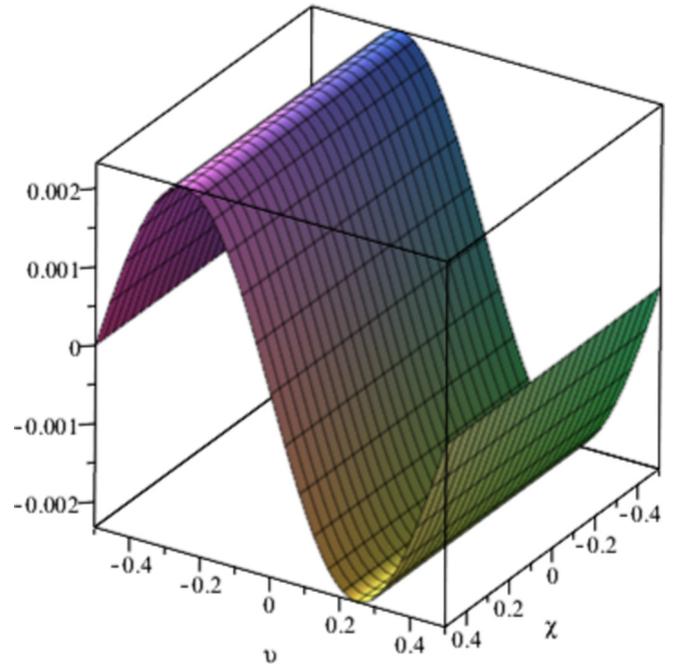


FIG. 9. Ratio $\lambda W^{(1)}/|W^{(0)}|$, for the interface described by Eq. (70), with $L_y = d = 300$ nm, $\epsilon_1 = 1$, $\epsilon_2 = 4$, and $\lambda = 1/10$. The vertical axis exhibits $\lambda W^{(1)}/|W^{(0)}|$, whereas the other axes represent $\chi = x/d$ and $\nu = y/d$.

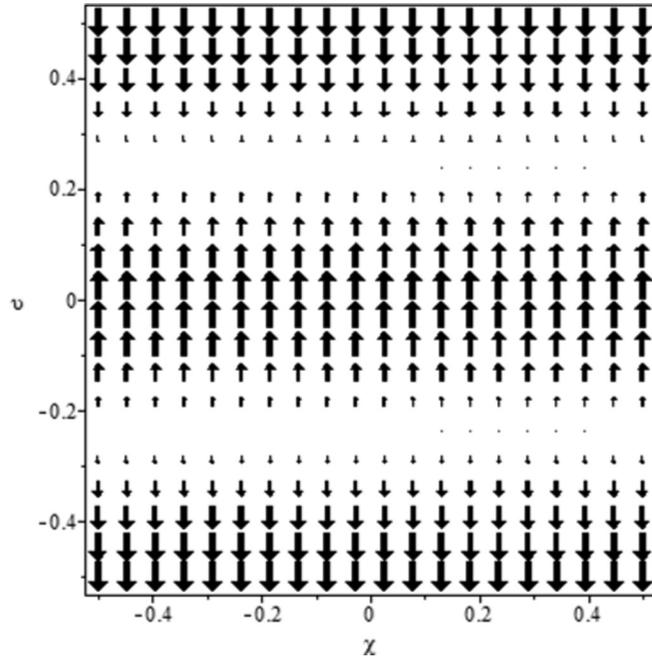


FIG. 10. Effective external force acting on an electron in the two-dimensional system, for the interface described by Eq. (70), with $L_y = d = 300$ nm, $\epsilon_1 = 1$, $\epsilon_2 = 4$, and $\lambda = 1/10$. The vectors represent exhibits $\mathbf{F}_{\parallel}^{(\text{ext})}/|\mathbf{F}_{\perp}^{(0)}|$, whereas the axes represent $\chi = x/d$ and $\nu = y/d$.

The effective potential related to a charge Q is given by Eq. (50), with

$$\phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = \frac{Q}{\pi} \int_0^{\infty} dq \frac{[\epsilon_{21}^- e^{-q2d} + \epsilon_{12}^+]}{2\epsilon_2 \epsilon_{12}^+} \times J_0(q|\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}|), \quad (73)$$

$$\begin{aligned} \phi^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) &= \epsilon_{21}^- \frac{Q}{\pi} \frac{d}{(\epsilon_{12}^+)^2} \int d\tilde{x} \int d\tilde{y} \sin(k_y \tilde{y}) \\ &\times \left\{ \frac{[(\tilde{x} - x)(\tilde{x} - x') + (\tilde{y} - y)(\tilde{y} - y')] + \frac{\epsilon_1}{\epsilon_2} d^2}{(|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|^2 + d^2)^{3/2} (|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|^2 + d^2)^{3/2}} \right\}. \end{aligned} \quad (74)$$

The term $\phi^{(0)}$ depends on the distance $|\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}|$, as expected, whereas the first correction $\phi^{(1)}$ depends on \mathbf{r}'_{\parallel} and \mathbf{r}_{\parallel} , separately.

