Geometric quench in the fractional quantum Hall effect: Exact solution in quantum Hall matrix models and comparison with bimetric theory

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We investigate the recently introduced geometric quench protocol for fractional quantum Hall (FQH) states within the framework of exactly solvable quantum Hall matrix models. In the geometric quench protocol, a FQH state is subjected to a sudden change in the ambient geometry, which introduces anisotropy into the system. We formulate this quench in the matrix models and then we solve exactly for the postquench dynamics of the system and the quantum fidelity (Loschmidt echo) of the postquench state. Next, we explain how to define a spin-2 collective variable $\hat{g}_{ab}(t)$ in the matrix models, and we show that for a weak quench (small anisotropy), the dynamics of $\hat{g}_{ab}(t)$ agrees with the dynamics of the intrinsic metric governed by the recently discussed bimetric theory of FQH states. We also find a modification of the bimetric theory such that the predictions of the modified bimetric theory agree with those of the matrix model for arbitrarily strong quenches. Finally, we introduce a class of higher-spin collective variables for the matrix model, which are related to generators of the W_∞ algebra, and we show that the geometric quench induces nontrivial dynamics for these variables.

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I. INTRODUCTION

Topological phenomena in gapped fractional quantum Hall (FQH) states, such as anyonic excitations, robust edge modes, and ground-state degeneracy on closed manifolds, are well-described by Chern-Simons topological quantum field theories [1]. These theories apply in the limit in which the bulk energy gap is sent to infinity and so, by their very nature, they are incapable of describing the dynamics of gapped excitations in FQH states. Nevertheless, FQH states support a bulk gapped collective excitation known as the *magnetoroton* or *Girvin-MacDonald-Platzman* (GMP) mode [2]. For small wave vectors $\mathbf{k}$ the GMP mode is characterized by a definite angular momentum equal to $2\hbar$, i.e., the GMP mode is a "spin-2" mode near $\mathbf{k}=0$. Recently, a new effective "bimetric" field theory was developed [3,4] to describe the gapped dynamics of this spin-2 mode. The fundamental degree of freedom in this theory is a dynamical unimodular metric $\hat{g}_{ab}(\mathbf{x},t)$, and the gapped fluctuations of this metric, which have spin-2, correspond to the dynamics of the GMP mode near $\mathbf{k}=0$. The development of the bimetric theory relied on the extensive body of work on geometry [5–26] and Hall viscosity [14–16,27–36] in quantum Hall states from the past two decades.

Given the existence of interesting gapped excitations in FQH states, it is natural to try to engineer a situation in which the gapped dynamics of FQH states could be observed, either in numerical simulations or in experiments. With this goal in mind, a quantum quench protocol for FQH states, dubbed a "geometric quench," was introduced in Ref. [37]. This geometric quench is designed for the express purpose of exciting the (neutral) gapped excitations in FQH systems and can be summarized briefly as follows. First, we prepare the system in an isotropic FQH ground state $|\psi_0\rangle$ of an isotropic Hamiltonian H_0. Next, we suddenly change the Hamiltonian to incorporate some anisotropy, $H_0 \to H'$. Finally, we evolve the initial state forward in time using the new anisotropic Hamiltonian, $|\psi(t)\rangle = e^{-i\frac{H' t}{\hbar}}|\psi_0\rangle$.

The authors of Ref. [37] investigated this geometric quench in two ways. First, they studied the quench analytically using the aforementioned bimetric theory. Second, they studied the quench numerically using the recently introduced anisotropic Haldane pseudopotentials [38]. For quadropolar anisotropy parametrized by a constant unimodular metric g_{ab}, this quench was shown to excite the gapped spin-2 mode near $\mathbf{k}=0$ (i.e., the small-$\mathbf{k}$ limit of the GMP mode). In addition, the dynamics of this mode in the case of *small* anisotropy was shown to be well-described by bimetric theory. Reference [37] also considered quenches with more complicated anisotropy, and these quenches were shown to excite exotic higher-spin modes, which have a larger excitation gap than the spin-2 mode. The existence of such higher-spin excitations in the FQH effect has been anticipated since early work on infinite-dimensional W_∞ symmetry in FQH states [39–43].

Our goal in this paper is to study the geometric quench protocol in more detail. To do so we consider this quench in the context of exactly solvable *matrix models* of FQH states. The exact solubility of these matrix models allows us to

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make significant analytical progress in studying the geometric quench. We focus most of our discussion on the matrix model for Laughlin states, known as the Chern-Simons matrix model (CSMM). The CSMM was introduced by Polychronakos [44], who proposed it as a concrete regularization of Susskind's noncommutative Chern-Simons theory of the Laughlin states [45], and the CSMM and noncommutative Chern-Simons theory were subsequently studied by many authors [46–55]. We will also explain how our results for the Laughlin states extend to a matrix model for the Blok-Wen series [56] of non-Abelian FQH states. This non-Abelian matrix model was introduced and studied in detail in Refs. [57,58] (this model also appeared in Ref. [47], but was given a different physical interpretation in that reference). In Refs. [59,60], it was shown that the matrix models accurately capture the geometric properties of the FQH states they describe. In particular, the correct value of the guiding center Hall viscosity of these FQH states can be recovered from the matrix model descriptions (see Refs. [14–16] for the concept of Landau orbit versus guiding center contributions to the Hall viscosity). The fact that the CSMM and its non-Abelian generalizations accurately describe the geometric response of FQH states suggest that these models are ideal testing grounds for the geometric quench of Ref. [37].

In this paper, we formulate the geometric quench protocol in the CSMM for the case of quadropolar anisotropy parametrized by a constant unimodular metric g_{ab}. We then solve exactly for the postquench state $|\psi(t)\rangle$ and compute the *quantum fidelity* $|\langle\psi_0|\psi(t)\rangle|^2$ (also known as the *Loschmidt echo*). We also define and compute the exact dynamics of a spin-2 collective variable that naturally emerges in the CSMM. We denote this collective variable by $\hat{g}_{ab}(t)$ because, as we show in the paper, this quantity is the analog in the CSMM of the dynamical metric in bimetric theory. We show that $\hat{g}_{ab}(t)$ undergoes nonlinear oscillations after the quench, with a period set by the gap E_2 for spin-2 excitations in the CSMM. In the case of *small* anisotropy, we show that the dynamics of $\hat{g}_{ab}(t)$ in the CSMM coincides with the postquench dynamics predicted by bimetric theory in Ref. [37]. We also generalize these results to the non-Abelian matrix model of Refs. [57,58]. These results imply that the quantum Hall matrix models can describe the numerical data of Ref. [37] for small anisotropy just as well as bimetric theory.

We then explore the connection between the matrix models and bimetric theory in more detail, and we show that there exists a modified potential energy term for bimetric theory such that the predictions of the matrix models for the geometric quench *exactly* match the predictions of bimetric theory with the alternative potential energy term. Finally, in the last part of the paper, we define a set of higher-spin collective variables for the CSMM and discuss their relation to previous work on higher-spin operators and W_∞ symmetry in the CSMM. We then show that the geometric quench considered in this paper induces nontrivial dynamics for these higher-spin variables.

The CSMM is closely related to the Calogero model of interacting particles in one dimension (see Ref. [44] for the connection). Consequently, there is a relation between the geometric quench in the CSMM and the quench of the harmonic trap frequency in the Calogero model that was considered in Ref. [61]. The main difference between the geometric quench for the CSMM and the work of Ref. [61] is that, in the language of the Calogero model, the geometric quench of the CSMM corresponds to a *simultaneous* quench of both the harmonic trap frequency *and* the mass of the Calogero particles (note that in the Calogero Hamiltonian the mass parameter appears as a coefficient in the kinetic energy term *and* the interaction term). Thus the dynamics induced by the geometric quench in the CSMM is qualitatively distinct from that studied in Ref. [61]. Another similar quench protocol was discussed in Ref. [62], where the harmonic trap frequency was quenched simultaneously with the interaction strength.

This paper is organized as follows. In Sec. II, we review the CSMM and introduce various important variables and notation. In Sec. III, we formulate and solve the geometric quench in the CSMM, and extend those results to the non-Abelian matrix model. In Sec. IV, we give a detailed comparison of the predictions of the CSMM and bimetric theory, and we also discuss the new potential energy term for bimetric theory that we mentioned in the previous paragraph. In Sec. V, we introduce a set of higher-spin collective variables for the CSMM, and we calculate their postquench dynamics. Section VI presents our conclusions. Finally, several important formulas are contained in Appendices A–C.

II. REVIEW OF THE CHERN-SIMONS MATRIX MODEL (CSMM)

A. Physical meaning of the model and summary of notation

In this section, we give a lightning review of the CSMM and its quantization. We also highlight some specific properties of the quantum ground state of the CSMM which we use later in the paper in the solution of the geometric quench. For more details on this model and its physical interpretation, we refer the reader to the original work [44], and to Refs. [59,60] for a recent discussion in the context of geometric response of quantum Hall states (our notation is essentially the same as Ref. [60]).

The degrees of freedom in the CSMM consist of two $N \times N$ Hermitian matrices $X^a(t)$, $a = 1, 2$, a complex length N vector $\varphi(t)$, and an additional $N \times N$ Hermitian matrix $A_0(t)$, which is a U(N) gauge field. All of these degrees of freedom are functions of time t. We denote the matrix elements of the matrix degrees of freedom by $(X^a)^j{}_k$, $j, k = 1, \dots, N$ (and likewise for A_0), and the components of φ by φ^j, $j = 1, \dots, N$.

The physical meaning of the CSMM can be briefly summarized as follows. The starting point for this interpretation is Susskind's noncommutative Chern-Simons theory description of the Laughlin states [45]. In that description, a quantum Hall state is modeled as a fluid on the "noncommutative plane," a deformation of the two-dimensional plane $\mathbb{R}^2$ in which the coordinates x^a are promoted to operators $\hat{x}^a$, which obey a nontrivial commutation relation $[\hat{x}^1, \hat{x}^2] = i\theta$, where θ is a constant with units of length squared. In Susskind's model, θ is quantized as

$$\theta = \ell_B^2 m, m \in \mathbb{Z}, \tag{2.1}$$

where $\ell_B^2 = \frac{\hbar}{eB}$ is the square of the magnetic length.[1] The integer m, which we take to be positive, is related to the filling fraction of the Laughlin state by

$$\nu = \frac{1}{m}. \tag{2.2}$$

This can be seen from the fact that the density of the fluid in Susskind's model is related to θ by

$$\overline{\rho} = \frac{1}{2\pi\theta} = \frac{1}{2\pi\ell_B^2 m}, \tag{2.3}$$

which is exactly the mean density of the $\nu = \frac{1}{m}$ Laughlin state. The physical interpretation of the parameter θ is that $2\pi\theta$ is the area occupied by a single electron in Susskind's model. In Ref. [45], it was argued that due to the finite value of the parameter θ, the noncommutative Chern-Simons theory accurately captures the "granularity" of a fluid composed of discrete particles (which are electrons in this case).

The CSMM can be viewed as a regularization of Susskind's noncommutative Chern-Simons theory. While the latter theory describes a constant density fluid occupying the entire noncommutative plane, the CSMM describes a finite droplet of fluid on the noncommutative plane consisting of N electrons. Indeed, the eigenvalues of the matrices X^a in the CSMM can be interpreted as the coordinates of electrons on the plane. Since the matrices X^a do not commute with each other in the CSMM (i.e., they are not simultaneously diagonalizable), the electrons described by the CSMM still live on the noncommutative plane. For further details on the physical interpretation of the CSMM, we refer the reader to Refs. [44,45,59].

