Berry curvature and Hall viscosities in an anisotropic Dirac semimetal

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We investigate parity-odd nondissipative transport in an anisotropic Dirac semimetal in two spatial dimensions. The analysis is relevant for interacting electronic systems with merging Dirac points at charge neutrality. For such systems the dispersion relation is relativistic in one direction and nonrelativistic in the other. We give a proposal of how to calculate the Berry curvature for this system and use it to derive more than one odd viscosities, in contrast to rotationally invariant systems. We observe that in such a model the odd part of viscosity tensor is parametrized by two independent transport coefficients and one that is identically zero.

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I. INTRODUCTION

Since the discovery of quantum Hall states, the topological response of these systems continues to be one of the emerging fields of research [1–6]. In particular, there has been a revived interest in understanding the interplay between geometry and quantum Hall states with fractional and integer fillings [7–12]. A key quantity that encodes the topological response to the geometry deformations is Hall viscosity [1,13] (see [14,15] for a review); a nondissipative part of the viscosity tensor that is odd under time reversal and hence nonvanishing when such a symmetry is absent.

When rotational symmetry is broken, the odd part of the two-dimensional viscosity tensor can have three nonzero components, in contrast to the usual single viscosity in a rotationally symmetric system. Despite extensive studies on both isotropic two-dimensional (2D) electron gas and Dirac materials [1–6,16,17], systems without rotational symmetry have received surprisingly little attention. This follows either from the scarcity of physically realizable examples or from the difficulty in the explicit calculation of the Hall viscosity tensor. Although some progress has been made in setups in which the anisotropy is introduced via the mass tensor or interaction tensor [18,19] in a 2D electron gas, the anisotropic case in 2D Dirac semimetals has not been explored so far.

The objective of this paper is to fill this gap by studying the Hall viscosity tensor in a new class of 2D anisotropic Dirac semimetals [20–22]. Such semimetals are known to exhibit a special phase, namely a critical semi-Dirac phase, which is characterized by electronic bands touching in a discrete set of nodes about which the bands disperse linearly in one direction and quadratically along the orthogonal direction. The low-energy Hamiltonian describing such materials is

$$\mathcal{H} = \mathbf{d}(\mathbf{p}) \cdot \boldsymbol{\sigma}, \tag{1}$$

where $\boldsymbol{\sigma}$'s are Pauli matrices. $\mathbf{d}(\mathbf{p}) = (\frac{p_x^2}{2m_0} - \delta_0, p_y, 0)$ with m_0 being a mass and δ_0 the gap parameter. This type of Hamiltonian has been argued to emerge in TiO_2/VO_2 heterostructures [23], $(BEDT\text{-}TTF)_2I_3$ organic salts under pressure [24], and photonic metamaterials [25]. However, the only experimental realization for such a dispersion has thus far been observed in optical lattices [26]. Because of the possibility to realize the semi-Dirac phases in real materials, it is natural to ask how this anisotropy can be leveraged to understand the values and universal properties of the Hall viscosity in such systems, a question that has received no attention to date.

The main difficulty in addressing the above problem comes from the following issue: How does a semi-Dirac material with electrons that have relativistic motion in one direction and nonrelativistic motion along the perpendicular direction couple to the underlying geometry? Given the nonrelativistic structure of the Hamiltonian [Eq. (1)], there is no straightforward answer to this question. To solve this problem, we propose a different path based on a generalized relativistic model that exhibits three distinct phases, including the critical semi-Dirac phase as a function of an anisotropic parameter. Writing this Hamiltonian on a torus in the presence of a magnetic field, we find the Landau levels and corresponding wave functions. We then derive the formula for Berry curvature with the help of these wave functions. We furthermore show how anisotropy leads to *two independent* Hall viscosity coefficients in this particular system. This is in contrast to the three independent components allowed by the symmetry arguments of [13]. Finally, we analyze the scaling of those coefficients as a function of the applied magnetic field and obtain a power-law behavior at the critical semi-Dirac phase. These constitute the central results of this paper.

