

## Hall and dissipative viscosity effects on edge magnetoplasmons

Roie Cohen and Moshe Goldstein

*Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv 6997801, Israel*



(Received 16 September 2018; published 3 December 2018)

Hydrodynamic and viscous effects in electronic liquids are at the focus of much current research. Most intriguing is perhaps the nondissipative Hall viscosity, which, due to its symmetry-protected topological nature, can help identify complex topological orders. In this work we study the effects of viscosity in general, and Hall viscosity in particular, on the dispersion relation of edge magnetoplasmons in two-dimensional electronic systems. Using an extension of the standard Wiener-Hopf technique, we derive a general solution to the problem, accounting for the long-range Coulomb potential. Among other features we find that on the edge viscosity affects the dispersion already to leading order in the wave vector, making edge modes better suited for its measurement compared to their bulk counterparts.

DOI: [10.1103/PhysRevB.98.235103](https://doi.org/10.1103/PhysRevB.98.235103)

### I. INTRODUCTION

Recently, hydrodynamic electron flow has been at the focus of much research effort, both theoretical and experimental [1–24]. In the hydrodynamic regime electronic dynamics is dominated by viscosity, rather than impurity scattering. Denoting the electronic stress tensor by  $T_{ij}$  and its strain by  $u_{ij}$ , one generally has [25]

$$T_{ij} = - \sum_{kl} C_{ijkl} u_{kl} + \eta_{ijkl} \partial_t u_{kl}, \quad (1)$$

where  $C_{ijkl}$  is the tensor of elastic moduli, which reduces to the compressibility for a fluid, and  $\eta_{ijkl}$  is the viscosity tensor. Viscosity is usually associated with energy dissipation. However, this is not always the case: The viscosity tensor  $\eta_{ijkl}$  can be decomposed into its symmetric and antisymmetric parts (with respect to exchanging the first and last pairs of indexes),

$$\eta_{ijkl} = \eta_{ijkl}^+ + \eta_{ijkl}^-, \quad (2)$$

with  $\eta_{ijkl}^\pm = \pm \eta_{klij}^\pm$ . The symmetric part  $\eta_{ijkl}^+$  is even under time reversal and indeed causes energy dissipation, but the antisymmetric part  $\eta_{ijkl}^-$  is odd under time reversal and therefore dissipationless. The latter part may thus arise when time-reversal symmetry is broken (either explicitly by, e.g., applying an external magnetic field or spontaneously) and is termed “Hall viscosity,” in analogy with Hall conductivity/resistance, which is also nondissipative [26–48].

In two-dimensional (2D) isotropic systems the symmetric viscosity has two independent components, the bulk viscosity  $\zeta$  and the shear viscosity  $\eta_s$ ,

$$\eta_{ijkl}^+ = \zeta \delta_{ij} \delta_{kl} + \eta_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl}), \quad (3)$$

whereas the Hall viscosity has a single independent component  $\eta_H$  [33],

$$\eta_{ijkl}^- = \eta_H (\delta_{jk} \epsilon_{il} - \delta_{il} \epsilon_{kj}). \quad (4)$$

Hall viscosity can exist in a 2D electron fluid in the presence of a strong magnetic field in the quantum Hall regime [49],

where dissipative effects are suppressed at low temperatures. Furthermore, in quantum Hall systems  $\eta_H$  is expected to be quantized (assuming rotational invariance) and equal to half the particle density times the orbital spin  $\bar{s}$  (one half of the shift quantum number) [37,39,43]. Hence, measuring the value of Hall viscosity in an experiment can help one distinguish between different theoretical proposals concerning the nature of quantum Hall states in certain fractional filling factors. Since some of these states have non-Abelian anyonic excitations (with potential applications for topological quantum computing [50]), knowing the value of Hall viscosity can help in their identification. However, despite recent progress in understanding the relation between viscosity and charge transport in the presence of nonuniform electric fields [42,43,51–57], so far its measurement has been possible only for very weak magnetic fields, far from the quantum Hall regime [24].

The elementary excitations of an ordinary fluid are sound waves; in charged fluid (such as an electron fluid) they become plasmons or magnetoplasmons in the presence of a magnetic field. Bulk magnetoplasmons are gapped excitations, and in the long-wavelength limit, the magnetoplasmon frequency approaches a finite value, which is determined by the cyclotron and plasma frequencies. In addition, there are also gapless edge magnetoplasmons (magnetoplasmons confined to the edge of the system) whose frequency vanishes as their wave vector goes to zero. In the quantum Hall regime the quantized version of these excitations gives rise to the edge modes which determine the low-energy transport properties of these systems. This allows one to extract their properties in an experiment [58–81]. Recently, analogous excitations induced by Berry curvature rather than magnetic field have attracted significant attention [82–89].