Before moving on, we summarize our notations. When the matrix model is quantized, the matrix elements of X^a and A_0, as well as the components of φ, become operators on a quantum Hilbert space. In what follows, we reserve the symbol "†" to denote Hermitian conjugation of quantum operators. For classical matrix and vector degrees of freedom, we use a superscript "T" to denote a transpose and an overline to denote complex conjugation. We also use the notation $[\cdot,\cdot]_M$ to denote the commutator of classical matrix degrees of freedom ("M" stands for matrix). The notation $[\cdot,\cdot]$ without any subscript will be used for the commutator of quantum operators. Finally, $\mathrm{Tr}\{\cdot\}$ always denotes the trace of classical matrices.

B. CSMM and its quantization

The action for the CSMM has the form[2]

$$S_0 = -\frac{eB}{2}\int_0^T dt\ \mathrm{Tr}\{\epsilon_{ab}X^a\mathcal{D}_0X^b + 2\theta A_0 + \omega\delta_{ab}X^aX^b\} + i\int_0^T dt\ \overline{\varphi}^T\mathcal{D}_0\varphi, \tag{2.4}$$

where the covariant derivatives $\mathcal{D}_0X^b$ and $\mathcal{D}_0\varphi$ are defined as

$$\mathcal{D}_0X^b = \dot{X}^b - i[A_0, X^b]_M, \tag{2.5a}$$
$$\mathcal{D}_0\varphi = \dot{\varphi} - iA_0\varphi, \tag{2.5b}$$

and the dot denotes an ordinary time derivative. Here we work on a time interval $t \in [0, T)$, and we impose periodic boundary conditions in time on all degrees of freedom. This turns the time-direction into a circle of circumference T, which we denote by S^1_T. Just as in Susskind's model, the parameter θ is quantized as $\theta = \ell_B^2 m,\ m \in \mathbb{Z}$ and we again choose $m > 0$. In this case, the CSMM describes the Laughlin state with $\nu = \frac{1}{m}$.

The quantization rule for θ comes from requiring the exponential $e^{i\frac{S_0}{\hbar}}$ of the action to be invariant under large U(N) gauge transformations. The action S_0 is nearly invariant under the U(N) gauge transformation

$$X^a \to VX^a\overline{V}^T, \tag{2.6a}$$
$$A_0 \to VA_0\overline{V}^T + iV\dot{\overline{V}}^T, \tag{2.6b}$$
$$\varphi \to V\varphi, \tag{2.6c}$$

where $V(t)$ is a time-dependent U(N) matrix. However, the term in the Lagrangian proportional to $\mathrm{Tr}\{A_0\}$ spoils this invariance. This is because of the existence of large gauge transformations in which the map $V : S^1_T \to \mathrm{U}(N)$ corresponds to a nontrivial element of the group $\pi_1(\mathrm{U}(N)) = \mathbb{Z}$. Requiring invariance of $e^{i\frac{S_0}{\hbar}}$ under these large gauge transformations then gives the quantization rule for θ.

In the CSMM, the gauge field A_0 enforces the constraint ($\mathbb{I}$ is the $N \times N$ identity matrix)

$$G := ieB[X^1, X^2]_M + eB\theta\mathbb{I} - \varphi\overline{\varphi}^T = 0, \tag{2.7}$$

and in the $A_0 = 0$ gauge the Hamiltonian takes the form

$$H_0 = \frac{eB\omega}{2}\mathrm{Tr}\{\delta_{ab}X^aX^b\}. \tag{2.8}$$

This Hamiltonian represents a harmonic trap for the noncommutative fluid described by the CSMM, and the strength of this trap is set by the frequency ω.

To quantize the model it is convenient to define a set of real scalar variables, which serve to completely specify the matrices X^a. In the quantized model, these variables then become Hermitian operators. To define these real scalar variables, we introduce a basis T^A, $A = 0, \ldots, N^2 - 1$, of generators of the Lie algebra of U(N) in the fundamental representation. Thus T^A are $N \times N$ Hermitian matrices, and we assume they are normalized so that $\mathrm{Tr}\{T^AT^B\} = \delta^{AB}$. A concrete choice for the generators T^A is to choose $T^0 = \frac{\mathbb{I}}{\sqrt{N}}$, and so T^0 is the generator of the U(1) part of U(N). For $A \neq 0$, we choose $T^A = \sqrt{2}t^A$, where t^A are a basis of conventionally normalized generators of SU(N), which satisfy $\mathrm{Tr}\{t^At^B\} = \frac{\delta^{AB}}{2}$ and $[t^A, t^B]_M = i\sum_C f^{ABC}t^C$, where f^{ABC} are the structure constants of SU(N) (we do not need to know the exact form of f^{ABC} in this paper). Using this basis, we then parametrize

[1] We use a convention in which electrons have charge $-e < 0$, and we choose a constant magnetic field with strength $B > 0$ (i.e., pointing in the positive z direction).

[2] We use a summation convention in which we sum over any index which appears once as a subscript and once as a superscript in any expression.

$X^a(t)$ as

$$X^a(t) = \sum_{A=0}^{N^2-1} x_A^a(t) T^A, \tag{2.9}$$

where we have introduced $2N^2$ real scalar variables $x_A^a(t)$. In the quantized CSMM, these scalar variables obey the commutation relations

$$\left[x_A^a, x_B^b\right] = i\ell_B^2 \epsilon^{ab}\delta_{AB}, \tag{2.10}$$

which are very similar to the commutation relations of *guiding center* coordinates in the quantum Hall problem.

Using these new scalar variables, we define the oscillator variables

$$z_A = \frac{1}{\ell_B\sqrt{2}}\left(x_A^1 + ix_A^2\right) \tag{2.11}$$

and $z_A^\dagger = \frac{1}{\ell_B\sqrt{2}}(x_A^1 - ix_A^2)$. We also define

$$b^j = \frac{1}{\sqrt{\hbar}}\varphi^j, \tag{2.12}$$

and $b_j^\dagger = \frac{1}{\sqrt{\hbar}}\overline{\varphi}_j$ (here $\overline{\varphi}_j$ are the components of the row vector $\overline{\varphi}^T$). In the quantized CSMM, these variables all obey the harmonic oscillator commutation relations

$$[z_A, z_B^\dagger] = \delta_{AB}, \tag{2.13a}$$

$$[b^j, b_k^\dagger] = \delta_k^j. \tag{2.13b}$$

For later use, we also define the matrix-valued operators $Z^\pm$ whose matrix elements are given by

$$(Z^-)^j{}_k = \sum_{A=0}^{N^2-1} z_A (T^A)^j{}_k, \tag{2.14a}$$

$$(Z^+)^j{}_k = \sum_{A=0}^{N^2-1} z_A^\dagger (T^A)^j{}_k. \tag{2.14b}$$

The commutation relations of z_A and $z_B^\dagger$ then imply that

$$[(Z^-)^j{}_k, (Z^+)^\ell{}_m] = \delta_m^j \delta_k^\ell. \tag{2.15}$$

If we quantize the CSMM in the $A_0 = 0$ gauge, then gauge invariance requires that all states in the physical Hilbert space of the model be annihilated by the matrix elements $G^j{}_k$ of the constraint G from Eq (2.7). A useful way to think about these constraints is to define a new set of constraints by taking the trace with the U(N) generators T^A, i.e., we define new constraints $G^A := \text{Tr}\{GT^A\}$. Let $|\text{phys}\rangle$ denote a state in the physical Hilbert space of the model. Then the constraints $G^A|\text{phys}\rangle = 0$ for $A \neq 0$ imply that all physical states transform as singlets under the SU(N) part of U(N). The remaining constraint $G^0|\text{phys}\rangle = 0$ can be shown to reduce to

$$b_j^\dagger b^j|\text{phys}\rangle = N(m-1)|\text{phys}\rangle. \tag{2.16}$$

This constraint implies that all physical states carry a total charge of $N(m-1)$ under the U(1) part of U(N). Note also that $G^A|\text{phys}\rangle = 0$ for all A implies that $G^j{}_k|\text{phys}\rangle = 0$ for all j, k, since the $G^j{}_k$ are linear combinations of the G^A.

Let $|0\rangle$ be the Fock vacuum state annihilated by the z_A and b^j operators. Then a complete basis of physical states for the CSMM consists of the states [48]

$$|\{c_1, \ldots, c_N\}\rangle = \text{Tr}\{Z^+\}^{c_1}\text{Tr}\{(Z^+)^2\}^{c_2}\cdots\text{Tr}\{(Z^+)^N\}^{c_N}|\psi_0\rangle, \tag{2.17}$$

where $c_j \in \mathbb{N}$ for $j = 1, \ldots, N$ and

$$|\psi_0\rangle = (\epsilon^{j_1\cdots j_N} b_{j_1}^\dagger [b^\dagger Z^+]_{j_2}\cdots[b^\dagger (Z^+)^{N-1}]_{j_N})^{(m-1)}|0\rangle. \tag{2.18}$$

In the $A_0 = 0$ gauge, the CSMM Hamiltonian can be rewritten in the form

$$H_0 = \hbar\omega\frac{N^2}{2} + \hbar\omega\sum_{A=0}^{N^2-1} z_A^\dagger z_A, \tag{2.19}$$

which is equal to a constant plus a term proportional to the total number operator for the z_A oscillators. From this it is clear that the lowest energy physical state is $|\psi_0\rangle$, with energy

$$E_0 = \hbar\omega\left[\frac{1}{2}mN^2 + \left(\frac{1-m}{2}\right)N\right]. \tag{2.20}$$

The other states $|\{c_1, \ldots, c_N\}\rangle$ can be seen to have an energy of

$$E(\{c_1, \ldots, c_N\}) = E_0 + \hbar\omega\sum_{j=1}^{N} c_j j. \tag{2.21}$$

For later use we also define a dimensionless ground-state energy ϵ_0 by

$$\epsilon_0 := \frac{E_0}{\hbar\omega}. \tag{2.22}$$

We also mention here that the angular momentum operator L_z for the CSMM takes the form

$$L_z = -\frac{eB}{2}\text{Tr}\{\delta_{ab}X^a X^b\}. \tag{2.23}$$

In particular, it is clear that $L_z = -\frac{1}{\omega}H_0$. It follows that the state $|\{c_1, \ldots, c_N\}\rangle$ has angular momentum

$$L_z(\{c_1, \ldots, c_N\}) = -\hbar\epsilon_0 - \hbar\sum_{j=1}^{N} c_j j. \tag{2.24}$$

C. $sl(2, \mathbb{R})$ generators

The matrices X^a can be interpreted as Lagrangian coordinates for a fluid on the noncommutative plane [45]. To investigate the response of this fluid to changes in the geometry, we need to identify the operators which generate area-preserving diffeomorphisms (APDs) of the fluid coordinates. The group SDiff($\mathbb{R}^2$) of APDs of the plane is an infinite-dimensional group whose elements are (smooth, invertible) functions $\eta : \mathbb{R}^2 \to \mathbb{R}^2$, which preserve the volume form vol $= dx^1 \wedge dx^2$ on $\mathbb{R}^2$, i.e., $\eta^*\text{vol} = \text{vol}$, where η^* denotes the pullback along the map η. This group has a finite-dimensional subgroup isomorphic to $SL(2, \mathbb{R})$, which consists of the functions η of the form

$$\eta^a(x) = A^a{}_b x^b, \tag{2.25}$$

where $A^a{}_b$ are the components of a 2×2 real matrix A with determinant 1, i.e., an element of $SL(2, \mathbb{R})$. Note here that $A^a{}_b$ has no x dependence. One can think of this subgroup of SDiff($\mathbb{R}^2$) as being equal to the subset of APDs which are uniform in space.

