

Pairing in quantum critical systems: Transition temperature, pairing gap, and their ratio

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We compute the ratio of the pairing gap Δ at $T = 0$ and T_c for a set of quantum-critical models in which the pairing interaction is mediated by a gapless boson with local susceptibility $\chi(\Omega) \propto 1/|\Omega|^\gamma$ (the γ model). The limit $\gamma = 0+$ [$\chi(\Omega) = \log|\Omega|$] describes color superconductivity, and models with $\gamma > 0$ describe superconductivity in a metal at the onset of charge or spin order. The ratio $2\Delta/T_c$ has been recently computed numerically for $0 < \gamma < 2$ within Eliashberg theory and was found to increase with increasing γ [T.-H. Lee *et al.*, *Phys. Rev. Lett.* **121**, 187003 (2018)]. We argue that the origin of the increase is the divergence of $2\Delta/T_c$ at $\gamma = 3$. We obtain an approximate analytical formula for $2\Delta/T_c$ for $\gamma \leq 3$ and show that it agrees well with the numerics. We also consider in detail the opposite limit of small γ . Here we obtain the explicit expressions for T_c and Δ , including numerical prefactors. We show that these prefactors depend on fermionic self-energy in a rather nontrivial way. The ratio $2\Delta/T_c$ approaches the BCS value 3.53 at $\gamma \rightarrow 0$.

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I. INTRODUCTION

The BCS theory of superconductivity [1] is rightly considered to be one of the most elegant theoretical works of the 20th century. It not only explains how to obtain the energy gap in the fermionic spectrum, Δ , and the transition temperature T_c as functions of material-dependent parameters, but it also predicts that the ratio $2\Delta/T_c = 3.53$ is a material-independent universal number. Measurements on ordinary superconductors, like aluminum, did find $2\Delta/T_c$ ratios consistent with BCS theory [2]. However, in other materials, including novel superconductors, $2\Delta/T_c$ is higher. The two obvious reasons, particularly applicable to the cuprates, are non- s -wave superconductivity [3,4] and pseudogap physics [5]. Another potential reason is the sensitivity of $2\Delta/T_c$ to strong coupling effects. They are often associated with Mott physics [6]; however, a large $2\Delta/T_c \sim 8$ –13 (depending how Δ is defined; see below) has been found in Eliashberg calculations of phonon-mediated s -wave superconductivity [2,7–10], in the limit when Debye frequency ω_D is vanishingly small but electron-phonon interaction g is finite (in this limit, both T_c and Δ scale with g ; see Refs. [11]).

Phonon-mediated pairing at $\omega_D \rightarrow 0$ is a specific realization of a more generic situation when the pairing is mediated by a massless boson with susceptibility $\chi(q, \Omega)$, minimally coupled to fermions. Other examples include pairing between fermions at a half-filled Landau level, when a massless boson is a gauge field with Landau overdamped propagator $\chi(q, \Omega) \propto 1/(q^2 + a|\Omega|/q)$ (e.g., Ref. [12]), and pairing in a metal at the onset of an instability towards a charge or a spin order either with $q = 0$ or with a finite lattice momentum [13–32]. The pairing problem in these systems is often considered within a computational scheme similar (but not identical) to the one originally used by Eliashberg in his analysis of phonon-mediated superconductivity [33]. Namely, the fully renormalized pairing vertex is obtained by summing

up series of ladder diagrams, like in BCS theory but with dynamical bosonic propagator $\chi(q, \Omega)$, and with fermionic propagators, which include one-loop fermionic self-energy. The latter comes from the same fermion-boson interaction and is computed self-consistently with the pairing vertex. Higher-order self-energy corrections and nonladder renormalizations of the pairing vertex are assumed to be small (a necessary condition is a requirement that a soft boson is a slow mode compared to a fermion; i.e., for the same momentum, a typical bosonic frequency must be smaller than a typical fermionic frequency). Within this approximation [34], the momentum integration in the Eliashberg equations can be performed exactly for a given pairing symmetry [35], and the problem reduces to the set of coupled one-dimensional (1D) integral equations for frequency dependent pairing vertex $\Phi(\omega)$ and fermionic self-energy $\Sigma(\omega)$ [2,7,10,13,14]. For spin-singlet pairing, which we consider here, the two equations are, in Matsubara frequencies,

$$\begin{aligned}\Phi(\omega_n) &= \pi T \sum_m \frac{\Phi(\omega_m)}{\sqrt{[\omega_m + \Sigma(\omega_m)]^2 + \Phi(\omega_m)^2}} \chi_L(\omega_n - \omega_m), \\ \Sigma(\omega_n) &= \pi T \sum_m \frac{\omega_m + \Sigma(\omega_m)}{\sqrt{[\omega_m + \Sigma(\omega_m)]^2 + \Phi(\omega_m)^2}} \chi_L(\omega_n - \omega_m).\end{aligned}\tag{1}$$

Here $\chi_L(\Omega_m)$ is the effective, local, dimensionless bosonic susceptibility [it is equal to $g^2\chi(q, \Omega_m)$ integrated over the Fermi surface with form factors for a given pairing channel s, p, d , etc.). For the electron-phonon problem at vanishing Debye frequency, $\chi_L(\Omega) = g^2/|\Omega_m|^2$. We consider a generic model with $\chi_L(\Omega) = g^\gamma/|\Omega_m|^\gamma$: the γ model. For nematic and Ising-ferromagnetic critical points, $\gamma = 1 - D/3$, where D is a spatial dimension; for the antiferromagnetic critical point, $\gamma = (3 - D)/2$; models with other values of γ have also been identified [15,16,36–38]. A similar set of equations

for the frequency dependent pairing vertex and fermionic self-energy emerges in the dynamical mean-field theory (DMFT) approach. It was argued that DMFT analysis of a Hund metal within the three-band Hubbard model for Fe-based superconductors yields $\chi_L(\Omega) \propto 1/|\Omega|^{1.2}$ in a wide range of frequencies [39,40]. As an additional complication, the form of $\chi_L(\Omega)$ may by itself depend on Φ due to feedback from superconductivity on the bosonic propagator [41,42]. This can be incorporated by treating γ below T_c as a temperature-dependent parameter.

The goal of our study is to extract some new physics from the analysis of $2\Delta/T_c$ in the γ model. The pairing gap $\Delta(\omega_n)$ is related to the pairing vertex as $\Delta(\omega_n) = \Phi(\omega_n)/(1 + \Sigma(\omega_n)/\omega_n)$, and the Eliashberg equation for $\Delta(\omega_n)$ is

$$\Delta(\omega_n) = \pi T \sum_m \frac{\Delta(\omega_m) - \frac{\omega_m}{\omega_n} \Delta(\omega_n)}{\sqrt{\omega_m^2 + \Delta(\omega_m)^2}} \frac{g^\gamma}{|\omega_n - \omega_m|^\gamma}. \quad (2)$$

T_c is obtained as the highest temperature at which Eq. (2) has a solution. Note that the term with $m = n$ in the right-hand side (r.h.s.) of (2) (the self-action term) can be neglected due to the vanishing numerator. To see this more clearly, one has to add a small mass term M to the interaction and take the limit $M \rightarrow 0$ only at the end of calculations. The numerator in (2) vanishes at $m = n$ for any M . This vanishing is the consequence of the cancellation between the contributions to the gap equation from the renormalization of the pairing vertex and the self-energy [43–45], and it has the same physics origin as the Anderson theorem: the independence of T_c on nonmagnetic impurities [46]. Indeed, the term with $m = n$ describes the scattering with zero frequency transfer, averaged over finite momentum transfers; i.e., its role in the gap equation is equivalent to that of elastic scattering by nonmagnetic impurities. We remind in this regard that we consider spin-singlet pairing. For spin-triplet pairing, the r.h.s. of the equation for the pairing vertex contains the extra overall factor $-1/3$, and the term with $m = n$ does not vanish, in analogy with the case when impurities are magnetic [47].

