

Integrability and duality in spin chains

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We construct a two-parametric family of integrable models and reveal their underlying duality symmetry. A modular subgroup of this duality is shown to connect noninteracting modes of different models. We apply this solution and duality to a Richardson-Gaudin model and generate a novel integrable system termed the *s-d* wave Richardson-Gaudin-Kitaev interacting chain, interpolating *s*- and *d*-wave superconductivity. The phase diagram of this interacting model has a topological phase transition that can be connected to the duality, where the occupancy of the noninteracting mode serves as a topological order parameter.

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I. INTRODUCTION

Integrable models (IM) play a crucial role in our understanding of low-dimensional statistical systems and condensed matter physics. The first problem of this kind was solved by Bethe [1], who described the solution for the wave function of the spin-1/2 Heisenberg chain in one dimension in terms of a system of coupled nonlinear equations. This method is now known as the analytic or continuum Bethe ansatz and its many extensions (such as the algebraic Bethe ansatz) have since been used to solve a plethora of low-dimensional spin chains and continuum models. Due to the relatively simple form and the linear scaling of the number of Bethe equations with particle number, numerical calculations are possible for systems containing a large number of particles, going beyond the reach of conventional numerical methods. Furthermore, in many cases the thermodynamic limit can be taken analytically, allowing for the direct observation of many interesting phenomena.

In particular, the 1D-Hubbard model [2] and its large-interaction limit, the spin-1/2 Heisenberg model [1,3,4], were instrumental in understanding the nature of quantum phase transitions in low-D, fractionalized excitations, spin-charge separation, and the importance of topological phenomena. Eventually, accumulation of this knowledge resulted in the *universal* paradigm of Luttinger liquids [5], a low-D counterpart of the concept of a Fermi liquid.

Another broad class of solvable many-body models, the so-called Richardson-Gaudin models [4,6–9], can be obtained as the so-called *quasiclassical* limit of the spin-1/2 model. This class of models embraces the Tavis-Cummings and Dicke models of superradiance [10], Richardson's reduced BCS model of superconductivity [6,7,11,12], central spin model of electrons in quantum dots and nitrogen vacancy centers in diamond [13], Lipkin-Meshkov-Glick models of nuclei [14], and many more. This class of models is popular because of

their relevance for solid-state based quantum computation, quantum decoherence, quantum information and excitation energy transfer.

Related continuum models can be solved using this method, including the one-dimensional Bose gas interacting via a short-range (δ function) contact potential, the so-called Lieb-Liniger (LL) model [15]. Various predictions of the Bethe ansatz solution of this model have now been extensively checked experimentally in cold atomic systems [16–20], and applications of this integrable model currently go far beyond the physics of 1D cold gases [21–27].

Furthermore, integrable models have attracted increasing interest in contemporary field and string theory. This interest emerged in a 2D context [28] and independently in 3D with invariants of knots [29], continuing in 4D with the AdS/CFT correspondence [30] and supersymmetric gauge theories [31], which led to contemporary developments in Ref. [32], where it was shown that integrable lattice models of 2D classical statistical mechanics can be understood in terms of quantum gauge theory in four dimensions. Dualities in various forms were always a close companion of these developments [33]. The recent echo of these ideas into the realm of condensed matter physics [34] may become a powerful tool for better understanding quantum criticality, correlated and topological states of matter [35].

Underlying these various models are solutions to the Yang-Baxter equation (YBE). Given a solution to this equation, it is possible to either construct Richardson-Gaudin spin models, integrable lattice models or a generalized LL model (which is treated in Ref. [36]). In this work, we present a rational solution to the YBE, allowing for the construction of such integrable models, and use it to realize a specific case of a Richardson-Gaudin model. The presented solution to the YBE exhibits a duality symmetry, and we reveal that the symmetry (which is reminiscent of the duality in certain string theories [37] and the fractional Hall effect [38]) is present in this

model as a result. Using this symmetry we reveal a novel integrable model termed s - d wave Richardson-Gaudin-Kitaev interacting chain, and we show how it displays a topological phase and phase transition. The topological phase transition can be directly connected to the duality symmetry, since the ground state of the resulting Hamiltonian can be mapped to two distinct limits of the rational Richardson model. Since the solution of the YBE also reduces to that of the rational Richardson model we can conclude that this duality symmetry connects the following different models: the s - d wave Richardson-Gaudin-Kitaev interacting chain and both limits of the rational Richardson model.

