Breaking of Goldstone modes in a two-component Bose-Einstein condensate

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We study the decay rate $\Gamma(k)$ of density excitations of two-component Bose-Einstein condensates at zero temperature. Those excitations, where the two components oscillate in phase, include the Goldstone mode resulting from condensation. While within Bogoliubov approximation the density sector and the spin (out-of-phase) sector are independent, they couple at the three-phonon level. For a Bose-Bose mixture we find that the Belyaev decay is slightly modified due to the coupling with the gapless spin mode. At the phase separation point the decay rate changes instead from the standard k^5 to a $k^{5/2}$ behavior due to the parabolic nature of the spin mode. If instead a coherent coupling between the two components is present, the spin sector is gapped and, away from the ferromagnetic-like phase transition point, the decay of the density mode is not affected. On the other hand, at the transition point, when the spin fluctuations become critical, the Goldstone mode is not well defined anymore since $\Gamma(k) \propto k$. As a consequence, we show that the friction induced by a moving impurity is enhanced—a feature which could be experimentally tested. Our results apply to every nonlinear 2-component quantum hydrodynamic Hamiltonian which is time-reversal invariant and possesses an $U(1) \times \mathbb{Z}_2$ symmetry.

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I. INTRODUCTION

The existence of Goldstone modes [1], i.e., gapless collective excitations, has crucial consequences on the thermodynamics and dynamics of systems with spontaneously broken continuous symmetries. While expected to be generically present in such systems, they can actually disappear in some specific situations. The most famous is the Anderson-Higgs mechanism [2,3], known in the relativistic context where, for instance, a scalar Higgs field gives a finite mass to the W- and Z-Bosons in electro-weak theory, i.e., three out of the four Goldstone modes associated with the four generators of $U(1) \times SU(2)$ become massive. This effect can be understood as due to the long-range interactions and is present also in nonrelativistic systems like superconductors [4], where the phase mode characterizing cooper-pair condensation disappears and the photons become massive—or jellium [5], where the Wigner crystal loses one of the three Goldstone modes corresponding to translational symmetry breaking.

Here we introduce a new scenario for the breaking of the Goldstone modes, where the modes do not become massive, but rather acquire a fast decay channel making them not well-defined excitations. This happens due to the coupling of the Goldstone modes with further gapless collective modes into which they can decay. These modes appear due to the spontaneous breaking of a further discrete symmetry. This mechanism carries analogies to the one predicted for systems possessing a Fermi surface [6], the last of these indeed showing gapless single-particle excitations into which the Goldstone modes can decay.

Our system consists of a two-component weakly interacting Bose-Einstein condensate (BEC) whose internal levels are coherently driven by an external electromagnetic field. The system shows both density (in-phase) and spin (out-of-phase) collective excitations [7–10]. The former are the $U(1)$ gapless phonons characterizing the condensation, while the latter are gapped and they become gapless at a ferromagnetic critical point for the spontaneous breaking of the $\mathbb{Z}_2$ symmetry corresponding to the exchange of the two components.

We show in the following that the vanishing of the gap makes the density modes decay into two spin modes with a rate of the same order of their energy, i.e., the density modes become not well-defined excitations. This implies, for instance, that a moving impurity would generate an enhanced friction, which we compute analytically.

Our results are more general than the two-component BEC studied here. They would namely apply to any nonlinear quantum hydrodynamic time-reversal-invariant Hamiltonian which couples density and spin, possessing an $U(1) \times \mathbb{Z}_2$ symmetry.

We also consider the case without the interconversion term, also known as a Bose-Bose mixture, which possesses a $U(1) \times U(1) \times \mathbb{Z}_2$ symmetry. Both the density and the spin excitations are gapless and linear. The system phase separates when the spin compressibility (susceptibility) diverges. Although enhanced, in this case the decay rate of density modes scales at a slower rate than their energy.

