Lieb-Schultz-Mattis type theorem with higher-form symmetry and the quantum dimer models

Ryohei Kobayashi,[1,*] Ken Shiozaki,[2,†] Yuta Kikuchi,[3,‡] and Shinsei Ryu[4,§]

[1]*Institute for Solid State Physics, The University of Tokyo, Kashiwa, Chiba 277-8581, Japan*
[2]*Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama, 351-0198, Japan*
[3]*RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*
[4]*James Franck Institute and Kadanoff Center for Theoretical Physics, University of Chicago, Illinois 60637, USA*

The Lieb-Schultz-Mattis theorem dictates that a trivial symmetric insulator in lattice models is prohibited if lattice translation symmetry and $U(1)$ charge conservation are both preserved. In this paper, we generalize the Lieb-Schultz-Mattis theorem to systems with higher-form symmetries, which act on extended objects of dimension $n > 0$. The prototypical lattice system with higher-form symmetry is the pure Abelian lattice gauge theory whose action consists only of the field strength. We first construct the higher-form generalization of the Lieb-Schultz-Mattis theorem with a proof. We then apply it to the $U(1)$ lattice gauge theory description of the quantum dimer model on bipartite lattices. Finally, using the continuum field theory description in the vicinity of the Rokhsar-Kivelson point of the quantum dimer model, we diagnose and compute the mixed 't Hooft anomaly corresponding to the higher-form Lieb-Schultz-Mattis theorem.

I. INTRODUCTION

Predicting low-energy properties of given many-body systems from a given kinematical data (spatial dimensions, symmetries, etc.) is of central importance, as majorities of strongly correlated systems do not admit exact analytical solutions. A prototypical example is the celebrated Lieb-Schultz-Mattis theorem [1] and its generalizations by Oshikawa and Hastings (LSMOH theorem) [2–4]. More recently, the LSMOH type theorem has been discussed for systems with various spatial lattice symmetries (space group symmetries) other than simple lattice translation [5–16].

The LSMOH type theorems provide a strong constraint on the possible low-energy spectrum of a lattice quantum many-body system for given input data of symmetries of a ground state. For example, when the lattice translation symmetry and $U(1)$ charge conservation are preserved in the ground state, the LSMOH theorem states that the system is gapless or it is gapped with ground state degenerate, if the $U(1)$ charge (the number of charged particle) per unit cell is not integral. This statement can be demonstrated, e.g., in the one-dimensional antiferromagnetic spin 1/2 XXZ chain, which is equivalent, by the Jordan-Wigner transformation, to a system of interacting fermions with particle number conservation at half filling. In this model, the system is in the gapless Tomonaga-Luttinger liquid (TLL) phase when the lattice translation symmetry is unbroken. When gapped, the LSMOH theorem dictates that ground states necessary breaks the symmetries: For example, the system may be in a Mott insulator phase with twofold

degenerate ground states, which spontaneously break the lattice translation symmetry and are related by the lattice translation. In higher dimensions, the degeneracy of ground states in a gapped system can also be accounted for by topological order [17].

The purpose of this paper is to generalize the LSMOH theorem for a wider class of systems, such as pure lattice gauge theories without matter. In particular, we will focus on and exploit the 1-form symmetry (or more generally n-form or higher-form symmetries), which is a generalization of global symmetries which act on point particles, e.g., the $U(1)$ global symmetry related to the particle number conservation [18–20]. Compared to ordinary global symmetries, where charged objects are (zero-dimensional, pointlike) particles, objects which are charged under higher-form symmetries have a dimension $n > 0$, i.e., charged objects are supported on a loop, brane, etc.

For example, consider the Maxwell theory on a $(d + 1)$-dimensional manifold X defined by the action:

$$S = -\frac{1}{2g^2} \int_X da \wedge *da, \tag{1}$$

where a is the $U(1)$ gauge field and g is the coupling constant. This theory possesses a symmetry that shifts a 1-form gauge field a by a flat connection λ: $a \mapsto a + \lambda$. This transformation defines a 1-form symmetry. The gauge invariant charged object under this symmetry is the Wilson loop operator supported on a closed loop C,

$$W(C) = \exp\left[i \int_C a\right]. \tag{2}$$

Here, the 1-form symmetry shifting the Wilson loop operator (2) defined on C is generated by the integration of $(d - 1)$-form $*da/g^2$ on a certain $(d - 1)$-dimensional manifold that intersects C once. This should be contrasted with the ordinary

*r.kobayashi@issp.u-tokyo.ac.jp
†ken.shiozaki@riken.jp
‡yuta.kikuchi@riken.jp
§ryuu@uchicago.edu

case of 0-form continuous symmetry, where the generator is given by integrating the d-form Noether current on a d-dimensional closed manifold regarded as a space. Similarly, in general, an operator that generates the transformation of an n-form symmetry in a $(d+1)$-dimensional spacetime is supported on a $(d-n)$-dimensional closed manifold.

One of the important concepts that can be generalized to higher-form symmetries is the spontaneous symmetry breaking, which is familiar from ordinary 0-form global symmetries [20–22]. Namely, the higher-form symmetry is spontaneously broken in the reference ground state $|0\rangle$, if there exists an operator $\mathcal{O}$ which is charged under the symmetry such that $\langle 0|\mathcal{O}|0\rangle \neq 0$. A typical example is the massless photon, which can be interpreted as a Nambu-Goldstone boson as a consequence of spontaneous breaking of 1-form $U(1)$ symmetry in the Maxwell theory.

More interestingly, the ordinary quantum anomaly has also been generalized to those for higher-form symmetries. The 't Hooft anomaly, which is an obstruction to promoting a global symmetry to a local gauge symmetry, has been known to impose rigorous constraints on ground state or infrared structure of quantum field theories by means of the anomaly matching argument [23,24]. Recently, new types of 't Hooft anomalies involving various symmetries including higher-form symmetries have been discovered and their consequences have been extensively studied and applied to constrain vacuum structures and phase structures of quantum field theories [12,20,25–36]. Along with the discussion on the LSMOH theorem we shall see its intimate relation with a 't Hooft anomaly [11–13,37].

A. Main results and outline

The main results and outline of the rest of the paper are summarized as follows. Section II: After summarizing basic properties of 1-form $U(1)$ symmetry in the continuum (Sec. II A), and on a lattice (Sec. II B), we review the basic properties of 1-form symmetry minimally required for the discussion of the LSMOH theorem, in Sec. II C, we construct the LSMOH theorem which is applied to systems invariant under 1-form $U(1)$ and lattice translation symmetries: It dictates the impossibility of having trivial gapped ground state without breaking the symmetries, when the "filling" of the higher-form symmetry is fractional (see "Theorem II.1"). (The generalization for n-form symmetry with $n > 1$ is straightforward, and discussed in Appendix A.)

Here, the filling for the charge of n-form symmetry is defined as follows: the $(d-n)$-dimensional hyperplane $M_{(d-n)}$ that supports the generator of the n-form symmetry is chosen so that $M_{(d-n)}$ is extended by $(d-n)$ unit lattice vectors among the d lattice vectors that constitute the whole system. With such choice, the filling is just defined as the charge per $(d-n)$-dimensional unit cell, measured on the hyperplane $M_{(d-n)}$.

The proof of the generalized theorem is in parallel with that of the original theorem by Oshikawa [2]: We first introduce the background gauge field coupled with the $U(1)$ global symmetry and consider the "adiabatic insertion" of the unit background magnetic flux, respecting the translation symmetry in the system. The unit magnetic flux can be eliminated by

the homotopically nontrivial gauge transformation (the large gauge transformation), which can change the lattice momentum of the ground state depending on the filling of the 1-form charge of the ground state. This leads to the degeneracy of the ground states with different momentum.

Section III: In Sec. III, as an interesting example that demonstrates the theorem, we consider the lattice gauge theory that simulates the dynamics of the $(2+1)$-dimensional quantum dimer model (QDM) on a bipartite lattice. This theory is a pure $U(1)$ lattice gauge theory whose Hamiltonian is analogous to the familiar compact quantum electrodynamics (CQED), but its Gauss law is modified from that of CQED due to the presence of background staggered charge density. This theory has a 1-form $U(1)$ global symmetry, which leads to the conservation of the number of dimers on a certain one-dimensional closed string in the QDM. Then, the LSMOH theorem based on the 1-form symmetry and the lattice translation symmetry implies that the system cannot be trivially gapped if the filling of the dimer on a deliberately chosen string is fractional. This result is explicitly demonstrated on the phase diagram of the QDM on the honeycomb and square lattice. For example, the filling of the 1-form symmetry is calculated as $\nu = 1/3$ in two neighboring gapped crystal (columnar and plaquette) phases of the QDM on the honeycomb lattice. In these phases the lattice translation symmetry is spontaneously broken, and the threefold degenerate ground states appear accordingly, which are related by the lattice translation to each other. A more interesting case is the incommensurate crystal found between two distinct crystal (plaquette and staggered) phases, where the gapless excitation called phason emerges. This gapless spectrum is enforced by the irrational filling of the 1-form charge realized in the incommensurate crystal.

A remarkable feature of the QDM is the existence of the special point called the Rokhsar-Kivelson (RK) point, where the exact ground state wave function can be obtained by the equal weight superposition of all dimer configuration states. When the lattice is bipartite, the RK point appears as a quantum criticality between the plaquette crystal and incommensurate ordered phase. The RK critical point on a bipartite lattice is described in the continuum by the quantum Lifshitz model, which is dual to a $U(1)$ gauge theory by the standard boson-vortex duality. In the continuum description, the LSMOH constraint is manifested in the form of the mixed 't Hooft anomaly afflicting the symmetries, by treating the lattice translation as an internal symmetry. We diagnose the 't Hooft anomaly for the 1-form $U(1)$ symmetry and the effective internal version of lattice translation, in the field theory which reproduces the vicinity of the RK critical point.