In this work we study the effects of Hall viscosity on the edge magnetoplasmon dispersion (related questions for neutral fluids were recently considered in Refs. [90,91]). We use a classical hydrodynamic model for a two-dimensional electron fluid with an edge. In contrast to most works in the field, we account for the long-range Coulomb interaction and the

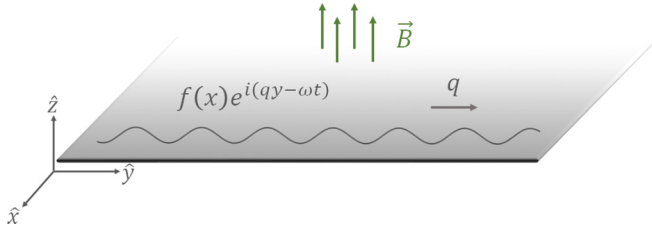


FIG. 1. A schematic depiction of the system: a 2D electronic system with an edge under the influence of a perpendicular magnetic field. See the text for further details.

resulting logarithmically divergent (at long wavelength) mode velocity [65,72,73] instead of using the Fetter approximation [59]. To the best of our knowledge, the interplay between the long-range Coulomb potential and viscosity has not been considered before. For this we develop a modification of the standard Wiener-Hopf technique [92]. We find that whereas in the bulk the viscosity affects only the dispersion through high-order terms in the wave vector, on the edge it modifies the leading-order term, making it easier to probe. In Sec. II we introduce our hydrodynamic model and then present our method and general solution for different boundary conditions in Sec. III. Then we discuss realistic parameter values and show examples of the resulting dispersions in Sec. IV, followed by the conclusions in Sec. V. Some technical details are relegated to the appendices.

## II. MODEL

We consider a 2D system on a half infinite plane with an edge (see Fig. 1) occupying the region  $x < 0$ , with a magnetic field  $\mathbf{B} = B\hat{z}$  perpendicular to the plane. We employ a linearized hydrodynamic model of a weakly damped, compressible, viscous fluid. The state of the system is characterized by the electron number density  $n = n_0 + \delta n$  ( $n_0$  is the equilibrium density and  $\delta n$  is a small deviation) and the electron velocity  $\mathbf{v}$ . The interaction between the electrons is taken into account through the electric potential  $\phi$ . These quantities are related through the 2D linearized continuity equation,

$$\partial_t \delta n + n_0 \nabla \cdot \mathbf{v} = 0, \quad (5)$$

the 2D linearized Navier-Stokes equation,

$$\begin{aligned} \partial_t \mathbf{v} + \frac{\mathbf{v}}{\tau} = & -\frac{s^2}{n_0} \nabla \delta n + \frac{e}{m} \nabla \phi + \omega_c \hat{z} \times \mathbf{v} \\ & + \frac{1}{n_0 m} [\eta_s \nabla^2 \mathbf{v} + \zeta \nabla (\nabla \cdot \mathbf{v}) - \eta_H \nabla^2 (\hat{z} \times \mathbf{v})], \end{aligned} \quad (6)$$

and the Poisson equation,

$$\nabla^2 \phi = \frac{4\pi e}{\epsilon} \delta n \Theta(-x) \delta(z), \quad (7)$$

where  $\tau$  is the electronic momentum relaxation time due to collisions with impurities,  $\omega_c = eB/mc$  is the cyclotron frequency ( $-e$  and  $m$  are the electron charge and effective mass, respectively, and  $c$  is the speed of light), and  $\epsilon$  is the dielectric constant of the three-dimensional (3D) medium. In

addition,  $\Theta$  is the Heaviside step function, and  $\delta$  is the Dirac delta function. Finally,  $s$  is the sound velocity in the absence of the Coulomb interaction,  $s^2 = (dP/dn)/m$ , where  $P$  is the noninteracting internal pressure. We note that the internal pressure is the equilibrium average of the diagonal elements of the stress tensor. It is different from the pressure on the boundaries, which contains the effects of the Lorentz force on the boundary currents. While the compressibility associated with the pressure on the boundaries vanishes in the quantum Hall regime, the internal pressure is associated with a finite internal compressibility, hence with a finite  $s^2$ . [43,93]. Thus, our model should apply to the quantum Hall regime, provided one plugs into it the appropriate (and generally frequency dependent) viscosities, etc.

As for boundary conditions, one of them is straightforward: the vanishing of the normal velocity at the edge,  $v_x(0, y) = 0$ . The other one is subtler [23]. To simplify the discussion we will start from taking the no-slip boundary condition,  $v_y(0, y) = 0$ . Later on we will show how the results are modified for the no-stress boundary condition (zero tangential force on the boundary),  $T_{xy}(0, y) = 0$ . As we will further show below, our method allows one to treat the more complicated electrostatic environment of the 2D system, including multiple dielectrics and metallic gates.