It was shown in Ref. [59] that the operators, which generate these $SL(2, \mathbb{R})$ transformations for the matrix coordinates X^a, are[3]

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{A=0}^{N^2-1} \{x_A^a, x_A^b\}, \tag{2.26}$$

where $\{\cdot, \cdot\}$ denotes an anticommutator, and one can check that these operators obey

$$[\Lambda^{ab}, \Lambda^{cd}] = \frac{i}{2}(\epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{ac} + \epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc}), \tag{2.27}$$

which are the commutation relations for the Lie algebra $sl(2, \mathbb{R})$.

Finite $SL(2, \mathbb{R})$ transformations of the noncommutative coordinates are implemented by conjugation by a unitary operator $U(\alpha) = e^{i\alpha_{ab}\Lambda^{ab}}$, where α_{ab} is a constant symmetric matrix which parametrizes the deformation. To first order in α_{ab}, we have (for all j, k)

$$U(\alpha)(X^a)^j{}_k U(\alpha)^\dagger = (X^a)^j{}_k + \epsilon^{ab}\alpha_{bc}(X^c)^j{}_k + \cdots . \tag{2.28}$$

Since the operators $U(\alpha)$ act identically on all matrix elements $(X^a)^j{}_k$ of the noncommutative coordinates X^a, the operators Λ^{ab} can indeed be interpreted as generating $SL(2, \mathbb{R})$ transformations of the noncommutative coordinates X^a.

It is convenient to introduce another basis for the generators of $sl(2, \mathbb{R})$, which have the form

$$K_0 = \tfrac{1}{2}(\Lambda^{11} + \Lambda^{22}), \tag{2.29}$$

$$K_- = \tfrac{1}{2}(\Lambda^{11} - \Lambda^{22}) + i\Lambda^{12}, \tag{2.30}$$

$$K_+ = (K_-)^\dagger. \tag{2.31}$$

This basis of generators obeys the algebra

$$[K_0, K_\pm] = \pm K_\pm, \tag{2.32a}$$

$$[K_-, K_+] = 2K_0, \tag{2.32b}$$

and in this form the $sl(2, \mathbb{R})$ algebra is also known as $su(1, 1)$.

One fact which will be useful later in the paper is that K_- annihilates the ground state $|\psi_0\rangle$ of the original CSMM,

$$K_-|\psi_0\rangle = 0, \tag{2.33}$$

and this can be shown using a proof by contradiction. Suppose instead that $K_-|\psi_0\rangle \neq 0$. Then, since K_- is invariant under the U(N) action in the CSMM (this can be seen by writing it as a trace, $K_- = \frac{1}{2}\text{Tr}\{(Z^-)^2\}$), the state $K_-|\psi_0\rangle$ is also a valid state in the physical Hilbert space of the matrix model. In addition, this state has energy $E_0 - 2\hbar\omega$ for the Hamiltonian H_0. Therefore $K_-|\psi_0\rangle$, if different from zero, would be a new physical state of the CSMM with lower energy than the ground state $|\psi_0\rangle$. This is a contradiction since it is already known that $|\psi_0\rangle$ has the lowest eigenvalue of H_0 among all of the physical states of the model. Therefore it must be that $K_-|\psi_0\rangle = 0$. Note that this proof also generalizes to a proof that $|\psi_0\rangle$ is annihilated by all the U(N)-invariant operators $\text{Tr}\{(Z^-)^p\}$ for $p = 1, \ldots, N$, with K_- corresponding to the case of $p = 2$.

[3]Note that in Refs. [59,60], these operators were referred to as "area-preserving deformation" generators. Here we refer to them as $sl(2, \mathbb{R})$ generators to make the connection with the full group of area-preserving diffeomorphisms of $\mathbb{R}^2$ more precise.

D. Introducing anisotropy into the CSMM

We now explain how to introduce anisotropy into the CSMM. One way to do this, following [59], is to deform the harmonic trap by replacing the Kronecker delta δ_{ab} with a constant unimodular metric g_{ab} (i.e., a constant metric with determinant equal to one). The nontrivial metric g_{ab} represents some externally imposed anisotropy in the problem. The action for this modified CSMM takes the form

$$S_g = -\frac{eB}{2}\int_0^T dt\ \text{Tr}\{\epsilon_{ab}X^a\mathcal{D}_0X^b + 2\theta A_0 + \omega g_{ab}X^aX^b\} + i\int_0^T dt\ \overline{\varphi}^T\mathcal{D}_0\varphi, \tag{2.34}$$

in which the only change to the action is the replacement $\delta_{ab} \to g_{ab}$ in the harmonic potential term. In the $A_0 = 0$ gauge, the Hamiltonian for this modified matrix model is

$$H_g = \frac{eB\omega}{2}\text{Tr}\{g_{ab}X^aX^b\}. \tag{2.35}$$

This model was solved in Ref. [59] and we mention here some of the important properties of this model. First, the entire energy spectrum of this model is identical to that of the CSMM with $g_{ab} = \delta_{ab}$ (this statement is only true because g_{ab} has determinant one). In particular, the quantum ground state $|\psi_g\rangle$ of this model has the same energy E_0 as the ground state $|\psi_0\rangle$ of the original CSMM. In addition, the expectation value of the $sl(2, \mathbb{R})$ generators Λ^{ab} in the state $|\psi_g\rangle$ is given by

$$\langle\psi_g|\Lambda^{ab}|\psi_g\rangle = \frac{\epsilon_0}{2}g^{ab}, \tag{2.36}$$

where g^{ab} is the inverse metric for g_{ab}. It was shown in Ref. [59] that, as a consequence of this relation, the Hall viscosity of this modified CSMM with $\theta = \ell_B^2 m$ is equal to the guiding center Hall viscosity of the Laughlin $\nu = \frac{1}{m}$ state with guiding center metric g_{ab} [14,15].

This calculation also suggests a way to define an intrinsic metric $\hat{g}_{ab}$ associated with any state $|\psi\rangle$ in the matrix model. We define $\hat{g}_{ab}$ by first defining its inverse $\hat{g}^{ab}$ to be proportional to the expectation value $\langle\psi|\Lambda^{ab}|\psi\rangle$. In the special case when $|\psi\rangle$ is chosen to be the ground state $|\psi_g\rangle$ of the CSMM with metric g_{ab}, Eq. (2.36) shows that the intrinsic metric $\hat{g}_{ab}$ associated with this state is locked to the externally imposed metric g_{ab}. Later in the paper, we use the intuition provided by this example to define a time-dependent intrinsic metric $\hat{g}_{ab}(t)$ in a time-dependent state $|\psi(t)\rangle$ obtained after performing a geometric quench in the CSMM.

Finally we note that the Hamiltonian H_g can be written in terms of the $sl(2,\mathbb{R})$ generators as

$$H_g = \hbar\omega g_{ab}\Lambda^{ab}. \tag{2.37}$$

In this form, the Hamiltonian of the CSMM resembles the Hamiltonian of the bimetric theory, as we discuss later.

III. GEOMETRIC QUENCH IN THE CSMM AND ITS EXACT SOLUTION

In this section, we formulate the geometric quench protocol in the CSMM, and we also define a time-dependent intrinsic metric in the postquench state. We then present the exact solution for the postquench state and the time-dependent intrinsic metric. We show that our results agree with the results obtained in Ref. [37] using bimetric theory in the limit of small anisotropy (we give a more detailed comparison with the bimetric theory results later in Sec. IV). Finally, at the end of this section, we show how our results for the CSMM can be extended to the case of the non-Abelian matrix model of Refs. [57,58].

A. Geometric quench protocol in the CSMM

The geometric quench of a FQH state was introduced in Ref. [37] and consists of a sudden change in the background geometry in a FQH system. This quench can be formulated in the CSMM as follows. We start with the system in the ground state $|\psi_0\rangle$ of the original CSMM. Then, at time $t=0$, we suddenly introduce anisotropy into the system by replacing $\delta_{ab}\to g_{ab}$ in the harmonic trap term of the CSMM (we still take g_{ab} to be a constant unimodular metric). As result, the initial state $|\psi_0\rangle$ evolves in time under the influence of the Hamiltonian H_g of the CSMM with nontrivial metric g_{ab} from Eq. (2.34). Mathematically, the postquench state at time t is related to the initial state as

$$|\psi(t)\rangle = e^{-i\frac{H_g t}{\hbar}}|\psi_0\rangle. \tag{3.1}$$

One of our main results in this section is an explicit expression for the postquench state $|\psi(t)\rangle$.

Given the postquench state $|\psi(t)\rangle$, we can define a time-dependent intrinsic metric using Eq. (2.36) as a guide. We denote this metric by $\hat{g}_{ab}(t)$ and we define it by first defining its inverse $\hat{g}^{ab}(t)$ as

$$\hat{g}^{ab}(t) := \frac{2}{\epsilon_0}\langle\psi(t)|\Lambda^{ab}|\psi(t)\rangle. \tag{3.2}$$

The normalization factor here can be understood by comparison with Eq. (2.36), and with this normalization $\hat{g}_{ab}(t)$ will also be a unimodular metric (this will be verified by an explicit computation). The "dynamical metric" $\hat{g}_{ab}(t)$ is a spin-2 collective variable, which (partially) characterizes the many-body dynamics of the CSMM.

For easy comparison with Ref. [37], we choose the anisotropy metric g_{ab} to be of the form

$$g = \begin{pmatrix} e^{A} & 0 \\ 0 & e^{-A}\end{pmatrix}, \tag{3.3}$$

where A is a real parameter which determines the anisotropy (g_{ab} are the components of the matrix g). This choice of metric stretches the system along one axis (the x^1 axis for $A>0$), while squashing the system along the other axis. The fact that g_{ab} is diagonal means that there is no additional rotation off of the main coordinate axes. This choice of g_{ab} makes our calculations in this section slightly easier, however, the case of a nondiagonal g_{ab} can be dealt with using the same methods.

To make contact with Ref. [37] we also parametrize the dynamical metric $\hat{g}_{ab}(t)$ using a a real parameter $Q(t)\geqslant 0$ and a real phase $\phi(t)$. In this parametrization, the metric takes the form considered in Ref. [37],

$$\hat{g} = \begin{pmatrix} \cosh(Q)+\cos(\phi)\sinh(Q) & \sin(\phi)\sinh(Q) \\ \sin(\phi)\sinh(Q) & \cosh(Q)-\cos(\phi)\sinh(Q)\end{pmatrix}. \tag{3.4}$$

One can check that this does indeed define a unimodular metric. We now proceed with the exact calculations of $|\psi(t)\rangle$ and $\hat{g}_{ab}(t)$.

B. Postquench state and the quantum fidelity (Loschmidt echo)

To determine the postquench state $|\psi(t)\rangle$ we first note that using expression (2.37), the Hamiltonian H_g for our specific choice of g_{ab} takes the form

$$H_g = \hbar\omega(\sinh(A)K_+ + 2\cosh(A)K_0 + \sinh(A)K_-). \tag{3.5}$$

Then, using the rearrangement identity (A1) from Appendix A, we can rewrite the time-evolution operator $e^{-i\frac{H_g t}{\hbar}}$ as

$$e^{-i\frac{H_g t}{\hbar}} = e^{-\beta(t)K_+}e^{\ln(\delta(t))K_0}e^{-\beta(t)K_-}, \tag{3.6}$$

where $\beta(t)$ and $\delta(t)$ are functions of t, ω, A and are given explicitly in Eqs. (A2) and (A3) of Appendix A. If we now use the fact that $K_-|\psi_0\rangle = 0$, then we find that

$$|\psi(t)\rangle = e^{-\beta(t)K_+}e^{\ln(\delta(t))K_0}|\psi_0\rangle. \tag{3.7}$$

In addition, from the definition of K_0 it is clear that $K_0|\psi_0\rangle = \frac{\epsilon_0}{2}|\psi_0\rangle$ (since $K_0 = \frac{1}{2}\frac{H_0}{\hbar\omega}$), and so our final answer for the time-evolved state is

$$|\psi(t)\rangle = [\delta(t)]^{\frac{\epsilon_0}{2}}e^{-\beta(t)K_+}|\psi_0\rangle. \tag{3.8}$$

We see that the quench excites all even spin excitations, since acting with K_+ changes the angular momentum of a state by $-2\hbar$ [recall that the angular momentum operator L_z for the CSMM has the form shown in Eq. (2.23)]. Indeed, we can write $K_+ = \frac{1}{2}\text{Tr}\{(Z^+)^2\}$ in terms of the matrix-valued operator Z^+, and $\text{Tr}\{(Z^+)^2\}$ is the operator that creates spin-2 excitations over the ground state $|\psi_0\rangle$ of the original CSMM [recall the form of the excited states $|\{c_1,\dots,c_N\}\rangle$ for the original CSMM from Eq. (2.17)].