II. LSMOH THEOREM WITH 1-FORM SYMMETRY

A. 1-form $U(1)$ symmetry in the continuum

Let us start by providing several general properties of 1-form symmetry in the continuum, focusing on the symmetry group $U(1)$ for concreteness. Consider a theory written in terms of a 1-form $U(1)$ connection a, which is a connection on a principal $U(1)$ bundle over a $(d+1)$-dimensional manifold X. Assume that the action $S[a]$ consists only of the curvature da. Then, the theory is invariant under the shift of a by a flat

connection

$$a(x) \mapsto a(x) + \omega(x), \quad d\omega = 0. \tag{3}$$

Objects charged under the 1-form $U(1)$ symmetry (3) are Wilson loop operator

$$W(C) = \exp\left[i \int_C a\right], \quad C \in Z_1(X), \tag{4}$$

which measures the holonomy along C.

Flat 1-form $U(1)$ field is classified up to gauge transformations by the first cohomology group

$$[\omega] \in H^1(X; \mathbb{R}/2\pi\mathbb{Z}). \tag{5}$$

Here, $H^1(X; \mathbb{R}/2\pi\mathbb{Z})$ is generated by elements $[\lambda] \in H^1(X; \mathbb{Z})$, and then we see that the theory has a global 1-form $U(1)$ symmetry

$$a(x) \mapsto a(x) + \theta\lambda(x), \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{6}$$

and $[\lambda] \in H^1(X; \mathbb{Z})$. In the case of $\theta \in 2\pi\mathbb{Z}$, the shift by $\theta\lambda(x)$ in (6) corresponds to large gauge transformations, which leave Wilson loop operators invariant.

The symmetry transformation is implemented by an operator $U_\theta(M^{(d-1)})$ supported on a $(d-1)$-dimensional manifold $M^{(d-1)}$. It has the following equal time commutation relation with Wilson loop operators:

$$U_\theta(M^{(d-1)})W(C)$$
$$= e^{i\theta \cdot \mathcal{I}(C, M^{(d-1)})} \cdot W(C)U_\theta(M^{(d-1)}) \text{ at equal time}, \tag{7}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $\mathcal{I}(C, M^{(d-1)})$ is the intersection number.

As 0-form global symmetries, 1-form global symmetries can be "gauged" or promoted to local (gauge) symmetries. What it means by gauging 1-form $U(1)$ symmetry is to introduce a flat 2-form background $U(1)$ field $B(x)$, $dB = 0$, and introduce the gauge equivalence relation

$$a(x) \mapsto a(x) + \theta(x)\lambda(x)$$
$$B(x) \mapsto B(x) - d(\theta(x)\lambda(x)) = B(x) - d\theta(x) \wedge \lambda(x), \tag{8}$$

so that the covariant derivative

$$D_B a := da + B \tag{9}$$

is invariant. Note that the flatness $dB = 0$ is retained under the transformation (8). The gauge equivalence classes of $B(x)$ are determined by the holonomy on a surface

$$\int_C B \in \mathbb{R}/2\pi\mathbb{Z}, \quad C \in Z_2(X), \tag{10}$$

which is classified by the second cohomology group

$$[B] \in H^2(X; \mathbb{R}/2\pi\mathbb{Z}). \tag{11}$$

B. 1-form $U(1)$ symmetry on a lattice

1-form symmetry can also be formulated for lattice systems. It can be most easily done by considering discrete spacetime by triangulating X. A 1-form field configuration is an assignment of $a \in \mathbb{R}/2\pi\mathbb{Z}$ on each edge (1-simplex), where we define $a_{(ab)} = -a_{(ba)}$ for each edge (ab). We assume that the action $S[a]$ depends on a only via $(da)_{(abc)} =$

$a_{(ab)} + a_{(bc)} + a_{(ca)}$ on a triangle (2-simplex) with vertices a, b, c. Then the action is invariant under the following 1-form global $U(1)$ transformation

$$a_{(ab)} \mapsto a_{(ab)} + \theta\lambda_{(ab)}, \tag{12}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a constant, $d\lambda = 0$, and

$$\sum_{(ab) \in C} \lambda_{(ab)} \in \mathbb{Z}, \quad C \in Z_1(X). \tag{13}$$

Upon gauging this $U(1)$ symmetry, we assign the background field $B \in \mathbb{R}/2\pi\mathbb{Z}$ to each triangle (2-simplex). B denotes a flat 2-form $U(1)$ gauge field on the lattice, subject to the following constraint. Given a tetrahedral (3-simplex) with vertices a, b, c, d, we have a flatness condition

$$B_{(abc)} - B_{(abd)} + B_{(acd)} - B_{(bcd)} = 0 \mod 2\pi. \tag{14}$$

Then, we replace the differential as

$$a_{(ab)} + a_{(bc)} + a_{(ca)} \mapsto a_{(ab)} + a_{(bc)} + a_{(ca)} + B_{(abc)}, \tag{15}$$

and the gauge transformation is introduced as

$$a_{(ab)} \mapsto a_{(ab)} + \beta_{(ab)},$$
$$B_{(abc)} \mapsto B_{(abc)} - (\beta_{(ab)} + \beta_{(bc)} + \beta_{(ca)}), \tag{16}$$

to maintain the $U(1)$ symmetry. B is a $U(1)$ 2-cocycle, and a gauge transformation (16) shifts $B_{(abc)}$ by a 2-coboundary. Hence the moduli space of flat background $U(1)$ 2-form gauge field B is identified as $H^2(X; \mathbb{R}/2\pi\mathbb{Z})$, and gauge equivalent classes of B are determined by the holonomy

$$\sum_{(abc) \in C} B_{(abc)} \in \mathbb{R}/2\pi\mathbb{Z}, \quad C \in Z_2(X). \tag{17}$$

C. LSMOH theorem with 1-form symmetry

In this subsection, we construct the generalized LSMOH theorem involving 1-form $U(1)$ symmetry. We derive an analog of filling constraint on low energy spectrum for a lattice system with 1-form $U(1)$ symmetry, employing an "adiabatic insertion" of flat 2-form field B. The case of n-form symmetry for $n > 1$ is given in Appendix A.

We consider a theory consisting of 1-form $U(1)$ field a living on edges of a d-dimensional cubic lattice with a periodic structure, whose vertices are labeled as $(x_1, x_2, \ldots, x_d) \in \mathbb{Z}/L_1\mathbb{Z} \times \mathbb{Z}/L_2\mathbb{Z} \times \cdots \times \mathbb{Z}/L_d\mathbb{Z}$. Time may be either continuous or discretized on a lattice. The theory is invariant under the following global $U(1)$ transformation (12). The theory is also invariant under the translation $\mathcal{T}_l$ about one unit cell along the lth direction, which acts on a as

$$a_j(\mathbf{x}) \mapsto a_j(\mathbf{x} + \mathbf{e}_l) \tag{18}$$

for $1 \leqslant j \leqslant d$, where $\mathbf{e}_l$ is a unit lattice vector in the lth direction, and $a_j(\mathbf{x})$ is a 1-form field on the edge $(\mathbf{x}, \mathbf{x} + \mathbf{e}_j)$.

Now we assume that neither $U(1)$ nor the translation symmetry is broken. Then we gauge the $U(1)$ symmetry (12) by coupling with a background flat 2-form gauge field B defined on faces. Consider the field configuration that corresponds to adiabatic flux insertion, represented as follows [see Fig. 1(a)]

$$\begin{aligned} B_{lm} &= 0 & t &< 0, \\ B_{lm}(\mathbf{x}; t) &= \delta(x_m) \cdot 2\pi t/L_l T & 0 &\leqslant t < T, \\ B_{lm}(\mathbf{x}; t) &= \delta(x_m) \cdot 2\pi/L_l & T &\leqslant t, \end{aligned} \tag{19}$$
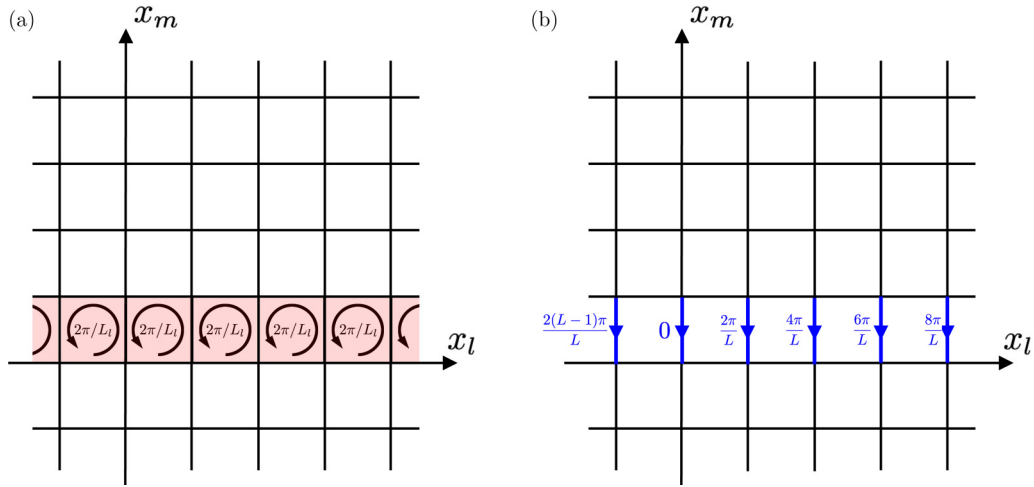
FIG. 1. (a) Configuration of 2-form field B on the $x_l x_m$ plane. (b) A large gauge transformation.

where m satisfies $m \neq l, 1 \leqslant m \leqslant d$, and $\delta(x)$ is a delta function such that

$$\begin{aligned} \delta(x) &= 1 \quad x = 0 \\ \delta(x) &= 0 \quad x \neq 0. \end{aligned} \tag{20}$$

$B_{lm}(\mathbf{x}; t)$ denotes B on a face whose vertices are $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_l, \mathbf{x} + \mathbf{e}_l + \mathbf{e}_m, \mathbf{x} + \mathbf{e}_m\}$ at time t. The other components of B are 0. For the configuration (19), the holonomy of B defined as (10) along the $x_l x_m$ plane grows gradually from 0 to 2π as time proceeds,

$$\sum_{f \in C} B_f = \frac{2\pi t}{T} \quad 0 \leqslant t < T, \tag{21}$$

where f is a label of a face, and C is a plane that includes vertices written as $(y_l \mathbf{e}_l + y_m \mathbf{e}_m)$ for $0 \leqslant y_l < L_l$, $0 \leqslant y_m < L_m$.