## III. GENERAL SOLUTION

Since the system is invariant under translations in the  $y$  direction, we can look for a solution which is a plane wave parallel to the edge,  $\propto e^{iqy}$ . In the following we do not write explicitly the  $q$  argument of the physical quantities. Fourier transforming in the  $x$  direction, we find

$$\left[ \frac{d^2}{dz^2} - (k^2 + q^2) \right] \bar{\phi}(k, z) = \frac{4\pi e}{\epsilon} \bar{\delta n}_+(k) \delta(z), \quad (8)$$

where  $\bar{\phi}(k, z) \equiv \int_{-\infty}^{\infty} \phi(x, z) e^{-ikx} dx$  and  $\bar{\delta n}_+(k) \equiv \int_{-\infty}^0 \delta n(x) e^{-ikx} dx$ ; similar notations will be used for the  $x < 0$  Fourier transform of other physical quantities, which are analytic in the upper complex  $k$  plane (provided the physical quantity does not diverge exponentially at large  $x$ ). The solution is then  $\bar{\phi}(k, z) = \bar{\phi}(k, 0) e^{-|z|(k^2 + q^2)^{1/2}}$ , where in the system plane  $z = 0$ :

$$\bar{\phi}(k, 0) = -\frac{4\pi e}{\epsilon} \bar{L}(k) \bar{\delta n}(k), \quad (9)$$

with the Poisson kernel  $\bar{L}(k) \equiv 1/2\sqrt{k^2 + q^2}$ . In the following calculations we will not rely on the specific form of  $\bar{L}(k)$ . This will make our results applicable to more complicated geometries (with dielectric media, gates, etc.), whose effect can be accounted for by a modified  $\bar{L}(k)$ .

In the bulk one may combine this results with the Fourier-transformed continuity equation (5) and Navier-Stokes equation (6) and look for a nontrivial solution for the resulting homogeneous linear algebraic equations for the Fourier coefficients. One may thus straightforwardly obtain an implicit equation for the dispersion relation  $\omega(\mathbf{p})$ , where  $\mathbf{p} = (k, q)$ ,

$$\frac{\omega}{\omega_s} (\omega_s^2 - \omega_H^2) = \Omega_p^2 + s_\zeta^2 p^2, \quad (10)$$

where  $\Omega_p = \sqrt{2\pi n_0 e^2 p / m \epsilon}$  is the 2D plasma velocity,  $\omega_H \equiv \omega_c + \gamma_H p^2$ ,  $\omega_s \equiv \tilde{\omega} + i\gamma_s p^2$  ( $\tilde{\omega} \equiv \omega + i/\tau$ ), and  $s_\zeta^2 \equiv s^2 - i\gamma_\zeta \omega$ , with  $\gamma_a \equiv \eta_a / n_0 m$  for  $a = H, s, \zeta$ . One thus sees that the bulk dispersion is affected by the viscosity only to second order in  $q$  [35,36].

The edge problem is much more involved. To simplify it the Fetter approximation [59] is often used (Appendix A); the Coulomb kernel  $\bar{L}(k)$  is replaced by a meromorphic function, allowing one to get an effective 2D ‘‘Poisson’’ equation for the potential. However, this approximation qualitatively affects the solution; most notably, it gives a finite group velocity at long wavelengths, as opposed to the true logarithmic divergence in the velocity, due to the long-range Coulomb interaction [65]. We will therefore refrain from such approximations and instead extend the Wiener-Hopf method [65,92] to the current system. We will later use the Fetter approximation as a test case for our general solution, comparing the direct solution it allows with the one obtained from our general solution when  $\bar{L}(k)$  is replaced by Fetter’s approximate kernel (Appendix A).

### A. No-slip boundary conditions

Let us start with the no-slip boundary conditions. Taking the Fourier transform of Eqs. (5) and (6) for  $x < 0$  and using the boundary conditions lead to

$$\begin{aligned} -i\omega_s \bar{v}_x^+ + \frac{s_\zeta^2}{n_0} [ik\bar{\delta}n_+ + \delta n(0)] - \frac{e}{m} [ik\bar{\phi}_+ + \phi(0)] \\ + \omega_H \bar{v}_y^+ - \gamma_H \partial_x v_y(0) - \gamma_s \partial_x v_x(0) = 0, \end{aligned} \quad (11a)$$

$$\bar{X}_\pm(k) = \exp \left( -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk'}{k' - k \mp i\epsilon} \ln \left\{ 1 + \frac{2\Omega_q^2 \omega_s(k')(k^2 + q^2) \bar{L}(k')}{q s_\zeta^2 \omega_s(k')(k^2 + q^2) + q\omega[\omega_H^2(k') - \omega_s^2(k')]} \right\} \right), \quad (13)$$

which further obeys

$$\frac{1}{\bar{X}_+} - \frac{1}{\bar{X}_-} = \frac{1}{\bar{X}_-} \frac{2\Omega_q^2 \omega_s(k^2 + q^2) \bar{L}}{q s_\zeta^2 \omega_s(k^2 + q^2) + q\omega(\omega_H^2 - \omega_s^2)}.$$

As a result, Eq. (12) can be rewritten as

$$\frac{\bar{\phi}_+}{\bar{X}_+} + \frac{\bar{\phi}_-}{\bar{X}_-} = \left( \frac{1}{\bar{X}_+} - \frac{1}{\bar{X}_-} \right) \frac{q s_\zeta^2 (k\omega_s + iq\omega_H) \tilde{A}}{2\Omega_q^2 \omega_s(k^2 + q^2)}. \quad (14)$$