For the gap Δ at $T = 0$ we will use $\Delta_0 = \Delta(\pi T)$ at the lowest temperature. One can show that $\Delta_0 = \Delta(\omega = 0)$ on the real axis. An alternative is to associate Δ with the frequency at which the density of states has a maximum, Δ_{DOS} . In BCS theory $\Delta_0 = \Delta_{\text{DOS}}$ and $2\Delta_0/T_c = 3.53$, but in the γ model, $\Delta_{\text{DOS}} > \Delta_0$ (this is probably true for all Eliashberg calculations, not necessary at a quantum critical point).

In a phonon superconductor with $\omega_D = 0$, $\Delta_{\text{DOS}} \approx (\pi/2)\Delta_0$. This accounts for the discrepancy in the reported $2\Delta/T_c$ ratio: $2\Delta_0/T_c \sim 8.3$, while $2\Delta_{\text{DOS}}/T_c \sim 12.9$ (Refs. [2,7]).

The ratio of $2\Delta/T_c$ in the γ model was recently analyzed numerically for $0 < \gamma \leq 2$ and was found to increase rapidly with increasing γ [40]. We obtained the same result (see Fig. 3) and also found that the increase of $2\Delta/T_c$ accelerates at larger γ . The goal of our work is to provide an explanation for the increase. We argue that $2\Delta/T_c$ actually diverges at $\gamma \rightarrow 3$. The divergence is the direct consequence of the fact that at $T = 0$, when Matsubara frequencies become continuous variables, the integral in the r.h.s. of the gap equation (2) becomes singular at $\omega_m \approx \omega_n$ ($\int dx x^2/|x|^\gamma$ diverges at

$\gamma \geq 3$). We obtain analytical formulas for T_c and Δ near $\gamma = 3$ and argue that they remain valid in a wide range of $\gamma < 3$.

Another goal of our study is to analyze the opposite limit of small γ . Here we explore the fact that for any $\gamma > 0$, $\chi_L(\Omega) = (g/|\Omega|)^\gamma$ is a decreasing function of Ω , in which case the r.h.s. of the gap equation is ultraviolet convergent, and there is no need to impose an upper cutoff in the frequency summation in (2). We obtain the explicit expressions for T_c and Δ in the small γ limit. We show that $T_c = Q_T \omega_0$ and $\Delta = Q_\Delta \omega_0$, where $\omega_0 = g(1.4458\gamma)^{1/\gamma}$ and Q_T and Q_Δ are numerical factors of order 1. The scale ω_0 has been identified before [25]. To obtain it, one can neglect fermionic self-energy, i.e., treat fermions as free quasiparticles, as in BCS theory. However, to obtain the factors Q_T and Q_Δ one needs to include the subleading terms in γ , and these additional terms do depend on the non-Fermi-liquid self-energy $\Sigma(\omega) \propto \omega_m^{1-\gamma}$. We show that the self-energy contributions to Q_T and Q_Δ are rather nontrivial, and the result is very different from the one in a weakly coupled Fermi liquid, where the self-energy changes the exponential factor $e^{-1/\lambda}$ into $e^{-(1+\lambda)/\lambda} = e^{-1/\lambda}/e$ [37,48,49]. Still, we show that self-energy equally affects T_c and Δ , such that $2Q_\Delta/Q_T = 3.53$, as in BCS theory. We computed T_c and Δ numerically at small γ , and found good agreement with our analytical results.

The structure of the paper is as follows. In Sec. II we briefly review how Δ and T_c are obtained in BCS theory. In Sec. III we study the case when γ is small and obtain explicit formulas for both T_c and Δ . The prefactors Q_T and Q_D are calculated both analytically and numerically. In Sec. IV we show the divergence of Δ when $\gamma \rightarrow 3$.

II. BCS THEORY

To set the stage for our calculations, we briefly outline how $2\Delta/T_c$ is obtained in BCS theory. Here, $\chi_L(\Omega) = \lambda$ is frequency independent, and $\Delta(\omega_n) = \Delta$. The frequency sum in the gap equation diverges at large ω_m and one has to set the upper cutoff Λ . We then have

$$1 = \lambda \sum_{m=0}^{\frac{\Lambda}{2\pi T_c}} \frac{1}{m + 1/2} = \text{Li}\left(\frac{3}{2} + \frac{\Lambda}{2\pi T_c}\right) - \text{Li}\left(\frac{1}{2}\right),$$

$$1 = \lambda \int_0^\Lambda \frac{d\omega}{\sqrt{\omega^2 + \Delta^2}} = \lambda \log \frac{2\Lambda}{\Delta}, \quad T = 0, \quad (3)$$

where $\text{Li}(z) = \int_0^z dx/\log x$ is a logarithmic integral. Using $\text{Li}[3/2 + \Lambda/(2\pi T)] - \text{Li}(1/2) = \log 2e^C \Lambda/(\pi T)$, where $C = 0.577216$ is Euler's constant, we immediately obtain $T_c = (2e^C \Lambda/\pi)e^{-1/\lambda}$, $\Delta = (2\Lambda)e^{-1/\lambda}$, and $2\Delta/T_c = 2\pi/e^C = 3.52775$. In Eliashberg theory with $\chi_L = \lambda$ one also has to include the self-energy $\Sigma = \lambda\omega$, Eq. (1), and then $T_c = (2e^{C-1} \Lambda/\pi)e^{-1/\lambda}$, $\Delta = (2\Lambda/e)e^{-1/\lambda}$. The ratio $2\Delta/T_c$ still remains 3.53.

III. SMALL γ

We first consider the case when $\chi_L(\Omega) = (g/|\Omega|)^\gamma$ with small but finite γ . As we said, for any finite γ , the pairing kernel $\chi_L(\omega_n - \omega_m)/|\omega_m|$ decreases faster than $1/|\omega_m|$; i.e., frequency summation over m in the r.h.s. of the gap equation

(2) converges. This eliminates the need to introduce an upper frequency cutoff Λ ; that is, T_c and Δ remain finite even when Λ is infinite.

The small γ limit has been considered before. Previous studies analyzed the pairing susceptibility at $T = 0$ and identified the large scale $\omega_0 = g(1.4458\gamma)^{-1/\gamma} \gg g$, at which this susceptibility diverges. We obtain $T_c \sim \omega_0$ explicitly by solving the linearized gap equation at a finite T and the nonlinear gap equation at $T = 0$, and find the proportionality factors.

A. Calculation of T_c

Consider first the linearized gap equation (the limit $\Delta_m \rightarrow 0$). Neglecting the term with $m = n$ in the r.h.s. of (2), one can reexpress (2) as

$$\Delta_n \left(1 + \frac{\tilde{\Sigma}_n}{n + 1/2}\right) = \frac{K_T}{2} \sum_{m \neq n} \frac{\Delta_m}{|m + 1/2|} \frac{1}{|n - m|^\gamma},$$

$$\tilde{\Sigma}_n = \frac{K_T}{2} \sum_{m \neq n} \frac{\text{sign}(m + 1/2)}{|n - m|^\gamma}, \quad (4)$$

where $\tilde{\Sigma}_n$ is the self-energy without the ‘‘self-action’’ term, and $K_T = (\frac{g}{2\pi T})^\gamma$. For $n = 0, -1$, $\tilde{\Sigma}_0 = \tilde{\Sigma}_{-1} = 0$; for $n \geq 1$, $\tilde{\Sigma}_n = K_T \sum_{l=1}^n \frac{1}{m^\gamma}$; and for $n < 0$, $\tilde{\Sigma}_{-n-1} = -\tilde{\Sigma}_n$ [44,50].