We close this section by computing the quantity $|\langle\psi_0|\psi(t)\rangle|^2$, which is also known as the *quantum fidelity* or *Loschmidt echo*. The result is [4]

$$\begin{aligned}|\langle\psi_0|\psi(t)\rangle|^2 &= [\delta(t)\overline{\delta(t)}]^{\frac{\epsilon_0}{2}} \\ &= [\cos^2(\omega t)+\cosh^2(A)\sin^2(\omega t)]^{-\epsilon_0} \\ &= [1+\sinh^2(A)\sin^2(\omega t)]^{-\epsilon_0},\end{aligned} \tag{3.9}$$

[4]Here and in the rest of the paper, we assume that $|\psi_0\rangle$ has been properly normalized.

where we plugged in for $\delta(t)$ using the explicit expression from Appendix A. Since $\sin^2(\omega t)$ has a period of $T = \frac{\pi}{\omega}$, we find that the quantum fidelity oscillates at the period

$$T = \frac{\pi}{\omega} \equiv \frac{2\pi\hbar}{E_2}, \tag{3.10}$$

where

$$E_2 := 2\hbar\omega \tag{3.11}$$

is the gap for spin-2 excitations in the CSMM. The actual magnitude of the overlap depends on the filling fraction $\nu = \frac{1}{m}$ through the power of ϵ_0. Note also that since $\sinh^2(A)\sin^2(\omega t) \geqslant 0$, the fidelity satisfies $|\langle\psi_0|\psi(t)\rangle|^2 \leqslant 1$.

Recall that the parameter ϵ_0 has the form $\epsilon_0 = \frac{1}{2}mN^2 + \frac{(1-m)}{2}N$, and so the quantum fidelity $|\langle\psi_0|\psi(t)\rangle|^2$ has a factor of N^2 appearing in the exponent. To eliminate this large factor, it is convenient to compare the values of the quantum fidelity between integer and fractional cases. Let $\mathcal{F}_{\frac{1}{m}}(t)$ denote the fidelity $|\langle\psi_0|\psi(t)\rangle|^2$ for the CSMM with $\theta = \ell_B^2 m$ corresponding to the $\nu = \frac{1}{m}$ Laughlin state. Then we consider the following ratio of $\mathcal{F}_{\frac{1}{m}}(t)$ with $\mathcal{F}_1(t)$ raised to the mth power:

$$\begin{aligned}\frac{[\mathcal{F}_1(t)]^m}{\mathcal{F}_{\frac{1}{m}}(t)} &= [1+\sinh^2(A)\sin^2(\omega t)]^{-(\frac{m-1}{2})N} \\ &= [1+\sinh^2(A)\sin^2(\omega t)]^{-\varsigma N},\end{aligned} \tag{3.12}$$

where

$$\varsigma = \frac{m-1}{2} \tag{3.13}$$

is the *anisospin* [3,4] for the $\nu = \frac{1}{m}$ Laughlin state, also called (minus) the *guiding center spin* [14,15]. For small anisotropy $A \ll 1$, this is ratio is approximately equal to

$$\frac{[\mathcal{F}_1(t)]^m}{\mathcal{F}_{\frac{1}{m}}(t)} \approx 1 - \varsigma N A^2 \sin^2(\omega t). \tag{3.14}$$

We see that by comparing the fidelity for $\nu = \frac{1}{m}$ with the fidelity for $\nu = 1$, we are able to extract the universal data ς which characterizes the $\nu = \frac{1}{m}$ Laughlin state. This type of comparison with the $\nu = 1$ state is very similar to the comparison which is used to extract the dipole moment per unit length at the edge of a FQH state [16,63] (the dipole moment also happens to be proportional to the same parameter ς as it is closely related to the guiding center part of the bulk Hall viscosity).

Finally, for comparison to numerics, it is useful to rewrite Eq. (3.12) in terms of the filling fraction $\nu = \frac{1}{m}$ and the energy gap $E_2 = 2\hbar\omega$ for the spin-2 mode, which gives

$$\frac{[\mathcal{F}_1(t)]^{\nu^{-1}}}{\mathcal{F}_\nu(t)} = \left[1+\sinh^2(A)\sin^2\left(\frac{E_2 t}{2\hbar}\right)\right]^{-\varsigma N} \tag{3.15a}$$

$$\approx 1 - \varsigma N A^2 \sin^2\left(\frac{E_2 t}{2\hbar}\right), \tag{3.15b}$$

where in the second line, we Taylor-expanded the result for small A. In this form, the expression for $\frac{[\mathcal{F}_1(t)]^{\nu^{-1}}}{\mathcal{F}_\nu(t)}$ suggests a way to extract the anisospin ς and the spin-2 gap E_2 for a general FQH state[5] with filling fraction ν by fitting numerical data from the simulation of a geometric quench for that FQH state to Eq. (3.15b). Indeed, preliminary numerical results [64] suggest that the formula (3.15b) is a good fit to the quantum fidelity for the geometric quench considered in Ref. [37]. Note also that for comparison to numerics E_2 is expected to equal the energy gap of the GMP mode at $\mathbf{k} = 0$.

C. Dynamics of the intrinsic metric

In this section, we present the exact calculation of the dynamical metric $\hat{g}_{ab}(t)$. We then show that for small anisotropy A, the CSMM result agrees with the bimetric theory results of Ref. [37]. We give a more detailed comparison with bimetric theory in Sec. IV.

We start by using the form of $|\psi(t)\rangle$ derived in the last section to write the formula for the inverse metric $\hat{g}^{ab}(t)$ in the form

$$\hat{g}^{ab}(t) = \frac{2}{\epsilon_0}[\delta(t)\overline{\delta(t)}]^{\frac{\epsilon_0}{2}}\langle\psi_0|e^{-\overline{\beta(t)}K_-}\Lambda^{ab}e^{-\beta(t)K_+}|\psi_0\rangle. \tag{3.16}$$

We know that the $sl(2,\mathbb{R})$ generators Λ^{ab} can be expressed in terms of the $su(1,1)$ generators $K_0, K_\pm$, and so we choose to proceed with this calculation by first calculating the expectation values $\langle\psi_0|e^{-\overline{\beta(t)}K_-}K_0 e^{-\beta(t)K_+}|\psi_0\rangle$ and $\langle\psi_0|e^{-\overline{\beta(t)}K_-}K_\pm e^{-\beta(t)K_+}|\psi_0\rangle$.

To calculate these expectation values we use a generating function technique. We define a function $f(a,b,c)$ of three variables $a,\ b,\ c$ by

$$f(a,b,c) := \langle\psi_0|e^{aK_-}e^{bK_0}e^{cK_+}|\psi_0\rangle. \tag{3.17}$$

Then the expectation values which we are interested in can be computed from $f(a,b,c)$ as

$$\langle\psi_0|e^{-\overline{\beta(t)}K_-}K_- e^{-\beta(t)K_+}|\psi_0\rangle = \left.\frac{\partial f(a,b,c)}{\partial a}\right|_{a=-\overline{\beta(t)},\ b=0,\ c=-\beta(t)}, \tag{3.18}$$

$$\langle\psi_0|e^{-\overline{\beta(t)}K_-}K_0 e^{-\beta(t)K_+}|\psi_0\rangle = \left.\frac{\partial f(a,b,c)}{\partial b}\right|_{a=-\overline{\beta(t)},\ b=0,\ c=-\beta(t)}, \tag{3.19}$$

$$\langle\psi_0|e^{-\overline{\beta(t)}K_-}K_+ e^{-\beta(t)K_+}|\psi_0\rangle = \left.\frac{\partial f(a,b,c)}{\partial c}\right|_{a=-\overline{\beta(t)},\ b=0,\ c=-\beta(t)}. \tag{3.20}$$

The function $f(a,b,c)$ itself can be calculated using the rearrangement identity Eq. (A5) from Appendix A, combined with the fact that $K_-|\psi_0\rangle = 0$. Using that information, we find that

$$f(a,b,c) = [b'(a,b,c)]^{\frac{\epsilon_0}{2}}, \tag{3.21}$$

where the new function $b'(a,b,c)$ is written down explicitly in Eq. (A7) of Appendix A.

[5] Or at least any FQH state which is well-described by projection into a single Landau level.

The calculation now proceeds in a straightforward manner and we find that the metric $\hat{g}_{ab}(t)$ [which is the inverse of $\hat{g}^{ab}(t)$] can be written in matrix form as

$$\hat{g}(t) = \frac{1}{1-|\beta(t)|^2} \times \begin{pmatrix} (1+\beta(t))(1+\overline{\beta(t)}) & -i(\beta(t)-\overline{\beta(t)}) \\ -i(\beta(t)-\overline{\beta(t)}) & (1-\beta(t))(1-\overline{\beta(t)}) \end{pmatrix}, \tag{3.22}$$

where $\beta(t)$ is again the function defined in Eq. (A2) of Appendix A. We also note here that in order to derive these expressions we needed to use the formula Eq. (A4).

From Eq. (A2), we can see that the parameter $\beta(t)$ oscillates with a period of $T = \frac{\pi}{\omega} = \frac{2\pi\hbar}{E_2}$ [$E_2 = 2\hbar\omega$ was defined in Eq. (3.11)], and its time average is

$$\langle\beta(t)\rangle = \frac{1}{T}\int_0^T dt\ \beta(t) = \tanh\left(\frac{A}{2}\right). \tag{3.23}$$

It is interesting to note that if we replace $\beta(t)$ with $\langle\beta(t)\rangle$ in the metric $\hat{g}_{ab}(t)$, then the dynamical metric reduces to the metric g_{ab} from Eq. (3.3) that we used for the quench Hamiltonian H_g.