Suppose that the Hamiltonian at $t = 0$ (written as H_0) has a finite excitation gap above the ground state and that the gap does not close during the process of adiabatic flux insertion. At $t = 0$, a ground state $|\psi_0\rangle$ is chosen (when the ground state are degenerate) so that it is an eigenstate of $\mathcal{T}_l$ and a $U(1)$ symmetry transformation operator Q_m,

$$\mathcal{T}_l |\psi_0\rangle = e^{ip_l} |\psi_0\rangle, \quad Q_m |\psi_0\rangle = \nu \prod_{1 \leqslant k \leqslant d, k \neq m} L_k |\psi_0\rangle, \tag{22}$$

where Q_m is a $U(1)$ charge operator associated with $(d-1)$-dimensional hyperplane characterized as $x_m = 0$, and ν denotes the $U(1)$ charge per unit cell. During the adiabatic process, the configuration of B is always translation symmetric, hence the state remains the eigenstate of $\mathcal{T}_l$ with the eigenvalue e^{ip_l}. When the holonomy (21) along C reaches the unit flux quantum 2π at $t = T$, the original ground state evolves into some ground state $|\psi'_0\rangle$ of the Hamiltonian at $t = T$ (written as H'_0), that satisfies $\mathcal{T}_l |\psi'_0\rangle = e^{ip_l} |\psi'_0\rangle$. The configuration of the flat background $U(1)$ gauge field (19) at $t = T$ with the holonomy 2π is gauge equivalent to that of $t = 0$, by the following gauge transformation [see Fig. 1(b)]

$$\begin{aligned} a_m(\mathbf{x}) &\mapsto a_m(\mathbf{x}) - \delta(x_m) \cdot 2\pi x_l / L_l, \\ B_{lm}(\mathbf{x}) &\mapsto B_{lm}(\mathbf{x}) - \delta(x_m) \cdot 2\pi / L_l. \end{aligned} \tag{23}$$

We write the symmetry operator corresponding to the above gauge transformation as U_{lm}. Then, it is found that $U_{lm}|\psi'_0\rangle$ is also a ground state of H_0, and we can see that there is the following commutation relation between U_{lm} and $\mathcal{T}_l$

$$U_{lm} \mathcal{T}_l U_{lm}^\dagger = \mathcal{T}_l \exp\left[-\frac{2\pi i}{L_l} Q_m \right]. \tag{24}$$

Now we obtain the action of $\mathcal{T}_l$ on $U_{lm}|\psi'_0\rangle$ using the commutation relation (24)

$$\begin{aligned} \mathcal{T}_l U_{lm}|\psi'_0\rangle &= e^{ip_l} \exp\left[\frac{2\pi i}{L_l} Q_m \right] \cdot U_{lm}|\psi'_0\rangle \\ &= \exp\left[ip_l + 2\pi i \nu \prod_{k \neq l, m} L_k \right] \cdot U_{lm}|\psi'_0\rangle. \end{aligned} \tag{25}$$

We have used that the gauge transformation U_{lm} commutes with the $U(1)$ charge Q_m. Thus, if we have $\nu = p/q$, with L_l integer multiple of q and $\prod_{k \neq l, m} L_k$ mutually prime with q, the momentum of $U_{lm}|\psi'_0\rangle$ is written as $p_l + 2\pi r/q$, using some integer r mutually prime with q. Therefore, we obtain at least q mutually orthogonal ground states $|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{q-1}\rangle$ with different momentum, such that

$$|\psi_{k+1}\rangle = U_{lm}|\psi'_k\rangle, \quad \mathcal{T}_l |\psi_k\rangle = \exp\left[i\left(p_l + \frac{2\pi kr}{q} \right) \right] |\psi_k\rangle. \tag{26}$$

Summarizing, we have proven the following [38]:

Theorem II.1. (LSMOH theorem for 1-form symmetry).

Consider a quantum many-body system defined on a d-dimensional periodic lattice, in the presence of a global 1-form $U(1)$ symmetry and a translation symmetry along the lth primitive lattice vector, and assume that both symmetries are not broken. Then, if the $U(1)$ charge (measured on a $(d-1)$-dimensional hyperplane characterized by $x_m = 0$ for $m \neq l$) per unit cell is $\nu = p/q$ at the ground state, there are only two possibilities for the low energy spectrum:

(1) The system is gapped, and the ground states are at least q-fold degenerate, or

(2) The system is gapless.

III. APPLICATION TO THE QUANTUM DIMER MODEL

In this section, we apply the generalized LSMOH theorem, obtained in Sec. II C, to the $(2+1)$-dimensional quantum dimer model (QDM) on a bipartite lattice. Previous applications of the original LSMOH theorem to the QDM are found in Refs. [39,40].

A. Quantum dimer model and $U(1)$ lattice gauge theory

The Hilbert space of the QDM is identified with the set of possible dimer coverings of a lattice, e.g., the square,

honeycomb lattice, etc. For each dimer covering $\mathcal{C}_{\text{dimer}}$, we define a corresponding quantum state $|\mathcal{C}_{\text{dimer}}\rangle$. The set of states $\{|\mathcal{C}_{\text{dimer}}\rangle\}$ are orthogonal

$$\langle \mathcal{C}_{\text{dimer}} | \mathcal{C}'_{\text{dimer}} \rangle = \delta_{\mathcal{C}_{\text{dimer}}, \mathcal{C}'_{\text{dimer}}} \qquad (27)$$

and complete. The Hamiltonian of the QDM model typically consists of two kinds of terms, one of which is diagonal in the basis $\{|\mathcal{C}_{\text{dimer}}\rangle\}$, and the other induces "hopping" between different dimer configurations. For the honeycomb and square lattices, the Hamiltonians are given by

$$H = -t \sum_{\{\bigcirc\}} \left(| \hexdimerA \rangle\langle \hexdimerB | + | \hexdimerB \rangle\langle \hexdimerA | \right) + v \sum_{\{\bigcirc\}} \left(| \hexdimerA \rangle\langle \hexdimerA | + | \hexdimerB \rangle\langle \hexdimerB | \right), \qquad (28)$$

$$H = -t \sum_{\{\square\}} \left(| \sqdimerA \rangle\langle \sqdimerB | + | \sqdimerB \rangle\langle \sqdimerA | \right) + v \sum_{\{\square\}} \left(| \sqdimerA \rangle\langle \sqdimerA | + | \sqdimerB \rangle\langle \sqdimerB | \right), \qquad (29)$$

respectively.

The QDM on a bipartite lattice can be formulated in terms of the $U(1)$ lattice gauge theory [41]. To derive this, we consider the enlarged Hilbert space where we introduce operators n on edges of the lattice taking their eigenvalues in $\mathbb{Z}$, which in the original QDM are $\mathbb{Z}_2$ variables representing the presence/absence of dimers on a given link. Conjugate to operators n on each edge, we also introduce operators θ taking their eigenvalues in $[0, 2\pi)$, which can raise or lower n on each edge (i.e., "create" or "annihilate" dimers). The enlarged Hilbert space is subject to the constraint, the "dimer constraint," that for each vertex, variables n for links emanating from it sum to 1. As we will see later, the local dimer constraint corresponds to the Gauss law in the $U(1)$ gauge theory.

In terms of n and θ, we consider the following Hamiltonian in the enlarged Hilbert space

$$H_{\text{eff}} = K \sum_{(ab)\in\text{edge}} \left(n_{(ab)} - \frac{1}{2} \right)^2 + H_0[n, \theta]. \qquad (30)$$

Here, for large positive K, the first term acts as the projector onto the physical Hilbert space $n \in \{0, 1\}$. The second term $H_0[n, \theta]$ reproduces the dynamics of the QDM in the physical Hilbert space. For example, in the case of the QDM on the square lattice, H_0 is given by

$$H_0 = -t \sum_{\mathbf{x}} [n_1(\mathbf{x})n_1(\mathbf{x} + \mathbf{e}_2) + n_2(\mathbf{x})n_2(\mathbf{x} + \mathbf{e}_1)] + 2v \sum_{\{\square\}} \cos[\theta_1(\mathbf{x}) - \theta_2(\mathbf{x} + \mathbf{e}_1) + \theta_1(\mathbf{x} + \mathbf{e}_2) - \theta_2(\mathbf{x})], \qquad (31)$$

where $n_j(\mathbf{x})$ and $\theta_j(\mathbf{x})$ are link variables on an edge $(\mathbf{x}, \mathbf{x} + \mathbf{e}_j)$.