We will see below that it will be sufficient to analyze Eq. (4) for large Matsubara number n ; however, we will need all internal m . At large n , $\tilde{\Sigma}_n \approx K_T n^{1-\gamma}$. Substituting this into (4), we obtain

$$\Delta_n \left(1 + K_T \frac{1}{(n + 1/2)^\gamma}\right) = \frac{K_T}{2} \sum_{m \neq n} \frac{\Delta_m}{|m + 1/2|} \frac{1}{|n - m|^\gamma}. \quad (5)$$

For internal $|m| < |n|$, the r.h.s. of (5) scales as $1/|n|^\gamma$. Substituting this dependence back into the r.h.s. of (5), we find that the summation over m converges and yields $O(1/\gamma)$. Matching $1/|n|^\gamma$ dependence on both sides of Eq. (5), we find $K_T \sim \gamma$, i.e., $T_c \propto g(1/\gamma)^{1/\gamma}$.

In order to find the prefactor in $T_c \propto g(1/\gamma)^{1/\gamma}$ we need to compute K_T to the second order γ . For this we search for the solution in the form

$$\Delta_n = \frac{1}{|n + 1/2|^\gamma} \sum_{p=0}^{\infty} \frac{a_p}{|n + 1/2|^{\gamma p}}. \quad (6)$$

Without loss of generality we set $a_0 = 1$, as the linear equation does not fix the overall magnitude of Δ_n . Substituting this Δ_n into (5) and matching the prefactors for $1/|n + 1/2|^{p\gamma}$ with $p = 1, 2, 3, \dots$, we obtain recursive relations for a_p ,

$$a_p = -Z a_{p-1} \left(\frac{1}{p!(p+1)!} + \gamma \right), \quad (7)$$

and the self-consistent condition on Z [which determines $T_c(\gamma)$],

$$\frac{1}{Z} = \sum_{p=0}^{\infty} a_p \left(\frac{1}{p+1} + \gamma \log 4e^C \right). \quad (8)$$

Here $Z = K_T/\gamma$. The terms $O(\gamma)$ in the r.h.s. of Eq. (7) are due to the self-energy, which mixes $1/|n|^{p\gamma}$ and $1/|n|^{(p+1)\gamma}$ gap components in Eq. (5); the $O(\gamma)$ term in the r.h.s. of Eq. (8) comes from the summation over Matsubara frequencies with $m = O(1)$.

Solving Eq. (7) we obtain

$$a_p = (-Z)^p \left(\frac{1}{p!(p+1)!} + \gamma \frac{p+2}{3p!(p-1)!} \right), \quad p \geq 1, \quad (9)$$

and we remind that $a_0 = 1$. Substituting the expressions for a_p into (8), we find that it reduces to

$$1 = - \sum_{p=1}^{\infty} \frac{(-Z)^p}{(p!)^2} - \gamma \log 4e^C \sum_{p=1}^{\infty} \frac{(-Z)^p}{p!(p-1)!} - \frac{\gamma}{3} \sum_{p=2}^{\infty} \frac{(-Z)^p (p+1)}{p!(p-2)!}. \quad (10)$$

The sums are expressed in terms of Bessel functions, and Eq. (10) becomes

$$J_0(2\sqrt{Z}) = \gamma \log 4e^C \sqrt{Z} J_1(2\sqrt{Z}) - \frac{\gamma}{3} (Z^{3/2} J_1(2\sqrt{Z}) + Z J_2(2\sqrt{Z})). \quad (11)$$

Without $O(\gamma)$ terms in the r.h.s, the condition on T_c is $J_0(2\sqrt{Z}) = 0$. This equation has multiple solutions, which is fundamentally important for the understanding of the phase diagram of the γ model [51,52]. For our current purposes, however, it is sufficient to consider only the solution with the highest T_c , i.e., with the smallest Z . The first zero of $J_0(2\sqrt{Z})$ is at $Z = Z_0 = 1.4458$. This yields $(g/2\pi T)^\gamma = 1.4458\gamma[1 + O(\gamma)]$, i.e., $T_c = Q_T \omega_0$, where, we remind, $\omega_0 = (1.4458\gamma)^{-1/\gamma}$ is the characteristic scale extracted from the analysis of the pairing susceptibility at $T = 0$ [25] and Q_T is the prefactor $O(1)$, which we determine below. The large n asymptotics of the corresponding eigenfunction Δ_n is

$$\Delta_n = \frac{1}{|n|^\gamma} \left(1 - \frac{2Z}{(2!)^2 |n|^\gamma} + \frac{3Z^2}{(3!)^2 |n|^{2\gamma}} + \dots \right)$$

$$= \frac{1}{|n|^\gamma} \sum_{m=0}^{\infty} \left(\frac{Z}{|n|^\gamma} \right)^m \frac{(-1)^m}{m!(m+1)!}$$

$$= \frac{1}{(Z|n|^\gamma)^{1/2}} J_1(2\sqrt{Z/|n|^\gamma}). \quad (12)$$

To obtain the prefactor Q_T , we need to include terms of order γ because $(1 + \alpha\gamma)^{1/\gamma} = e^{\alpha} [1 + O(\gamma)]$. For this we expand near $Z = Z_0$ using $J_0(2\sqrt{Z}) = J_0(2\sqrt{Z_0}) - (Z - Z_0) J_1(2\sqrt{Z_0})/\sqrt{Z_0}$. Substituting this expansion into (11), we obtain

$$Z = Z_0 \left[1 - \gamma \left(\log \frac{4e^C}{2.25978} \right) \right] \quad (13)$$

Using $Z = K_T/\gamma = [g/(2\pi T)]^\gamma/\gamma$ and $Z_0 = 1.4458$, we obtain from Eq. (13)

$$T_c = Q_T \omega_0 = Q_T g(1.4458\gamma)^{-1/\gamma} [1 + O(\gamma)], \quad (14)$$

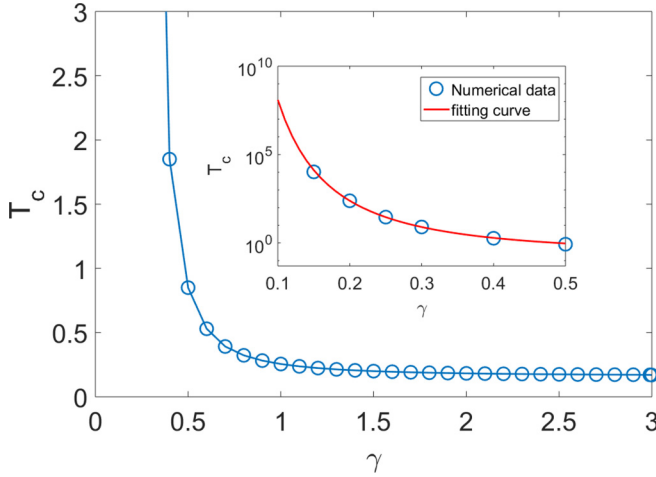


FIG. 1. Numerical results of T_c in units of g for different γ . The inset shows our numerical data for small γ and the red curve is the fitting curve based on (14) with a prefactor $Q_T = 0.4934$.

where

$$Q_T = \frac{1}{2\pi} \left(\frac{4e^C}{2.25978} \right) = 0.5018. \quad (15)$$

In Q_T , the term $4e^C$ comes from the summation over Matsubara frequencies $m = O(1)$, and the factor 2.25978 comes from the self-energy. The $4e^C$ term is the same as in BCS theory, but the self-energy contribution 2.25978 is different from $e = 2.71828$ in BCS theory. This is because even for the smallest γ , T_c is determined by large Matsubara numbers, for which $\tilde{\Sigma}_n/\omega_n = K_T/|n|^\gamma$ cannot be approximated by a constant.