II. MODEL

We consider a dilute (weakly interacting) atomic Bose gas at zero temperature, whose atoms of mass m have two internal levels $|a\rangle$ and $|b\rangle$. The last of these are typically magnetically trappable hyperfine levels. An external field is applied that couples the $|a\rangle$ to the $|b\rangle$ state via a two-photon transition, characterized by a Rabi splitting Ω that we take real and positive. The atoms interact via short range interactions described by the strengths, g_{aa}, g_{bb}, and g_{ab} corresponding to the intra and the interspecies collisions, respectively. Introducing the fields $\hat{\psi}_j$, with $j = a, b$ the microscopic Hamiltonian can be written as

$$H = \int d\mathbf{r} \left[\sum_{j=a,b} \frac{\hbar^2}{2m} |\nabla \hat{\psi}_j|^2 + \sum_{i,j} \frac{g_{ij}}{2} \hat{\psi}_i^\dagger \hat{\psi}_j^\dagger \hat{\psi}_j \hat{\psi}_i \right] + \int d\mathbf{r}\, \frac{\hbar\Omega}{2} (\hat{\psi}_a^\dagger \hat{\psi}_b + \hat{\psi}_b^\dagger \hat{\psi}_a). \tag{1}$$

The system has an $U(1)$ symmetry for $\Omega \neq 0$, corresponding to the total atom number $n = n_a + n_b$ being conserved, and an $U(1) \times U(1)$ symmetry for $\Omega = 0$, the relative particle number $n_a - n_b$ being also conserved. At $T = 0$ the system is a Bose-Einstein condensate (BEC) described by the complex spinor order parameter $(\Psi_a(\mathbf{r}, t), \Psi_b(\mathbf{r}, t))$, where Ψ_j, $j \in \{a, b\}$ is the wave function macroscopically occupied by atoms in the internal state $|j\rangle$. For the sake of clarity we consider $g_{aa} = g_{bb} \equiv g$ in which case the system possesses a further $\mathbb{Z}_2$ symmetry, corresponding to the exchange of the two components.

Due to the diluteness condition allows to write the order parameter in terms of the gas densities as $\Psi_j = \sqrt{n_i} \exp(i\phi_j)$ and the mean-field energy functional reads

$$E_{MF} = \sum_{j=a,b} \int d\mathbf{r} \left(\frac{\hbar^2}{2m} |\nabla \sqrt{n_j}|^2 + \frac{\hbar^2 n_j}{2m} |\nabla \phi_j|^2 + \frac{1}{2} g n_j^2 \right) + \int d\mathbf{r} (g_{ab} n_a n_b - \hbar\Omega \sqrt{n_a n_b} \cos(\phi_a - \phi_b)), \tag{2}$$

where at this level the interaction strengths are given in terms of the experimentally known s-wave scattering length a_{ij}: $g_{ij} = 4\pi\hbar^2 a_{ij}/m$.

The ground state of the system is homogeneous with a fixed relative phase $\phi_a^0 - \phi_b^0 = 0$—- due to the last term in Eq. (2) — and, as already mentioned, can be either an unpolarized paramagnetic phase with $n_a^0 = n_b^0 = n$ or a partially polarized ferromagnetic phase $n_a^0 \neq n_b^0$, which breaks the $\mathbb{Z}_2$ symmetry. The transition between the two phases is second order and occurs for $\hbar\Omega = \hbar\Omega_c = (g_{ab} - g)n$ (see, e.g., Ref. [11] and references therein). The phase transition between the unpolarized and polarized phase has been experimentally observed in Ref. [12]. A sketch of the phase diagram is reported in Fig. 1, where the singular nature of the $\Omega = 0$ ferromagnetic transition is also put in evidence.

To describe the energy and the lifetime of the excitations we use the quantization scheme for hydrodynamics based on the mean-field energy Eq. (2) [13,14], which allows also for a rather direct interpretation of the various results. Indeed for a dilute gas the beyond mean-field correction to the energy functional, the so-called Lee-Huang-Yang corrections [15] are of subleading order. To obtain the quantum Hamiltonian used for our perturbative approach we introduce the fluctuation fields Π_j and ϕ_j, $j = a, b$, respectively, describing the amplitude and phase fluctuations above the ground-state values n_j^0 and $\phi_j^0 = 0$ and promote them to operator fields.