II. INTEGRABILITY AND THE YANG-BAXTER EQUATION

A key ingredient of the integrability of all the above-mentioned models is the fact that any three-body interaction factorizes into two-body interactions, which can be parameterized by R or S matrices (respectively, for spin chains or continuum models). Because the representations of these matrices can be taken the same in both cases, the integrability conditions can be expressed by a single relation on R matrices: the celebrated Yang-Baxter equation (YBE), $R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v)$, which expresses the factorization property of scattering between three particles (labeled by 1,2,3) with respective rapidities u, v, w . The simplest known solution, suggested by Yang [39,40], is the *rational* R matrix given by

$$R_{12}(u, v) = b(u, v)I_{12} + c(u, v)P_{12}, \quad (1)$$

where I_{12} and P_{12} are the identity and permutation operators acting in the direct product of the two Hilbert spaces for particles 1 and 2, with the *rational* functions defined as

$$b(u, v) = \frac{u - v}{u - v + \eta}, \quad c(u, v) = \frac{\eta}{u - v + \eta}, \quad (2)$$

depending on a *single* free parameter $\eta \in \mathbb{C}$ and satisfying $b(u, v) + c(u, v) = 1$. This R matrix then gives rise to the XXX Heisenberg model, the Lieb-Liniger model, and the rational class of Richardson-Gaudin models in the quasiclassical limit $\eta \rightarrow 0$ [41].

III. GENERALIZATION OF THE KNOWN SOLUTION

In this paper we are interested in a *more general solution* for the R matrix of the rational form. For that we parametrize it as [42]

$$R_{12}(u, v) = I_{12} + F(u, v)P_{12}. \quad (3)$$

Substituting this ansatz into the YBE leads to the following functional equation for $F(u, v)$:

$$F(u, v)F(u, w) + F(u, w)F(v, w) = F(u, v)F(v, w). \quad (4)$$

Manifestly, the well-known form of Eq. (2) for $F(u, v) \equiv c(u, v)/b(u, v) = \eta/(u - v)$ is a direct solution of this equation. The *first* important observation is that a *general two parameter solution* to Eq. (4) in the class of *rational* functions can be found:

$$F(u, v) = \frac{c_0^2 + c_0 c_1(u + v) + c_1^2 uv}{u - v}. \quad (5)$$

This solution is parametrized by two free couplings c_0 and c_1 and could generate new integrable many-body 1D systems with possibly as many physical applications as the Lieb-Liniger or spin models before.

This solution was obtained by considering the following rational *ansatz* for the function $F(u, v)$:

$$F(u, v) = \frac{1}{u - v} \sum_{p, q=0}^N c_{p, q} u^p v^q, \quad (6)$$

and imposing the condition Eq. (4). The coefficients $c_{p, q}$ can then be determined by solving for each order separately (see Supplemental Material Ref. [41]). Here, the order N of the polynomial is assumed to be finite [43]. Higher order poles were also considered (up to third order), but these ansätze all reduced to the above mentioned result.

IV. $SL(2)$ DUALITY OF INTEGRABLE MODELS

The *second* important observation is an intrinsic $SL(2)$ duality symmetry associated with this solution. Namely, one can notice that if both rapidities u and v of the matrix Eq. (3) are simultaneously transformed according to the fractional-linear conformal transformation $k = (u, v)$,

$$\tilde{k}_i = \frac{\alpha k_i + \beta}{\gamma k_i + \delta}, \quad \alpha\delta - \gamma\beta = 1, \quad (7)$$

the solution Eq. (5) remains the same *iff* the couplings c_0 and c_1 are simultaneously transformed as

$$\begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{pmatrix} = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}. \quad (8)$$

Here, the unimodularity condition $\alpha\delta - \gamma\beta = 1$ is essential. The rapidities $k = (u, v)$ are, in principle, allowed to take arbitrary complex values, so the parameters $\alpha, \beta, \gamma, \delta$ could be complex as well, thus transforming under the group $SL_k(2, \mathbb{C})/\mathbb{Z}_2$ (the subscript k denotes that they act on the rapidities). While at the moment the couplings $c_{0,1}$ can be considered complex as well, we however restrict ourselves to the real domain.