To cast the above rotor model in the language of the $U(1)$ gauge theory, we assign an orientation to each edge of the bipartite lattice following the "all-in all-out" rule: For a bipartite lattice consisting of A and B sublattices, each edge is oriented from a vertex on A sublattice to the other vertex on B sublattice. Then, we define gauge and electric fields by

$$A_{(ab)} = \theta_{(ab)}, \qquad E_{(ab)} = n_{(ab)}, \qquad (32)$$

where $a \in $ A and $b \in $ B, and we impose $A_{(ab)} = -A_{(ba)}$, $E_{(ab)} = -E_{(ba)}$. In terms of these variables, the dimer constraint can be written as the Gauss law constraint

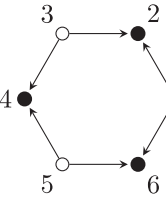$$(\text{div } E(\mathbf{x}) - \rho(\mathbf{x}))|\text{Phys.}\rangle = 0, \qquad (33)$$

on physical states, where the lattice divergence is defined as

$$\text{div } E(\mathbf{x}) \equiv \sum_{\substack{\mathbf{x}' \\ (\mathbf{x}\mathbf{x}')\in\text{edge}}} E_{(\mathbf{x}\mathbf{x}')}, \qquad (34)$$

and the staggered charge density $\rho(\mathbf{x})$ is defined as

$$\rho(\mathbf{x}) = 1\,\mathbf{x} \in \mathsf{A}, \quad \rho(\mathbf{x}) = -1\,\mathbf{x} \in \mathsf{B}. \qquad (35)$$

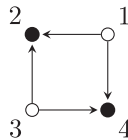Now we can express the effective Hamiltonian (30) in terms of $U(1)$ gauge fields. On the honeycomb lattice

$$H_{\text{eff}} = K \sum_{\substack{\mathbf{x}\in\mathsf{A},\mathbf{x}'\in\mathsf{B} \\ (\mathbf{xx}')\in\text{edge}}} \left(E_{(\mathbf{xx}')} - \frac{1}{2} \right)^2 - t \sum_{\{\bigcirc\}} (E_{(12)}E_{(34)}E_{(56)} - E_{(23)}E_{(45)}E_{(61)}) + 2v \sum_{\{\bigcirc\}} \cos[\text{rot}\, A], \qquad (36)$$

where the vertices labeled as A (resp. B) are described as white circles (resp. black circles). The lattice rotation on a plaquette is defined as summation of link variables around the plaquette counterclockwise. For example, in the case of the honeycomb lattice

$$\text{rot}\, A = A_{(12)} + A_{(23)} + A_{(34)} + A_{(45)} + A_{(56)} + A_{(61)}. \qquad (37)$$

Similarly, on the square lattice

$$H_{\text{eff}} = K \sum_{\substack{\mathbf{x}\in\mathsf{A},\mathbf{x}'\in\mathsf{B} \\ (\mathbf{xx}')\in\text{edge}}} \left(E_{(\mathbf{xx}')} - \frac{1}{2} \right)^2 - t \sum_{\{\square\}} (E_{(12)}E_{(34)} + E_{(23)}E_{(41)}) + 2v \sum_{\{\square\}} \cos[\text{rot}\, A]. \qquad (38)$$

In the expressions (36), (38), and the Gauss law constraint (33), we obtain faithful representations of QDM Hamiltonians (28), (29) by taking the limit $K \to \infty$.

B. 1-form symmetry and LSMOH theorem

The lattice gauge theories (36), (38) clearly have the lattice translation symmetry which leaves the sublattice structure invariant. In addition, these theories are also invariant under the following 1-form global $U(1)$ transformation

$$U(1)_{[1]} : A \mapsto A + \omega, \qquad (39)$$

where ω represents a flat field, i.e., $\text{rot}\,\omega = 0$. The operator U_ω which implements the $U(1)_{[1]}$ transformation (39) is expressed as

$$U_\omega = \exp(iQ_\omega); \quad Q_\omega = \sum_{\substack{\mathbf{x}\in\mathsf{A},\mathbf{x}'\in\mathsf{B} \\ (\mathbf{xx}')\in\text{edge}}} \omega_{(\mathbf{xx}')} E_{(\mathbf{xx}')}. \qquad (40)$$

Since U_ω commutes with both E and $\text{rot}\, A$, one can easily verify that U_ω commutes with the whole Hamiltonian (36), (38). Thus, we can apply the LSMOH theorem for 1-form symmetry (Theorem 1.1) to the gauge theories and deduce that the system cannot be trivially gapped (i.e., the ground state is degenerate or gapless) if the filling of the $U(1)_{[1]}$ charge is fractional. Below, we will demonstrate the LSMOH is consistent with the known phases that exist in the honeycomb lattice QDM. (From now on, we will mostly discuss the case of the honeycomb lattice in the main text. The discussion on the square lattice is found in Appendix C [42].)

First, let us clarify the meaning of the filling fraction for the 1-form charge. The $U(1)_{[1]}$ charge in the honeycomb lattice QDM (36) measured on a line $x_2 = 0$ is

$$Q_2(x_2 = 0) \equiv \sum_{x_1=0}^{L_1-1} E_\alpha(x_1, 0). \qquad (41)$$

Here, we employed the Cartesian coordinate (x_1, x_2) defined on the honeycomb lattice, whose x_1 axis is vertical to one edge in the honeycomb lattice. The scale is chosen such that the distance between two neighboring parallel edges is 1. $E_\alpha(\mathbf{x})$

is an electric field on an edge $(\mathbf{x}, \mathbf{x} + \mathbf{e}_\alpha)$, where $\mathbf{e}_\alpha$ is a lattice vector connecting neighboring vertices which is perpendicular to the x_1 axis (see Fig. 2). We assume that the lattice is periodic as required for applying the LSMOH theorem on the system, with L_1 being the length of the system in the x_1 direction. The operator Q_2 generates a shift of the Wilson loop extended in the x_2 direction, via the canonical commutation relation between E and A. In the QDM, Q_2 is simply the sum of the number of dimers vertical to the x_1 axis touching the vertices on a line $x_2 = 0$, since using (32) we see that

$$Q_2(x_2 = 0) = \sum_{x_1=0}^{L_1-1} n_\alpha(x_1, 0). \qquad (42)$$

Then, the filling $\nu \equiv Q_2/L_1$ is the number of dimers per the unit length.

Now let us refer to several ordered phases in QDM. It is known that there are three distinct ordered phases in the QDM on the honeycomb lattice [43] (see Fig. 3). In the region $\nu \ll t$ the system lies in the "columnar" crystal phase, which gives way to the "plaquette" crystal phase by a first order transition. In these two ordered phases the filling is calculated
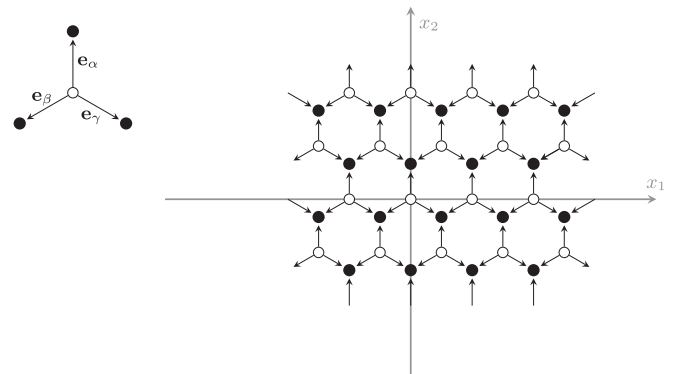
FIG. 2. Cartesian coordinate on the honeycomb lattice. Lattice vectors connecting two neighboring vertices are labeled as $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, and $\mathbf{e}_\gamma$, respectively.
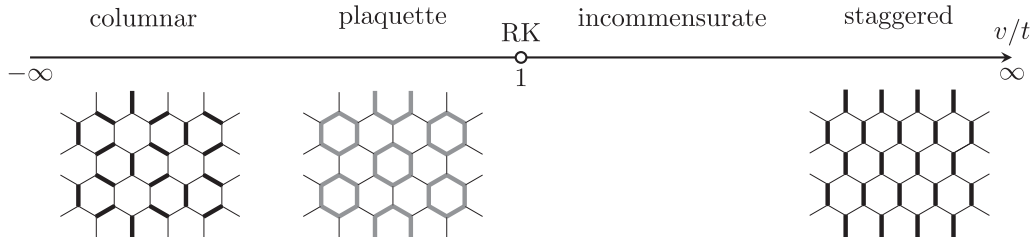
FIG. 3. The schematic phase diagram of the QDM on the honeycomb lattice. The 1-form filling $\nu = 1$ in the staggered phase. $\nu = 1/3$ in the columnar and plaquette phase, which terminate at the RK critical point. There is a sequence of the incommensurate crystal and commensurate ordered phase between the RK point and the staggered phase [44,45], where ν increases continuously from 1/3 to 1.

as $\nu = 1/3$, then the generalized LSMOH theorem based on $U(1)_{[1]}$ and the lattice translation symmetry in the x_1 direction dictates that the lattice translation symmetry must be broken to have a gapped phase. Here, we note that the continuous 1-form symmetry such as $U(1)_{[1]}$ cannot be spontaneously broken in a $(2 + 1)$-dimensional system, as guaranteed by the generalized version of the Coleman-Mermin-Wagner theorem. In the columnar and plaquette crystal phase, the lattice translation is indeed spontaneously broken, and threefold degenerate ground states appear accordingly, which are related by the lattice translation. The third ordered phase is a "staggered" phase which appears in the region $\nu > t$, where the filling is calculated as $\nu = 1$ if dimers in the staggered phase are vertical to the x_1 axis, otherwise $\nu = 0$. At any rate ν is integral and we deduce that a gapped phase can be realized without breaking the lattice translation symmetry, which is consistent with symmetries of the staggered phase.

We remark that the 1-form filling ν is allowed to take distinct values for different phases of the QDM. This should be contrasted with the 0-form filling of lattice models which preserve the particle number [46]. The LSMOH theorem is applied to each sector of the Hilbert space with the specific 1-form charge. More generally, in gauge theories we usually sum over all configurations of gauge fields in path integral, without fixing specific topological sector.