IV. FINAL COMMENTS AND IMPLICATIONS OF THE RESULTS

Two-dimensional materials, for instance graphene and transition metal dichalcogenide monolayers (TMD), are very important systems in condensed matter physics. The behavior of 2D systems between substrates, or in the presence of other material media (for example, conducting materials), is a relevant problem, since these external media affect, for instance, the Coulomb interaction between electrons in the 2D systems

which, in turn, influences various electronic properties of these systems. In this context, there is a class of papers devoted to the problem of the effective electrical interaction between charges in a 2D system in the presence of planar interfaces, also discussing the consequences of this interaction for the properties of 2D systems [7,10–13]. However, these papers do not include the presence of nonplanar interfaces. This is, however, a relevant problem since there are many situations involving gratings or nonplanar substrates whose physical dimensions often vary on demand or due to intrinsic limitations in the manufacturing process.

In the literature, another class of papers is devoted to the problem of the interaction between an electrical point charge and a nonplanar interface [15,18,20]. These papers do not discuss the charge as part of a 2D system of charges, neither the effect of the nonplanar interface on the effective interaction between these charges and its consequences for the properties of the 2D system.

In the present paper, we combined both classes of problems mentioned above, investigating the problem of the interaction between charges living in a planar 2D system, put between two media with different dielectric constants, and in the presence of a third dielectric medium separated from those by a nonplanar interface.

We found that a physical implication of the nonplanar interface is an effective potential of the electron-electron interaction in the 2D system dependent not only on the distance to the source charge (as it occurs for the effective potentials for the case of planar interfaces found in the literature) but also on the position of the charge itself. Thus the presence of a nonplanar interface implies, for instance, a local renormalization of the Fermi velocity, which, in turn, can lead to a local increase of the optical conductivity. Since the Fermi velocity becomes dependent on the coordinates, this can also lead to a scattering of electrons in the flat 2D conductor.

We showed that another physical implication of the presence of a nonplanar interface is that on each charge in the 2D planar system acts along the plane an effective external force, which depends on the magnitude of the charge but not on its sign, so that positive and negative charges are affected in a same manner. This can contribute, for instance, to a redistribution of electrons in a graphene sheet. As far as we know, no other work highlighted the existence of such effective external field induced by a nonplanar interface near a planar 2D system. Moreover, this effective external field depends on the position and can point to the next peak of the nonplanar interface or to the next valley, depending on the relations between the dielectric constants separated by the nonplanar interface. For example, the electrons in a graphene sheet can be more concentrated near the peaks or near the valleys of the nonplanar interface.

More specifically, in the present paper, we have extended the perturbative method for solving Poisson's equation for a point charge in the presence of a nonplanar conducting interface, proposed by Clinton, Esrick, and Sacks [20], to the problem of a point charge between two media with different dielectric constants and in the presence of a third dielectric medium separated from those by a nonplanar interface. Up to the first order $\lambda G^{(1)}$, we obtained the effective potential, effective electrostatic field, dielectric constant, and showed

the existence of an effective external field acting along the 2D system.

The results for $G^{(1)}$, from Eq. (22) to (24), generalize those found in the literature for the case of a vacuum-conductor situation [20]. The results for a vacuum-dielectric interface, given by Eqs. (27) and (28), also generalize those found in the literature [20]. Moreover, Eqs. (30) and (31) recover the vacuum-conducting result found in the literature [20] which, in turn, is formally identical to Hadamard's theorem for Green's functions [20,25].

In the case where all interfaces are flat, $G = G^{(0)}$ and it coincides with the results found in the literature [7,13]. The effective potential, dielectric constant, and electric field are given, respectively, in Eqs. (51), (54), and (59). The first terms in the right-hand sides of these equations correspond to results found in the literature [7,12,13], whereas the second terms (proportional to λ) correspond to the first order correction from the nonplanar behavior, obtained here.

From Eq. (51), we obtained that the effective potential is affected locally (term $\lambda\phi^{(1)}$) by the presence of the nonplanar interface. This means that, as mentioned above, if a graphene sheet is put on the plane $z/d = 1$ (see Figs. 2, 5, and 8), a local change in the electron-electron interaction caused by the presence of a nonplanar interface implies in a local renormalization of the Fermi velocity, which leads to a local increasing of the optical conductivity in graphene. From Eq. (59), we obtained another physical implication of the presence of a nonplanar interface, namely the effective electric field is affected locally by the presence of the nonplanar interface and does not point along the line from the source charge to the point where the field is considered.