We now study the CSMM solution for the dynamical metric in the case of small anisotropy $A \ll 1$, because in this case, we can compare to the results of Ref. [37] obtained using the linearized equations of motion of bimetric theory. To compare our dynamical metric $\hat{g}_{ab}(t)$ with the one obtained in Ref. [37], we write the complex parameter $\beta(t)$ in terms of a real parameter $Q(t)$ and a real phase $\phi(t)$ as

$$\beta(t) = \tanh\left(\frac{Q(t)}{2}\right)e^{i\phi(t)}. \tag{3.24}$$

With this parametrization the dynamical metric $\hat{g}_{ab}(t)$ takes the form shown in Eq. (3.4) and used in Ref. [37]. Note that in this parametrization, one should always choose $Q(t) \geqslant 0$ so that there is no redundancy in the description (all information about the phase of $\beta(t)$ should be packaged in the parameter $\phi(t)$). In this case, we find that $Q(t)$ is related to $\beta(t)$ as

$$Q(t) = 2\tanh^{-1}[\sqrt{|\beta(t)|^2}], \quad \phi(t) = \arg[\beta(t)]. \tag{3.25}$$

For small anisotropy, the parameter $Q(t)$ in the solution for the dynamical metric is expected to be small, and so in this case, we can write

$$\beta(t) \approx \frac{Q(t)}{2}e^{i\phi(t)}. \tag{3.26}$$

On the other hand, for small A, the exact solution for $\beta(t)$ from the CSMM takes the form

$$\beta(t) \approx \frac{A}{1-i\cot(\omega t)} = A\sin(\omega t)e^{-i\omega t + i\frac{\pi}{2}}. \tag{3.27}$$

By comparing these two expressions for $\beta(t)$, we obtain the solution for $Q(t)$ and $\phi(t)$ for the case of small A (we assume $A > 0$),

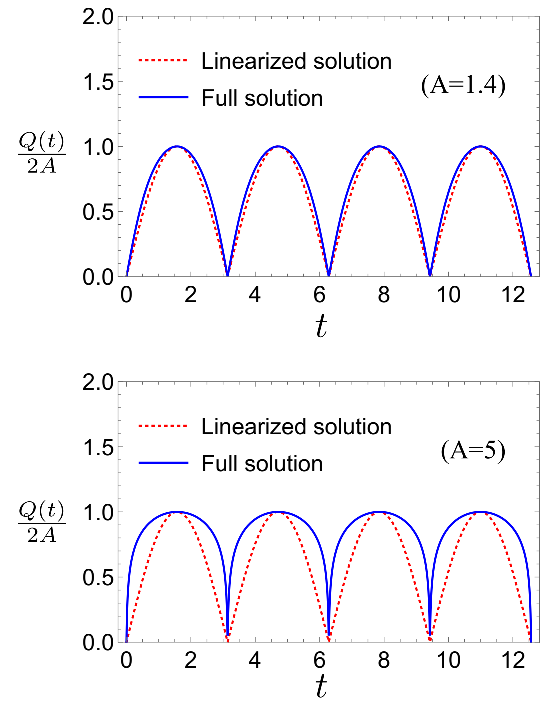

FIG. 1. Plots of the linearized solution for $Q(t)$ (dotted red line) and the full nonlinear solution (blue line) for the case of medium anisotropy $A = 1.4$ and large anisotropy $A = 5$, both normalized by dividing by $2A$. The time axis is plotted in units with $\omega = 1$. The maximum value of $Q(t)/(2A)$ is 1 and occurs (for $\omega = 1$) when $t = \pi/2 + \pi n$ for any integer n. For smaller A, the linearized solution is close to the full solution, while for larger A, the full nonlinear solution has a much rounder profile than the linearized solution.

$$Q(t) = 2A\left|\sin\left(\frac{E_2}{2\hbar}t\right)\right|, \tag{3.28}$$

$$\phi(t) = \pi - \frac{E_2}{2\hbar}t - \frac{\pi}{2}\text{sgn}\left[\sin\left(\frac{E_2}{2\hbar}t\right)\right], \tag{3.29}$$

where $E_2 = 2\hbar\omega$ is the gap for the spin-2 mode in the CSMM. These equations exactly match the predictions of bimetric theory, as these solutions are identical to Eq. (5) of Ref. [37], which is a solution to the linearized equations of motion Eqs. (20) and (21) of Ref. [37] for the geometric quench in bimetric theory.[6]

For the case of arbitrary anisotropy A, the CSMM predicts that the dynamical metric $\hat{g}_{ab}(t)$ undergoes *nonlinear* oscillations, in the sense that the amplitude of the function $\beta(t)$ is a nonlinear function of A. However, these oscillations still have a definite period $T = \frac{2\pi\hbar}{E_2}$ set by the energy gap E_2 of the spin-2 mode in the CSMM, so the period of the oscillations is independent of the amplitude.

[6]For comparison to Ref. [37] note that $\hbar = 1$ in that paper. Also, here we use the convention that $Q(t) \geqslant 0$ to avoid redundancy in the parametrization of $\beta(t)$ in terms of $Q(t)$ and $\phi(t)$. This explains the slight difference between our linearized solution and Eq. (5) of Ref. [37].

For small anisotropy $A \ll 1$, the linearized solution and the full solution are nearly identical. However, the difference between these two solutions can be seen clearly in the case of a quench with large anisotropy (see Fig. 1 for details).

D. Extension to the non-Abelian Blok-Wen states

We close this section with a discussion of how our results extend to the matrix model for the non-Abelian Blok-Wen series of FQH states [57,58] (see also Ref. [60] for the calculation of the Hall viscosity in this matrix model).

The main difference between the CSMM and the non-Abelian matrix model (NAMM) is that instead of having just one complex vector φ, the NAMM has p complex vectors φ_α, $\alpha = 1, \dots, p$, for some positive integer p. The action for the NAMM takes the form

$$S_0^{(\mathrm{NA})} = -\frac{eB}{2}\int_0^T dt\ \mathrm{Tr}\{\epsilon_{ab}X^a\mathcal{D}_0X^b + 2\theta A_0 + \omega\delta_{ab}X^aX^b\} + i\sum_{\alpha=1}^{p}\int_0^T dt\ \overline{\varphi}_\alpha^T\mathcal{D}_0\varphi_\alpha, \quad (3.30)$$

and one can see that this action has an additional SU(p) global symmetry which rotates the different φ_α into each other. In this model, it is also convenient to parametrize θ (which is still quantized to be an integer) as

$$\theta = \ell_B^2(k+p), \quad (3.31)$$

for some other integer k, and we assume that k is chosen so that $k + p \geqslant 0$. The NAMM then describes the subset of the Blok-Wen states at filling

$$\nu = \frac{p}{k+p}, \quad (3.32)$$

and the $\nu = 1/(k+1)$ Laughlin state is recovered from this model upon setting $p = 1$ (so for $p = 1$, set $k + 1 = m$ to compare with our previous results on the CSMM). In addition, it is known [60] that the anisospin ς for these states is independent of p and given by

$$\varsigma = \frac{k}{2}. \quad (3.33)$$

We now give a brief summary of the quantization of this model. First, the b_j variables from earlier acquire an additional index α, so that we now have pN oscillator variables b^j_α and their Hermitian conjugates $b^\dagger_{\alpha,j}$, and these obey $[b^j_\alpha, b^\dagger_{\beta,\ell}] = \delta^j_\ell\delta_{\alpha\beta}$. Next, the constraint enforced by A_0 is now modified to

$$G := ieB[X^1, X^2]_M + eB\theta\mathbb{I} - \sum_{\alpha=1}^{p}\varphi_\alpha\overline{\varphi}_\alpha^T = 0. \quad (3.34)$$

The SU(N) part of this constraint still requires physical states to be SU(N) singlets, but the U(1) part of the constraint now takes the form

$$\sum_{\alpha=1}^{p} b^\dagger_{\alpha,j}b^j_\alpha|\mathrm{phys}\rangle = Nk|\mathrm{phys}\rangle \quad (3.35)$$

for all physical states $|\mathrm{phys}\rangle$. On the other hand, the Hamiltonian H_0 (in the $A_0 = 0$ gauge) and angular momentum operator L_z for the NAMM are identical to those in the CSMM. Thus, anisotropy parametrized by g_{ab} is introduced into the Hamiltonian in the same way as for the CSMM and the geometric quench protocol for the NAMM is exactly the same as for the CSMM.

Finally, we come to the construction of the quantum ground state of the NAMM. Here we consider only the case where N is divisible by p, because in this case the ground state is unique (see Refs. [57,58] for more details and the general case). To construct the ground state, we first construct, for any integer $r \geqslant 0$, the operator

$$\mathcal{B}^\dagger(r)_{j_1\cdots j_p} := \epsilon^{\alpha_1\cdots\alpha_p}[b^\dagger_{\alpha_1}(Z^\dagger)^r]_{j_1}\cdots[b^\dagger_{\alpha_p}(Z^\dagger)^r]_{j_p}. \quad (3.36)$$

This operator is a singlet under the global SU(p) symmetry of the model, but it is not invariant under the SU(N) gauge symmetry. An operator which is invariant under both the SU(p) global symmetry and the SU(N) gauge symmetry can then be constructed from the $\mathcal{B}^\dagger(r)_{j_1\cdots j_p}$ operators as

$$\tilde{\mathcal{B}} := \epsilon^{j_1\cdots j_N}\mathcal{B}^\dagger(0)_{j_1\cdots j_p}\mathcal{B}^\dagger(1)_{j_{p+1}\cdots j_{2p}} \cdots\mathcal{B}^\dagger(N/p-1)_{j_{N-p+1}\cdots j_N}. \quad (3.37)$$

Finally, the unique ground state of the NAMM can be constructed using $\tilde{\mathcal{B}}$ as

$$|\psi_0^{(\mathrm{NA})}\rangle = \tilde{\mathcal{B}}^k|0\rangle. \quad (3.38)$$

In particular, the power of k here ensures that Eq. (3.35) is satisfied. The energy of the ground state is

$$E_0^{(\mathrm{NA})} = \hbar\omega\left[\left(\frac{k+p}{p}\right)\frac{N^2}{2} - \frac{k}{2}N\right], \quad (3.39)$$

and we again define the dimensionless quantity

$$\epsilon_0^{(\mathrm{NA})} := \frac{E_0^{(\mathrm{NA})}}{\hbar\omega}. \quad (3.40)$$

We are now ready to explain how our results generalize to the NAMM. The key point is that one can still construct the $su(1,1)$ generators $K_0, K_\pm$ as before and, crucially, we still have the property that $K_-|\psi_0\rangle = 0$. The proof of this fact is exactly the same as the proof we gave in the CSMM case. This fact implies that our results for the geometric quench in the CSMM carry over to the NAMM with the trivial replacement $\epsilon_0 \to \epsilon_0^{(\mathrm{NA})}$ in all formulas. For the postquench state in the NAMM, we find

$$|\psi^{(\mathrm{NA})}(t)\rangle = [\delta(t)]^{\frac{\epsilon_0^{(\mathrm{NA})}}{2}}e^{-\beta(t)K_+}|\psi_0^{(\mathrm{NA})}\rangle. \quad (3.41)$$

Here we emphasize that even though the NAMM has an excitation spectrum which is much more complicated than the CSMM, the geometric quench still only excites the spin-2 excitations, which are created by K_+. For the quantum fidelity, we find (again, assuming that $|\psi_0^{(\mathrm{NA})}\rangle$ has been properly normalized)

$$|\langle\psi_0^{(\mathrm{NA})}|\psi^{(\mathrm{NA})}(t)\rangle|^2 = [1 + \sinh^2(A)\sin^2(\omega t)]^{-\epsilon_0^{(\mathrm{NA})}}. \quad (3.42)$$

In particular, Eqs. (3.15) still hold in this case, with the appropriate values $\nu = \frac{p}{k+p}$ and $\varsigma = \frac{k}{2}$ for the Blok-Wen states. Finally, we define the dynamical metric in the NAMM as [compare to Eq. (3.2)]

$$\hat{g}^{ab}(t) := \frac{2}{\epsilon_0^{(\mathrm{NA})}}\langle\psi^{(\mathrm{NA})}(t)|\Lambda^{ab}|\psi^{(\mathrm{NA})}(t)\rangle, \tag{3.43}$$

and with this definition we find that $\hat{g}_{ab}(t)$ for the NAMM is identical to the answer found for the CSMM.

We conclude that the geometric quench excites the same dynamics in the Laughlin and Blok-Wen states, despite the fundamentally different topological order. Indeed, both states support the gapped GMP mode, and the geometric quench excites the same dynamics for this mode in both sets of states.