The right-hand side of the last equation can be expressed as the difference  $\bar{\psi}_+(k) - \bar{\psi}_-(k)$  of two functions analytic in the upper and lower  $k$  planes, respectively. The solution of this additional Riemann-Hilbert problem is

$$\begin{aligned} \bar{\psi}_\pm(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk'}{k' - k \mp i\epsilon} \left( \frac{1}{\bar{X}_+(k')} - \frac{1}{\bar{X}_-(k')} \right) \\ \times \frac{q s_\zeta^2 [k'\omega_s(k') + iq\omega_H(k')] \tilde{A}(k')}{2\Omega_q^2 \omega_s(k')(k^2 + q^2)}. \end{aligned} \quad (15)$$

By rearranging Eq. (14) we get the relation

$$\frac{\bar{\phi}_+}{\bar{X}_+} - \bar{\psi}_+ = -\frac{\bar{\phi}_-}{\bar{X}_-} - \bar{\psi}_-. \quad (16)$$

$$\begin{aligned} -i\omega_s \bar{v}_y^+ + iq s_\zeta^2 \frac{\bar{\delta}n_+}{n_0} - iq \frac{e}{m} \bar{\phi}_+ - \omega_H \bar{v}_x^+ \\ + \gamma_H \partial_x v_x(0) - \gamma_s \partial_x v_y(0) = 0, \end{aligned} \quad (11b)$$

$$-i\omega \bar{\delta}n_+ + n_0 (ik\bar{v}_x^+ + iq\bar{v}_y^+) = 0. \quad (11c)$$

Combining these equations with Eq. (9) and using  $\phi(\bar{k}, 0) = \bar{\phi}_+(k, 0) + \bar{\phi}_-(k, 0)$ , where  $\bar{\phi}_-(k, 0) \equiv \int_0^\infty \phi(x, 0) e^{-ikx} dx$  is analytic in the lower half of the complex  $k$  plane, we find

$$\begin{aligned} \bar{\phi}_- + \left( 1 + \frac{2\Omega_q^2 \omega_s(k^2 + q^2) \bar{L}}{q s_\zeta^2 \omega_s(k^2 + q^2) + q\omega(\omega_H^2 - \omega_s^2)} \right) \bar{\phi}_+ \\ = \frac{s_\zeta^2 (k\omega_s + iq\omega_H) \tilde{A} \bar{L}}{s_\zeta^2 \omega_s(k^2 + q^2) + \omega(\omega_H^2 - \omega_s^2)}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \tilde{A}(k) = -\frac{4\pi i n_0 e}{s_\zeta^2 \epsilon} \left\{ s_\zeta^2 \frac{\delta n(0)}{n_0} - \frac{e}{m} \phi(0) \right. \\ \left. + \gamma_H \left[ \frac{iq\omega_s + k\omega_H}{ik\omega_s - q\omega_H} \partial_x v_x(0) - \partial_x v_y(0) \right] \right\}. \end{aligned}$$

The coefficient of  $\bar{\phi}_+$  can be expressed as the ratio  $\bar{X}_- / \bar{X}_+$  of two functions which are analytic and have no zeros in the lower/upper half of the complex  $k$  plane, respectively. The solution of this Riemann-Hilbert problem is

According to Liouville’s theorem, every bounded entire function must be a constant. By our assumptions the functions appearing in Eq. (16) decay to zero at infinity in the complex  $k$  plane, so both sides of Eq. (16) must vanish. This gives us  $\bar{\phi}_+ = \bar{X}_+ \bar{\psi}_+$ . Evaluating the integrals in Eq. (15) leads to

$$\begin{aligned} \frac{e}{m} \bar{\phi}_+ = -i \left\{ \frac{1}{k^2 + q^2} \left[ Ak + iqB + (iqA + kB) \frac{\omega_H}{\omega_s} \right] \right. \\ + \frac{\bar{X}_+}{2|q|} \left( \frac{C_-}{k + i|q|} - \frac{C_+}{k - i|q|} \right) \\ \left. + \frac{\bar{X}_+}{2k_s} \frac{\omega_H(k_s)}{\tilde{\omega}} \left( \frac{D_+}{k - k_s} - \frac{D_-}{k + k_s} \right) \right\}, \end{aligned} \quad (17)$$

where

$$A \equiv \left( \frac{s_\zeta^2}{i\omega} - \gamma_s \right) \partial_x v_x(0) - \frac{e}{m} \phi(0) - \gamma_H \partial_x v_y(0),$$

$$B \equiv i[\gamma_s \partial_x v_y(0) - \gamma_H \partial_x v_x(0)],$$

$$C_\pm = \frac{\tilde{\omega}(Bq \pm A|q|) + \omega_c(Aq \pm B|q|)}{\tilde{\omega} \bar{X}_\pm(\pm i|q|)},$$

$$D_\pm = \frac{iqA \pm Bk_s}{\bar{X}_\pm(\pm k_s)},$$

and  $k_s$  is the root of  $\omega_s(k_s) = \omega + i/\tau + i\gamma_s(k_s^2 + q^2) = 0$  in the upper complex plain.