Our numerical results for $T_c(\gamma)$ are shown in Fig. 1. The calculation requires care because for small γ the solution of (2) still depends on the number of Matsubara points N even when $N \sim 10^4$. We obtained T_c by solving the gap equation on mesh of $N = m \times 10^3$ points, with m ranging between 4 and 16, and then extrapolating to $N \rightarrow \infty$ (see the Appendix for details). We see from Fig. 1 that the numerical results for T_c/g are well described by $(1.4458\gamma)^{-1/\gamma}$ dependence in a surprisingly broad range of γ (roughly up to $\gamma \approx 0.5$). The numerical prefactor Q_T , extracted from the data at small g , is 0.4934, very close to $Q_T = 0.5018$ in (15). We went even further and computed the next terms in the expansion in γ . We found (see Appendix B for details) that the $O(\gamma^2)$ correction to Z is quite small even for $\gamma \leq 1$, due to small prefactor. We believe this is the reason why, even for $\gamma = 1$, the analytical $T_c = 0.35g$ is reasonably close to the numerical value $T_c = 0.26g$.

B. Calculation of the gap at $T = 0$

We next consider the non-linear gap equation at $T = 0$. We follow the same line of reasoning as above and search for the solution for the gap at high frequencies in the form

$$\Delta(\omega) = \Delta f\left(\frac{\omega}{\Delta}\right). \quad (16)$$

Substituting into the gap equation, rescaling $\omega/\Delta = \bar{\omega}$, and introducing $\bar{K} = (g/\Delta)^\gamma$, we obtain from (5)

$$f(\bar{\omega}) \left(1 + \frac{\bar{K}}{|\bar{\omega}|} \right) = \frac{\bar{K}}{2} \int \frac{d\bar{\omega}' f(\bar{\omega}')}{\sqrt{(\bar{\omega}')^2 + f^2(\bar{\omega}')}} \frac{1}{|\omega - \omega'|^\gamma}. \quad (17)$$

As before, we search for $f(\bar{\omega})$ in the form

$$f(x) = \frac{1}{x^\gamma} \left(1 + \frac{a}{x^\gamma} + \frac{b}{x^{2\gamma}} + \dots \right) \quad (18)$$

For each component of $f(x)$ we represent $\int \frac{d\bar{\omega}'}{|\bar{\omega}'|^{p\gamma}} \frac{1}{\sqrt{(\bar{\omega}')^2 + f^2(\bar{\omega}')}} \frac{1}{|\omega - \omega'|^\gamma}$ as $\bar{A} + \bar{B}$, where

$$\begin{aligned} \bar{A} &= \int \frac{d\bar{\omega}'}{|\bar{\omega}'|^{p\gamma}} \frac{1}{\sqrt{(\bar{\omega}')^2 + f^2(\bar{\omega}')}} \frac{1}{|\omega - \omega'|^\gamma} \\ &\quad - \int_1^\infty \frac{d\bar{\omega}'}{|\bar{\omega}'|^{1+p\gamma}} \left(\frac{1}{|\omega - \omega'|^\gamma} + \frac{1}{|\omega + \omega'|^\gamma} \right), \\ \bar{B} &= \int_1^\infty \frac{d\bar{\omega}'}{|\bar{\omega}'|^{1+p\gamma}} \left(\frac{1}{|\omega - \omega'|^\gamma} + \frac{1}{|\omega + \omega'|^\gamma} \right). \end{aligned} \quad (19)$$

In \bar{A} , the contribution from large $\bar{\omega}'$ cancels out, and the remaining integral reduces to $\bar{A} = 2\bar{C}/|\omega|^\gamma$, where

$$\bar{C} = \int \frac{d\bar{\omega}'}{|\bar{\omega}'|^{p\gamma}} \frac{1}{\sqrt{(\bar{\omega}')^2 + f^2(\bar{\omega}')}} - \int_1^\infty \frac{d\bar{\omega}'}{|\bar{\omega}'|^{1+p\gamma}}. \quad (20)$$

The integral does not contain $1/\gamma$, and its leading, γ -independent piece can be computed right at $\gamma = 0$, where $f(\bar{\omega}') = 1$. This piece is $\bar{C} = \int_0^\infty dx/\sqrt{x^2 + 1} - \int_1^\infty dx/x = \log 2$.

For the term \bar{B} , we obtain to order $O(\gamma)$

$$\bar{B} = \frac{2}{|\bar{\omega}|^\gamma} \int_1^{\bar{\omega}} \frac{d\bar{\omega}'}{|\bar{\omega}'|^{1+p\gamma}} + 2 \int_{\bar{\omega}}^\infty \frac{d\bar{\omega}'}{|\bar{\omega}'|^{1+(p+1)\gamma}} \quad (21)$$

Evaluating the integrals and matching the prefactors for $1/|\bar{\omega}|^{p\gamma}$ in the r.h.s. and the l.h.s. of the gap equation, we obtain

$$\begin{aligned} \Delta &= Q_\Delta \omega_0 = Q_\Delta g (1.4458\gamma)^{-1/\gamma}, \\ Q_\Delta &= \left(\frac{2}{2.25978} \right) = 0.885. \end{aligned} \quad (22)$$

Combining (14) and (22), we obtain $2\Delta/T_c = 2Q_D/Q_T = 2\pi/e^C = 3.53$, as in BCS theory. In the inset of Fig. 2 we show our numerical results for Δ at $\gamma < 0.5$. The numerical $\Delta(\gamma)$ indeed scales with ω_0 . The prefactor Q_Δ , extracted from numerical data, is 0.87, very close to the analytical $Q_\Delta = 0.885$. The ratio $2\Delta/T_c$ is plotted in Fig. 3. It clearly approaches the BCS value when $\gamma \rightarrow 0$.

IV. T_c AND Δ IN THE γ MODEL WITH $\gamma \leq 3$

We now consider the γ model with exponent $\gamma > 1$. We show that the ratio $2\Delta/T_c$ increases with γ and diverges at $\gamma = 3$. We argue that this divergence is the primary reason why earlier works [2,7,9,10] found a very large (but finite) $2\Delta/T_c$ by solving Eliashberg equations for a phonon superconductor with effective phonon-mediated pairing interaction

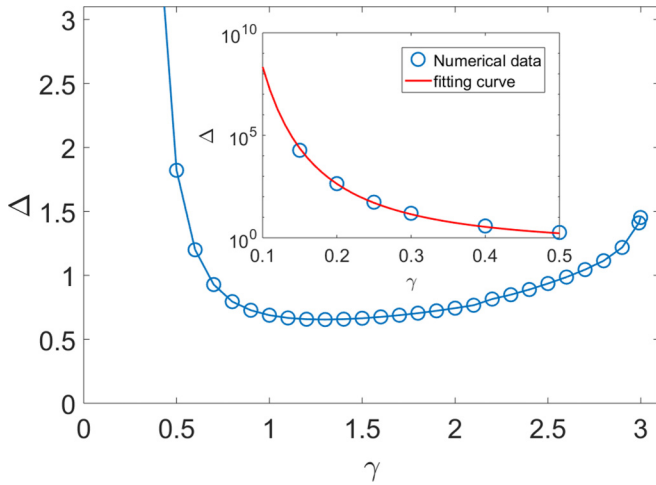


FIG. 2. Numerical results for Δ (in units of g) for different γ . The inset shows numerical data for small γ , and the red curve is the fitting based on (22). The prefactor Q_Δ , extracted from the fit, is $Q_\Delta = 0.87$.