IV. COMPARISON WITH BIMETRIC THEORY

In this section, we present a more detailed comparison between the geometric quench in the CSMM[7] and in bimetric theory. We derive the differential equation obeyed by the dynamical metric $\hat{g}_{ab}(t)$ in the CSMM, and we show that this differential equation is not an exact match to the differential equations obtained within bimetric theory in Ref. [37]. We then suggest an alternative (and simpler) potential energy term for the bimetric theory Lagrangian, and we show that the equations of motion for bimetric theory with this simpler potential energy exactly match the equations of motion for $\hat{g}_{ab}(t)$ in the CSMM.

A. Differential equations obeyed by the dynamical metric

To derive the differential equation obeyed by the dynamical metric in the CSMM, we return to the relation

$$\hat{g}^{ab}(t) = \frac{2}{\epsilon_0}\langle\psi(t)|\Lambda^{ab}|\psi(t)\rangle \tag{4.1}$$

and differentiate with respect to time,

$$\begin{aligned}\dot{\hat{g}}^{ab}(t) &= \frac{2}{\epsilon_0}\frac{i}{\hbar}\langle\psi(t)|[H_g,\Lambda^{ab}]|\psi(t)\rangle\\ &= -i\frac{2\omega}{\epsilon_0}g_{cd}\langle\psi(t)|[\Lambda^{ab},\Lambda^{cd}]|\psi(t)\rangle,\end{aligned} \tag{4.2}$$

where we used $H_g = \hbar\omega g_{ab}\Lambda^{ab}$. Next, we use the commutation relations of the $sl(2,\mathbb{R})$ generators [Eq. (2.27)] to find that $\hat{g}^{ab}(t)$ obeys the *linear* differential equation

$$\begin{aligned}\dot{\hat{g}}^{ab}(t) = \frac{\omega}{2}g_{cd}[&\epsilon^{bc}\hat{g}^{ad}(t) + \epsilon^{bd}\hat{g}^{ac}(t) + \epsilon^{ac}\hat{g}^{bd}(t)\\ &+ \epsilon^{ad}\hat{g}^{bc}(t)].\end{aligned} \tag{4.3}$$

We now choose the anisotropy metric g_{ab} as in Eq. (3.3) and we parametrize $\hat{g}_{ab}(t)$ in terms of two variables $Q(t)$ and $\phi(t)$ as in Eq. (3.4). This means that the inverse metric $\hat{g}^{-1}(t)$ takes the form

$$\hat{g}^{-1}(t) = \begin{pmatrix}\cosh(Q)-\cos(\phi)\sinh(Q) & -\sin(\phi)\sinh(Q)\\ -\sin(\phi)\sinh(Q) & \cosh(Q)+\cos(\phi)\sinh(Q)\end{pmatrix}. \tag{4.4}$$

In this case, the linear differential equation (4.3) for $\hat{g}^{ab}(t)$ reduces to two coupled nonlinear differential equations for ϕ and Q,

$$\dot{Q} = 2\omega\sinh(A)\sin(\phi), \tag{4.5a}$$

$$\begin{aligned}\dot{\phi}\sinh(Q) = 2\omega(&\sinh(A)\cos(\phi)\cosh(Q)\\ &-\cosh(A)\sinh(Q)).\end{aligned} \tag{4.5b}$$

B. Comparison with bimetric equations

We now compare Eq. (4.5) to the predictions of bimetric theory. Here we briefly recall the form of the Lagrangian for bimetric theory (as considered in the quench calculation of Ref. [37]). For more details on bimetric theory, we refer the reader to Refs. [3,4].

The degree of freedom in bimetric theory is a dynamical unimodular metric, which we denote here by $\hat{g}_{ab}(\mathbf{x},t)$, where $\mathbf{x} = (x^1, x^2)$ are coordinates on two-dimensional space, and t is the time.[8] Physically, the field $\hat{g}_{ab}(\mathbf{x},t)$ corresponds to the gapped spin-2 mode, which is equal to the long-wavelength (small $\mathbf{k}$) limit of the gapped GMP mode [2] (recall that the GMP mode has a definite angular momentum equal to $2\hbar$ near $\mathbf{k}=0$). Note also that because of the constraint that $\hat{g}_{ab}(\mathbf{x},t)$ is a *unimodular* metric (i.e., it has determinant equal to one), the bimetric theory of Refs. [3,4] does not contain a spin-0 "dilaton" mode.

In the specific case of the geometric quench problem, in which anisotropy is represented by the constant metric g_{ab} of Eq. (3.3), the dynamical metric in bimetric theory can be taken to be independent of space, $\hat{g}_{ab}(\mathbf{x},t)\to\hat{g}_{ab}(t)$, and the Lagrangian of bimetric theory consists of two terms

$$\mathscr{L} = \mathscr{L}_{\mathrm{top}} + \mathscr{L}_{\mathrm{pot}}. \tag{4.6}$$

The first term $\mathscr{L}_{\mathrm{top}}$ is the topological term in the bimetric theory Lagrangian, and it has the form [here we assume a

[7]For brevity, in the rest of the paper, we mostly refer to our results on the CSMM. However, the reader should keep in mind that we have demonstrated in the previous section that our results on the geometric quench in the CSMM also apply to the non-Abelian matrix model for the Blok-Wen states.

[8]References [3,4,37] use $i,j,k,\ldots$ for spatial indices and $a,b,c,\ldots$ for internal SO(2) indices on frame and coframe fields. Here we depart from their convention and use $a,b=1,2$ for spatial indices in order to match our conventions for the CSMM. No confusion should arise as our discussion here does not require the introduction of frame or coframe fields.

parametrization of $\hat{g}_{ab}(t)$ as in Eq. (3.4)]

$$\mathscr{L}_{\text{top}} = \frac{\varsigma \overline{\rho}}{2}(1 - \cosh(Q))\dot{\phi}, \tag{4.7}$$

where ς is the *anisospin* of the FQH state and $\overline{\rho} = \frac{\nu}{2\pi \ell_B^2}$ is the mean particle density of the state. The potential energy term incorporates the anisotropy metric g_{ab} and takes the form

$$\mathscr{L}_{\text{pot}} = -\frac{m}{2}\left[\frac{1}{2}g^{ab}\hat{g}_{ab} - \gamma\right]^2, \tag{4.8}$$

where $m > 0$ and γ are parameters appearing in the bimetric theory. In particular, the parameter γ allows for the possibility to realize the nematic quantum Hall transition within bimetric theory, and this transition occurs at $\gamma = 1$ (the gapped FQH phase corresponds to $\gamma < 1$).

The differential equations for the geometric quench in bimetric theory, which were obtained in Ref. [37] by varying the Lagrangian $\mathscr{L} = \mathscr{L}_{\text{top}} + \mathscr{L}_{\text{pot}}$, take the form (Eqs. (15) and (16) of Ref. [37])

$$\begin{aligned}\dot{Q} = 2\Omega \sinh(A)\sin(\phi)(&-\gamma - \sinh(A)\sinh(Q)\cos(\phi)\\ &+ \cosh(A)\cosh(Q))\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\dot{\phi}\sinh(Q) = 2\Omega(&\sinh(A)\cos(\phi)\cosh(Q) - \cosh(A)\sinh(Q))\\ &\times(-\gamma - \sinh(A)\sinh(Q)\cos(\phi)\\ &+ \cosh(A)\cosh(Q)),\end{aligned} \tag{4.10}$$

where $\Omega = \frac{m}{\overline{\rho}\varsigma}$. The only difference between these equations and Eqs. (4.5) for the quench in the CSMM is that the constant factor of 2ω in Eqs. (4.5) is replaced by the large factor

$$\begin{aligned}&2\Omega(-\gamma - \sinh(A)\sinh(Q)\cos(\phi) + \cosh(A)\cosh(Q))\\ &\quad = 2\Omega\big(\tfrac{1}{2}g^{ab}\hat{g}_{ab} - \gamma\big),\end{aligned} \tag{4.11}$$

which has explicit dependence on the dynamical fields $Q(t)$ and $\phi(t)$, which parametrize $\hat{g}_{ab}(t)$.

For small anisotropy (small A and, hence, small Q), we have

$$2\Omega\big(\tfrac{1}{2}\hat{g}_{ab}g^{ab} - \gamma\big) \to 2\Omega(1 - \gamma), \tag{4.12}$$

and

$$E_\gamma := 2\Omega(1 - \gamma) \tag{4.13}$$

is interpreted in bimetric theory as the gap of the spin-2 mode at $\mathbf{k} = 0$. On the other hand, we know that $E_2 = 2\omega$ (we set $\hbar = 1$ here to compare with Ref. [37]) is the gap for the spin-2 excitation in the CSMM. Thus it appears that while the CSMM has a constant gap of 2ω for the spin-2 mode, the bimetric theory with potential $\mathscr{L}_{\text{pot}}$ can be interpreted as having a *field-dependent* gap $2\Omega(\frac{1}{2}g^{ab}\hat{g}_{ab} - \gamma)$, and this field-dependent gap only reduces to a constant in the regime of small anisotropy and small fluctuations of the dynamical metric. This field-dependent gap can be thought of as arising from the nontrivial interaction in bimetric theory with the potential $\mathscr{L}_{\text{pot}}$, which is *quadratic* in the dynamical metric and, therefore, *quartic* in the coframe field which is the true degree of freedom in bimetric theory.

These findings suggest that the main difference between the predictions of the CSMM and of bimetric theory stems from the particular choice of potential energy term $\mathscr{L}_{\text{pot}}$ for bimetric theory. This raises the question of whether there exists a different choice of potential energy term, say $\mathscr{L}'_{\text{pot}}$, such that the equations of motion in the bimetric theory with this new potential energy term coincide with the equations derived from the CSMM. We construct such a potential energy term in the next section.

C. A new potential energy term for bimetric theory, and an exact match with the CSMM

In this section, we show that the differential equations for the intrinsic metric $\hat{g}_{ab}(t)$ derived in the CSMM can be reproduced by a variant of the bimetric theory which features a different potential energy term than the one used in Ref. [37]. The modified potential energy term that we consider has the form

$$\mathscr{L}'_{\text{pot}} = -\frac{m'}{2}g^{ab}\hat{g}_{ab}, \tag{4.14}$$

where $m' > 0$ is a new phenomenological parameter with units of (length)$^{-2}$(time)$^{-1}$. This term is chosen to mimic the form of the Hamiltonian $H_g = \hbar\omega g_{ab}\Lambda^{ab}$ in the CSMM. The main difference between $\mathscr{L}_{\text{pot}}$ and $\mathscr{L}'_{\text{pot}}$ is that the latter allows for a single, isotropic phase, whereas the former supports two phases: isotropic and (gapless) nematic phase, which spontaneously breaks rotational symmetry. In terms of Q, ϕ, and A this term takes the form

$$\mathscr{L}'_{\text{pot}} = m'(\sinh(A)\sinh(Q)\cos(\phi) - \cosh(A)\cosh(Q)). \tag{4.15}$$

The equations of motion for the modified bimetric theory with Lagrangian

$$\mathscr{L}' = \mathscr{L}_{\text{top}} + \mathscr{L}'_{\text{pot}} \tag{4.16}$$

exactly match the CSMM equations (4.5) if the parameters of bimetric theory are related to the parameter ω in the CSMM as

$$\omega = \frac{m'}{\overline{\rho}\varsigma}. \tag{4.17}$$

Therefore we find that there exists an alternative potential energy function for the bimetric theory such that the bimetric theory and the CSMM give identical answers for the dynamics of the metric $\hat{g}_{ab}(t)$ after a geometric quench.

Finally, we emphasize that the main qualitative difference between the two potentials considered here is that $\mathscr{L}'_{\text{pot}}$ does not support a nematic transition. This has to be the case since the CSMM describes only the gapped quantum Hall phase. Implementing the nematic transition within the CSMM is presently an open problem.