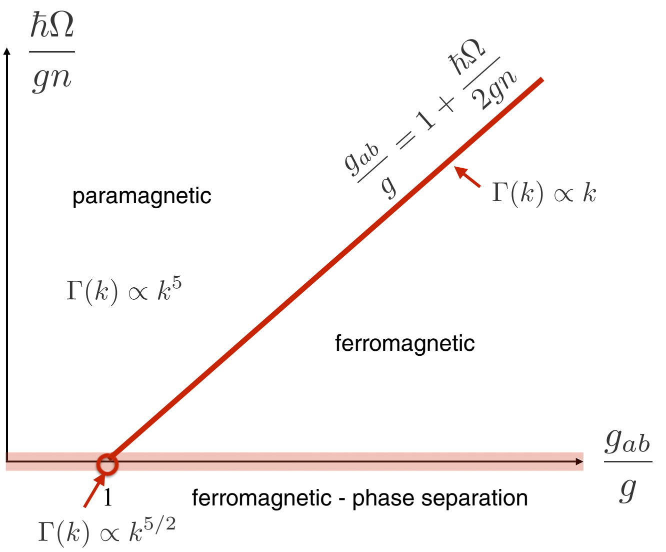

FIG. 1. Sketch of the phase diagram of two component Bose-Einstein condensates with density n in presence of both intra and interspecies interactions, g and g_{ab}, respectively, as well as a coherent interconversion term Ω between the two species. The system exhibits a ferromagnetic-like phase transition for strong enough interspecies interaction. For $\Omega = 0$ since the total magnetization is preserved the transition has a different character with respect to the $\Omega \neq 0$ case. In particular, Belyaev decay $\Gamma(k)$ strongly differs in the two cases (see text).

Inserting them in the Eq. (2) we obtain a quantum Hamiltonian. Expanding the Hamiltonian operator in the fluctuation fields one obtains the various processes needed to describe the excitations.

The first nontrivial order is given by the quadratic Hamiltonian, i.e. the Bogoliubov approximation, in the fluctuation fields. It decomposes into two sectors $H^{(2)} = H_d^{(2)} + H_s^{(2)}$, with

$$H_d^{(2)} = \int d\mathbf{r} \left[\frac{\hbar^2 |\nabla \Pi_d|^2}{4mn} + g_d \Pi_d^2 + \frac{\hbar^2 n |\nabla \phi_d|^2}{4m} \right], \tag{3}$$

$$H_s^{(2)} = \int d\mathbf{r} \left[\frac{\hbar^2 |\nabla \Pi_s|^2}{4mn} + g_s \Pi_s^2 + \frac{\hbar^2 n |\nabla \phi_s|^2}{4m} + \frac{\hbar\Omega n}{2} \phi_s^2 \right], \tag{4}$$

where we introduce the in-phase (density) $\Pi_d = (\Pi_a + \Pi_b)/2$, $\phi_d = \phi_a + \phi_b$ and out-of-phase (spin) $\Pi_s = (\Pi_a - \Pi_b)/2$, $\phi_s = \phi_a - \phi_b$ fluctuations, as well as the coupling constants $g_d = g + g_{ab}$ and $g_s(\Omega) = g - g_{ab} + \hbar\Omega/2n$. The quadratic Hamiltonians Eqs. (3) and (4) can be easily diagonalized by introducing the annihilation (creation) operators for the density $d_\mathbf{k}$ ($d_\mathbf{k}^\dagger$) and spin mode $s_\mathbf{k}$ ($s_\mathbf{k}^\dagger$) at momentum $\mathbf{k}$ as

$$\Pi_\alpha(\mathbf{r}) = \sqrt{\frac{n}{2}} \sum_\mathbf{k} U_{\alpha,k} \left(\alpha_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} + \alpha_\mathbf{k}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right), \tag{5}$$

$$\phi_\alpha(\mathbf{r}) = i\sqrt{\frac{1}{2n}} \sum_\mathbf{k} U_{\alpha,k}^{-1} \left(\alpha_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} - \alpha_\mathbf{k}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right), \tag{6}$$

TABLE I. Belyaev decay of the density Bogoliubov mode for three-dimensional spinor Bose gases. For completeness the single component case is also reported.