$g^2/(\Omega^2 + \omega_D^2)$ in the limit when the Debye frequency vanishes, but the coupling g remains finite (this corresponds to $\gamma = 2$ in our notations).

A. Calculation of T_c

The onset temperature of the pairing at large γ was earlier analyzed by the two of us and collaborators [44]. T_c decreases with increasing γ and saturates at $T_c = g/(2\pi)$ in the formal limit $\gamma \rightarrow \infty$. At large γ , the gap equation becomes local in the sense that the largest contribution to the r.h.s. of the gap equation (5) for a given n comes from $m = n \pm 1$, i.e., Δ_n is predominantly coupled to Δ_{n-1} and Δ_{n+1} . This local pairing problem can be solved exactly, and the result is $K \equiv [g/(2\pi T_c)]^\gamma = 1/s$, where $s = 1.1843$ is the solution of $J_{3/2+1/s}/J_{1/2+1/s} = s - 1$. Then $T_c = s^{1/\gamma} g/(2\pi) = \frac{g}{2\pi} (1 + \frac{1}{\gamma} \log s + \dots)$. The dots stand for $O(1/\gamma^2)$ terms, which cannot be obtained within a local approach. T_c ,

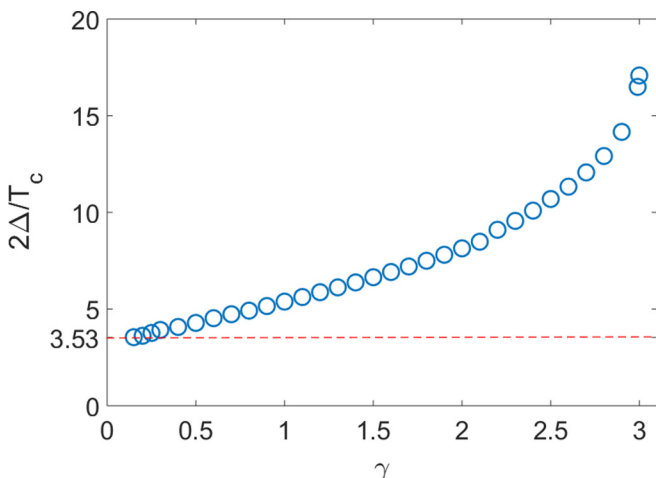


FIG. 3. Numerical results for $\frac{2\Delta}{T_c}$ for different γ .

obtained numerically (Fig. 1), indeed saturates at $g/(2\pi)$ at large γ and is actually rather close to this value for all $\gamma > 1$.

B. Calculation of Δ at $T = 0$

It is convenient to write the gap equation at $T = 0$ as

$$\begin{aligned} \Delta(\omega)\omega &= \frac{g^\gamma}{2} \int \frac{d\omega'}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} \frac{\Delta(\omega')\omega - \Delta(\omega)\omega'}{|\omega - \omega'|^\gamma} \\ &= \frac{g^\gamma}{2} \int d\omega' \frac{\Delta(\omega')}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} \frac{\text{sign}(\omega - \omega')}{|\omega - \omega'|^\gamma} \\ &\quad + \frac{g^\gamma}{2} \int d\omega' \frac{\omega'}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} \frac{\Delta(\omega') - \Delta(\omega)}{|\omega - \omega'|^\gamma}. \end{aligned} \quad (23)$$

The first contribution to the r.h.s. of (23) can be re-expressed by shifting the integration variable as

$$\frac{g^\gamma}{2} \int_0^\infty \frac{dx}{x^{\gamma-1}} \left(\frac{1}{\sqrt{1 + \left(\frac{x-\omega}{\Delta(x-\omega)}\right)^2}} - \frac{1}{\sqrt{1 + \left(\frac{x+\omega}{\Delta(x+\omega)}\right)^2}} \right). \quad (24)$$

The second contribution can be reexpressed by collecting the terms with positive and negative ω' as

$$\begin{aligned} &\frac{g^\gamma}{2} \int_0^\infty \frac{d\omega' \omega'}{\sqrt{(\omega')^2 + \Delta^2(\omega')}} [\Delta(\omega') - \Delta(\omega)] \\ &\quad \times \left(\frac{1}{|\omega - \omega'|^\gamma} + \frac{1}{|\omega + \omega'|^\gamma} \right). \end{aligned} \quad (25)$$

Both contributions have infrared divergencies $\int dx/x^{\gamma-2}$ at $\gamma > 3$, as one can easily verify. However, the integral in (24), diverges already if we approximate the gap $\Delta(\omega)$ as a constant Δ at low frequencies, while in the second contribution, Eq. (25), the divergent piece contains $\partial^2 \Delta(\omega)/\partial \omega^2$. We assume that this second contribution is smaller and focus on the first one. We set external ω in (23) and approximate its l.h.s. as $\Delta\omega$, where $\Delta = \Delta(\omega = 0)$. The equation on Δ is then obtained by expanding in Eq. (24) to linear order in ω . Neglecting again the derivatives of $\Delta(\omega)$ we obtain

$$\Delta = \int_0^\infty \frac{dx}{x^{\gamma-2}} \frac{\Delta(x)}{(\Delta_x^2 + x^2)^{3/2}}. \quad (26)$$

For γ close to 3 the integral is determined by small x , and we can approximate $\Delta(x)$ by Δ . The remaining integration over x can be carried out exactly, and we obtain

$$\Delta = g \left(\frac{\Gamma(\frac{\gamma}{2}) \Gamma(\frac{3-\gamma}{2})}{\sqrt{\pi}} \right)^{\frac{1}{\gamma}}. \quad (27)$$

When γ approaches 3, Δ diverges as $(1/(3-\gamma))^{1/3}$. For $\gamma = 2$, Eq. (27) yields $\Delta = g$. We note in passing that Δ given by (27) also diverges as $(1/\gamma)^{1/\gamma}$ at small γ ; however, the assumption that the integral in (24) is determined by small x obviously does not work there.