II. MODEL AND PHASES

We propose a low-energy Hamiltonian for an anisotropic 2D Dirac semi-metal:

$$H = \gamma^0(\mathbf{p} \cdot \boldsymbol{\gamma} + \mathbf{b} \cdot \boldsymbol{\gamma}\gamma^5 + m_0). \tag{2}$$

Here $\gamma^\mu = (\tau^x, -i\tau^y\boldsymbol{\sigma})$, $\gamma^5 = \tau^z$ are 4×4 Dirac matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_4$, where $\eta = (1, -1, -1, -1)$; τ and σ's are the Pauli matrices in spin and pseudo-spin space, respectively, $\mathbf{p} = (p_x, p_y, 0)$; m_0 denotes mass gap and $\mathbf{b} = (b, 0, 0)$ is the anisotropic parameter of the Hamiltonian.

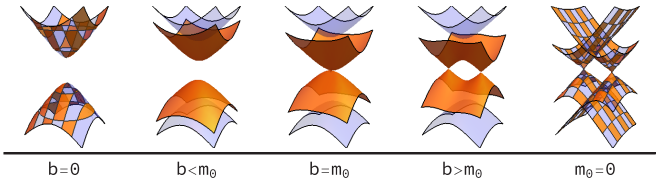

FIG. 1. Evolution of energy dispersion of a two-dimensional anisotropic Dirac semimetal [Eq. (2)] for different values of parameter b/m_0. For $b = 0$ the spectrum is gapped, whereas for $m_0 = 0$ we see two gapless Dirac nodes. In both limiting cases the bands are doubly degenerate as presented with cross-shaded colors. For $b/m_0 = 1$, the two Dirac nodes merge, leading to a semi-Dirac point as discussed in the main text.

The energy spectrum of Eq. (2) is given by

$$E(p_x, p_y) = r\sqrt{p_x^2 + p_y^2 + b^2 + m_0^2 + 2s\sqrt{b^2\left(m_0^2 + p_x^2\right)}}, \quad (3)$$

where $r, s = \pm$. Note that $r = \pm 1$, $s = -1$ correspond to lowest conduction and highest valence band, respectively. The competition between mass gap m_0 and anisotropy b leads to three distinct phases (cf. Fig. 1). For $b > m_0$, the spectrum is gapless, with two-Dirac nodes at $(\pm\sqrt{b^2 - m_0^2}, 0, 0)$, while $b < m_0$ corresponds to a gapped insulating phase. On the other hand, for $b = m_0$, we obtain a critical phase where two Dirac nodes merge and lead to a semi-Dirac phase. Thus, the variation of b/m_0 changes the Fermi surface topology, leading to a Lifshitz transition. Note that redefining $b^2 = m_0^2 + 2\delta$ and assuming a large gap $\delta_0 \equiv \delta/m_0 \ll 1$ and $p_x/m_0 \ll 1$, the highest valence and lowest conduction bands behave as $E \approx \pm\sqrt{p_y^2 + (\frac{1}{2m_0}p_x^2 - \delta_0)^2 + \mathcal{O}(p_x)^5}$, which agrees with the spectrum of the Hamiltonian in Eq. (1). In the Supplemental Material [27] we discuss in more detail the equivalence of both Hamiltonians (1) and (2) after including a magnetic field.