We have shown that on each charge Q in the 2D planar system acts along the plane an effective external force, given by Eq. (47), which depends on the magnitude of the charge (specifically, on Q^2) and whose direction depends on $\epsilon_2 - \epsilon_1$. This force can point to the next peak of the nonplanar interface (if $\epsilon_2 < \epsilon_1$, as illustrated in Fig. 7), or to a valley (if $\epsilon_2 > \epsilon_1$, as illustrated in Fig. 10). The possibility of the effective external force moving the charge in the 2D system to a valley or to a peak, depending on $\epsilon_2 - \epsilon_1$, generalizes the result found in the literature for the case vacuum-conductor, where the charge is always attracted to a position of the plane which is over the next elevated part of the interface [15]. This effective external field, induced by a nonplanar interface, can contribute to the redistribution of the charges in the 2D system, for instance, of electrons in a graphene sheet.

Then, as mentioned above, our results generalize several found in the literature [7,12,13,15,20,25] and can be applied in a wide range of other problems. For instance, in the context of the pseudoquantum electrodynamics (PQED) [8,9], an effective quantum field theory describing 2D systems in the presence of nonplanar interfaces (as illustrated in Fig. 1) needs to be built taking into account an effective static potential which is not a Coulombic potential, but in the one given by Eq. (51). In addition, the effective 2D quantum field theory should take into account the presence of an effective external field [see Eq. (47)] induced by the nonplanar interface. The formulas obtained in the present paper can also be useful, for example, for problems of finding the quantum states of electrons localized at surfaces of materials which

exhibit negative electron affinity, in realistic contexts, since the effects of corrugations on the image potential can be relevant because it is almost impossible to create perfectly planar interfaces [18].

We have obtained the first perturbative correction $G^{(1)}$ in Eq. (5), which is the solution of Eq. (4) in the presence of three dielectric regions, as presented in Eq. (1). On the other hand, the calculations and their subtleties shown in the present paper may be useful for those who want to extend them to find other orders of corrections, enhancing the accuracy of the analytical results, or to investigate systems with a larger number of dielectric regions.

We discussed here the situation where a flat 2D system is in the presence of a nonplanar interface, but, for example, a graphene sheet is not exactly flat. The perturbative analytical calculations presented here can also be used to get insights on the problem where the 2D system is itself nonplanar. Similarly, we expect that the Fermi velocity for a nonplanar 2D material becomes dependent on the coordinates, so that coordinate dependent optical conductivity and scattering of electrons would be present in a nonplanar 2D conductor material, as well as they are present in a planar 2D material in the presence of a nonplanar interface.

Finally, we remark that, beyond the physical implications revealed by the analytical perturbative results presented here, these results can also be used as a reference by researchers who want to develop numerical methods to address these kind of problems with more accurate results, since, when the distance of the 2D system from the nonplanar interface grows up, or when the amplitude of the nonplanar interface becomes smaller, numerical calculations and the analytical perturbative approach presented here must be in agreement.

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APPENDIX A: SOLUTIONS FOR $G^{(0)}(\mathbf{q}, z, \mathbf{r}'_||, d)$ AND $G^{(1)}(\mathbf{q}, z, \mathbf{r}'_||, d)$

When we have a plane interfaces between the media ϵ_3 and ϵ_2 and between ϵ_2 and ϵ_1 , namely,

$$\epsilon(\mathbf{r}) = \epsilon(z) = \epsilon_3\theta[z - d] + \epsilon_2\theta[z]\theta[d - z] + \epsilon_1\theta[-z], \quad (\text{A1})$$

the solution $\phi^{(0)}$ of Eq. (4) [or the solution of Eq. (11)] can be obtained directly via image method or solving directly this

equation. The correspondent Fourier version (noting that only \mathbf{r}_{\parallel} is involved in the Fourier transform) of Eq. (4) for this case is [7]

$$q^2 \epsilon(z) G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) - \frac{\partial}{\partial z} \left[\epsilon(z) \frac{\partial G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d)}{\partial z} \right] = 4\pi \delta(z-d) e^{-iq \cdot \mathbf{r}'_{\parallel}}. \quad (\text{A2})$$

Integrating this equation in z , between $d - \eta$ and $d + \eta$ and sending $\eta \rightarrow 0$ we get

$$\epsilon_3 \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=d+\eta} - \epsilon_2 \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=d-\eta} = -4\pi e^{-iq \cdot \mathbf{r}'_{\parallel}}. \quad (\text{A3})$$

Integrating again, now between $-\eta$ and $+\eta$, we get

$$\epsilon_2 \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, z') \right]_{z=+\eta} \quad (\text{A4})$$

$$- \epsilon_1 \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, z') \right]_{z=-\eta} = 0. \quad (\text{A5})$$