As independent verification, we computed $\Delta(\pi T)$ at a finite temperature for $\gamma > 3$ and indeed found that it diverges as $T \rightarrow 0$. We show the results in Fig. 4 along with

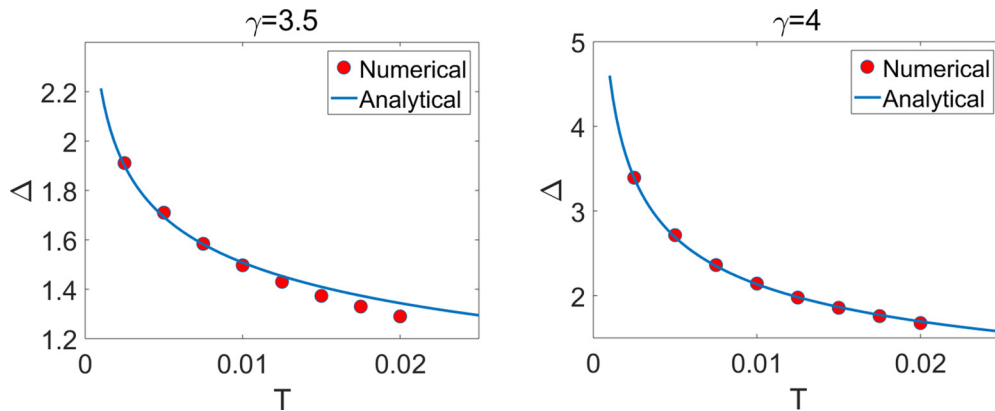


FIG. 4. The behavior of $\Delta(\pi T)$ vs T for $\gamma = 3.5$ and $\gamma = 4$. The red dots are numerical results and the blue lines are the scaling forms $\Delta \sim (\frac{1}{T})^{\frac{\gamma-3}{3}}$.

$\Delta(\pi T) \propto (1/T)^{\frac{\gamma-3}{3}}$ obtained by a straightforward scaling analysis. We see from Fig. 4 that numerical results reproduce this behavior quite well.

Combining $T_c \approx g/(2\pi)$ and Eq. (27) we obtain, for $\gamma \leq 3$,

$$\frac{2\Delta}{T_c} = 4\pi \left(\frac{\Gamma(\frac{\gamma}{2})\Gamma(\frac{3-\gamma}{2})}{\sqrt{\pi}} \right)^{\frac{1}{\gamma}}. \quad (28)$$

Near $\gamma = 3$,

$$\frac{2\Delta}{T_c} \approx 4\pi \left(\frac{1}{3-\gamma} \right)^{1/3}. \quad (29)$$

In Figs. 1, 2, and 3 we show the numerical results for $2\Delta/T_c$ in the full range of γ . We see that T_c monotonically decreases with increasing γ and saturates at $T_c = g/(2\pi)$ at large γ , while Δ is nonmonotonic: it diverges at $\gamma \rightarrow 0$ and $\gamma \rightarrow 3$ and has a minimum at $\gamma \approx 1$. The ratio $2\Delta/T_c$ monotonically increases with increasing γ and diverges at $\gamma = 3$. At $\gamma = 2$, $2\Delta/T_c = 8.3$, is already quite large, consistent with earlier works [2,7,9,10]. We see that the large $2\Delta/T_c$ for $\gamma = 2$ reflects the fact that, at this γ , $2\Delta/T_c$ already accelerates towards the divergence at $\gamma = 3$. If we substitute $\gamma = 2$ into our analytical formula for $\gamma \leq 3$, Eq. (28), we find $2\Delta/T_c \sim 12.5$, which is larger than the numerical result, but not too far from it.

V. CONCLUSIONS

In this paper we analyzed the superconducting T_c and the $2\Delta/T_c$ ratio in a metal at the verge of an instability towards a spin or a charge order. Near the instability, the dominant interaction between fermions is the exchange of soft bosonic fluctuations of spin or charge order parameter. In spatial dimension $D \leq 3$ this interaction gives rise to a non-Fermi-liquid behavior either on a whole Fermi surface or in hot regions, but also provides a strong attraction in at least one pairing channel. We considered a subset of such systems, in which soft bosons can be regarded as slow modes compared to electrons, and the pairing can be treated within Eliashberg theory with an effective local interaction $\chi_L(\Omega_m) = (g/|\Omega_m|)^\gamma$ (the γ model). The same effective theory emerges for the

pairing between fermions at the half-filled Landau level and in models studied within DMFT.

The γ model with $\gamma = 2$ describes electron-phonon superconductivity in the special limit when Debye frequency vanishes but fermion-boson coupling g remains finite, i.e., the boson-mediated interaction is $(g/|\Omega|)^2$. This problem has been extensively studied in the past [2,7,9–11]. It was well established that $T_c \approx 0.18g$ and $\Delta(0) \approx 0.75g$ remain finite, but their ratio $2\Delta/T_c \approx 8.3$ is much larger than in BCS theory. T.-H. Lee *et al.* recently analyzed numerically $2\Delta/T_c$ in the γ model for $\gamma < 2$ [40] and found that the ratio monotonically increases with increasing γ .

One goal of our work was to provide an explanation for this increase. We considered a larger range of $\gamma \leq 3$ and found that $2\Delta(0)/T_c$ diverges at $\gamma \rightarrow 3$ (T_c remains finite in this limit, but Δ_0 diverges). We obtained analytical formulas for T_c and Δ near $\gamma = 3$ and argued that they remain valid in a relatively wide range of $\gamma < 3$. We also computed T_c and Δ numerically and found good agreement between analytical and numerical results.

Another goal of our work was to analyze the opposite limit of small γ . Here we obtained the exact expressions for T_c and Δ with numerical prefactors. We emphasize that, for any nonzero γ , the normal state self-energy has a non-Fermi-liquid form at small frequencies, and non-Fermi-liquid behavior does affect the values of T_c and Δ .

We offer some words of caution: In our analysis we focused on the solution of the linearized gap equation with the highest T_c and on the “conventional” solution of the nonlinear equation at $T = 0$, for which $\Delta(\omega_m)$ is a regular function of frequency with no sign changes. There exist other solutions of the gap equation, for which $\Delta(\omega_m)$ oscillates. For $\gamma < 2$, there is little doubt that the conventional solution with non-nodal $\Delta(\omega_m)$ has the largest condensation energy. However, for $\gamma > 2$, it is possible that the largest condensation energy is for an unconventional solution with oscillating $\Delta(\omega_m)$. This would affect the ratio of $2\Delta/T_c$. Still, even if this is the case, our analysis is applicable to $\gamma \leq 2$, and it explains why $2\Delta/T_c$ increases with γ . Also, T_c in our analysis is the onset temperature for the pairing instability. In the absence of fluctuations it coincides with the actual superconducting