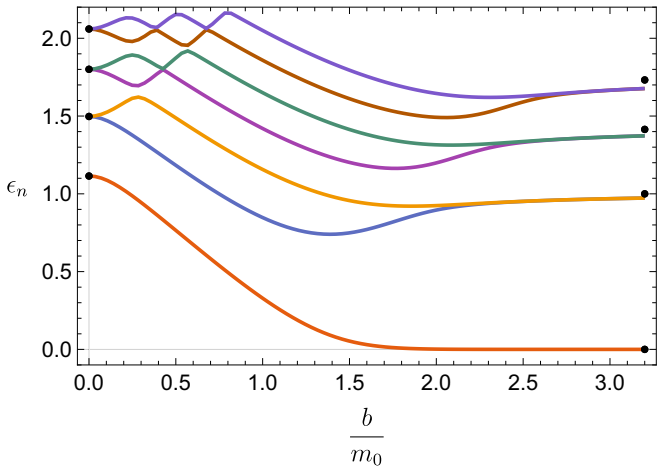

FIG. 2. Positive energy Landau levels (n). The zeroth Landau level is nondegenerate irrespective of the values of b for finite m_0. The black dots for $b = 0$ correspond to the Landau energies $\epsilon_n \sim \sqrt{n + M^2}$, while black dots for $b/m_0 \gg 1$ correspond to $\epsilon_n \sim \sqrt{n}$, showing agreement between numerical and analytical results.

III. LANDAU LEVELS AND WAVE FUNCTIONS ON A TORUS

Let us now focus on finding out the Landau spectrum and corresponding wave functions of Eq. (2) on a torus. The metric of the torus is given by

$$ds^2 = \frac{V}{\tau_2}(dx^2 + 2\tau_1 dx\, dy + |\tau|^2 dy^2), \quad (4)$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter and V is the volume of the torus. With this, the Landau Hamiltonian in the presence of a constant perpendicular magnetic field $B = \epsilon^{ij}\partial_i A_j$ [28] is obtained to be

$$H_L = \gamma^0(\Pi_i e_a{}^i \gamma^a + m_0 + b_i e_a{}^i \gamma^a \gamma^5), \quad (5)$$

where $a, i, j \in (1, 2)$, $\Pi_i = p_i - eA_i$, and $e_a{}^i$'s are the frame vectors satisfying $g^{ij} = e_a{}^i \delta^{ab} e_b{}^j$. With this construction, the kinematical momenta Π_i satisfy $[\Pi_i, \Pi_j] = i\epsilon_{ij} l_B^{-2}$, where $l_B = \hbar/(eB)$ is the magnetic length. For simplicity, we set $\hbar = e = 1$ for the rest of this study.

To diagonalize Eq. (5), we introduce ladder operators a, $a^\dagger$, satisfying $[a, a^\dagger] = 1$. This leads to

$$H_L = \omega[(a\sigma_+ + a^\dagger\sigma_-)\tau_z + (\bar{d}\sigma_+ + d\sigma_-)\tau_0 + M\sigma_x\tau_0], \quad (6)$$

where a bar denotes complex conjugation, $\sigma_\pm = \sigma_x \pm \sigma_y$, $\omega = \sqrt{2}l_B^{-1}$, $a = i(V\tau_2)^{-1/2}(\Pi_y - \tau\Pi_x)/\omega$, $d = -i(V\tau_2)^{-1/2}\tau b/\omega$, and $M = m_0/\omega$.

For nonzero b and m_0, Eq. (6) cannot be exactly diagonalized. Thus, we choose an algebraic semianalytic method to diagonalize Eq. (6) and obtain the Hall viscosities. To do so, let us introduce new shifted ladder operators $\mathfrak{a}_d = a + d$, $\mathfrak{a}_d^\dagger = a^\dagger + \bar{d}$. This leads to a set of basis states $|n, \alpha, d, \tau\rangle$, satisfying

$$\mathfrak{a}_d^\dagger \mathfrak{a}_d |n, \alpha, d, \tau\rangle = n|n, \alpha, d, \tau\rangle. \quad (7)$$