V. HIGHER-SPIN COLLECTIVE MODES

In this section, we show that in addition to the spin-2 collective mode $\hat{g}^{ab}(t)$ in the CSMM, it is possible to introduce an infinite tower of higher-spin collective modes, $\hat{g}^{abcd\cdots}(t)$. We then show that the higher-spin modes with even spin are excited by the geometric quench and undergo oscillations at frequencies determined by their gap. Higher-spin modes in the FQH effect have quite a long history [2,37,39,45,46,48,65–68]. Despite previous theoretical efforts, the dynamics of these modes is still not well understood.

A. Dynamics of the higher-spin modes

The higher spin collective modes are introduced by generalizing the $K_\pm$ and K_0 operators studied in the previous sections. Specifically, we will consider the single trace operators

$$K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots} = \mathrm{Tr}\{Z^{\alpha_1}Z^{\alpha_2}Z^{\alpha_3}Z^{\alpha_4}\cdots\}\,, \tag{5.1}$$

where each $\alpha_j = \pm$. To connect these operators with $K_\pm$ and K_0, we simply note that

$$K_\pm = \frac{1}{2}K^{\pm\pm}, \tag{5.2a}$$

$$K_0 = \frac{1}{2}K^{+-} + \frac{N^2}{4}. \tag{5.2b}$$

The extra constant factor in the relation between K_0 and $K^{+-} = K^{-+}$ is not important since K_0 and $\frac{1}{2}K^{+-}$ still have identical commutators with any other operator. The total spin of the operator $K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}$ is given by the number of indices equal to "+" minus the number of indices equal to"−". More concretely, acting with $K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}$ on a state will change the angular momentum of that state by $-\hbar\sum_j \alpha_j$. In particular, it is clear that operators with greater than two indices can have spin higher than 2.

For every state $|\chi\rangle$ in the Hilbert space of the CSMM, we can define intrinsic higher-spin collective variables according to

$$\hat{g}_\chi^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots} = \langle\chi|K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}|\chi\rangle\,. \tag{5.3}$$

Our objective is to quantify the dynamics of $\hat{g}_\chi^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}(t)$, for a particular choice of $|\chi\rangle$, namely, the quenched state $|\psi(t)\rangle = e^{-i\frac{H_g t}{\hbar}}|\psi_0\rangle$. We will assume that the quenched Hamiltonian is given by (3.5)[9]. It turns out that finding $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}(t)$ is already quite a formidable task because the operators $K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}$ do not possess simple commutation relations with each other, with the exception of the spin-2 $sl(2,\mathbb{R})$ subalgebra formed by $\{K^{++}, K^{--}, K^{+-}\}$. It is believed that when properly defined the operators $K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}$ should obey a W_∞ algebra. Identifying the "right" basis in the set of $K^{\alpha_1\alpha_2\alpha_3\alpha_4\cdots}$ that leads to the W_∞ algebra is an unsolved problem [54]. We give some further discussion of this issue in Appendix C.

Given these complications, we limit our considerations in this section to the spin-4 collective variable

$$\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t) := \langle\psi(t)|K^{\alpha_1\alpha_2\alpha_3\alpha_4}|\psi(t)\rangle, \tag{5.4}$$

where $|\psi(t)\rangle = e^{-i\frac{H_g t}{\hbar}}|\psi_0\rangle$ is the postquench state. The equation of motion for $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t)$ takes the form

$$\dot{\hat{g}}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t) = \frac{i}{\hbar}\langle\psi(t)|[H_g, K^{\alpha_1\alpha_2\alpha_3\alpha_4}]|\psi(t)\rangle. \tag{5.5}$$

To evaluate the right-hand side of this equation, recall that the quench Hamiltonian H_g can be written in terms of the $su(1,1)$ generators as in Eq. (3.5). Then we can evaluate the commutators $[H_g, K^{\alpha_1\alpha_2\alpha_3\alpha_4}]$ using the following commutation relations:

$$[K_0, (Z^\pm)^j{}_k] = \pm\tfrac{1}{2}(Z^\pm)^j{}_k, \tag{5.6}$$

$$[K_-, (Z^+)^j{}_k] = (Z^-)^j{}_k, \tag{5.7}$$

$$[K_+, (Z^-)^j{}_k] = -(Z^+)^j{}_k, \tag{5.8}$$

which are easily derived from the commutation relations of $Z^\pm$ and the definition of the $su(1,1)$ generators. These commutation relations make it clear that the Hamiltonian H_g mixes the 16 operators $K^{\alpha_1\alpha_2\alpha_3\alpha_4}$ among themselves, but *does not* mix them with any other operators. This is because taking the commutator of $(Z^\pm)^j{}_k$ with any of the $su(1,1)$ generators does not have any effect on the $\mathrm{U}(N)$ indices j and k.

The resulting evolution equations for the 16 variables $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t)$ can be written in a matrix form. To write down this equation we first define a 16-dimensional vector whose components $V^J(t)$, $J = 1,\ldots,16$, are defined in Eq. (B1) of Appendix B. We also define a 16×16 matrix M, which is displayed in Eq. (B2) of Appendix B. Using $V(t)$ and M, the evolution equations for the 16 variables $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t)$ can be written in the concise form

$$\dot{V}(t) = i\omega M V(t). \tag{5.9}$$

Let us pause here to discuss some properties of the matrix M. This matrix is too big to be manipulated by hand, but it can be handled using MATHEMATICA [69]. We find that M has eigenvalues ± 4 with multiplicity one for each sign, ± 2 with multiplicity four for each sign, and 0 with multiplicity six. In addition, one can show that M has sixteen linearly independent eigenvectors.[10] It seems, however, that the eigenvectors of M cannot be chosen to be orthogonal while still remaining eigenvectors of M.

The fact that M possesses a set of 16 linearly independent eigenvectors means that we can decompose M as

$$M = SDS^{-1}, \tag{5.10}$$

where D is a diagonal matrix whose entries are the eigenvalues of M and S is an invertible (but in general not orthogonal) matrix whose columns are the eigenvectors of M. We can

[9] In principle, we could have studied more complicated Hamiltonians which depend on higher-spin operators as well as the spin-2 operators. However, such Hamiltonians appear to lead to very complicated dynamics, which is beyond the scope of the present paper.

[10] MATHEMATICA's "Eigenvectors" command yields 16 eigenvectors for this matrix which are clearly not orthonormal. However, one can check that these eigenvectors are linearly independent by studying the determinant of the matrix whose rows are these eigenvectors. We have checked that this determinant is nonzero for any value of A, and so M really does have a full set of 16 linearly independent eigenvectors.

use this decomposition to solve the differential equation by defining a new vector

$$W(t) = S^{-1}V(t). \tag{5.11}$$

Then one can show that $\dot{W}(t) = i\omega DW(t)$ and so

$$W(t) = e^{i\omega Dt}W(0). \tag{5.12}$$

Since D is diagonal it follows that the components $W^J(t)$ of the new vector $W(t)$ evolve in time by simply being multiplied by a phase $e^{id_J\omega t}$, where d_J are the elements on the diagonal of D (i.e., the eigenvalues of M, which are 0, ±2, and ±4). The components $W^J(t)$ are all linear combinations of the original collective variables $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(t)$, and we can think of them as a new set of collective variables with especially simple time dependence. The presence of the frequency 4ω shows that the quench has indeed excited higher-spin collective variables with angular momentum $\pm4\hbar$.

The reader may wonder about a certain difference between our present study of higher-spin excitations in the CSMM and the previous numerical study of higher-spin excitations in Ref. [37]. In the CSMM we find that the matrix M discussed above has eigenvalues 0, ±2, and ±4, indicating that excitations with angular momentum 0, $\pm2\hbar$, and $\pm4\hbar$ are excited by the quench. On the other hand, in Ref. [37] the authors investigated a quench involving the anisotropic Haldane pseudopotential $\hat{V}_{0,4}$ and found that modes with angular momentum $\pm2\hbar$ were *not* excited, but higher-spin modes were. The difference between these two studies is the following. In Ref. [37], the pseudopotential $\hat{V}_{0,4}$ has an octopolar structure in momentum space [see their Fig. 2(b)], indicating that $\hat{V}_{0,4}$ excites a pure angular momentum $\pm4\hbar$ mode. Therefore, in a quench driven by the introduction of $\hat{V}_{0,4}$, one expects to only see modes with angular momentum that is a multiple of $\pm4\hbar$. On the other hand, the anisotropy that we introduce in the geometric quench in the CSMM, which is parametrized by g_{ab}, has a dipolar structure, and it excites angular momentum ±2 modes. As we can see from Eq. (3.8), in the CSMM, the postquench state $|\psi(t)\rangle$ is a superposition of states with all possible numbers of spin-2 quanta excited, and so this state has nonzero overlap with states of any even angular momentum. This is why the quench that we considered in the CSMM is capable of exciting modes with angular momentum $\pm2\hbar$, $\pm4\hbar$, etc.

One final comment is in order regarding the dynamics of these higher-spin observables. The initial values $W^J(0)$ of the components of $W(t)$ are determined by the initial values $V^J(0)$, which are in turn determined by $\hat{g}^{\alpha_1\alpha_2\alpha_3\alpha_4}(0)$. It follows that if $W^J(0) = 0$ for a particular J, then Eq. (5.12) implies that $W^J(t) = 0$ for all time.

Let us assume that we have ordered the eigenvectors of M in S in such a way that $d_1 = 4$ and $d_2 = -4$. Then the components $W^1(t)$ and $W^2(t)$ evolve in time with the phase factors $e^{i4\omega t}$ and $e^{-i4\omega t}$, respectively. We would like to check that $W^1(0)$ and $W^2(0)$ are not both zero. If they *were* both zero, then we would have $W^1(t) = W^2(t) = 0$ for all time and we could not legitimately claim that the geometric quench had excited the collective variables with spin 4.

We now perform a simple check which gives evidence that $W^1(0)$ and $W^2(0)$ are not zero. Specifically, we will check this for the case $N = 1$ (i.e., the matrix model with one-component matrices). In this case, we just have $Z^- = z_0$, $Z^+ = z_0^\dagger$, and the normalized ground state $|\psi_0\rangle$ takes the form

$$|\psi_0\rangle = \frac{1}{\sqrt{(m-1)!}}(b_1^\dagger)^{m-1}|0\rangle, \tag{5.13}$$

where $b_1^\dagger = \frac{1}{\sqrt{\hbar}}\overline{\varphi}_1$ is proportional to the single component of the row vector $\overline{\varphi}^T$, and $|0\rangle$ is again the Fock vacuum satisfying $z_0|0\rangle = b_1|0\rangle = 0$. We find that in this initial state, the only nonzero components of $V(0)$ are $V^{11}(0) = 1$ and $V^{13}(0) = 2$. We have checked numerically for several values of the anisotropy parameter A that $W^1(0)$ and $W^2(0)$ are not zero in this case. Since we do not expect any sudden changes in the properties of the CSMM when we increase N to values $N > 1$, we believe that this check is good evidence that $W^1(0)$ and $W^2(0)$ are not zero for the CSMM with $N > 1$, and so we expect that the geometric quench really does excite these spin 4 observables in the CSMM.