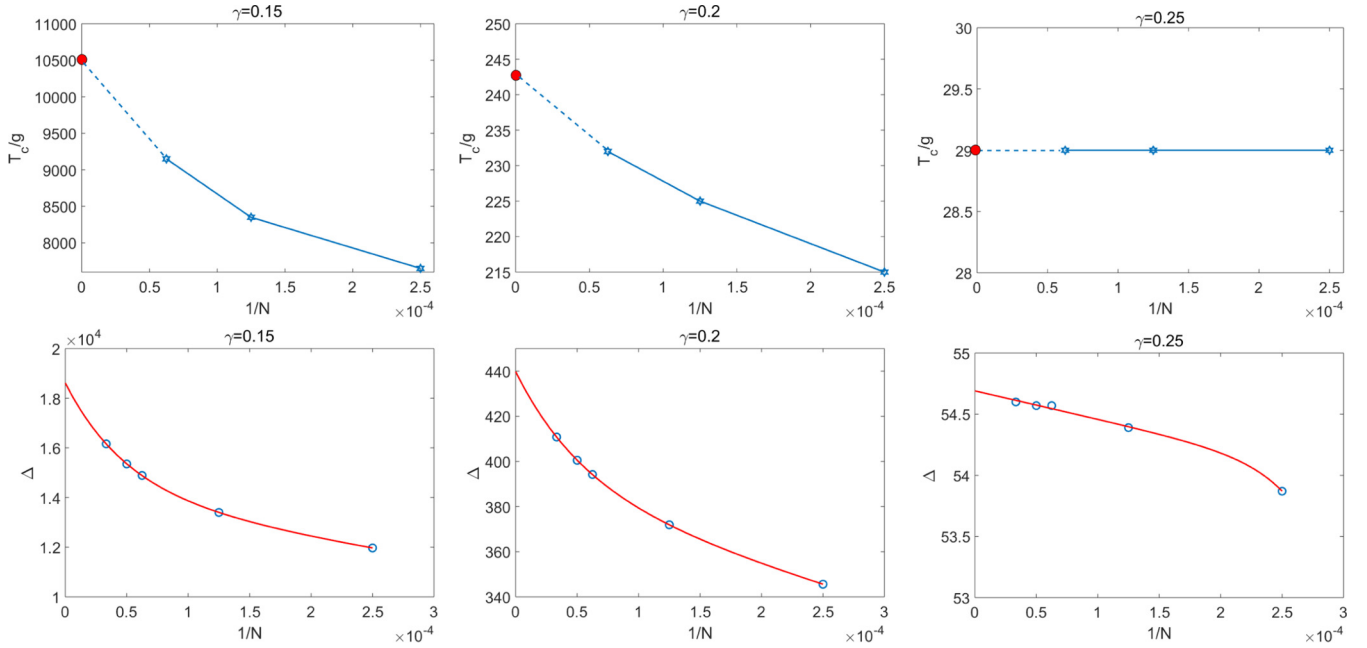


FIG. 5. Extrapolation procedure for T_c and Δ (in units of g) for $\gamma = 0.15, 0.2$, and 0.25 . By extrapolating to $N \rightarrow \infty$, we obtained $T_c = 10\,500, 243$, and 29 , respectively, and $\Delta = 18\,620, 440$, and 54.6 respectively. The red lines for $\Delta(1/N)$ are double exponential fits ($a_1 e^{-\alpha_1/N} + a_2 e^{-\alpha_2/N}$).

T_c , but when fluctuations are present the actual $T_{c,act}$ likely gets smaller, while our mean-field T_c marks the onset of pseudogap behavior. Our $2\Delta/T_c$ should then be understood as the ratio of the gap at $T = 0$ to the onset temperature for pseudogap behavior. And, finally, in our analysis we neglected the feedback from the gap opening on the form of $\chi_L(\Omega)$ (e.g., the development of the resonant peak in the spin-fluctuation propagator due to the opening of a d -wave or s^{+-} gap). Within the γ model this last effect can be cured by allowing γ to vary with T below T_c towards a smaller value.

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APPENDIX

1. Details of numerical calculations at small γ

The results of numerical calculations of T_c and Δ at small γ are presented in the insets in Figs. 1 and 2. The analysis requires care, as at small γ the numerical results depend on the number of Matsubara points N , and to obtain reliable results one should properly extrapolate to $N = \infty$. We solved the linearized gap equation on sets of $N = 4000, 8000$, and $16\,000$, identified T_c with the temperature when the largest eigenvalue crosses 1, and extrapolated the results to $N = \infty$. We show the extrapolation procedure in Fig. 5(a). In our calculation of $\Delta(\pi T)$ as the solution of the nonlinear gap equation, we used the fact that Δ rather quickly saturates below T_c , set the temperature to be $0.3T_c$, computed $\Delta(\pi T)$ for $N = 4000, 8000, 16\,000, 20\,000$, and $30\,000$ Matubara points, and then

extrapolated to $N = \infty$. We show the extrapolation procedure in Fig. 5(b).

2. The calculation of T_c at small γ to order $O(\gamma^2)$

In this subsection we extend our analysis from Sec. III A to include terms of next order in γ . The specific goal here is to understand whether corrections there are small for $\gamma = 0.2$, which is the smallest γ for which the comparison between analytic and numerical data is possible.

The calculations follow the same path as the ones we reported in Sec. III A in the main text, i.e., we write Δ as the sum of partial components $\Delta_n = (1/|n + 1/2|^\gamma) \sum_{p=0}^{\infty} a_p / |n + 1/2|^{\gamma p}$, as in Eq. (7), obtain recursive relations for a_p , and obtain T_c from a self-consistent equation on $Z = K_T/\gamma = [g/(2\pi T)]^\gamma/\gamma$. However, now at each step we extend the analysis to next order in γ . We skip the details of the calculations and report the results. The recursive relations are

$$a_p = -Z a_{p-1} \left(\frac{1}{p!(p+1)!} + \gamma^* \right), \quad (\text{A1})$$

where $\gamma^* = \gamma(1 + 0.165\gamma)$. The self-consistent condition on Z is

$$\frac{1}{Z} = \sum_{p=0}^{\infty} a_p \left(\frac{1}{p+1} + \gamma \log 4e^C + 1.353\gamma^2(p+1) \right). \quad (\text{A2})$$

The solution of the recursive relation (A1) to order γ^2 is

$$a_p = (-Z)^p \left(\frac{1}{p!(p+1)!} + \gamma^* \frac{p+2}{3p!(p-1)!} + \gamma^2 \frac{b_p}{90(p+1)!(p-2)!} \right), \quad (\text{A3})$$

where

$$b_p = 6 + 31p + 16p^2 + 26p^3 + 5p^4. \quad (\text{A4})$$

The last term in the self-consistency equation (A2) is already of order γ^2 , and it can be computed using the leading, γ -independent terms in a_p . The corresponding sum over p reduces to

$$\sum_{p=0}^{\infty} \frac{(-Z)^p}{(p!)^2} = J_0(2\sqrt{Z}). \quad (\text{A5})$$

At $T = T_c$, $J_0(2\sqrt{Z})$ is by itself of order γ , hence this term is actually of order γ^3 and can be neglected. Evaluating the

remaining sums analytically and numerically,

$$J_0(2\sqrt{Z}) = \gamma \log 4e^C \sqrt{Z} J_1(2\sqrt{Z}) - \frac{\gamma}{3} (Z^{3/2} J_1(2\sqrt{Z}) + Z J_2(2\sqrt{Z})) - 0.30246\gamma^2. \quad (\text{A6})$$

The solution of (A6) to order γ^2 is

$$Z = Z_0(1 - \gamma \log 3.15265 + 0.036\gamma^2), \quad (\text{A7})$$

where we remind that $Z_0 = 1.4458$ is the smallest solution of $J_0(2\sqrt{Z}) = 0$. We see that the γ^2 term has a very small prefactor. Hence the critical value of K_T is only weakly affected by the $O(\gamma)^2$ term.