The index $\alpha = 1, \ldots, N$ labels the magnetic degeneracy, which for notational simplicity we ignore in the rest of the paper. For a detailed discussion on the existence of these eigenvectors and how to impose the proper boundary conditions on the torus, see Ref. [8,29]. Having these bases, we proceed to expand each Landau level eigenstate as follows:

$$|\psi\rangle = \sum_n \mathbf{c}_n |n, d, \tau\rangle, \quad (8)$$

where $\mathbf{c}_n$ is a set of four-component constant fermions, depending only on the values of τ, V. At this point, the problem of diagonalizing Eq. (6) is translated into the eigenvalues problem of the infinite matrix

$$H_{nm}\mathbf{c}_m = \epsilon_n \mathbf{c}_n \quad \text{where } H_{nm} = \langle n, d, \tau|H_L|m, d, \tau\rangle. \quad (9)$$

In general $\mathbf{c}_n$'s have to be obtained numerically by truncating the series at some large enough values of n. However, there are two limiting cases in which the diagonalization process of Eq. (6) can be done analytically; the first and simplest case corresponds to $m_0 = 0$ (see Fig. 1). In this case, the Hamiltonian decouples into two 2-band subsystems which do not interact with each other. The eigenenergies turn out to be $\epsilon_n = \pm\omega\sqrt{n}$ for each subsystem. Then the wave functions for

the zeroth Landau level of the two subsystems aref

$$\psi_0^1 = (1,0,0,0)|0,d,\tau\rangle,$$
$$\psi_0^2 = (0,0,1,0)|0,-d,\tau\rangle. \tag{10}$$

For transparency, the higher excited wave functions are presented in the Supplemental Material.

In contrast, the second analytically solvable case, $b=0$, is slightly more involved because there is no decoupling. After a careful calculation, the eigenenergies are found to be $\epsilon_n = \pm\omega\sqrt{n+M^2}$. The zero mode here turns out to be nondegenerate and the states with $n>0$ are doubly degenerate. The zeroth Landau level wave function is

$$\psi_0 = \frac{1}{\sqrt{2}}(1,0,\pm 1,0)|0,0,\tau\rangle. \tag{11}$$

As before, the degenerate excited wave functions can be obtained easily, and they are presented in the Supplemental Material for simplicity. These limiting behaviors of Eq. (6) are expected to be reflected in the Hall viscosities (discussed in the next section) for both $b \gg m_0$ and $b \ll m_0$, in which a single viscosity exists and can be computed analytically. Finally, we computed for arbitrary values of the ratio b/m_0 the spectra and we show in Fig. 2 some of the lowest Landau levels.

IV. BERRY CURVATURE

According to the adiabatic response theorem by Feynman and Hellmann, the expectation value of the variation of the Hamiltonian, with respect to some set of parameters x_i, is given by

$$\left\langle \frac{\partial H}{\partial x_j} \right\rangle = \frac{\partial E}{\partial x_j} - \Omega_{ij}\dot{x}_j. \tag{12}$$

The first term is a result of the energy change of the ground state deformation. The second term is the adiabatic Berry curvature

$$\Omega_{ij} = i\left[\frac{\partial}{\partial x_i}\left\langle\psi\middle|\frac{\partial\psi}{\partial x_j}\right\rangle - \frac{\partial}{\partial x_j}\left\langle\frac{\partial\psi}{\partial x_i}\middle|\psi\right\rangle\right], \tag{13}$$

which is nonzero if the phase of the state ψ changes along a closed path in the space of deformations. Plugging the eigenstates (8) into Eq. (13), the total Berry curvature can be readily obtained by

$$\Omega = id(\mathbf{c}_n^\dagger \cdot d\mathbf{c}_n) + d(\mathbf{c}_n^\dagger \cdot \mathbf{c}_m)\wedge \mathcal{A}^{nm} + \mathbf{c}_n^\dagger \cdot \mathbf{c}_m \mathcal{F}^{nm}. \tag{14}$$