While this symmetry is preserved at the level of the scattering matrix and the YBE equation, it acts, however, nontrivially on a system with fixed external boundary conditions (say, periodic). The condition of single-valuedness of the wave function then leads to the Bethe equations, which are derived in the general form in the Supplemental Material [41]. The solutions of the Bethe equations are therefore not symmetric with respect to the transformations Eqs. (7) and (8) but rather generate a *duality* between different models, since they relate different physical models. In this case, the $SL(2, \mathbb{R})$ symmetry is similar to the duality in string theory (e.g., in IIB superstring theory) relating theories with different coupling constants [37].

It is important to note that there are special points in the space of rapidities when the scattering matrix trivializes and $R(u, v) \equiv I$. This happens when one of the rapidities satisfies $u = -c_0/c_1$. In this case the Bethe equations reduce to quantization conditions for noninteracting particles. This observation combined with the $SL(2, \mathbb{R})$ duality implies that a subgroup of the latter, the *modular* group $SL(2, \mathbb{Z})$, connects,

in a sense of duality transformation all free, noninteracting modes of the model.

V. RICHARDSON-GAUDIN LIMIT

Starting from a solution to the Yang-Baxter equation, exactly solvable models can be constructed containing either Bosonic or spin degrees of freedom, depending on the representation of the algebra. In this regard, a solution to the functional Eq. (4) is also known to determine integrable Richardson-Gaudin spin systems [4,6–9,12]. A class of integrable spin systems can be obtained by plugging in the solution to Eq. (5) into the conserved charges Eq. (11) as described below, and the influence of the $SL(2, \mathbb{R})$ symmetry can be made apparent.

Richardson-Gaudin models arise as the so-called quasiclassical $\eta \rightarrow 0$ limit of integrability, where the R -matrix Eq. (1) can be expanded as

$$R_{12}(u, v) = I_{12} - \eta \mathcal{R}_{12}(u, v) + \mathcal{O}(\eta^2), \quad (9)$$

leading to the quasiclassical \mathcal{R} matrix as

$$\begin{aligned} \mathcal{R}_{12}(u, v) &= F(u, v)(I_{12} - P_{12}) \\ &= -F(u, v)(S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^z S_2^z - \frac{1}{2}), \end{aligned} \quad (10)$$

where the permutation operator has been written in a spin-1/2 representation. Through the usual transfer matrix construction, this gives rise to a set of conserved charges (see Supplemental Material [41] for a detailed derivation),

$$Q_i = S_i^z - \sum_{j \neq i} F_{ij} \left[\frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z \right], \quad (11)$$

where the demand $[Q_i, Q_j] = 0$ leads precisely to Eq. (4) with $F_{ij} \equiv F(\epsilon_i, \epsilon_j)$ with arbitrary inhomogeneities $\epsilon_{i,j} \in \mathbb{R}$.

Inserting Yang's solution for F_{ij} into Eq. (10) leads to the rational Richardson model, best-known in the context of superconductivity and the BEC-BCS crossover [8], while plugging the solution Eq. (5) leads to the rational limit of a parametrization proposed in Ref. [11]. For this parametrization, the conserved charges are explicitly invariant under the $SL(2, \mathbb{R})$ symmetry. An integrable Hamiltonian can then be constructed by taking a linear combination of these conserved charges, satisfying $[H, Q_i] = 0$ by construction. However, the choice of linear combination can break the $SL(2, \mathbb{R})$ invariance, similar to the choice of boundary conditions in the generic Bethe ansatz model [36], possibly leading to a nontrivial phase diagram. This is reflected in the fact that, while the conserved charges will be invariant under the duality transformation, the coefficients in the linear combination need not be, leading to a different Hamiltonian with the same conserved charges. The resulting eigenstates will clearly be independent of the linear combination, but the eigenstate corresponding to the ground state will depend strongly on the choice of Hamiltonian.