	$\Gamma(\mathbf{k})$	dominant term
1-comp. Bose gas	k^5	$\Pi\|\nabla\phi\|^2$
2-comp. Bose gas	k^5	$\Pi_d\|\nabla\phi_d\|^2$
noncritical		$\Pi_s\nabla\phi_s\nabla\phi_d$, $\Pi_d\|\nabla\phi_s\|^2$ (if $\Omega = 0$)
$\Omega = 0$ PS point	$k^{5/2}$	$\Pi_s\nabla\phi_s\nabla\phi_d$
$\Omega \neq 0$ FM transition	k	$\Pi_d\Pi_s^2$

with $\alpha = d, s$ and where we defined (see also Ref. [10] for the most general case $g_a \neq g_b$)

$$U_{d,k} = \left(\frac{k^2}{k^2 + 4mg_d n}\right)^{\frac{1}{4}}, \quad U_{s,k} = \left(\frac{k^2 + 2m\hbar\Omega}{k^2 + 4mg_s n}\right)^{\frac{1}{4}}. \tag{7}$$

The density and spin Hamiltonians now simply read

$$H_d^{(2)} = \sum_{\mathbf{k}} \omega^d_{\mathbf{k}} d^\dagger_{\mathbf{k}} d_{\mathbf{k}}, \ \omega^d_{\mathbf{k}} = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2g_d n\right)},$$
$$H_s^{(2)} = \sum_{\mathbf{k}} \omega^s_{\mathbf{k}} s^\dagger_{\mathbf{k}} s_{\mathbf{k}}, \ \omega^s_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} + \hbar\Omega\right)\left(\frac{\hbar^2 k^2}{2m} + 2g_s n\right)}. \tag{8}$$

The density mode is gapless and linear at small momenta, $\omega^d_{\mathbf{k}} \simeq c_d|\mathbf{k}|$, with a speed of sound $c_d = \sqrt{ng_d/m}$ independent of the coupling Ω, while the spin mode has a gap $\Delta_s = \sqrt{2\hbar\Omega n g_s}$.

At the transition point $g_s(\Omega_c) = 0$, the gap closes, the low energy spin-mode is *linear* and dominated by relative amplitude fluctuations Π_s as it is clear already from Eq. (4). This last becomes critical since the instability is due to the system breaking $\mathbb{Z}_2$ and building a finite polarization.

The case $\Omega = 0$ behaves very differently. The density and the spin sector are both gapless and the low momentum excitations are phase-like, as it has to be for Goldstone modes of the $U(1) \times U(1)$ broken symmetries. The speeds of sound are c_d and $c_s = \sqrt{ng_s(0)/m}$ for the density and the spin sector, respectively. On the verge of phase separation, i.e., $g_s(0) = 0$, the spin mode becomes *quadratic* at low momenta and it acquires an amplitude contribution, being now both the relative phase and the relative amplitude fluctuations finite at low momenta.

III. BELYAEV DECAY FOR A TWO-COMPONENT BOSE GAS

At the quadratic Bogoliubov level discussed above the modes are well defined. The finite lifetime comes from higher order terms which represent interaction among various modes. In particular, the third order term represents the so-called Belyaev decay of one excitation into two new excitations and which is the dominant process at low temperatures [16]. In a single component weakly interacting Bose gas the decay rate Γ of phonons at low momentum $\mathbf{k}$ is very small $\Gamma(\mathbf{k}) \propto k^5$ in three spatial dimensions (see, e.g., Refs. [13,16] and Table I).

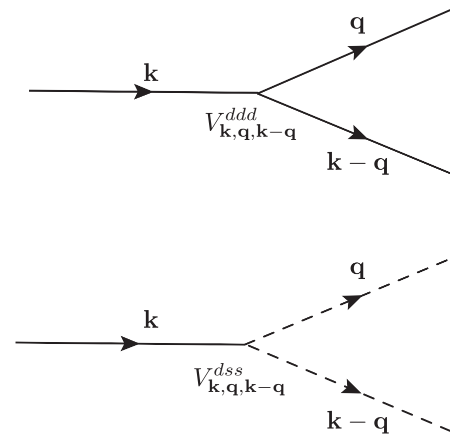

FIG. 2. Schematic representation of the vertices for the decay of a Goldstone phonon (continuous line) of momentum $\mathbf{k}$ into two Goldstone modes (upper vertex) or into two spin modes represented by dashed lines (lower panel).