Typically, in the Wiener-Hopf technique, the next step would be to impose the boundary conditions [65,92]. However, in the current case these are satisfied automatically

(see Appendix B). Interestingly, as opposed to the common situation, what is not immediately ensured is the analyticity in the appropriate half of the complex  $k$  plane of the transformed variables. It turns out to be sufficient to consider the electron number density (Appendix B). Using Eqs. (11) we can extract

$$\frac{\overline{\delta n_+}}{n_0} = \frac{\omega_s \frac{e}{m} (k^2 + q^2) \bar{\phi}_+ + i(\omega_s k + iq\omega_H)A + i(i\omega_s q + k\omega_H)B}{\omega_s s_\zeta^2 (k^2 + q^2) + \omega(\omega_H^2 - \omega_s^2)}. \quad (18)$$

It has potential poles in the upper plane for  $k$  values for which the denominator in the last equation vanishes, giving

$$[i\gamma_s s_\zeta^2 + \omega(\gamma_H^2 + \gamma_s^2)]k^4 + [s_\zeta^2(\tilde{\omega} + 2i\gamma_s q^2) + 2\omega\{\omega_c \gamma_H - i\gamma_s \tilde{\omega} + (\gamma_H^2 + \gamma_s^2)q^2\}]k^2 + \tilde{\omega} s_\zeta^2 q^2 + i\gamma_s s_\zeta^2 q^4 + \omega(\omega_H^2 - \omega_s^2) = 0. \quad (19)$$

Let us denote the two roots of this equation in the upper complex  $k$  plane as  $k_{H1}$  and  $k_{H2}$ . The analyticity of  $\overline{\delta n_+}(k)$  in the upper half complex  $k$  plain amounts to the vanishing of the residues at  $k_{H1}$  and  $k_{H2}$ , giving

$$[\omega_s(k_H)k_H + iq\omega_H(k_H)]A + [i\omega_s(k_H)q + k_H\omega_H(k_H)]B - i\frac{e}{m}\omega_s(k_H)(k_H^2 + q^2)\bar{\phi}_+(k_H) = 0 \quad (20)$$

for  $k_H = k_{H1}, k_{H2}$ . Substitution of  $\bar{\phi}_+$  in the last equation gives

$$\frac{1}{2|q|} \left( \frac{C_-}{k_H + i|q|} - \frac{C_+}{k_H - i|q|} \right) + \frac{1}{2k_s} \frac{\omega_H(k_s)}{\tilde{\omega}} \left( \frac{D_+}{k_H - k_s} - \frac{D_-}{k_H + k_s} \right) = 0. \quad (21)$$

The vanishing of the determinant for these two linear equations (for  $k_H = k_{H1}, k_{H2}$ ) gives the dispersion relation:

$$\begin{aligned} & [-k_s(k_{H1} + k_s)(k_{H1} - k_s)(k_{H1} - iq)\bar{X}_+(iq)(\tilde{\omega} - \omega_c) - k_s(k_{H1} + k_s)(k_{H1} - k_s)(k_{H1} + iq)\bar{X}_+(-iq)(\tilde{\omega} + \omega_c) \\ & + \omega_H(k_s)(k_{H1} + iq)(k_{H1} - iq)(k_{H1} + k_s)\bar{X}_+(-k_s)iq - \omega_H(k_s)(k_{H1} + iq)(k_{H1} - iq)(k_{H1} - k_s)\bar{X}_+(k_s)iq] \\ & \times [(k_{H2} + k_s)(k_{H2} - k_s)(k_{H2} - iq)\bar{X}_+(iq)(\tilde{\omega} - \omega_c) - (k_{H2} + k_s)(k_{H2} - k_s)(k_{H2} + iq)\bar{X}_+(-iq)(\tilde{\omega} + \omega_c) \\ & + \omega_H(k_s)(k_{H2} + iq)(k_{H2} - iq)(k_{H2} + k_s)\bar{X}_+(-k_s) + \omega_H(k_s)(k_{H2} + iq)(k_{H2} - iq)(k_{H2} - k_s)\bar{X}_+(k_s)] - \{H1 \leftrightarrow H2\} = 0. \end{aligned} \quad (22)$$

This last equation, together with Eqs. (19) and (13), constitutes the full solution of the problem. We have checked that when the Fetter approximation [59] is substituted into the general solution, it agrees with the direct solution which one can obtain in that case (see Appendix A).