These equations, together with the continuity condition for $G^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, z')$ and the requirement of $\lim_{z \rightarrow \pm\infty} G^{(0)} = 0$, lead to solution in the form shown in Eq. (16), with:

$$G_{\text{III}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = D^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{-qz}, \quad (\text{A6})$$

$$G_{\text{II}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = B^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{qz} + C^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{-qz}, \quad (\text{A7})$$

$$G_{\text{I}}^{(0)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = A^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{qz}, \quad (\text{A8})$$

$$A^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_2 8\pi e^{-iq \cdot \mathbf{r}'_{\parallel}}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A9})$$

$$B^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_{12}^+ 4\pi e^{-iq \cdot \mathbf{r}'_{\parallel}}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A10})$$

$$C^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_{21}^- 4\pi e^{-iq \cdot \mathbf{r}'_{\parallel}}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A11})$$

$$D^{(0)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{[\epsilon_{21}^- e^{-qd} + \epsilon_{12}^+ e^{qd}] 4\pi e^{-iq \cdot \mathbf{r}'_{\parallel}}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-2qd} + \epsilon_{23}^+ \epsilon_{12}^+]}. \quad (\text{A12})$$

Now, let us focus on the solution form $G^{(1)}$. When we have a plane interface between the media ϵ_3 and ϵ_2 , but a nonplanar interface between ϵ_2 and ϵ_1 , as described by Eq. (2), we obtain the solution for ϕ in Eq. (4) via perturbative method, according to Eq. (5). The first correction to $G^{(0)}$, namely $G^{(1)}$, can be obtained by solving Eq. (12) (in coordinate space) or Eq. (15) in Fourier space. The procedures to solve this latter equation are described in Sec. II, with the functions mentioned in Eq. (21) given by

$$G_{\text{III}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = D^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{-qz}, \quad (\text{A13})$$

$$G_{\text{II}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = B^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{qz} + C^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{-qz}, \quad (\text{A14})$$

$$G_{\text{I}}^{(1)}(\mathbf{q}, z, \mathbf{r}'_{\parallel}, d) = A^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) e^{qz}, \quad (\text{A15})$$

$$A^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_{23}^- (T_2 \epsilon_2 + T_1 \epsilon_2 q - T_2 \epsilon_1) e^{-qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]} - \frac{\epsilon_{23}^+ (-T_2 \epsilon_2 + T_1 \epsilon_2 q + T_2 \epsilon_1) e^{qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A16})$$

$$B^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_{23}^- (-T_2 \epsilon_1 + T_2 \epsilon_2 + T_1 \epsilon_1 q) e^{-qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A17})$$

$$C^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_{23}^+ (-T_2 \epsilon_1 + T_2 \epsilon_2 + T_1 \epsilon_1 q) e^{qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A18})$$

$$D^{(1)}(\mathbf{q}, \mathbf{r}'_{\parallel}, d) = \frac{2\epsilon_2 (-T_2 \epsilon_1 + T_2 \epsilon_2 + T_1 \epsilon_1 q) e^{qd}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]}, \quad (\text{A19})$$

with

$$T_1 = -\frac{\epsilon_{12}^-}{\epsilon_2} \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' h(\mathbf{q} - \mathbf{q}') q' G^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, d), \quad (\text{A20})$$

$$T_2 = \int \frac{1}{(2\pi)^2} d^2 \mathbf{q}' h(\mathbf{q} - \mathbf{q}') (\mathbf{q}' \cdot \mathbf{q}) G^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, d), \quad (\text{A21})$$

$$G^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, d) = \frac{\epsilon_2 8\pi e^{-iq' \cdot \mathbf{r}'_{\parallel}}}{q[\epsilon_{23}^- \epsilon_{12}^- e^{-q'd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{q'd}]}. \quad (\text{A22})$$

APPENDIX B: BOUNDARY CONDITIONS

Let us start, considering

$$G(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) = \begin{cases} G_{\text{III}}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d), & d < z \\ G_{\text{II}}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d), & \lambda h(\mathbf{r}_{\parallel}) < z < d. \\ G_{\text{I}}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d), & z < \lambda h(\mathbf{r}_{\parallel}) \end{cases} \quad (\text{B1})$$

Requiring the continuity of the Green function, we have

$$G_{\text{II}}(\mathbf{r}_{\parallel}, \lambda h(\mathbf{r}_{\parallel}), \mathbf{r}'_{\parallel}, d) = G_{\text{I}}(\mathbf{r}_{\parallel}, \lambda h(\mathbf{r}_{\parallel}), \mathbf{r}'_{\parallel}, d), \quad (\text{B2})$$

from which we get

$$\begin{aligned} G_{\text{II}}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d) + \left[\frac{\partial}{\partial z} G_{\text{II}}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0} \lambda h(\mathbf{r}_{\parallel}) + \dots \\ = G_{\text{I}}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d) + \left[\frac{\partial}{\partial z} G_{\text{I}}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0} \lambda h(\mathbf{r}_{\parallel}) + \dots \end{aligned} \quad (\text{B3})$$