C. Continuum field theory description

The special point $\nu = t$ which appears at the transition between the plaquette and staggered phases, is called the Rokhsar-Kivelson (RK) point [47,48]. The RK point is remarkable in the sense that one can obtain exact ground states as the equal weight superpositions of states in each sector of dimer configurations connected by the resonance term in (28). The vicinity of the RK point has a field theoretical description in the continuum. The degree of freedom in the effective field theory is a scalar field ϕ with the compactification radius 2π, which is introduced as the height field. In terms of ϕ, the underlying theory believed to control the vicinity of the RK point is given by the following Lagrangian [48]

$$\mathcal{L} = \frac{1}{2}\partial_t\phi\partial_t\phi - \frac{\rho}{2}\nabla_i\phi\nabla_i\phi - \frac{\kappa^2}{2}\nabla^2\phi\nabla^2\phi$$
$$= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\tilde{\kappa}^2}{2}\partial_i\partial^i\phi\partial_j\partial^j\phi$$
$$= \mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{\tilde{\kappa}^2}{2}(\Delta\phi)^2, \tag{43}$$

where $\partial_\mu := (\partial_0, \sqrt{\rho}\nabla_i)$, $\Delta = \partial_i\partial^i$ and $\tilde{\kappa} := \kappa/\rho$, with $\rho \propto -(\nu/t) + 1$ changing its sign precisely at the RK point. An identification of the dimer variables n and the scalar field ϕ of the field theory is [44,49]

$$n_\alpha - \frac{1}{3} = \frac{1}{2\pi}\partial_1\phi + \frac{1}{2}\left[e^{i\phi}e^{\frac{4\pi i x_1}{3}} + \mathrm{H.c.}\right],$$

$$n_\beta - \frac{1}{3} = \frac{1}{2\pi}\left(-\frac{1}{2}\partial_1 + \frac{\sqrt{3}}{2}\partial_2\right)\phi + \frac{1}{2}\left[e^{i\phi}e^{\frac{4\pi i x_1}{3} + \frac{4\pi i}{3}} + \mathrm{H.c.}\right],$$

$$n_\gamma - \frac{1}{3} = \frac{1}{2\pi}\left(-\frac{1}{2}\partial_1 - \frac{\sqrt{3}}{2}\partial_2\right)\phi + \frac{1}{2}\left[e^{i\phi}e^{\frac{4\pi i x_1}{3} - \frac{4\pi i}{3}} + \mathrm{H.c.}\right]. \tag{44}$$

1. $U(1)_{[1]} \times \mathcal{T}_1$ symmetry

Let us refer to symmetry in the continuum description (43). One can read off the action of the translation symmetry in x_1 direction (denoted by $\mathcal{T}_1$) on ϕ from (44) by imposing that n transforms correctly under the translation. Then, in the continuum limit $\mathcal{T}_1$ acts on ϕ as an internal symmetry

$$\mathcal{T}_1 : \phi \mapsto \phi - \frac{2\pi}{3}. \tag{45}$$

Besides the $\mathcal{T}_1$ symmetry (45), this theory has a $U(1)$ 1-form symmetry, whose generator is given by (42) in the lattice model. The continuum description of the charge operator of $U(1)_{[1]}$ (42) in the QDM is expressed as

$$Q_2 = \sum_{x_1=0}^{L_1-1} n_\alpha(x_1, 0)$$
$$= \sum_{x_1=0} \frac{1}{2\pi}\partial_1\phi(x_1, 0) + \frac{1}{2}\left[e^{i\phi(x_1,0)}e^{\frac{4\pi i x_1}{3}} + \mathrm{H.c.}\right] + \frac{L_1}{3}$$
$$\approx \int_{x_2=0} dx_1\partial_1\phi + \frac{L_1}{3}, \tag{46}$$

where we used the identification (44) and dropped the summation of the staggered part in the last equation.

Equations of motion are read off from the Lagrangian (43),

$$\partial_\mu\partial^\mu\phi + \tilde{\kappa}^2\Delta\Delta\phi = 0, \quad \epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\phi = 0. \tag{47}$$

Conserved currents for 0-form and 1-form $U(1)$ symmetries are, respectively, given by

$$\mathrm{d}j_A = 0, \quad (*j_A)^\mu = (\partial^0\phi, \ \partial^i\phi + \tilde{\kappa}^2\Delta\Delta\phi), \tag{48}$$

$$\mathrm{d}j_B = 0, \quad (*j_B)^{\mu\nu} = \epsilon^{\mu\nu\rho}\partial_\rho\phi. \tag{49}$$

Then, (46) is identified as a generator of 1-form symmetry given by integrating the current on a line up to constant,

$$Q_2 = \int_{x_2=0} j_B + \frac{L_1}{3}. \tag{50}$$

On the other hand, $\mathbb{Z}_3$ translation symmetry (45) is a subgroup of the $U(1)$ 0-form symmetry.

According to (46), the filling measured relative to $1/3$ is identified as the gradient of the height ϕ measured in x_1 direction per unit lattice,

$$\nu - \frac{1}{3} = \frac{Q_2 - L_1/3}{L_1} \approx \int_{x_2=0} dx_1 \partial_1 \phi \tag{51}$$

which is sometimes called "tilt" [50]. The flat tilt is observed in the columnar and plaquette phase reflecting $\nu = 1/3$, while the staggered phase is fully tilted. On the tilted side of the RK transition $v/t > 1$, it is argued [44,45,51] that the tilt increases in the "incomplete devil's staircase" fashion. Namely, the increase of the tilt is continuous at least in the vicinity of the RK point on the tilted side, and there is a sequence of commensurate gapped crystal and incommensurate points. It is also argued [44] that the incommensurate region has finite measure in the parameter space. Here, the incommensurate region is characterized as the irrational tilt, which corresponds to the limit $q \to \infty$ for $\nu = p/q$. We remark that in the incommensurate region it is guaranteed nonperturbatively to have gapless spectrum by the LSMOH theorem based on $U(1)_{[1]}$ symmetry.

2. Quantum anomaly in continuum description

Next, we move on to the continuum description of the LSMOH theorem based on $U(1)_{[1]} \times \mathcal{T}_1$ symmetry. It is known [7,11–13] that the LSMOH theorem is manifested in the form of a quantum anomaly afflicting the symmetry in continuum field theory. This is analogous to 't Hooft anomaly which appears on the boundary of symmetry-protected topological phases. There is however a subtle difference between the lattice models subject to the LSM type theorem, and the boundaries of symmetry-protected topological phases—see, for example, Refs. [11,12].

In our system, such anomaly involves the combination of 1-form $U(1)_{[1]}$ and the translation symmetry $\mathcal{T}_1$. Here, we diagnose the 't Hooft anomaly involving $U(1)_{[1]} \times \mathcal{T}_1$ symmetry by looking at the action (43) under the background gauge fields for the symmetry.

We introduce a background $U(1)_{[1]}$-gauge field B by coupling to the conserved current $j_B = \mathrm{d}\phi$,

$$\mathcal{L} = \mathrm{d}\phi \wedge *\mathrm{d}\phi + \mathrm{d}\phi \wedge B - \frac{\tilde{\kappa}^2}{2}(\Delta\phi)^2, \tag{52}$$

which is invariant under a 1-form gauge transformation

$$B \mapsto B + \mathrm{d}\lambda. \tag{53}$$

Next, we introduce a background $U(1)$-gauge fields (A, C) to gauge $\mathbb{Z}_3$ symmetry by forming the covariant derivative

$\mathrm{d}\phi - A$ [19],

$$S = \int (\mathrm{d}\phi - A) \wedge *(\mathrm{d}\phi - A) + (\mathrm{d}\phi - A) \wedge B$$
$$- \frac{\tilde{\kappa}^2}{2}(\partial_i(\partial^i\phi - A^i))^2 \mathrm{d}^2 x + F \wedge (3A - \mathrm{d}C). \tag{54}$$

where A is a 1-form $U(1)$ gauge field, C is a 2π-periodic scalar field, and F is a 2-form field. Integration over F yields $3A = dC$ and makes A into a $\mathbb{Z}_3$ gauge field.

The gauged action (54) is not invariant under the 1-form gauge transformation (53) but

$$S \mapsto S - \frac{2\pi}{3}k \quad (\mathrm{mod}\ 2\pi), \tag{55}$$

with $k \in \mathbb{Z}$. This is an expected $\mathbb{Z}_3$ 't Hooft anomaly that signals the phase shift of the partition function in the presence of $\mathbb{Z}_3$ twist, under $U(1)_{[1]}$ large gauge transformation. The counterpart of this 't Hooft anomaly is observed in the lattice model, as the nontrivial commutation relation between the lattice translation $\mathcal{T}_1$ and a large $U(1)_{[1]}$ gauge transformation dependent on the filling of $U(1)_{[1]}$ charge, which leads to the degeneracy of ground states. Namely, this anomaly (55) is a continuum description of (24) with the filling at the vacuum $\nu = Q_2/L_1 = 1/3$, as realized in the plaquette phase lying in the vicinity of the RK point.

3. Gauge invariant operators

Finally, we refer to possible perturbations to the theory (43) near the RK point. There are two types of gauge invariant observables in this theory, both of which are forbidden by requiring $U(1)_{[1]} \times \mathcal{T}_1$ symmetry, respectively: One of them is a vertex operator of a magnetic charge n,

$$V_n(\mathbf{x}) = \exp[in\phi(\mathbf{x})]. \tag{56}$$

The quantization $n \in \mathbb{Z}$ follows from the compactness of the scalar field $\phi \sim \phi + 2\pi$. The lattice translation symmetry $\mathcal{T}_1$ (45) forbids V_n with $n = 3l + 1, 3l + 2$ for $l \in \mathbb{Z}$, hence the leading perturbation becomes $\cos(3\phi)$.

The other is made manifest in $U(1)$ gauge theory which is dual to (43), by standard particle-vortex duality. The dual action is written as [49]

$$S = \int \mathrm{d}^2 x \left(\frac{1}{4} \mathrm{d}a \wedge *\mathrm{d}a - \frac{\tilde{\kappa}^2}{16}(\epsilon^{ij}\partial_i f_{j0})^2 \right), \tag{57}$$

where a is $U(1)$ gauge field. Then, we find Wilson loop operator for an electric charge of charge m is gauge invariant,

$$W_m(C) = \exp\left[im \int_C a\right], \tag{58}$$

where C is a closed loop. Like the case of $V_n(\mathbf{x})$, m is also quantized as $m \in \mathbb{Z}$ when C is chosen to be noncontractible, which follows from invariance under the large gauge transformation. $W_m(C)$ for a noncontractible C is forbidden by $U(1)_{[1]}$ symmetry for arbitrary m.