### B. No-stress boundary conditions

The above derivation assumed the tangential component of the velocity vanishes at the boundary. An alternative commonly employed boundary condition is the vanishing of the tangential force on the boundary,  $T_{xy}(x=0) = 0$  [23]. From Eqs. (1)–(4) we have

$$T_{xy} = -\eta_s(\partial_x v_y + \partial_y v_x) + \eta_H(\partial_x v_x - \partial_y v_y); \quad (23)$$

hence,  $T_{xy}(x=0) = 0$  gives

$$v_y(0) = \frac{1}{iq} \left( \partial_x v_x(0) - \frac{\eta_s}{\eta_H} \partial_x v_y(0) \right). \quad (24)$$

Using this above relation [instead of  $v_y(x=0) = 0$ ] and following the steps of the derivation in the previous section, once again we arrive at a pair of equations similar to Eq. (21) for  $k_H = k_{H1}, k_{H2}$ . The no-slip case had the simplifying but nongeneric property that the three boundary values,  $\phi(0)$ ,  $\partial_x v_x(0)$ , and  $\partial_x v_y(0)$ , only entered the solution through the two combinations  $A$  and  $B$ , so Eq. (21) for  $k_H = k_{H1}, k_{H2}$  was sufficient to obtain the dispersion relation. For the no-stress boundary condition this does not happen, and a third equation is required. Such an equation is provided

by demanding the boundary value of the potential matches the one obtained from its Fourier transform (a condition which was automatically obeyed in the no-slip case; see Appendix B).

$$\begin{aligned} \phi(0) &= \lim_{\text{Im}k \rightarrow \infty} [-ik\bar{\phi}_+(k)] \\ &= -\frac{m}{e} \left[ A + B \frac{\gamma_H}{i\gamma_s} - \frac{(\gamma_s^2 + \gamma_H^2)}{\gamma_s} \left( \partial_x v_x(0) - \frac{\gamma_s}{\gamma_H} \partial_x v_y(0) \right) \right. \\ &\quad \left. + \frac{1}{2|q|} (C_- - C_+) + \frac{\omega_H(k_s)}{2k_s \tilde{\omega}} (D_+ - D_-) \right], \end{aligned} \quad (25)$$

where all the parameters appearing are as defined previously. This, together with Eq. (21) for  $k_H = k_{H1}, k_{H2}$ , is a set of three homogeneous linear algebraic equations for the boundary values  $\phi(0)$ ,  $\partial_x v_x(0)$ , and  $\partial_x v_y(0)$ . To ensure a nontrivial solution we equate the determinant of the resulting coefficient to zero, giving us the dispersion relation.

Let us finally note that one may treat more complicated boundary conditions (intermediate between no slip and no stress [23]) in a similar manner.



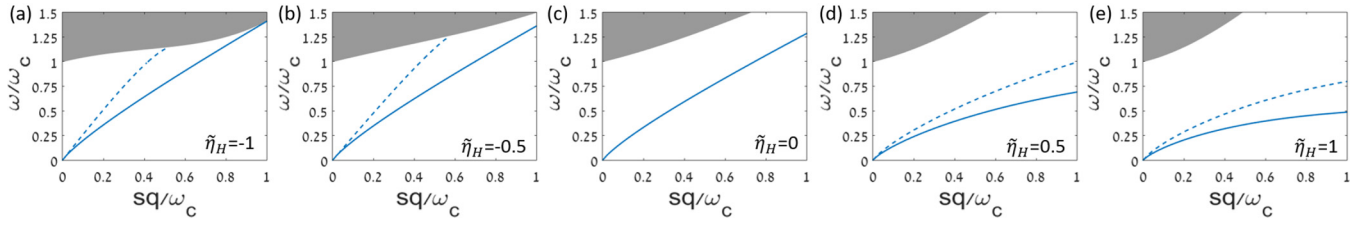


FIG. 2. Dissipationless magnetoplasmons: The dispersion relation for the edge magnetoplasmons in the dissipationless limit ( $\tau \rightarrow \infty$ ,  $\zeta = \eta_s = 0$ ) for  $\tilde{\Omega} = 1$  and different values of the dimensionless Hall viscosity,  $\tilde{\eta}_H \equiv \omega_c \eta_H / n_0 m s^2$ . The solid and dashed lines correspond to no-slip and no-stress boundary conditions, respectively. The gray area denotes the bulk magnetoplasmon spectrum.

## IV. RESULTS

### A. Parameter values

Let us now estimate the values of the parameters appearing in Eqs. (5)–(7). The electron density  $n_0$  is related to the Fermi wave vector  $k_F$  by  $n_0 = k_F^2 / 2\pi$ . The number of filled Landau levels  $\nu$  is then  $\nu = n_0 \Phi_0 / B$ , where  $\Phi_0 = hc/e$  is the flux quantum ( $h = 2\pi \hbar$  is Planck's constant). As for the pressure, at low temperatures and without interactions or magnetic fields, it is given by (the grand-canonical energy per unit area)

$$P = -2 \int^{k_F} \frac{d^2 k}{(2\pi)^2} (\epsilon_k - \mu) = \frac{\pi \hbar^2}{m} n_0^2. \quad (26)$$

The noninteracting  $B = 0$  sound velocity is then

$$s^2 = \frac{1}{m} \frac{\partial P}{\partial n_0} = \frac{2\pi \hbar^2}{m^2} n_0. \quad (27)$$

Let us note that, as discussed above, in the presence of a magnetic field one actually needs to use the internal pressure [93]. Using the results of Ref. [43], we find that in the integer quantum Hall regime,

$$s^2 = \frac{1}{mn_0} \left( n_0 \frac{\partial P}{\partial n_0} \right) = \frac{1}{mn_0} (n_0 \bar{s} \hbar \omega_c) = \frac{2\pi \hbar^2}{m^2} n_0 \frac{2\bar{s}}{\nu}. \quad (28)$$

This result coincides with the former one for the integer quantum Hall case, where  $\bar{s} = \nu/2$  [34,37,39,43], so we will stick to Eq. (27) below.