In the case of a 2-component Bose gas further decay processes are, in principle, possible since, e.g., a density mode can decay into two spin modes. At the phase transition point the spin modes change their character, leading to a strong enhancement of the Belyaev decay rate. We anticipate here (see also Table I) that the Goldstone mode is still well defined for a mixture $\Omega = 0$ with a decay rate which scales like $k^{5/2}$, while for $\Omega \neq 0$ the Goldstone mode is not properly defined. In particular, in three spatial dimensions where our perturbation theory is well justified, the decay rate of the Goldstone mode scales like ist energy, i.e., $\Gamma(\mathbf{k}) \propto k$.

A. Symmetries and the general structure of the three-mode vertices

To obtain the vertices of the possible decay processes we have to expand Eq. (2) to third order. The number of nonzero terms is pretty small due to the symmetries of the system. In the paramagnetic phase due to the $\mathbb{Z}_2$ symmetry all the terms with an odd number of spin fields have to be zero. Therefore, the density mode can decay either in (i) two density modes or in (ii) two spin modes, as schematically represented in Fig. 2. Moreover, due to the total-density $U(1)$ symmetry the process (i) can occur only via $\Pi_d|\nabla\Pi_d|^2$, Π_d^3, and $\Pi_d|\nabla\phi_d|^2$, which leads to the standard Belyaev decay. The possible terms related to process (ii) are $\Pi_d|\nabla\phi_s|^2$ and $\Pi_s\nabla\phi_d\nabla\phi_s$ for $\Omega = 0$, due to the spin $U(1)$ symmetry, while also the terms $\Pi_d\Pi_s^2$ and $\Pi_d\phi_s^2$ are present for $\Omega \neq 0$. For instance, the term $\Pi_d\Pi_s^2$ gives rise to the vertex

$$V^{dss}_{\mathbf{q1},\mathbf{q2},\mathbf{q3}} = -\frac{\Omega}{2n^2} U_{d,\mathbf{q}_1} U_{s,\mathbf{q}_2} U_{s,\mathbf{q}_3}, \tag{9}$$

which is responsible for the breaking of the Goldstone mode at the critical point for the ferromagnetic-like transition.

B. Results

The decay rate is given by the imaginary part of the self-energy for the density mode. We calculate the self-energy at the one-loop level, which coincides with a Fermi's golden rule calculation. The general expression for anyone of the above-mentioned processes reads

$$\Gamma(\mathbf{k}) = \frac{\mathtt{Pm}_V}{(2\pi)^2} \int d^3\mathbf{q} |V_{\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q}}|^2 \delta\big(\omega_k^d - \omega_q^s - \omega_{|\mathbf{k}-\mathbf{q}|}^s\big), \quad (10)$$

where V is the vertex of the process and $\mathtt{Pm}_V$ the number of possible equivalent diagrams.

Since we are interested in the decay rate at low momentum we can consider only the most relevant contributions in the different regimes as reported in Table I.

1. Mixture ($\Omega = 0$)

In the case of a mixture $\Omega = 0$ and away from the phase separation $g \neq g_{12}$ one has the ordinary Belyaev decay, where the prefactor is renormalized due to the decay of density in two spin phonons. The most relevant terms at low momentum are $\Pi_d(\nabla\phi_d)^2$ for the three density phonon vertex and $\Pi_d(\nabla\phi_s)^2$ and $\Pi_s\nabla\phi_s\nabla\phi_d$ for the density into two spin phonon vertex. The decay rate reads

$$\Gamma(\mathbf{k}) \simeq \frac{3k^5(1 + h(c_d/c_s))}{640 nm\pi}, \quad (11)$$

where $h(r) = 7r/12 + 43/72r - 11r^3/24 + 5r^5/18$, which for two noninteracting species reduces to $h(1) = 1$. At the phase separation point the most relevant term is only $\Pi_s\nabla\phi_s\nabla\phi_d$ as can be seen by putting Ω and g_s to zero in Eq. (7) and one gets a strong enhancement of the phonon decay which now reads

$$\Gamma(\mathbf{k}) = \frac{(mc_d k)^{5/2}}{48 nm\pi}. \quad (12)$$

Still, the phononic Goldstone mode is well defined at low momenta since $\Gamma_k/\omega_k \simeq k^{3/2} \to 0$.