This situation is analogous to the case of one-dimensional antiferromagnetic spin-1/2 XXZ chain, which is mapped to a half-filled fermion system with (0-form) $U(1)$ charge conservation. The continuum field theory for the XXZ chain is the TLL in terms of the bosonic scalar field ϕ, with (0-form)

global symmetry $U(1)_{[0]}$ and the lattice translation symmetry $\mathcal{T}$. The possible perturbations for the TLL are $V_n = e^{in\phi}$ and $\tilde{V}_m = e^{im\theta}$ for $n, m \in \mathbb{Z}$, where θ is the dual field of ϕ. Like the perturbations in the QDM, V_n is forbidden up to $\cos(2\phi)$ by the lattice translation symmetry $\mathcal{T} : \phi \mapsto \phi + \pi$, and $\tilde{V}_m$ is forbidden by the $U(1)_{[0]}$ symmetry for arbitrary m.

IV. CONCLUSION AND OUTLOOK

In this paper, we have studied the LSMOH type theorem based on the combination of $U(1)$ higher-form symmetry and lattice translations, with particular focus on 1-form symmetries. Our result is applied for pure $U(1)$ lattice gauge theories which simulate the QDM on bipartite lattices in $2 + 1$ dimension. The QDM on a bipartite (square, honeycomb) lattice has a gapless deconfined phase called the incommensurate crystal next to the RK critical point. We observed that the deconfinement in the incommensurate crystal phase is enforced by the irrational filling of 1-form charge. The LSMOH theorem is manifested as a mixed 't Hooft anomaly for $U(1)_{[1]} \times \mathcal{T}_1$ symmetry near the RK critical point. We explicitly diagnosed this 't Hooft anomaly by calculating the partition function in the presence of a background gauge field.

One direction to extend the studies here is to apply our results to the QDM on a bipartite lattice in higher dimensions, which can be realized, for example, as an effective model of the spin-1/2 antiferromagnetic Heisenberg model on a pyrochlore lattice [52]. It would be interesting to look for the possibility of the deconfined phase enforced by the fractional 1-form filling in such systems, which is left for future investigation. Finally, in this work we have not considered spatial symmetries other than simple lattice translation and reflection, so we leave the refinement of our result for additional crystal symmetries for future work.

ACKNOWLEDGMENTS

APPENDIX A: n-FORM SYMMETRY

In this appendix, we discuss the generalized version of the LSMOH theorem based n-form symmetry. It is straightforward to generalize the logic introduced in Section II to n-form $U(1)$ symmetry.

1. n-form $U(1)$ symmetry in the continuum

Consider a theory written in terms of a n-form $U(1)$ field h and assume that the action $S[h]$ consists only of dh. The

theory is invariant under the shift of h by a flat field

$$h(x) \mapsto h(x) + \omega(x), \quad d\omega = 0. \tag{A1}$$

Gauge equivalence classes of flat n-form $U(1)$ field are classified by the cohomology group

$$[\omega] \in H^n(X; \mathbb{R}/2\pi\mathbb{Z}). \tag{A2}$$

Then we see that the theory has a global n-form $U(1)$ symmetry

$$h(x) \mapsto h(x) + \theta\lambda(x), \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{A3}$$

and $[\lambda] \in H^n(X; \mathbb{Z})$.

Objects charged under the n-form $U(1)$ symmetry (A3) are operators defined on n-dimensional surfaces,

$$V(C) = \exp\left[i \int_C h\right], \quad C \in Z_n(X), \tag{A4}$$

which measures a kind of holonomy along C. The n-form symmetry transformation is implemented by an operator $U_\theta(M^{(d-n)})$ supported on $(d-n)$-dimensional manifold $M^{(d-n)}$. We have the equal time commutation relation as

$$U_\theta(M^{(d-n)})V(C)$$
$$= e^{i\theta(C, M^{(d-n)})} \cdot V(C)U_\theta(M^{(d-n)}) \quad \text{at equal time,} \tag{A5}$$

where $(C, M^{(d-n)})$ is the intersection number.

Gauging n-form $U(1)$ symmetry means introducing the flat $(n+1)$-form background $U(1)$ field $c(x)$, $dc = 0$, and introduce the gauge equivalence relation

$$h(x) \mapsto h(x) + \theta(x)\lambda(x)$$
$$c(x) \mapsto c(x) - d(\theta(x)\lambda(x)) = c(x) - d\theta(x) \wedge \lambda(x), \tag{A6}$$

so that the covariant derivative

$$D_c h := dh + c \tag{A7}$$

is invariant. The gauge equivalence classes of $c(x)$ are determined by a kind of holonomy

$$\int_C c \in \mathbb{R}/2\pi\mathbb{Z}, \quad C \in Z_{n+1}(X), \tag{A8}$$

i.e.,

$$[c] \in H^{n+1}(X; \mathbb{R}/2\pi\mathbb{Z}). \tag{A9}$$

2. LSMOH theorem with n-form symmetry

We formulate the above theory on the periodic lattice, whose vertices are labeled as $(x_1, x_2, \ldots, x_d) \in \mathbb{Z}/L_1\mathbb{Z} \times \mathbb{Z}/L_2\mathbb{Z} \times \cdots \times \mathbb{Z}/L_d\mathbb{Z}$, and repeat the same logic as Sec. II C to derive LSMOH-type theorem for higher form symmetry. In this case, n-form $U(1)$ field h is assigned on each n-dimensional hypercube. The theory is invariant under the global $U(1)$ transformation

$$h \mapsto h + \theta\lambda, \tag{A10}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a constant, $d\lambda = 0$, and

$$\sum_{\gamma_n \in C} \lambda_{\gamma_n} \in \mathbb{Z}, \quad C \in Z_n(X), \tag{A11}$$

where γ_n is a label of n-dimensional hypercube. The theory is also invariant under the translation $\mathcal{T}_l$ about one unit cell along the lth direction and assume that neither $U(1)$ nor the translation symmetry is broken.

Then we gauge the $U(1)$ symmetry (A10) by coupling with a background flat $(n + 1)$-form gauge field c defined on $(n + 1)$-dimensional hypercube. Consider the field configuration that corresponds to adiabatic insertion

$$c_{lm_1...m_n} = 0 \quad t < 0,$$

$$c_{lm_1...m_n}(\mathbf{x}; t) = \prod_{i=1}^{n} \delta(x_{m_i}) \cdot 2\pi t / L_l T \quad 0 \leqslant t < T, \quad \text{(A12)}$$

$$c_{lm_1...m_n}(\mathbf{x}; t) = \prod_{i=1}^{n} \delta(x_{m_i}) \cdot 2\pi / L_l \quad T \leqslant t,$$

and the other components of c are 0. For the configuration (A12), the holonomy of c along $(n + 1)$-dimensional $x_l x_{m_1} \ldots x_{m_n}$ hyperplane grows gradually from 0 to 2π as time proceeds:

$$\sum_{\gamma_{n+1} \in C} c_{\gamma_{n+1}} = \frac{2\pi t}{T} \quad 0 \leqslant t < T, \quad \text{(A13)}$$

where γ_{n+1} is a label of $(n + 1)$-dimensional hypercube, and C is some $x_l x_{m_1} \ldots x_{m_n}$ hyperplane.

Suppose that the Hamiltonian at $t = 0$ (written as H_0) has finite excitation gap above the ground state and that the gap does not close during the process of adiabatic flux insertion. At $t = 0$, the ground state $|\psi_0\rangle$ is chosen (when the ground state are degenerate) so that it is an eigenstate of $\mathcal{T}_l$ and a $U(1)$ symmetry transformation operator $Q_{m_1...m_n}$:

$$\mathcal{T}_l|\psi_0\rangle = e^{ip_l}|\psi_0\rangle, \quad Q_{m_1...m_n}|\psi_0\rangle = \nu \prod_{k \neq m_1...m_n} L_k|\psi_0\rangle, \quad \text{(A14)}$$

where $Q_{m_1...m_n}$ is a $U(1)$ charge operator associated with $(d - n)$-dimensional hyperplane characterized as $x_{m_1} = \cdots = x_{m_n} = 0$, and ν denotes the $U(1)$ charge per unit cell. When the holonomy (A13) along C reaches the unit flux quantum 2π at $t = T$, the original ground state evolves into some ground state $|\psi_0'\rangle$ of the Hamiltonian at $t = T$ (written as H_0') that satisfies $\mathcal{T}_l|\psi_0'\rangle = e^{ip_l}|\psi_0'\rangle$. The configuration of the flat background $U(1)$ gauge field (A12) at $t = T$ with the holonomy 2π is gauge equivalent to that of $t = 0$, by the following gauge transformation

$$h_{m_1...m_n}(\mathbf{x}) \mapsto \quad h_{m_1...m_n}(\mathbf{x}) - \prod_{i=1}^{n} \delta(x_{m_i}) \cdot 2\pi x_l / L_l,$$

$$c_{lm_1...m_n}(\mathbf{x}) \mapsto \quad c_{lm_1...m_n}(\mathbf{x}) - \prod_{i=1}^{n} \delta(x_{m_i}) \cdot 2\pi / L_l. \quad \text{(A15)}$$

We write the symmetry operator corresponding to the above gauge transformation as $U_{lm_1...m_n}$. Then, $U_{lm_1...m_n}|\psi_0'\rangle$ is also a ground state of H_0, and there is the following commutation relation between $U_{lm_1...m_n}$ and $\mathcal{T}_l$

$$U_{lm_1...m_n} \mathcal{T}_l U_{lm_1...m_n}^{\dagger} = \mathcal{T}_l \exp\left[-\frac{2\pi i}{L_l} Q_{m_1...m_n}\right]. \quad \text{(A16)}$$