Now we can calculate the dimensionless quantities:

$$\tilde{\Omega}^2 \equiv \frac{\Omega_q^2}{sq\omega_c} = \frac{\nu}{\sqrt{2\pi n_0 a_B^2}}, \quad (29)$$

where  $a_B = \hbar^2 \epsilon / m e^2$  is the Bohr radius in the material and [37,39,43]

$$\tilde{\eta}_H \equiv \frac{\omega_c \eta_H}{n_0 m s^2} = \frac{\omega_c}{n_0 m s^2} \frac{\hbar n_0 \bar{s}}{2} = \frac{\bar{s}}{2\nu}. \quad (30)$$

For a 2D electron gas (2DEG) at the GaAs/AIGaAs interface the effective mass (that is, the effective mass of the conduction band of GaAs) is  $m \approx 0.067 m_e$  ( $m_e$  is the bare electron mass), and the dielectric constant is  $\epsilon \approx 13$ , so the Bohr radius is  $a_B \approx 10$  nm. A typical number density is  $n_0 \sim 10^{13} \text{ cm}^{-2}$ . Thus, both  $\tilde{\Omega}^2$  and  $\tilde{\eta}_H$  are of the order of unity. We will also define the dimensionless shear viscosity  $\tilde{\eta}_s \equiv \omega_c \eta_s / n_0 m s^2$ . It should be close to zero in the quantum Hall regime at low frequencies (long wavelengths), but we will also examine its effect when non-negligible.

### B. Edge magnetoplasmon dispersion

We will now show some examples of the dispersions coming out of our results. We start from the dissipationless case ( $\tau \rightarrow \infty$ ,  $\zeta = \eta_s = 0$ ) and display the edge magnetoplasmon dispersion in Fig. 2 for  $\tilde{\Omega} = 1$  for both the no-slip and no-stress boundary conditions. For some values of the parameters the edge dispersion may terminate by overlapping with the bulk magnetoplasmon spectrum. In Fig. 3 we go on to examine the effect of finite shear viscosity on the dispersion, leaving the other parameters unchanged. The finite dissipation gives the dispersion a negative imaginary part, which is also plotted. Let us also note that the viscosities are expected to display frequency dependence [43,45,47], which can straightforwardly be incorporated into our formalism.

Finally, we numerically examine the functional form of the dispersion at small  $q$  and find that it is well described by a formula of the form

$$\omega \sim v_{mp} q \ln \left( \frac{a\omega_c}{vq} \right), \quad (31)$$

with ‘‘velocity’’  $v_{mp}$  and dimensionless coefficients  $a$ . The leading term gives rise to magnetoplasmon velocity, which

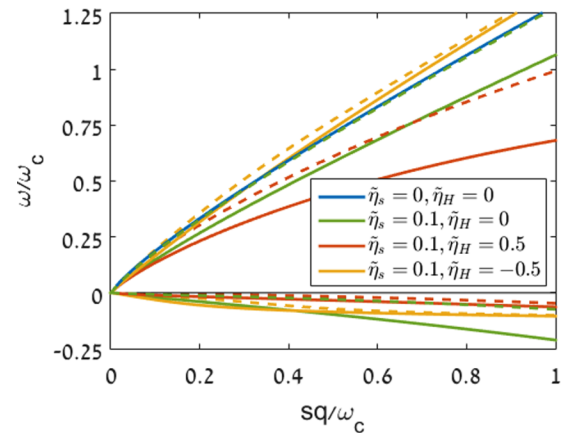


FIG. 3. Dissipative magnetoplasmons: The edge magnetoplasmon dispersion for different values of the dimensionless Hall and shear viscosities,  $\tilde{\eta}_{s/H} \equiv \omega_c \eta_{s/H} / n_0 m s^2$ . We took  $\tilde{\Omega} = 1$ ,  $\tau \rightarrow \infty$ , and  $\zeta = 0$ . The solid and dashed lines correspond to no-slip and no-stress boundary conditions, respectively. Different colors correspond to different values of the shear and Hall viscosities, as indicated in the legend. For  $\eta_s > 0$  the frequency has a negative imaginary part (decay), which is plotted with the same markings below the horizontal axis.