A large variety of resulting integrable Hamiltonians can be obtained depending on the parametrization and the choice of a Hamiltonian, but we will illustrate some of the resulting physics with a specific model based on the Richardson-Gaudin-Kitaev chain describing topological superconductivity in fermion chains [44].

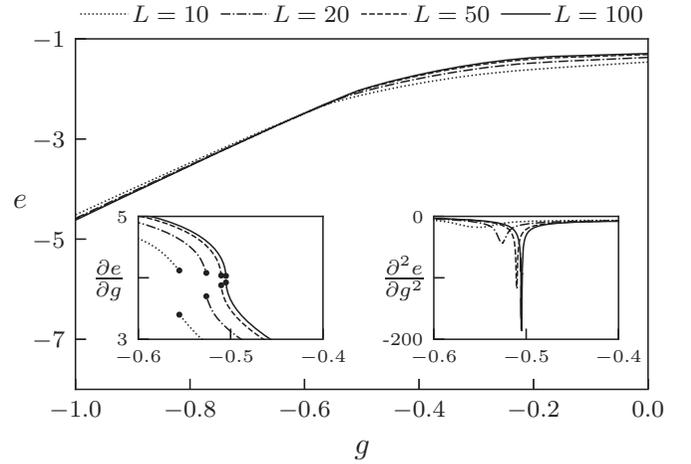


FIG. 1. The energy per site $e = E/L$ and its first and second derivative as a function of the interaction strength $g = GL$ for systems of length L at half-filling and periodic boundary conditions. The system parameters are $t_1 = 1$ and $t_2 = 0$, with $g_c = G_c L$ moving toward $g_c^{-1} = -2(t_1 + t_2)$ in the thermodynamic limit (here $-1/2$).

By mapping the spin $su(2)$ algebras to fermion pairing operators, an integrable Hamiltonian can be found as

$$H = \sum_k \epsilon_k c_k^\dagger c_k - \frac{G}{2}(C^\dagger C + CC^\dagger) - \frac{G}{4}BB, \quad (12)$$

$$C^\dagger = \sum_{k>0} \eta_k c_k^\dagger c_{-k}^\dagger, \quad B = \sum_k \eta_k c_k^\dagger c_k, \quad (13)$$

for one-dimensional fermions with momentum k and single-particle spectrum $\epsilon_k = -2t_1 \cos k - 2t_2 \cos 2k$, and interactions modulated by $\eta_k = 4 \sin^2(k/2)[t_1 + 4t_2 \cos^2(k/2)]$ with interaction strength G . These are related to the previous parametrization through $\sqrt{G}\eta_k = c_0 + c_1 \epsilon_k$ with $2(t_1 + t_2) = c_0/c_1$ and $G^{-1} = 2c_1^{-2} - \sum_k \eta_k$; see Supplemental Material Ref. [41].

This model is integrable for any choice of the momentum k -distribution, and describes a chain with either nearest neighbor-interactions ($t_2 = 0$) or long-range interactions ($t_2 \neq 0$) in real space, on top of which pairing interactions ($C^\dagger C$) and long-range interactions (B) have been added. This model can be seen as a variant on the Richardson-Gaudin-Kitaev chain [44] with pairing interactions interpolating between s - and d -wave pairing [45], also motivating the choice of parametrization. In the determination of the phase diagram, a crucial element is that the interaction vanishes at certain *finite* values of the momentum (here $k = 0$, for which $\eta_k = 0$ and $\epsilon_k = -c_0/c_1$ by construction). Equivalently, this corresponds to trivial particle scattering with $F_{ij} = 0$ (5). The existence of noninteracting modes in Richardson-Gaudin models has previously been linked to the existence of Majorana fermions in particle-number conserving models [44,46] and topological phase transitions [45,47,48]. This can be illustrated on the model at hand, since the ground-state energy for any Richardson-Gaudin model can be obtained directly from knowledge of the conserved charges Eq. (11) (see Ref. [49] for details).