VI. CONCLUSION

We have investigated the geometric quench protocol for FQH states proposed in Ref. [37] in the context of exactly solvable matrix models of the Laughlin and Blok-Wen FQH states [44,57,58]. We were able to leverage the algebraic properties of these models to solve the quench exactly, and we then compared the exact solution to previous results obtained using the bimetric theory of FQH states. Our exact result for the postquench dynamics of the spin-2 collective variable $\hat{g}_{ab}(t)$ in the matrix models agrees with the results of bimetric theory in the case of small anisotropy, and we also showed how the bimetric theory Lagrangian could be altered so that the matrix models and bimetric theory results match *exactly* for any anisotropy. Beyond the comparison with bimetric theory, we also presented an exact calculation for the quantum fidelity $|\langle\psi_0|\psi(t)\rangle|^2$ after the geometric quench in the matrix models, and the expression that we derived seems to be in good agreement with preliminary results of numerical simulations of the geometric quench [64]. We also initiated an investigation of the dynamics of higher-spin observables in the matrix models, and we showed that the geometric quench leads to a nontrivial dynamics for those observables. Our results here also give further confirmation for the general picture put forward by two of us in Ref. [59], which is that quantum Hall matrix models are capable of describing geometric properties of FQH states which are of current interest.

The major open problem that was partially addressed in the present paper is the dynamics of the higher-spin collective modes. It is clear both from numerical work of Ref. [37] and the present considerations that there are well-defined collective modes of higher angular momentum in FQH states. However, the theoretical description of these modes is plagued by the technical difficulties which we have reviewed in Appendix C. Presently it is not clear what is the fundamental origin of these difficulties. Development of a unified approach to the higher-spin modes in the language of quantum Hall matrix models, effective field theory, and trial quantum Hall states is an important open problem.

It is also important to generalize the matrix model description of FQH states to the paired states of Moore-Read

[70] and Read-Rezayi [71]. These are major candidates for the real-world realization of non-Abelian topological order. Consequently, developing solvable microscopic models that capture both topological and geometric features of these states is an important unsolved problem.

Finally, electrons in a magnetic field support a variety of spatially-ordered phases known as quantum Hall liquid crystals [72]. It would be interesting to implement these phases within the matrix model framework or, more generally, in the framework of noncommutative fluids. This possibility is particularly intriguing since both bimetric theory and general noncommutative scalar field theories [73] support spatially ordered phases.

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APPENDIX A: SOME USEFUL FORMULAS FOR SU(1,1)

In this appendix, we present several "rearrangement" identities for exponentials of the $su(1,1)$ generators $K_\pm$ and K_0. We use these identities in Sec. III of the main text to solve the geometric quench in the CSMM. These identities are essentially the same as those used to manipulate squeezed coherent states of harmonic oscillators (see, for example, Ref. [74]).

The first rearrangement identity is

$$e^{-i\omega t(\sinh(A)K_+ + 2\cosh(A)K_0 + \sinh(A)K_-)} = e^{-\beta(t)K_+}e^{\ln(\delta(t))K_0}e^{-\beta(t)K_-} \tag{A1}$$

with

$$\beta(t) = \frac{\sinh(A)}{\cosh(A) - i\cot(\omega t)}, \tag{A2}$$

$$\delta(t) = \frac{1}{[\cos(\omega t) + i\cosh(A)\sin(\omega t)]^2}. \tag{A3}$$

In addition, in this case, the functions $\beta(t)$ and $\delta(t)$ obey the relation (an overline denotes complex conjugation)

$$\frac{\sqrt{\delta(t)\overline{\delta(t)}}}{1 - |\beta(t)|^2} = 1. \tag{A4}$$

The second rearrangement identity is

$$e^{aK_-}e^{bK_0}e^{cK_+} = e^{a'K_+}e^{\ln(b')K_0}e^{c'K_-}, \tag{A5}$$

where a', b', c' are functions of a, b, c and are given explicitly by

$$a'(a,b,c) = \frac{ce^b}{1 - ace^b}, \tag{A6}$$

$$b'(a,b,c) = \frac{e^b}{(1 - ace^b)^2}, \tag{A7}$$

$$c'(a,b,c) = \frac{ae^b}{1 - ace^b}. \tag{A8}$$

The trick to proving these identities is to explicitly check them in a specific representation of SU(1,1) which is easy to work with. They are then guaranteed to hold in any other representation (since the operators obey the same algebra in any representation). The specific representation we use to check these is the (nonunitary) 2×2 representation in which $K_0 = \frac{1}{2}\sigma^z$ and

$$K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{A9}$$

$$K_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{A10}$$

APPENDIX B: DETAILS OF THE CALCULATIONS FOR SEC. V

Here we give the explicit formulas for the vector $V(t)$ and matrix M used in Sec. V. The components of $V(t)$ are defined as

$$V^1(t) = \hat{g}^{++++}(t), \tag{B1a}$$
$$V^2(t) = \hat{g}^{+++-}(t), \tag{B1b}$$
$$V^3(t) = \hat{g}^{++-+}(t), \tag{B1c}$$
$$V^4(t) = \hat{g}^{++--}(t), \tag{B1d}$$
$$V^5(t) = \hat{g}^{+-++}(t), \tag{B1e}$$
$$V^6(t) = \hat{g}^{+-+-}(t), \tag{B1f}$$
$$V^7(t) = \hat{g}^{+--+}(t), \tag{B1g}$$
$$V^8(t) = \hat{g}^{+---}(t), \tag{B1h}$$
$$V^9(t) = \hat{g}^{-+++}(t), \tag{B1i}$$
$$V^{10}(t) = \hat{g}^{-++-}(t), \tag{B1j}$$
$$V^{11}(t) = \hat{g}^{-+-+}(t), \tag{B1k}$$
$$V^{12}(t) = \hat{g}^{-+--}(t), \tag{B1l}$$
$$V^{13}(t) = \hat{g}^{--++}(t), \tag{B1m}$$
$$V^{14}(t) = \hat{g}^{--+-}(t), \tag{B1n}$$
$$V^{15}(t) = \hat{g}^{---+}(t), \tag{B1o}$$
$$V^{16}(t) = \hat{g}^{----}(t). \tag{B1p}$$

The matrix M has the form

$$M = \begin{pmatrix}
4c_A & s_A & s_A & 0 & s_A & 0 & 0 & 0 & s_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-s_A & 2c_A & 0 & s_A & 0 & s_A & 0 & 0 & 0 & s_A & 0 & 0 & 0 & 0 & 0 & 0 \\
-s_A & 0 & 2c_A & s_A & 0 & 0 & s_A & 0 & 0 & 0 & s_A & 0 & 0 & 0 & 0 & 0 \\
0 & -s_A & -s_A & 0 & 0 & 0 & 0 & s_A & 0 & 0 & 0 & s_A & 0 & 0 & 0 & 0 \\
-s_A & 0 & 0 & 0 & 2c_A & s_A & s_A & 0 & 0 & 0 & 0 & 0 & s_A & 0 & 0 & 0 \\
0 & -s_A & 0 & 0 & -s_A & 0 & 0 & s_A & 0 & 0 & 0 & 0 & 0 & s_A & 0 & 0 \\
0 & 0 & -s_A & 0 & -s_A & 0 & 0 & s_A & 0 & 0 & 0 & 0 & 0 & 0 & s_A & 0 \\
0 & 0 & 0 & -s_A & 0 & -s_A & -s_A & -2c_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_A \\
-s_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_A & s_A & s_A & 0 & s_A & 0 & 0 & 0 \\
0 & -s_A & 0 & 0 & 0 & 0 & 0 & 0 & -s_A & 0 & 0 & s_A & 0 & s_A & 0 & 0 \\
0 & 0 & -s_A & 0 & 0 & 0 & 0 & 0 & -s_A & 0 & 0 & s_A & 0 & 0 & s_A & 0 \\
0 & 0 & 0 & -s_A & 0 & 0 & 0 & 0 & 0 & -s_A & -s_A & -2c_A & 0 & 0 & 0 & s_A \\
0 & 0 & 0 & 0 & -s_A & 0 & 0 & 0 & -s_A & 0 & 0 & 0 & 0 & s_A & s_A & 0 \\
0 & 0 & 0 & 0 & 0 & -s_A & 0 & 0 & 0 & -s_A & 0 & 0 & -s_A & -2c_A & 0 & s_A \\
0 & 0 & 0 & 0 & 0 & 0 & -s_A & 0 & 0 & 0 & -s_A & 0 & -s_A & 0 & -2c_A & s_A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_A & 0 & 0 & 0 & -s_A & 0 & -s_A & -s_A & -4c_A
\end{pmatrix}, \tag{B2}$$

where to save space we used a shorthand notation $s_A := \sinh(A)$ and $c_A := \cosh(A)$.

APPENDIX C: ON THE COMMUTATION RELATIONS FOR THE HIGHER-SPIN OPERATORS IN THE CSMM

In this appendix, we comment on how the operators $K^{\alpha_1\alpha_2\alpha_3\alpha_4}$ that we introduced in Sec. V are related to previous work on the W_∞ algebra in the CSMM [54]. The authors of Ref. [54] considered higher-spin operators $\mathcal{O}_{n,m}$ in the matrix model of the form

$$\mathcal{O}_{n,m} = \mathrm{Tr}\{(Z^+)^{n+1}(Z^-)^{m+1}\}, \tag{C1}$$

for $m, n \geqslant -1$. For reasons that we explain below, they found it necessary to also include operators $\mathcal{P}_{n,m}$ which depend on the vector φ and which are defined as

$$\mathcal{P}_{n,m} = \overline{\varphi}^T (Z^+)^{n+1}(Z^-)^{m+1}\varphi, \tag{C2}$$

where again we always have $n, m \geqslant -1$. For any $n < m$, one can show that $\mathcal{O}_{n,m}$ and $\mathcal{P}_{n.m}$ annihilate the ground state $|\psi_0\rangle$ of the CSMM (the proof is identical to our proof in Sec. II that $K_-|\psi_0\rangle = 0$). This fact, which expresses the incompressibility of the CSMM ground state, is one piece of evidence that these operators generate the W_∞ algebra in the CSMM. However, the algebra obeyed by these operators is not exactly the W_∞ algebra, and the authors of Ref. [54] were unable to identify a set of operators in the CSMM, which obey the W_∞ algebra exactly.

To understand what goes wrong in the algebra of these operators, it is useful to study a specific example. We consider the commutator

$$[\mathcal{O}_{0,2}, \mathcal{O}_{1,1}] = [K^{+---}, K^{++--}]. \tag{C3}$$

When evaluating this commutator one finds many different terms. In some of these terms, the quantum operators and the matrix indices are in the correct order so that the term can be expressed in terms of the original operators $\mathcal{O}_{n,m}$. For example, we find a term proportional to $\mathcal{O}_{1,3} = K^{++----}$. In other terms, the matrix indices are in the correct order so that the term can be expressed as a trace, but the operators Z^+ and Z^- (whose matrix elements do not commute as quantum operators) are in the wrong order for the operator to be identified with one of the $\mathcal{O}_{n,m}$. For example, we find a term proportional to $K^{+---+--}$. Finally, we find a third kind of term in which both the matrix ordering and the quantum ordering prevent one from writing the term in terms of any of the operators we previously defined. For example, we find a term of the form

$$(Z^+)^i{}_j\{(Z^-)^2\}^k{}_i\{(Z^+(Z^-)^2\}^j{}_k, \tag{C4}$$

in which the ordering of the quantum operators clashes with the matrix ordering so that the term cannot be identified with any of the operators $K^{\alpha_1\cdots\alpha_6}$ or $\mathcal{O}_{n,m}$. In Ref. [54], the authors proposed that within the physical Hilbert space of the CSMM the constraint (2.7) could be used to simplify complicated terms like this one which arise in commutators of the $\mathcal{O}_{n,m}$. After using the CSMM constraint one finds that the commutator of two $\mathcal{O}_{n,m}$ operators now contains terms involving the $\mathcal{P}_{n,m}$ operators, and this is why the authors of Ref. [54] introduced the $\mathcal{P}_{n,m}$ operators in the first place. It was conjectured in Ref. [54] that a proper linear combination of $\mathcal{O}_{n,m}$ and $\mathcal{P}_{n,m}$ should satisfy the W_∞ algebra exactly.

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