Using Eq. (5) in Eq. (B3), we have

$$\begin{aligned} G_{\text{II}}^{(0)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') + \lambda G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') \\ + \left[\frac{\partial}{\partial z} G_{\text{II}}^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') \right]_{z=0} \lambda h(\mathbf{r}_{\parallel}) + \mathcal{O}(\lambda^2) \\ = G_{\text{I}}^{(0)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') + \lambda G_{\text{I}}^{(1)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') \\ + \left[\frac{\partial}{\partial z} G_{\text{I}}^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') \right]_{z=0} \lambda h(\mathbf{r}_{\parallel}) + \mathcal{O}(\lambda^2), \end{aligned} \quad (\text{B4})$$

from which we obtain the two boundary conditions written next. First, for $G^{(0)}$, we have

$$G_{\text{II}}^{(0)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z') = G_{\text{I}}^{(0)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, z'), \quad (\text{B5})$$

whose Fourier version is

$$G_{\text{II}}^{(0)}(\mathbf{q}, 0, \mathbf{r}'_{\parallel}, z') = G_{\text{I}}^{(0)}(\mathbf{q}, 0, \mathbf{r}'_{\parallel}, z'), \quad (\text{B6})$$

which we also write as

$$G^{(0)}(\mathbf{q}, 0^+, \mathbf{r}'_{\parallel}, z') = G^{(0)}(\mathbf{q}, 0^-, \mathbf{r}'_{\parallel}, z'), \quad (\text{B7})$$

used in Eq. (17). Second, for $G^{(1)}$, we obtain

$$\begin{aligned} & G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d) - G_{\text{I}}^{(1)}(\mathbf{r}_{\parallel}, 0, \mathbf{r}'_{\parallel}, d) \\ &= - \left\{ \left[\frac{\partial}{\partial z} G_{\text{II}}^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0} \right. \\ & \quad \left. - \left[\frac{\partial}{\partial z} G_{\text{I}}^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0} \right\} h(\mathbf{r}_{\parallel}), \end{aligned} \quad (\text{B8})$$

which can be written as

$$\begin{aligned} & G^{(1)}(\mathbf{r}_{\parallel}, 0^+, \mathbf{r}'_{\parallel}, d) - G^{(1)}(\mathbf{r}_{\parallel}, 0^-, \mathbf{r}'_{\parallel}, d) \\ &= - \left\{ \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0^+} \right. \\ & \quad \left. - \left[\frac{\partial}{\partial z} G^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) \right]_{z=0^-} \right\} h(\mathbf{r}_{\parallel}), \end{aligned} \quad (\text{B9})$$

whose Fourier version is shown in Eq. (18).

For the region $z = d$, we require the following continuity condition for the Green function:

$$G_{\text{III}}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') = G_{\text{II}}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z'). \quad (\text{B10})$$

Expanding this equation, we have

$$\begin{aligned} & G_{\text{III}}^{(0)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') + \lambda G_{\text{III}}^{(1)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') + \mathcal{O}(\lambda^2) \\ &= G_{\text{II}}^{(0)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') + \lambda G_{\text{II}}^{(1)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') + \mathcal{O}(\lambda^2), \end{aligned} \quad (\text{B11})$$

from which we obtain the other two boundary conditions. For $G^{(0)}$, we get

$$G_{\text{III}}^{(0)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') = G_{\text{II}}^{(0)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z'), \quad (\text{B12})$$

which can be written in the notation

$$G^{(0)}(\mathbf{q}, d^+, \mathbf{r}'_{\parallel}, z') = G^{(0)}(\mathbf{q}, d^-, \mathbf{r}'_{\parallel}, z'). \quad (\text{B13})$$

For $G^{(1)}$, we get

$$G_{\text{III}}^{(1)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z') = G_{\text{II}}^{(1)}(\mathbf{q}, d, \mathbf{r}'_{\parallel}, z'), \quad (\text{B14})$$

which can be written in the notation

$$G^{(1)}(\mathbf{q}, d^+, \mathbf{r}'_{\parallel}, z') = G^{(1)}(\mathbf{q}, d^-, \mathbf{r}'_{\parallel}, z'), \quad (\text{B15})$$

used in Eq. (17).

APPENDIX C: OBTAINING $G^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d)$

The solution for $G^{(1)}$ in terms of x and y can be obtained using Eqs. (6), (21), and (A13)–(A22). This leads to