Here repeated indices denote Einstein's notation and the exterior derivative d acts on the space expanded by the parameters τ, V. The detailed derivation is shown in the Supplemental Material. The explicit forms of $\mathcal{A}$ and $\mathcal{F}$ evaluated at nondeformed torus $\tau = i$ are

$$\mathcal{A}^{mn} = -\frac{1}{4}(\sqrt{m(m-1)}\,\delta_{m,n-2}d\bar{\tau} + \sqrt{(m+1)(m+2)}\,\delta_{m,n+2}d\tau), \tag{15}$$

$$\mathcal{F}^{mn} = -\frac{i}{4}\left(m+\frac{1}{2}\right)\delta_{m,n}d\tau\wedge d\bar{\tau}. \tag{16}$$

For $m_0 = 0$, the first and second terms in Eq. (14) identically vanish since $\mathbf{c}_n$ are independent of V, τ. The only surviving contribution produces the value for the Berry curvature at $\tau = i$ [27]:

$$\Omega^{pq} = -\frac{i}{4}\left(n+\frac{1}{2}\delta_{n,0}\right)d\tau\wedge d\bar{\tau}\delta^{pq}, \tag{17}$$

where $p,q=1,2$ label the degenerate subspaces associated with each subsystem as pointed out before. Evidently, Ω is diagonal in the subsystem subspace. Thus, we recover the Berry curvature for isotropic Dirac systems using Eq. (14) [30,31].

Similarly, for $b=0$ and for the zeroth Landau level, $\mathbf{c}_n \sim \delta_{n,0}$, leading to $\Omega = -\frac{i}{8}\frac{d\tau\wedge d\bar{\tau}}{\tau_2^2}$. For higher Landau levels, the calculation of Berry curvature is subtle due to twofold degenerate Landau levels (not to be confused with magnetic degeneracy) as discussed in the preceding sections. These subtleties, however, do not change the message we want to convey, because the degeneracy is not present around the critical point. For simplicity, we focus mainly on the zeroth Landau level in our analysis.

For nonzero b and m_0, $\mathbf{c}_n \neq \delta_{n,0}$, thus we may have nonzero contribution from the first and second terms of Eq. (14), which in turn may lead to more than one Hall coefficient, as will be evident shortly. We would like to mention that Eq. (14) is a generic formula and can be applied to any system where rotational symmetry is broken. Thus, this is one of the main results of this study.

V. ANISOTROPY AND HALL VISCOSITY

Armed with the derivation of Berry curvature, we now relate different component of Ω to the viscosity components and show how anisotropy in a Dirac system leads to more than one Hall viscosity coefficient. The odd transport coefficients are the most readily visible at the level of constitutive relations. One can expand the average stress tensor in time derivatives of the strain,

$$T_{ij} = -\sum_{kl}\lambda_{ijkl}u_{kl} - \sum_{kl}\eta_{ijkl}\frac{\partial u_{kl}}{\partial t} + \cdots, \tag{18}$$

where the strain is expressed in terms of a deformation vector $u_{kl} = \partial_k u_l + \partial_l u_k$. The first term in that expansion λ_{ijkl} corresponds to a generalized Hooke's elasticity tensor and the second term η_{ijkl} corresponds to the viscosity tensor. Note that η_{ijkl} is symmetric under exchange of i with j and k with l [32]. In general, η can be divided into $\eta = \eta_S + \eta_A$, where η_S is symmetric with respect to interchanging first pair (ij) with (kl) whereas η_A is antisymmetric under exchange of (ij) with (kl). Since the antisymmetric part is odd under time reversal, $\eta_A \neq 0$ only when time reversal symmetry is broken. As the antisymmetric part of the viscosity tensor is nondissipative, it may survive at zero temperature. From now on we only focus on the antisymmetric nondissipative part and remove the label A for brevity and clarity.