2. Coherent coupling $\Omega \neq 0$

When the coherent coupling is on, the spin sector is gapped, therefore, away from the transition point and at zero temperature it does not contribute to the phonon decay which is simple due the standard Belyaev process $\Pi_d(\nabla\phi_d)^2$, leading to $\Gamma(\mathbf{k}) = 3k^5/(640nm\pi)$.

At the ferromagnetic transition the situation is very different. The gap in the spin channel closes and the spectrum becomes linear at small momentum, i.e., $\omega_s(k) = c_{s,PT}|k|$ with $mc_{s,PT}^2 = (g_{12} - g)n = \Omega_c$ where Ω_c is the value of the coherent coupling at the transition point. A density phonon can now decay into two spin ones. Thses are critical at the transition and, as already mentioned, dominated by the relative amplitude fluctuations, since the system is on the verge of polarization. The most relevant term becomes $\Pi_d\Pi_s^2$, whose contribution leads to a critical decay rate

$$\Gamma(\mathbf{k}) = \frac{(mc_{s,PT})^4 k}{4nm\pi}, \quad (13)$$

making the Goldstone mode a not well-defined excitation.

IV. DYNAMIC STRUCTURE FACTOR: BRAGG SPECTROSCOPY AND FRICTION

The decay rate of the density excitations can be measured having access to the dynamic structure factor, which, accounting for the finite lifetime $\Gamma(\mathbf{k})$ of on-shell phonons, can be written as

$$S(\mathbf{k}, \omega) = n|U_d(\mathbf{k})|^2 \frac{\Gamma(\mathbf{k})/\pi}{\big(\omega - \omega_{\mathbf{k}}^d\big)^2 + \Gamma(\mathbf{k})^2}. \quad (14)$$

A measurement of such quantity in cold gases can be done using two-photon optical Bragg spectroscopy [17–19]. Two photonic beams properly intersecting on the atomic cloud introduce an external perturbation $V_{\mathrm{Bragg}} = V_B/2\cos(qz - \omega t)\theta(t)$ from the arbitrary time $t = 0$. Experimentally it is then rather easy to measure the total momentum transferred by the lasers. Within linear response theory the momentum transferred along the direction given by q at long enough time t reads

$$P_z(t) = \frac{\pi V_B^2}{2\hbar} qt(S(q, \omega) - S(-q, -\omega)). \quad (15)$$

For very small damping the Lorentzian Eq. (14) reduces to a delta function and the momentum will increase as soon as the Bragg frequency ω is close to the density mode frequency ω_q^d, in analogy with the single component condensate [20]. At low momenta the dispersion can be properly measured although looking at the interferometry after the Bragg pulse [21].

When the Bragg excited cloud separates from the original BEC the Belyaev decay has been directly measured showing the typical suppression at low-k due to the vertex for a single component BEC [20].

The very same kind of experiments can be performed on the 2-component BEC close to the ferromagnetic transition point leading to the absence of clear peaks and especially to a linear dependence of the strong Belyaev damping.

Force on an impurity: Friction

In this section we describe a different and more indirect effect of the short lifetime of the phonons, namely the response of the system to a *local* density perturbation.

Landau theory of superfluidity leads to the existence of a finite critical velocity below which the flow is dissipationless. A moving object weakly interacting with the fluid feels a friction force only if its speed is larger than the Landau critical velocity. For homogeneous ultracold gases the situation is quite clear and the critical velocity is due to Cherenkov phonon emission [22]. If phonons have a finite life-time a friction force is present for any speed of the moving impurity.

The dissipation of energy due to a time-dependent potential can be generally written in terms of $S(\mathbf{k}, \omega)$ as

$$\dot{E} = -\int \frac{d\mathbf{k}}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} \omega S(\mathbf{k}, \omega)|W(\mathbf{k}, \omega)|^2, \quad (16)$$

where $W(\mathbf{k}, \omega)$ is the Fourier transform of the external perturbation. Considering a delta-like infinite mass impurity moving at a constant speed $\mathbf{V}$, we can write $W(\mathbf{r}, t) = \lambda\delta(\mathbf{r} - \mathbf{V}t)$ where λ is the coupling between the impurity and the gas, which leads to $W(q, \omega) = 2\pi\lambda\delta(\omega - \mathbf{q}\cdot\mathbf{V})$. Using Eq. (14) the expression for dissipated energy per unit time reads

$$\dot{E} = \frac{2\pi}{\hbar}\lambda^2 \int \frac{d\mathbf{k}}{(2\pi)^3} n|U_d(\mathbf{k})|^2 \frac{\Gamma(\mathbf{k})}{\big(\mathbf{k}\cdot\mathbf{V} - \omega_{\mathbf{k}}^d\big)^2 + \Gamma(\mathbf{k})^2} \mathbf{k}\cdot\mathbf{V}. \quad (17)$$