$$\begin{aligned} G_{\text{III}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') &= \epsilon_{21}^- \frac{1}{4\pi} \int \frac{1}{(2\pi)^2} d^2\mathbf{q} \int \frac{1}{(2\pi)^2} d^2\mathbf{q}' h(\mathbf{q} - \mathbf{q}') \left\{ \nabla_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}, d - z, -\mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, z') \right. \\ & \quad \left. + \frac{\epsilon_1}{\epsilon_2} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, z') \right]_{\tilde{z}=d-z} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}', \tilde{z}, \mathbf{r}'_{\parallel}, z') \right]_{\tilde{z}=0} \right\}, \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} G_{\text{II}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') &= \frac{1}{8\pi} \frac{1}{\epsilon_2} \epsilon_{21}^- \int \frac{1}{(2\pi)^2} d^2\mathbf{q} \int \frac{1}{(2\pi)^2} d^2\mathbf{q}' h(\mathbf{q} - \mathbf{q}') \left\{ \epsilon_{23}^- \left[\nabla_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=z-d} \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, z') \right. \\ & \quad \left. + \frac{\epsilon_1}{\epsilon_2} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=z-d} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}', \tilde{z}, \mathbf{r}'_{\parallel}, z') \right]_{\tilde{z}=0} \right] \\ & \quad \left. + \epsilon_{23}^+ \left[\nabla_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=d-z} \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, z') \right. \\ & \quad \left. + \frac{\epsilon_1}{\epsilon_2} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=d-z} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}', \tilde{z}, \mathbf{r}'_{\parallel}, z') \right]_{\tilde{z}=0} \right] \right\}, \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} G_{\text{I}}^{(1)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, d) &= \frac{1}{8\pi} \frac{\epsilon_{21}^-}{\epsilon_2} \int \frac{1}{(2\pi)^2} d^2\mathbf{q} \int \frac{1}{(2\pi)^2} d^2\mathbf{q}' h(\mathbf{q} - \mathbf{q}') \left\{ \epsilon_{23}^- \left[\nabla_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}, z - d, -\mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, d) \right. \right. \\ & \quad \left. \left. + \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=z-d} \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}', \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right] \right. \\ & \quad \left. + \epsilon_{23}^+ \left[\nabla_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}, z + d, -\mathbf{r}_{\parallel}, d) \cdot \nabla'_{\parallel} G_{\text{I}}^{(0)}(\mathbf{q}', 0, \mathbf{r}'_{\parallel}, d) - \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}, \tilde{z}, -\mathbf{r}_{\parallel}, d) \right]_{\tilde{z}=z+d} \right. \right. \\ & \quad \left. \left. \times \left[\frac{\partial}{\partial \tilde{z}} G_{\text{I}}^{(0)}(\mathbf{q}', \tilde{z}, \mathbf{r}'_{\parallel}, d) \right]_{\tilde{z}=0} \right] \right\}. \end{aligned} \quad (\text{C3})$$

Now, considering the symmetry

$$G_1^{(0)}(\mathbf{r}_{\parallel}, z, \mathbf{r}'_{\parallel}, z') = G_1^{(0)}(-\mathbf{r}_{\parallel}, z, -\mathbf{r}'_{\parallel}, z'), \quad (\text{C4})$$

we get, after manipulations, the formulas (22)–(26).

APPENDIX D: ENERGY OF INTERACTION

When we put together a set of macroscopic (real) charges [described by $\rho(\mathbf{r})$], in the presence of dielectric media, we have to take into account the state of polarization induced in these media [26]. The total work W to assemble the system described by $\rho(\mathbf{r})$ includes the work done on the dielectric media. If the behavior of the media is linear, then we can use the formula [26]

$$W = \frac{1}{2} \int d^3r \rho(\mathbf{r}) \phi(\mathbf{r}). \quad (\text{D1})$$

Let us consider the total potential ϕ divided into two parts:

$$\phi(\mathbf{r}) = \phi_{\rho}(\mathbf{r}) + \phi_{\text{ind}}(\mathbf{r}), \quad (\text{D2})$$

where ϕ_{ρ} is the potential associated with the distribution $\rho(\mathbf{r})$, whereas ϕ_{ind} is the potential produced by the averaged induced charges on the dielectric media. Considering

$$\rho(\mathbf{r}) = Q\delta(\mathbf{r}-\mathbf{r}'), \quad (\text{D3})$$

and using the notation $\phi_{\rho} \rightarrow \phi_Q$, we have

$$W = \frac{1}{2} Q\phi_Q(\mathbf{r}') + \frac{1}{2} Q\phi_{\text{ind}}(\mathbf{r}'). \quad (\text{D4})$$

The term $\frac{1}{2} Q\phi_Q(\mathbf{r}')$ can be seen as the work to build the point charge Q , which is divergent and will be discarded. Then, effectively, we will consider just the second term in the right-hand side of Eq. (D4), which leads to Eq. (35).

APPENDIX E: POTENTIAL AND ELECTRIC FIELDS IN COORDINATE REPRESENTATION

The functions $\phi^{(0)}$ and $\phi^{(1)}$ in Eq. (50) are given explicitly by

$$\phi^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = \frac{Q}{\pi} \int_0^{\infty} dq \frac{[\epsilon_{21}^- e^{-q2d} + \epsilon_{12}^+]}{[\epsilon_{23}^- \epsilon_{12}^- e^{-2qd} + \epsilon_{23}^+ \epsilon_{12}^+]} J_0(q|\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}|), \quad (\text{E1})$$