Now we obtain the action of $\mathcal{T}_l$ on $U_{lm_1...m_n}|\psi_0'\rangle$ using the commutation relation (A16)

$$\mathcal{T}_l U_{lm_1...m_n}|\psi_0'\rangle = e^{ip_l} \exp\left[\frac{2\pi i}{L_l} Q_{m_1...m_n}\right] \cdot U_{lm_1...m_n}|\psi_0'\rangle$$

$$= \exp\left[ip_l + 2\pi i\nu \prod_{k \neq l, m_1,...,m_n} L_k\right]$$

$$\cdot U_{lm_1...m_n}|\psi_0'\rangle. \quad \text{(A17)}$$

Thus, if we have $\nu = p/q$, with L_l integer multiple of q and $\prod_{k \neq l, m_1,...,m_n} L_k$ mutually prime with q, the momentum of $U_{lm_1...m_n}|\psi_0'\rangle$ is written as $p_l + 2\pi r/q$, using some integer r mutually prime with q. Therefore, we obtain at least q mutually orthogonal ground states $|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{q-1}\rangle$ with different momentum, such that

$$|\psi_{k+1}\rangle = U_{lm_1...m_n}|\psi_k'\rangle,$$

$$\mathcal{T}_l|\psi_k\rangle = \exp\left[i\left(p_l + \frac{2\pi kr}{q}\right)\right]|\psi_k\rangle. \quad \text{(A18)}$$

Then, we have proven that

*Theorem A.1. (LSMOH theorem for **n**-form symmetry).*

Consider a quantum many-body system defined on a d-dimensional periodic lattice, in the presence of a global n-form $U(1)$ symmetry and a translation symmetry about the lth primitive lattice vector, and assume that both symmetries are not broken. Then, if the $U(1)$ charge (measured on a $(d - n)$-dimensional hyperplane characterized as $x_{m_1} = \cdots = x_{m_n} = 0$ for $m_1, \ldots, m_n \neq l$) per unit cell $\nu = p/q$ at the ground state, only two possibilities are possible for the low energy spectrum:

(1) The system is gapped, and the ground states are at least q-fold degenerate, or

(2) The system is gapless.

APPENDIX B: 1-FORM LSMOH THEOREM IS AVAILABLE IN THE QUANTUM DIMER MODEL

Here, we give a simple proof that the result of LSMOH theorem (Theorem 1.1) is true for (36), even if we take the limit $K \to \infty$ before taking the thermodynamic limit. To do this, we see if $|\psi_0\rangle$ is a ground state of the gauge theory (36), the state $U_{12}|\psi_0\rangle$ also lies in the low energy sector, whose energy splitting from $|\psi_0\rangle$ is independent of K and bounded by $O(1/L_1)$. Here, the operator U_{12} is defined as

$$U_{12} \equiv \exp\left(\frac{2\pi i}{L_1} \sum_{x_1=0}^{L_1-1} x_1 E_\alpha(x_1, 0)\right). \quad \text{(B1)}$$

This statement can be proven in the same manner as the original proof of the LSM theorem (for one-dimensional spin system) by Lieb, Schultz, and Mattis. We evaluate the difference of the energy expectation values for $|\psi_0\rangle$ and $U_{12}|\psi_0\rangle$ as

$$\delta E[K] = \langle\psi_0|(U_{12}^{\dagger} H_{\text{eff}}[K] U_{12} - H_{\text{eff}}[K])|\psi_0\rangle$$

$$\leqslant \langle\psi_0|(U_{12}^{\dagger} H_{\text{eff}}[K] U_{12} - H_{\text{eff}}[K])|\psi_0\rangle$$

$$+ \langle\psi_0|(U_{12} H_{\text{eff}}[K] U_{12}^{\dagger} - H_{\text{eff}}[K])|\psi_0\rangle$$
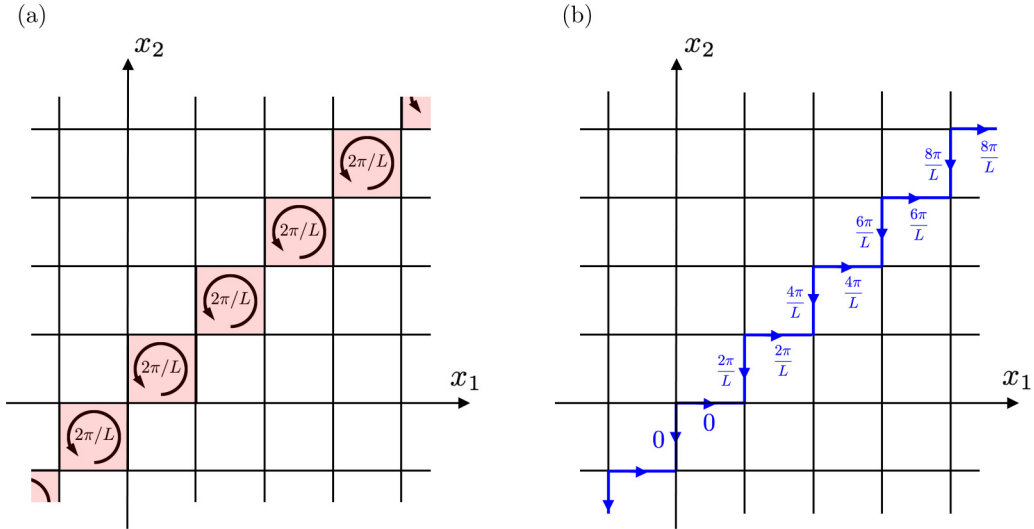
FIG. 4. (a) Configuration of 2-form field B on the x_1x_2 plane at the end of the insertion process. (b) A large gauge transformation.

$$= 8v \left(\cos\left(\frac{2\pi}{L_1}\right) - 1 \right) \langle \psi_0 | \left(\sum_{\substack{\{\bigcirc\} \\ x_2=0}} \cos[\mathrm{rot}A] \right) | \psi_0 \rangle$$

$$\leqslant 8v \cdot \frac{1}{2}\left(\frac{2\pi}{L_1}\right)^2 \cdot L_1 = \frac{16\pi^2 v}{L_1}, \tag{B2}$$

where we simply added the term $\langle \psi_0 | (U_{12} H_{\mathrm{eff}}[K] U_{12}^\dagger - H_{\mathrm{eff}}[K]) | \psi_0 \rangle$ in the second line, which is non-negative due to the variational principle. Using the similar logic, we find that the variational energy of $U_{12}^n | \psi_0 \rangle$ is bounded by $16\pi^2 vn/L_1$. With help of the commutation relation between the lattice translation in x_1 direction like (24), we find at least q mutually orthogonal ground states with distinct momentum when the filling $\nu \equiv Q_2/L_1 = p/q$. Here the upper bound of energy splitting $16\pi^2 vq/L_1$ is independent of K, therefore

$$\lim_{L_1,L_2\to\infty} \lim_{K\to\infty} \delta E[K] \leqslant \lim_{L_1,L_2\to\infty} \lim_{K\to\infty} \frac{16\pi^2 vq}{L_1} = 0, \tag{B3}$$

which assures the availability of LSMOH theorem after taking the limit $K \to \infty$. It is straightforward to generalize the logic to the case of square lattice.

APPENDIX C: THE QUANTUM DIMER MODEL ON THE SQUARE LATTICE

1. Symmetries and LSMOH theorem

In this appendix, we discuss the quantum dimer model on the square lattice, which is largely in parallel with the case of the honeycomb lattice. The main difference is that the gauge theory is not invariant under translations by the unit lattice vectors $\mathbf{e}_1$, $\mathbf{e}_2$ of the square lattice due to the presence of the staggered background charges. I.e., the unit cells are enlarged. The spatial symmetry is thus generated by the translations $\mathcal{T}_\pm$ by the lattice vectors $\mathbf{e}_\pm := \mathbf{e}_1 \pm \mathbf{e}_2$. The LSMOH theorem can be applied based on $U(1)_{[1]} \times \mathcal{T}_+$ symmetry.

To see how this works, first we assume that $L_1 = L_2 = L$ to make the system periodic in $\mathbf{e}_+$ direction. Then, one

performs the adiabatic insertion of 2-form background field B coupled with A from $B = 0$ to the final configuration

$$B_{12}^{\mathrm{final}}(\mathbf{x}) = \frac{2\pi}{L} \delta_{x_1 x_2} \tag{C1}$$

[see Fig. 4(a)]. The configuration of background field (C1) is gauge equivalent to the initial configuration $B = 0$, by the following large gauge transformation [see Fig. 4(b)]

$$U_{+-} = \exp\left(\frac{2\pi i}{L} \sum_x x E_1(x,x) - \frac{2\pi i}{L} \sum_x x E_2(x,x-1) \right). \tag{C2}$$

One can see that the commutation relation between the large gauge transformation U_{+-} and $\mathcal{T}_+$ is given by

$$U_{+-} \mathcal{T}_+ U_{+-}^\dagger = \mathcal{T}_+ \exp\left[-\frac{2\pi i}{L} Q_+ \right], \tag{C3}$$

where Q_+ is the charge operator that operates on Wilson loops extended in $\mathbf{e}_-$ direction:

$$Q_+ = \sum_x E_1(x,x) - \sum_x E_2(x,x-1). \tag{C4}$$

Using (32) we see that Q_+ is the number of dimers measured on the string, when the vertices (x,x) belong to the sublattice A.

The phase diagram of the square lattice QDM is qualitatively similar to the honeycomb lattice QDM. There are three kinds of ordered phases: columnar, plaquette, and staggered phases. The filling $\nu = Q_+/L$ takes value $1/2$ in the columnar and plaquette phase where the $\mathcal{T}_+$ symmetry is broken, while in the staggered phase with $\nu = 0$ or 1 the $\mathcal{T}_+$ symmetry is preserved.