Considering that at low speed $|\mathbf{V}|$ the most relevant contribution comes from momenta $k < \bar{k} \ll 1/\xi_d$ with $\xi_d = \hbar/mc_d$ the density healing length, we find that the dissipated energy depends quadratically on the speed of the impurity and scales very differently far from the transition and at the transition point, namely

$$\dot{E} = -\frac{\lambda^2}{12\pi^2\xi_d^6}\left(\frac{V}{c_d}\right)^2 \begin{cases} \frac{3}{160}(\bar{k}\xi_d)^8, & \Omega > \Omega_c, \\ \frac{(c_s c_d)^4}{\left(c_d^4+c_s^4\right)^2}(\bar{k}\xi_d)^4, & \Omega = \Omega_c. \end{cases} \tag{18}$$

This strongly enhanced energy dissipation via a moving obstacle close to the transition might offer a practicable means of experimentally testing our predictions [23].

V. EFFECTIVE ϕ^4-THEORY AND THE ROLE OF THE DIMENSIONALITY

Our results are based on the assumption that also at the phase transition point the propagator for spin waves is linear at low momenta, i.e., that the effective theory close to the phase transition point has a dynamical critical exponent $z = 1$.

To discuss the validity of such an assumption we expand Eq. (2) to fourth order. Again due to symmetry reasons the only possible terms have to contain an even number of spin operators. Contributions arise from the kinetic part

$$\propto (\Pi_d + \Pi_s)^2|\nabla(\Pi_d + \Pi_s)|^2 + (\Pi_d - \Pi_s)^2|\nabla(\Pi_d - \Pi_s)|^2 \tag{19}$$

and from the Rabi part

$$\propto 3n\Pi_s^2\left(4\Pi_d^2+\Pi_s^2\right)+3n^2\left(\Pi_s^2+2\Pi_d\Pi_s - \Pi_d^2\right)\phi_s^2+n^4\phi_s^4 \tag{20}$$

of the Hamiltonian. Similar arguments as the ones used above to identify the most relevant third-order terms contributing to the decay of Goldstone phonons, from the mode expansion Eqs. (5) and (6) leads to the conclusion that Π_s^4 is the most relevant forth-order contribution at the phase-transition point $g_s = 0$. Indeed it introduces a phonon-phonon interactions which contains $U_{d,\mathbf{q}_1}U_{s,\mathbf{q}_2}U_{s,\mathbf{q}_3}U_{s,-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3}$. Since at low momenta $U_s(k) \simeq 1/\sqrt{k}$, the interaction Π_s^4 leads to a term in the self-energy scaling like k^{-2} at small momenta (see, e.g., Ref. [24] for more details). Therefore, close to critical point, the low energy $\mathbb{Z}_2$ effective theory corresponds to a standard ϕ^4-theory, which is well known to have an upper critical spatial dimension $D = 3$ [24]. As it has been shown by Irkhin and Katanin [25], the log-corrections at the upper critical dimension do not affect the mean-field critical exponents. In particular, the dynamical critical exponent is $z = 1$ and the dispersion relation for spin waves gives rise to the Gaussian propagator employed in our previous analysis. Therefore, the main result of our paper remains unchanged: the critical spin waves induce a damping of the density Goldstone mode which is linear in momentum $\Gamma \sim k$. On the other hand, due to the log-corrections the prefactor in the damping rate Eq. (13) as well as in the expression of the friction given in Eq. (18) would change.

A few remarks are due here. The previous argument just reinforces the fact that Bose-Bose mixtures do not belong to the same universality class of coherently coupled Bose gases since the Π_s^4 term arises only in presence of a Rabi coupling.