$$\begin{aligned} \phi^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) &= 4\epsilon_{21}^- \epsilon_{12}^+ \frac{Q}{\pi} \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \int_0^{\infty} d\tilde{q} \frac{\tilde{q}}{[\epsilon_{23}^- \epsilon_{12}^- e^{-\tilde{q}d} + \epsilon_{23}^+ \epsilon_{12}^+ e^{\tilde{q}d}]} \int_0^{\infty} dq \frac{q}{[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]} \\ &\times \left\{ J_1(\tilde{q}|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|) J_1(q|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|) \mathcal{A}^{(1)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) + \frac{\epsilon_1}{\epsilon_2} J_0(\tilde{q}|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|) J_0(q|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|) \right\}, \quad (\text{E2}) \end{aligned}$$

where

$$\mathcal{A}^{(1)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) = \frac{(\tilde{x} - x)}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|} \frac{(\tilde{x} - x')}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|} \quad (\text{E3})$$

$$+ \frac{(\tilde{y} - y)}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|} \frac{(\tilde{y} - y')}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|}. \quad (\text{E4})$$

The fields $\mathbf{E}_{\parallel}^{(0)}$ and $\mathbf{E}_{\parallel}^{(1)}$ are given by

$$\mathbf{E}_{\parallel}^{(0)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) = -Q \frac{1}{\pi} \int_0^{\infty} dq \frac{[\epsilon_{21}^- e^{-q2d} + \epsilon_{12}^+]}{[\epsilon_{23}^- \epsilon_{12}^- e^{-2qd} + \epsilon_{23}^+ \epsilon_{12}^+]} q J_1(q|\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}|) \frac{\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}}{|\mathbf{r}'_{\parallel} - \mathbf{r}_{\parallel}|}, \quad (\text{E5})$$

$$\begin{aligned} \mathbf{E}_{\parallel}^{(1)}(\mathbf{r}_{\parallel}, d, \mathbf{r}'_{\parallel}, d) &= \epsilon_{21}^- \epsilon_{12}^+ \frac{4}{\pi} Q \int d^2 \tilde{\mathbf{r}} h(\tilde{\mathbf{r}}_{\parallel}) \int_0^{\infty} d\tilde{q} \frac{\tilde{q}}{[\epsilon_{23}^- \epsilon_{12}^- e^{-\tilde{q}d} + \epsilon_{23}^+ \epsilon_{12}^+ e^{\tilde{q}d}]} \int_0^{\infty} dq \frac{q}{[\epsilon_{23}^- \epsilon_{12}^- e^{-qd} + \epsilon_{23}^+ \epsilon_{12}^+ e^{qd}]} \\ &\times \left\{ \tilde{q} J_0(\tilde{q}|\beta_1|) J_1(q|\beta_2|) \mathcal{A}^{(1)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) \frac{\beta_1}{|\beta_1|} - J_1(\tilde{q}|\beta_1|) J_1(q|\beta_2|) \mathcal{A}^{(1)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) \frac{\beta_1}{|\beta_1|^2} \right. \\ &\left. + J_1(\tilde{q}|\beta_1|) J_1(q|\beta_2|) \mathcal{A}^{(2)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) \beta_3 - \frac{\epsilon_1}{\epsilon_2} \tilde{q} J_1(\tilde{q}|\beta_1|) J_0(q|\beta_2|) \frac{\beta_1}{|\beta_1|} \right\}, \quad (\text{E6}) \end{aligned}$$

where

$$\beta_1 = \tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}, \quad (\text{E7})$$

$$\beta_2 = \tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}, \quad (\text{E8})$$

$$\beta_3 = -(\tilde{y} - y)\hat{\mathbf{x}} + (\tilde{x} - x)\hat{\mathbf{y}}, \quad (\text{E9})$$

$$\mathcal{A}^{(2)}(\tilde{\mathbf{r}}_{\parallel}, \mathbf{r}_{\parallel}, \mathbf{r}'_{\parallel}) = \frac{(\tilde{x} - x')}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|} \frac{(\tilde{y} - y)}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|^3} - \frac{(\tilde{y} - y')}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}'_{\parallel}|} \frac{(\tilde{x} - x)}{|\tilde{\mathbf{r}}_{\parallel} - \mathbf{r}_{\parallel}|^3}. \quad (\text{E10})$$