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- [1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **106**, 162 (1957); **108**, 1175 (1957).
- [2] D. Scalapino, in *Superconductivity*, edited by R. Parks (CRC, Boca Raton, FL, 1969).
- [3] S. Maiti and A. V. Chubukov, *Phys. Rev. B* **83**, 220508 (2011).
- [4] K. A. Musaelian, J. Betouras, A. V. Chubukov, and R. Joynt, *Phys. Rev. B* **53**, 3598 (1996).
- [5] M. R. Norman, D. Pines, and C. Kallin, *Adv. Phys.* **54**, 715 (2005).
- [6] P. A. Lee, N. Nagaosa, and X.-G. Wen, *Rev. Mod. Phys.* **78**, 17 (2006).
- [7] J. P. Carbotte, *Rev. Mod. Phys.* **62**, 1027 (1990).
- [8] F. Marsiglio and J. P. Carbotte, *Phys. Rev. B* **43**, 5355 (1991).
- [9] R. Combescot, *Phys. Rev. B* **51**, 11625 (1995).
- [10] F. Marsiglio and J. P. Carbotte, in *Superconductivity: Conventional and Unconventional Superconductors*, edited by K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2008), pp. 73–162.
- [11] P. B. Allen and R. C. Dynes, *Phys. Rev. B* **12**, 905 (1975); G. Bergmann and D. Rainer, *Z. Phys.* **263**, 59 (1973); A. Karakozov, E. Maksimov, and A. Mikhailovsky, *Solid State Commun.* **79**, 329 (1991); Z. Leong, K. Limtragoon, C. Setty, and P. W. Phillips, *Phys. Rev. B* **98**, 184501 (2018).
- [12] N. E. Bonesteel, I. A. McDonald, and C. Nayak, *Phys. Rev. Lett.* **77**, 3009 (1996).
- [13] A. Abanov, A. V. Chubukov, and A. M. Finkel'stein, *Europhys. Lett.* **54**, 488 (2001).
- [14] A. Abanov, A. V. Chubukov, and J. Schmalian, *Adv. Phys.* **52**, 119 (2003).
- [15] D. T. Son, *Phys. Rev. D* **59**, 094019 (1999).
- [16] A. V. Chubukov and J. Schmalian, *Phys. Rev. B* **72**, 174520 (2005).
- [17] S.-S. Lee, *Phys. Rev. B* **80**, 165102 (2009); D. Dalidovich and S.-S. Lee, *ibid.* **88**, 245106 (2013).
- [18] S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, *Phys. Rev. B* **80**, 155129 (2009); E. G. Moon and S. Sachdev, *ibid.* **80**, 035117 (2009).
- [19] E.-G. Moon and A. Chubukov, *J. Low Temp. Phys.* **161**, 263 (2010).
- [20] M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **82**, 075127 (2010); **82**, 075128 (2010).
- [21] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, *Phys. Rev. B* **82**, 045121 (2010).
- [22] R. Mahajan, D. M. Ramirez, S. Kachru, and S. Raghu, *Phys. Rev. B* **88**, 115116 (2013); A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, *ibid.* **88**, 125116 (2013); **89**, 165114 (2014); G. Torroba and H. Wang, *ibid.* **90**, 165144 (2014); A. L. Fitzpatrick, G. Torroba, and H. Wang, *ibid.* **91**, 195135 (2015), and references therein.
- [23] E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein, and A. P. Mackenzie, *Annu. Rev. Condens. Matter Phys.* **1**, 153 (2010).
- [24] P. Monthoux, D. Pines, and G. G. Lonzarich, *Nature (London)* **450**, 1177 (2007); D. J. Scalapino, *Rev. Mod. Phys.* **84**, 1383 (2012).
- [25] M. A. Metlitski, D. F. Mross, S. Sachdev, and T. Senthil, *Phys. Rev. B* **91**, 115111 (2015).
- [26] H. Meier, C. Pépin, M. Einenkel, and K. B. Efetov, *Phys. Rev. B* **89**, 195115 (2014); K. B. Efetov, *ibid.* **91**, 045110 (2015).
- [27] S. Raghu, G. Torroba, and H. Wang, *Phys. Rev. B* **92**, 205104 (2015).
- [28] S. Lederer, Y. Schattner, E. Berg, and S. A. Kivelson, *Phys. Rev. Lett.* **114**, 097001 (2015).
- [29] J. M. Bok, J. J. Bae, H.-Y. Choi, C. M. Varma, W. Zhang, J. He, Y. Zhang, L. Yu, and X. J. Zhou, *Sci. Adv.* **2**, e1501329 (2016).
- [30] S. A. Hartnoll, D. M. Hofman, M. A. Metlitski, and S. Sachdev, *Phys. Rev. B* **84**, 125115 (2011); S. A. Hartnoll, R. Mahajan, M. Punk, and S. Sachdev, *ibid.* **89**, 155130 (2014); A. V. Chubukov, D. L. Maslov, and V. I. Yudson, *ibid.* **89**, 155126 (2014).
- [31] S. A. Maier and P. Strack, *Phys. Rev. B* **93**, 165114 (2016); S. P. Ridgway and C. A. Hooley, *Phys. Rev. Lett.* **114**, 226404 (2015).
- [32] R. Haslinger and A. V. Chubukov, *Phys. Rev. B* **68**, 214508 (2003).
- [33] G. M. Eliashberg, *J. Exp. Theor. Phys.* **11**, 696 (1960).
- [34] In several $D = 2$ systems the corrections to Eliashberg formula for the self-energy are singular at the lowest frequencies. In this case, Eliashberg theory is valid if T_c is higher than a typical frequency at which these singular corrections become dominant.
- [35] In the cases, when the interaction is peaked for fermions in hot regions on the Fermi surface, the dependence of the pairing vertex and the self-energy on the angle θ along the Fermi surface is coupled to frequency dependence, and, strictly speaking, one has to solve the set of two 2D coupled integral equations for $\Phi(\omega_m, \theta)$ and $\Sigma(\omega_m, \theta)$. This does not lead to qualitative changes in the T_c value compared to the corresponding γ

- models, in which Φ and Σ are assumed to be independent of θ at frequencies relevant to T_c , but may affect the $2\Delta/T_c$ ratio.
- [36] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, *Phys. Rev. B* **52**, 5563 (1995).
- [37] Y. Wang and A. Chubukov, *Phys. Rev. B* **88**, 024516 (2013).
- [38] D. Bergeron, D. Chowdhury, M. Punk, S. Sachdev, and A.-M. S. Tremblay, *Phys. Rev. B* **86**, 155123 (2012).
- [39] K. M. Stadler, Z. P. Yin, J. von Delft, G. Kotliar, and A. Weichselbaum, *Phys. Rev. Lett.* **115**, 136401 (2015).
- [40] T.-H. Lee, A. Chubukov, H. Miao, and G. Kotliar, *Phys. Rev. Lett.* **121**, 187003 (2018).
- [41] A. Abanov and A. V. Chubukov, *Phys. Rev. Lett.* **83**, 1652 (1999).
- [42] M. Eschrig, *Adv. Phys.* **55**, 47 (2006).
- [43] A. J. Millis, S. Sachdev, and C. M. Varma, *Phys. Rev. B* **37**, 4975 (1988).
- [44] Y. Wang, A. Abanov, B. L. Altshuler, E. A. Yuzbashyan, and A. V. Chubukov, *Phys. Rev. Lett.* **117**, 157001 (2016).
- [45] A. Abanov, A. V. Chubukov, and M. R. Norman, *Phys. Rev. B* **78**, 220507 (2008).
- [46] P. Anderson, *J. Phys. Chem. Solids* **11**, 26 (1959).
- [47] Z. Wang, W. Mao, and K. Bedell, *Phys. Rev. Lett.* **87**, 257001 (2001); R. Roussev and A. J. Millis, *Phys. Rev. B* **63**, 140504 (2001); A. V. Chubukov, A. M. Finkel'stein, R. Haslinger, and D. K. Morr, *Phys. Rev. Lett.* **90**, 077002 (2003).
- [48] O. V. Dolgov, I. I. Mazin, A. A. Golubov, S. Y. Savrasov, and E. G. Maksimov, *Phys. Rev. Lett.* **95**, 257003 (2005).
- [49] F. Marsiglio, *Phys. Rev. B* **98**, 024523 (2018).
- [50] A. V. Chubukov and D. L. Maslov, *Phys. Rev. B* **86**, 155136 (2012).
- [51] A. Klein, Y. Wu, and A.V. Chubukov, [arXiv:1812.00521](https://arxiv.org/abs/1812.00521).
- [52] Ar. Abanov *et al.* (unpublished).