For generic two-dimensional (2D) systems without time reversal, η_{ijkl} has in principle three independent components η_{xxxy}, η_{xyyy}, and η_{xxyy}. However, for rotationally invariant systems, the number of independent quantities decreases, and η_{ijkl} is solely determined by a single viscosity, denoted as η_H since $\eta_H = \eta_{xxxy} = \eta_{xyxx}$ and $\eta_{xxyy} = 0$. This single object

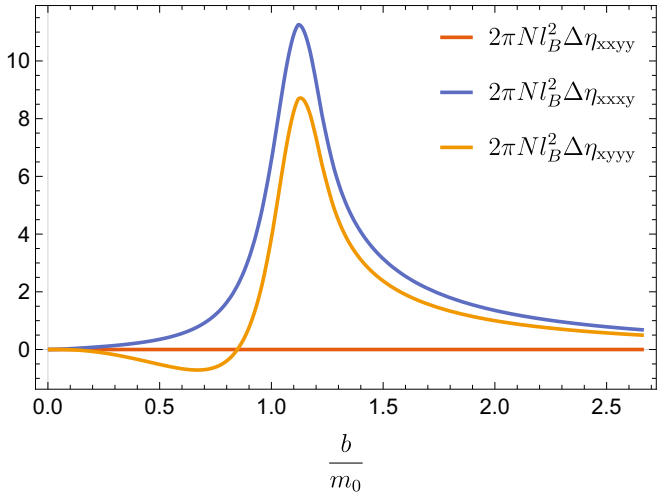

FIG. 3. Subtracted viscosity coefficients ($\Delta\eta = \eta - \eta_{\rm iso}$) as a function of the ratio b/m_0.

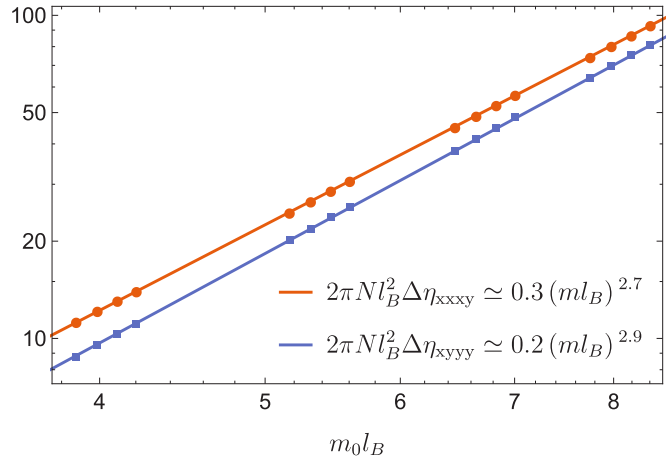

FIG. 4. Scaling of the subtracted viscosity tensor as a function of the magnetic length. Dots correspond to the numerical data whereas solid lines represent the fitting.

turns out to be an universal quantity $\eta_H = \bar{s}\rho/2$, where $\bar{s}$ is the orbital angular momentum and ρ is the average number density.

Since the odd viscosity tensor is a multicomponent tensor for a generic 2D system, we relate different components of Berry curvature to the viscosity coefficients. Together with Eqs. (12) and (18), and evaluating the strain rate on the nondeformed torus $\tau = i, V = 1$, we obtain

$$\begin{aligned} 2\pi N l_B^2 \eta_{xxxy} &= \Omega_{\tau_1\tau_2} - \Omega_{\tau_1 V}, \\ 2\pi N l_B^2 \eta_{xyyy} &= \Omega_{\tau_1\tau_2} + \Omega_{\tau_1 V}, \\ 2\pi N l_B^2 \eta_{yyxx} &= \Omega_{\tau_2 V}, \end{aligned} \tag{19}$$

where each component of Ω can be extracted from Eq. (14) (see Supplemental Material). In the isotropic case, $\Omega_{\tau_2 V}$ and $\Omega_{\tau_1 V}$ turn out to be identically zero. Thus $\eta_{xxxy} = -\eta_{yyxy}$ is the only parameter that determines the response to the geometry of the QH states. However, due to the anisotropic nature of the Dirac system, each term in Eqs. (19) contributes to the viscosities except for $\Omega_{\tau_2 V}$, which turns out to be zero (see Fig. 3).