In Fig. 1 we plot the energy per site and its first and second derivative for varying interaction strength. For finite system sizes, the discontinuity in the first derivative points to a first-order quantum phase transition. In the thermodynamic limit this discontinuity vanishes and the second derivative diverges, indicating a second-order phase transition, consistent with recent results on the thermodynamics of topological phases [50]. Furthermore, this is a phase transition between a topologically nontrivial phase ($G > G_c$) and a topologically trivial phase ($G < G_c$). This can be obtained from mean-field theory in the thermodynamic limit [45] or from the recently proposed characterization of topological superconductivity in number-conserving systems for finite system sizes [44,46]. Following this last route, the occupation of the single-particle levels for vanishing momentum k can be seen as the one-dimensional equivalent of a *winding number* [44,46,51], and it can be checked that $\lim_{k \rightarrow 0} \hat{n}_k = 1$ in the topological phase and $\lim_{k \rightarrow 0} \hat{n}_k = 0$ in the trivial phase.

This transition can be understood through symmetry arguments. The Hamiltonian undergoes a quantum phase transition at $G_c^{-1} = -\sum_k \eta_k$, which can be mapped back to a limit where $c_1 \rightarrow \infty$ while c_0/c_1 remains fixed. In this limit, the conserved charges reduce to those of the Gaudin model, which has an additional $su(2)$ total spin-symmetry compared to the Richardson-Gaudin models [4]. This symmetry results in level crossings in the spectrum between states with different particle numbers (spin-projection). Combined with the existence of a noninteracting level, this leads to level crossings between states with the same particle number at the symmetric point, since changing the occupation of the noninteracting level does not influence the energy. The quantum phase transition is then caused by a level crossing of two levels with different occupations of this level, leading to a vanishing chemical potential and a change in topological invariant [51]. Similarly, the rapidity distribution in complex space also undergoes a transition, as shown in the Supplemental Material and which is expected for a quantum phase transition [52]. The $SL(2, \mathbb{R})$ symmetry can again be compared to the field theory duality which relates theories with different coupling constants, since the ground state can be mapped to eigenstates of the Richardson model in different regimes (c_1^2 and c_0^2 change sign

at the phase transition and using the duality to map the system to one where either $\tilde{c}_0 = 0$ or $\tilde{c}_1 = 0$ results in a change of \tilde{c}_1^2 respectively \tilde{c}_0^2 between $-\infty$ and $+\infty$ at the phase transition). These limiting cases return the rational Richardson model with interaction strength $\tilde{c}_{0,1}^2$ [6,7], corresponding to the standard solution of the YBE, and the phase transition is then a transition between the ground states of this model at infinite interaction strengths, but with different signs of the interaction strength.

It should be stressed that the existence of such a transition does not depend on the specific choice of Hamiltonian, but is a fundamental property of the new parametrization (with $c_1 \neq 0$) and the resulting combination of total spin $su(2)$ -symmetry and the $SL(2, \mathbb{R})$ symmetry.

VI. DISCUSSION AND CONCLUSIONS

In this paper we presented a general rational solution of the Yang-Baxter relation in the fundamental (spin-1/2) representation. A hidden duality of this solution was demonstrated, connecting models different couplings, and a modular subgroup $SL(2, \mathbb{Z})$ of this duality connects special noninteracting modes of different systems. Constructing a Richardson-Gaudin model from this solution, it was shown how in these models the noninteracting modes serve as a topological order parameter and are responsible for a topological phase transition. While here we mostly focused on a Richardson-Gaudin limit of this solution it can be directly extended to inhomogeneous spin chains, higher spin models and Lieb-Liniger-type models.

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