2. Continuum field theory calculations

As in the honeycomb lattice QDM, one can identify the LSMOH anomaly corresponding to (C3) near the RK point, that involves the combination of $U(1)_{[1]}$ and the translation symmetry $\mathcal{T}_+$. The underlying field theory also has identical

form to the case of honeycomb lattice, (43). An identification of the dimer variables $n_j(\mathbf{x})$ and the height variables of the field theory are [44,49]

$$
\begin{aligned}
n_1 - \frac{1}{4} &= \frac{1}{2\pi}(-1)^{x_1+x_2}\partial_2\phi + \frac{1}{2}[(-1)^{x_1}e^{i\phi} + \text{H.c.}], \\
n_2 - \frac{1}{4} &= \frac{1}{2\pi}(-1)^{x_1+x_2+1}\partial_1\phi + \frac{1}{2}[(-1)^{x_2}ie^{i\phi} + \text{H.c.}].
\end{aligned}
\tag{C5}
$$

One can read off the action of $\mathcal{T}_+$ on ϕ from (C5) by imposing that n_j transforms correctly under the translation. Then, in the continuum limit we see that $\mathcal{T}_+$ acts on ϕ as

$$
\mathcal{T}_+ : \phi \mapsto \phi - \pi.
\tag{C6}
$$

The charge operator Q_+ of $U(1)_{[1]}$ symmetry (C4) in the quantum dimer model is translated into the charge of 1-form symmetry in the field theory as

$$
\begin{aligned}
Q_+ &= \sum_x E_1(x,x) - \sum_x E_2(x,x-1) \\
&= \sum_x \left(\frac{1}{2\pi}\partial_2\phi(x,x) - (-1)^x\cos\phi(x,x) \right) \\
&\quad + \sum_x \left(\frac{1}{2\pi}\partial_1\phi(x,x-1) \right. \\
&\quad \left. - (-1)^x\sin\phi(x,x-1) \right) + \frac{L}{2} \\
&\approx \int_{x_1=x_2} d\phi + \frac{L}{2},
\end{aligned}
\tag{C7}
$$

where we used the identification (C5) and dropped the summation of the staggered part in the last equation. Now we diagnose the mixed LSMOH anomaly using the same logic as in Sec. III, i.e., first we gauge the $U(1)_{[1]}$ symmetry and then calculate the partition function twisted by $\mathcal{T}_+$, in the presence of the 2-form background field coupled with a. Both $\mathcal{T}_+$ symmetry and $U(1)_{[1]}$ act only on the zero mode part of

fields. The action of $\mathcal{T}_+$ is

$$
\mathcal{T}_+ : \alpha_0 \mapsto \alpha_0 - \pi,
\tag{C8}
$$

leaving the other operators invariant. The 2-form flux insertion in terms of $U(1)_{[1]}$ corresponds to shifting the fractional part λ_0 of $\beta_0 = N_0 + \lambda_0$, where $N_0 \in \mathbb{Z}$ is the untwisted integral winding number. In conclusion, the LSMOH anomaly in this case is diagnosed as

$$
Z[\lambda_0 + 1] = -Z[\lambda_0],
\tag{C9}
$$

reflecting the filling $\nu = 1/2$ at the plaquette phase.

3. CR symmetry

Besides the translation symmetry $\mathcal{T}_+$, the system has the CR symmetry changing the sublattice represented as

$$
\begin{aligned}
\mathsf{CR} : \quad &E_1(x_1,x_2) \mapsto E_1(-x_1,x_2), \\
&E_2(x_1,x_2) \mapsto -E_2(1-x_1,x_2), \\
&A_1(x_1,x_2) \mapsto A_1(-x_1,x_2), \\
&A_2(x_1,x_2) \mapsto -A_2(1-x_1,x_2),
\end{aligned}
\tag{C10}
$$

and we can apply the LSMOH theorem that is based on the $U(1)_{[1]} \times \mathsf{CR}$ symmetry. In this case, the commutation relation between the large gauge transformation for the $U(1)_{[1]}$ (written as U_{12}) and CR is analogous to (24)

$$
U_{12}(\mathsf{CR})U_{12}^\dagger = (\mathsf{CR})\exp\left(-\frac{2\pi i}{L_1}Q_2\right),
\tag{C11}
$$

where the large gauge transformation and Q_2 are defined on a line $x_2 = \text{const}$.

$$
U_{12} = \exp\left(\frac{2\pi i}{L_1}\sum_{x_1}x_1 E_2(\mathbf{x})\right),
\tag{C12}
$$

and

$$
Q_2 = \sum_{x_1} E_2(\mathbf{x}).
\tag{C13}
$$

According to (C11), we see that the ground state cannot be trivially gapped if the filling (the eigenvalue of Q_1/L_1 at the ground state) is fractional.

---

[1] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (NY) **16**, 407 (1961).

[2] M. Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000).

[3] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).

[4] M. Hastings, Europhys. Lett. **70**, 824 (2005).

[5] S. A. Parameswaran, A. M. Turner, D. P. Arovas, and A. Vishwanath, Nat. Phys. **9**, 299 (2013).

[6] R. Roy, arXiv:1212.2944.

[7] S. C. Furuya and M. Oshikawa, Phys. Rev. Lett. **118**, 021601 (2017).

[8] H. Watanabe, H. C. Po, A. Vishwanath, and M. Zaletel, Proc. Natl. Acad. Sci. USA **112**, 14551 (2015).

[9] H. Watanabe, H. C. Po, M. P. Zaletel, and A. Vishwanath, Phys. Rev. Lett. **117**, 096404 (2016).

[10] X. Yang, S. Jiang, A. Vishwanath, and Y. Ran, Phys. Rev. B **98**, 125120 (2018).

[11] G. Y. Cho, C.-T. Hsieh, and S. Ryu, Phys. Rev. B **96**, 195105 (2017).

[12] M. A. Metlitski and R. Thorngren, Phys. Rev. B **98**, 085140 (2018).

[13] C.-M. Jian, Z. Bi, and C. Xu, Phys. Rev. B **97**, 054412 (2018).

[14] Y. Qi, C. Fang, and L. Fu, arXiv:1705.09190.

[15] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, Phys. Rev. B **96**, 205106 (2017).

[16] M. Cheng, arXiv:1804.10122.

[17] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford Univ. Press, New York, 2004).

[18] A. Kapustin and R. Thorngren, Algebra, Geometry, and Physics in the 21st Century, 177 (2017).

[19] A. Kapustin and N. Seiberg, J. High Energy Phys. 04 (2014) 001.

[20] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energy Phys. 02 (2015) 172.

[21] E. Lake, arXiv:1802.07747.

[22] D. M. Hofman and N. Iqbal, arXiv:1802.09512.

[23] G. 't Hooft, in *Recent Developments in Gauge Theories. Proceedings, Nato Advanced Study Institute, Cargese, France, August 26 - September 8, 1979*, Vol. 59 (1980), pp. 135–157.

[24] A. Kapustin and R. Thorngren, Phys. Rev. Lett. **112**, 231602 (2014).

[25] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, Phys. Rev. X **7**, 031051 (2017).

[26] Y. Tanizaki and Y. Kikuchi, J. High Energy Phys. 06 (2017) 102.

[27] Z. Komargodski, A. Sharon, R. Thorngren, and X. Zhou, arXiv:1705.04786.

[28] Z. Komargodski, T. Sulejmanpasic, and M. Ünsal, Phys. Rev. B **97**, 054418 (2018).

[29] H. Shimizu and K. Yonekura, Phys. Rev. D **97**, 105011 (2018).

[30] D. Gaiotto, Z. Komargodski, and N. Seiberg, J. High Energy Phys. 01 (2018) 110.

[31] M. Yamazaki, J. High Energy Phys. 10 (2018) 172.

[32] A. Cherman and M. Unsal, arXiv:1711.10567.

[33] Y. Tanizaki, T. Misumi, and N. Sakai, J. High Energy Phys. 12 (2017) 056.

[34] Y. Tanizaki, Y. Kikuchi, T. Misumi, and N. Sakai, Phys. Rev. D **97**, 054012 (2018).

[35] M. Guo, P. Putrov, and J. Wang, Ann. Phys. **394**, 244 (2018).

[36] T. Sulejmanpasic and Y. Tanizaki, Phys. Rev. B **97**, 144201 (2018).

[37] Y. Yao, C.-T. Hsieh, and M. Oshikawa, arXiv:1805.06885.

[38] In the following theorem, we assume that the ground state degeneracy does not depend on the system size.

[39] N. E. Bonesteel, Phys. Rev. B **40**, 8954 (1989).

[40] G. Misguich, C. Lhuillier, M. Mambrini, and P. Sindzingre, Eur. Phys. J. B **26**, 167 (2002).

[41] E. Fradkin and S. A. Kivelson, Mod. Phys. Lett. B **4**, 225 (1990).

[42] One may worry that the LSMOH constraint becomes unavailable in the limit $K \to \infty$ in (36) and (38): It is good to prove that the energy splitting of two states, which decays in the thermodynamic limit at finite K, still decays even if we first take the limit $K \to \infty$ and then the thermodynamic limit. This can be actually done in $(2+1)$ dimensions rigorously as discussed in Appendix B.

[43] R. Moessner, S. L. Sondhi, and P. Chandra, Phys. Rev. B **64**, 144416 (2001).

[44] E. Fradkin, D. A. Huse, R. Moessner, V. Oganesyan, and S. L. Sondhi, Phys. Rev. B **69**, 224415 (2004).

[45] A. Vishwanath, L. Balents, and T. Senthil, Phys. Rev. B **69**, 224416 (2004).

[46] Thus, one may vary the filling ν at the ground state by introducing a kind of chemical potential term in the QDM Hamiltonian like

$$H_{\text{chem}} = \mu \sum_{x_2} Q_2(x_2) = \mu \sum_{\mathbf{x}} n_\alpha(\mathbf{x}).$$

[47] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[48] E. Ardonne, P. Fendley, and E. Fradkin, Ann. Phys. **310**, 493 (2004).

[49] E. Fradkin, *Field Theories of Condensed Matter Physics* (Cambridge University Press, England, 2013).

[50] R. Moessner and K. Raman, arXiv:0809.3051.

[51] S. Papanikolaou, K. S. Raman, and E. Fradkin, Phys. Rev. B **75**, 094406 (2007).

[52] M. Hermele, M. P. A. Fisher, and L. Balents, Phys. Rev. B **69**, 064404 (2004).