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- [1] Y.-R. Jang and B. I. Min, Renormalization constant and effective mass for the two-dimensional electron gas, *Phys. Rev. B* **48**, 1914 (1993).
- [2] L. Zheng and A. H. MacDonald, Correlation in double-layer two-dimensional electron-gas systems: Singvri-Tosi-Land-Sjölander theory at $b=0$, *Phys. Rev. B* **49**, 5522 (1994).
- [3] J. González, F. Guinea, and M. A. H. Vozmediano, Non-Fermi liquid behavior of electrons in the half-filled honeycomb lattice (A renormalization group approach), *Nucl. Phys. B* **424**, 595 (1994).
- [4] M. A. H. Vozmediano and F. Guinea, Effect of Coulomb interactions on the physical observables of graphene, *Phys. Scr.* **2012**, 014015 (2012).
- [5] D. C. Elias, R. V. Gorbachev, A. S. Mayorov, S. V. Morozov, A. A. Zhukov, P. Blake, L. A. Ponomarenko, I. V. Grigorieva, K. S. Novoselov, F. Guinea, and A. K. Geim, Dirac cones reshaped by interaction effects in suspended graphene, *Nat. Phys.* **7**, 701 (2011).
- [6] E. J. G. Santos and E. Kaxiras, Electric-field dependence of the effective dielectric constant in graphene, *Nano Lett.* **B 13**, 898 (2013).
- [7] P. A. D. Gonçalves and N. M. R. Peres, *An Introduction to Graphene Plasmonics* (World Scientific, 2016).
- [8] E. C. Marino, *Quantum Field Theory Approach to Condensed Matter Physics* (Cambridge University Press, Cambridge and New York, 2017).
- [9] E. C. Marino, Quantum electrodynamics of particles on a plane and the Chern-Simons theory, *Nucl. Phys. B* **408**, 551 (1993).
- [10] J. D. L. Silva, A. N. Braga, W. P. Pires, V. Sérgio Alves, D. T. Alves, and E. C. Marino, Inhibition of the Fermi velocity renormalization in a graphene sheet by the presence of a conducting plate, *Nucl. Phys. B* **920**, 221 (2017).
- [11] W. P. Pires, J. D. L. Silva, A. N. Braga, V. Sérgio Alves, D. T. Alves, and E. C. Marino, Cavity effects on the Fermi velocity renormalization in a graphene sheet, *Nucl. Phys. B* **932**, 529 (2018).
- [12] R. E. V. Profumo, M. Polini, R. Asgari, R. Fazio, and A. H. MacDonald, Electron-electron interactions in decoupled graphene layers, *Phys. Rev. B* **82**, 085443 (2010).
- [13] M. I. Katsnelson, Coulomb drag in graphene single layers separated by a thin spacer, *Phys. Rev. B* **84**, 041407 (2011).
- [14] M. Carrega, T. Tudorovskiy, A. Principi, M. I. Katsnelson, and M. Polini, Theory of coulomb drag for massless dirac fermions, *New J. Phys.* **14**, 063033 (2012).
- [15] W. L. Clinton, M. Esrick, H. Ruf, and W. S. Sacks, Image potential for stepped and corrugated surfaces, *Phys. Rev. B* **31**, 722 (1985).
- [16] P. M. Echenique, F. Flores, and R. H. Ritchie, Image potential effects for low and high energy electrons, *Surf. Sci.* **251**, 119 (1991).
- [17] P. M. Echenique, F. Flores, and R. H. Ritchie, Properties of image-potential-induced surface states of insulators, *Phys. Rev. B* **2**, 4239 (1970).
- [18] T. S. Rahman and A. A. Maradudin, Effect of surface roughness on the image potential, *Phys. Rev. B* **21**, 504 (1980).
- [19] H. Sun and G. Shi-Wei, Effective image potential and surface electronic states outside stepped dielectric surfaces, *Phys. Rev. B* **41**, 3145 (1990).
- [20] W. L. Clinton, M. A. Esrick, and W. S. Sacks, Image potential for nonplanar metal surfaces, *Phys. Rev. B* **31**, 7540 (1985).
- [21] D. A. Iranzo, S. Nanot, E. J. C. Dias, I. Epstein, C. Peng, D. K. Efetov, M. B. Lundeberg, R. Parret, J. Osmond, J.-Y. Hong, J. Kong, D. R. Englund, N. M. R. Peres, and F. H. L. Koppens, Probing the ultimate plasmon confinement limits with a van der waals heterostructure, *Science* **360**, 291 (2018).
- [22] B. Amorim, P. A. D. Gonçalves, M. I. Vasilevskiy, and N. M. R. Peres, Impact of graphene on the polarizability of a neighbor nanoparticle: a dyadic green's function study, *Appl. Sci.—Basel* **7**, 1158 (2017).
- [23] E. J. C. Dias, D. A. Iranzo, P. A. D. Gonçalves, Y. Hajati, Y. V. Bludov, A.-P. Jauho, N. A. Mortensen, F. H. L. Koppens, and N. M. R. Peres, Probing nonlocal effects in metals with graphene plasmons, *Phys. Rev. B* **97**, 245405 (2018).
- [24] A. J. Chaves, N. M. R. Peres, D. R. da Costa, and G. A. Farias, Localized surface plasmons in a continuous and flat graphene sheet, *Phys. Rev. B* **97**, 205435 (2018).
- [25] J. Hadamard, *Leçons sur le Calcul des Variations* (Librairie Scientifique A. Hermann et Fils, Paris, 1910).
- [26] J. D. Jackson, *Classical Electrodynamics* (Wiley, 1962).