Figure 3 illustrates the different components of the viscosity tensor after subtracting the isotropic value ($\Delta\eta = \eta - \eta_{\rm iso}$) as a function of the ratio b/m_0. It is evident that the different components of η start to deviate from the universal isotropic value as we increase b for fixed m_0 and become maximum near the ideal semi-Dirac phase ($b/m_0 = 1$). Thus, the anisotropy $b \neq 0$ leads to more than single viscosity coefficients, in contrast to the isotropic case ($b = 0$). If we further increase b/m_0, both nonzero components of η start to decrease and merge again toward the isotropic value. This is attributed to the fact that, in the large $b \gg m_0$ limit, we obtain two well-separated Dirac nodes, in conjunction with the earlier discussion. Consequently, the wave function behaves approximately as in Eqs. (10), which in turn gives the isotropic value of Hall viscosity.

We next aim to find the dependence of viscosity coefficients on the magnetic field B near the semi-Dirac phase. It is known for typical isotropic 2D system, $\eta \sim B$ [1], irrespective of the relativistic or nonrelativistic nature of the electrons. In contrast, we find a different scaling behavior of the subtracted viscosity near the semi-Dirac point. Figure 4 illustrates the maximum of $\Delta\eta$ for several values of $m_0 l_B$. In the given range of analyzed data, we can fit the power-law scaling as seen in Fig. 4. We observe that $\Delta\eta$ goes to zero for large magnetic fields. This confirms an intuitive picture that, for large enough energy scale, the system behaves isotropically.

VI. CONCLUSION

We have introduced a framework for studying the nondissipative transport in anisotropic Dirac semimetals, where the anisotropy is present due to a preferred direction. This distinguishes this model from the previous cases studied in the literature, where isotropy is broken by a tensor [18,19]. We have introduced a relativistic model with an anisotropic vector that reproduces the spectrum of the nonrelativistic semi-Dirac system Eq. (1) at low energies for certain values of parameters. We have derived a universal formula for a Berry curvature in this model that succinctly captures the anisotropy for the semi-Dirac phase. Using the formula, we have numerically investigated how the anisotropy leads to the departure from one Hall viscosity coefficient for the zeroth Landau level. We have shown that, at the critical semi-Dirac point, the odd stress tensor has two nonequal entries. In addition to that, we have shown that the third entry is identically zero for this model.

The general formalism developed here is a step forward in the understanding of odd transport in anisotropic systems. In the realm of quantum Hall effect, the anisotropic effects are of great interests, and only recently appropriate methods have started to be developed. A first step in this direction was given by the authors of Ref. [18] for 2D electron gasses, using general effective field theory techniques. They classified and explored possible universalities in systems where the rotational invariance is broken by a symmetric tensor. In our situation the rotational invariance is broken by a vector. Moreover, recent progress in two-dimensional hydrodynamics motivates studies of Hall transport in electronic systems. To date odd transport has been only studied in isotropic models

[33,34]. Thus the studies presented here can be useful to generalize the existing isotropic, parity-odd hydrodynamic solutions to anisotropic systems. This would supplement existing analysis that takes into account dissipative viscosities [35]. In order to facilitate the experimental analysis it will be useful to study finite momentum corrections to anisotropic conductivities [36]. If relations between Hall viscosities and conductivities generalize to anisotropic systems, measurements of conductivities are potentially the most feasible ones to reveal aspects of odd viscosities in anisotropic materials.

In three spatial dimensions, some models for Weyl semimetals show a similar anisotropic semi-Dirac behavior. In fact, the Hall viscosity for an anisotropic toy model for a strongly coupled Weyl semimetal was computed in [37], showing a puzzling relation between the Hall viscosity tensor and the mixed gauge-gravitational anomaly. Above all, our results pave the way for a detailed analysis of the Hall transport in anisotropic Dirac systems, including the special semi-Dirac phase, and subsequently for understanding their hydrodynamic regime.

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