It would be also interesting to check the effect of the neglected terms coupling spin and density in the actual behavior close to the phase transition. If indeed the density mode is very damped we expect to be left with just the spin degrees of freedom and therefore a pure ϕ^4 theory only for the Π_s as argued above. More importantly, for spatial dimension $D < 3$ the ferromagnetic transition will exhibit the same features of the quantum Ising model. In particular, even the transition point would be strongly modified with respect to the mean field value $g_s(\Omega) = 0$.

The only theoretical results so far trying to address the previous remarks have been obtained for one-dimensional Hubbard models. It has been numerically shown that both in the insulating, where a mapping to XXZ model in the transverse field is possible, and in the superfluid phase the β critical exponent for the ferromagnetic transition is consistent with the one of an one-dimensional Ising model in transverse field [26–28]. Also the transition point appears indeed very far from the prediction of the mean-field analysis. Experimentally, the non-mean-field behavior should be possible to observe with present technology, by confining strongly in one or two dimension a driven Bose gas. A recent experiment with coherently driven Bose gases in a very elongated trap has been already carried out [29]. The outcomes of the experiment are, however, consistent with mean field results for the critical exponents since the confinement was not large enough and the physics was therefore three dimensional.

A special mention for the one-dimensional homogeneous case is due here. In the dilute regime a single component Bose gas is well described by the Lieb-Liniger model and therefore exactly solvable, implying that the excitations have infinite life time. The 2-component gas instead is not integrable. However, the simple one-loop approximation for the decay into two density modes fails in this case since energy and momentum conservation coincide. More accurate analysis lead for single-component BECs to a decay rate proportional to k^2 [30–32]. On the other hand, for the decay of a density mode in two spin modes, the energy and the momentum conservation are distinct and one-loop analysis works. Close to the phase transition instead the mean field description of the critical point fails completely and therefore the fate of the Goldstone mode cannot be determined following the arguments used in the present work. We stress again that the problem of expanding around a mean field solution is present both in $D = 1, 2$.

VI. CONCLUSION

In conclusion, we show that two-component Bose gases present an interesting scenario for the breaking of Goldstone modes. If the system has a $U(1) \times \mathbb{Z}_2$ symmetry, the Goldstone mode related to the breaking of the global phase symmetry $U(1)$ in the condensed phase becomes not well defined at the critical point for the breaking the discrete symmetry $\mathbb{Z}_2$ due to decay into critical spin amplitude modes. When the system has instead a $U(1) \times U(1) \times \mathbb{Z}_2$ symmetry, the Goldstone mode related to the global phase (density mode) is strongly affected by the spin modes at the $\mathbb{Z}_2$ transition point, but still well defined in the limit of large wave lengths.

We also discuss how, for coherently coupled gases, the ferromagnetic transition is described by an usual ϕ^4-theory while this is not the case for a Bose-Bose mixture. Although sometimes put on the same footing our results show clearly that 2-component Bose-Einstein condensates with and without interconversion term behave very differently concerning the $\mathbb{Z}_2$ phase transition.

The main effects here presented can be experimentally studied with present technology using trapped ultracold Bose gases with two hyperfine levels. Coherently coupled Bose gases have been indeed realized for the first time many years ago in the context of atom optics by the group of Cornell [33,34] where interesting coherence properties related to the superfluid stiffness have been observed. A few years ago the spontaneous magnetization of the gas has been (although indirectly) observed in the group of Oberthaler [12] and very recently the same laboratory reported the first study about the dynamics of the phase transition and its critical exponents [29].

While measuring the Goldstone mode life time could be rather demanding, the main qualitative signature of strong spin wave fluctuations could be simply obtained by perturbing the system via a density probe. If indeed around the phase transition point there exists a strong coupling between the Goldstone and the spin modes we expect that a density perturbation should produce an emission of spin waves when the gas is prepared close enough to the phase transition point since also any finite (low) temperature would enlarge the region of quantum criticality.

The system we address would also open new possibilities for the investigation of quantum phase transitions in low dimensions. Indeed, as we argued in Sec. V, in two- and one-spatial dimension the mean field starting point is not appropriate in studying the ferromagnetic transition (not the phase separation at zero temperature).

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