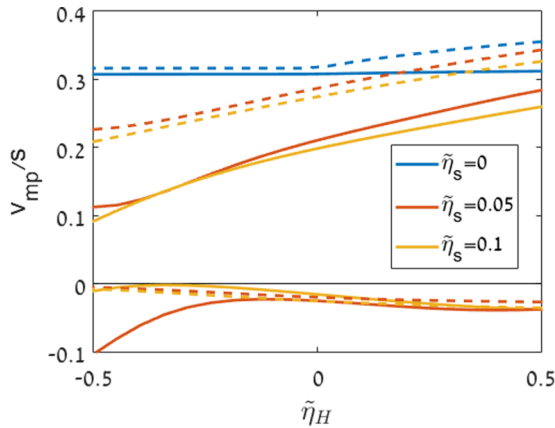


FIG. 4. Coefficient of the leading term in the dispersion,  $q \ln(\omega_c/sq)$  [see Eq. (31)], as a function of the dimensionless Hall viscosity,  $\tilde{\eta}_H \equiv \omega_c \eta_H / n_0 m s^2$ , for different values of the dimensionless shear viscosity (similarly defined). We took  $\tilde{\Omega} = 1$ ,  $\tau \rightarrow \infty$ ,  $\zeta = 0$ . The solid and dashed lines correspond to no-slip and no-stress boundary conditions, respectively. For  $\eta_s > 0$  the coefficient has a negative imaginary part (decay), which is plotted with the same markings below the horizontal axis.

diverges logarithmically as the wave vector goes to zero, a result of the long-range Coulomb potential [65,72,73] if the Fetter approximation [59] is not employed. It is sensitive to the viscosity, as shown in Fig. 4. This is in contrast to the bulk dispersion, Eq. (10), where the viscosity effect enters only to order  $q^2$  [35,36]. This behavior is a result of the finite decay length of the edge modes into the bulk, which implies that even for small  $q$  the electron density and velocity have non-negligible 2D Fourier components.

## V. CONCLUSIONS

To conclude, we have investigated the effect of viscosity (both the standard dissipative bulk and shear viscosities, as well as the symmetry-protected topological, nondissipative Hall viscosity) on the dispersion of edge magnetoplasmons in 2D electronic systems under the influence of a magnetic field. Extending the Wiener-Hopf technique, we have been able to find a general solution to the problem, which can be applied for arbitrary boundary conditions or the frequency dependence of the viscosities. As opposed to bulk magnetoplasmons, whose dispersion is only affected by viscosity starting at second order in the wave vector [35,36], on the edge the dispersion is already affected by the viscosity to lowest order in the wave vector, making it easier to extract experimentally. Our calculations indicate that the relevant frequencies ( $\lesssim \omega_c$ ) are in the microwave to terahertz regime, the wavelengths ( $\gtrsim s/\omega_c$ ) are in the tens of nanometers range and above, and the velocities ( $\sim s$ ) are of the order of  $10^3$  km/s in both semiconductor heterostructures and graphene. These are thus within reach of present edge magnetoplasmon measurements [58,61,66–71,74–81] or, possibly, adaptations of bulk techniques [94–97]. In the future it would be interesting to use the tools developed here to study the effects of viscosity on the analogous modes driven by nonzero Berry curvature instead of magnetic field [82–89].

## ACKNOWLEDGMENTS

We would like to thank L. I. Glazman for suggesting that we investigate this problem. Support from the Israel Science Foundation (Grant No. 227/15), the German Israeli Foundation (Grant No. I-1259-303.10), the U.S.-Israel Binational Science Foundation (Grant No. 2014262), and the Israel Ministry of Science and Technology (Contract No. 3-12419) is gratefully acknowledged.

## APPENDIX A: THE FETTER APPROXIMATION

The need to resort to the Wiener-Hopf method arose since the solution of the Poisson equation led to a potential-density relation, Eq. (9), with a kernel  $\bar{L}(k)$  which is not a meromorphic function. As a result, one cannot write down a local 2D partial differential equation relating the potential and the charge density. To simplify the problem, Fetter [59] suggested replacing  $\bar{L}(k)$  by a meromorphic function,  $\bar{L}_0(k)$ , which agrees with  $\bar{L}(k)$  to second order in  $k$ . It is given by

$$\bar{L}_0(k) = \frac{q}{k^2 + 2q^2}. \quad (\text{A1})$$

In real space, this amounts to replacing the exact kernel,

$$L(x) = \frac{1}{2\pi} K_0(|qx|), \quad (\text{A2})$$

with  $K_0$  being the modified Bessel function [98], by

$$L_0(x) = 2^{-3/2} e^{-\sqrt{2}q|x|}. \quad (\text{A3})$$

With the Fetter approximation,  $-L_0/q$  is the Green's function of the operator  $\partial_x^2 - 2q^2$ , allowing one to replace the 3D Poisson equation (7) by the 2D local linear equation

$$(\partial_x^2 - 2q^2)\phi(x) = \frac{4\pi e}{\epsilon} q \delta n(x). \quad (\text{A4})$$

Combining this with the 2D local linearized continuity and Navier-Stokes equations (5) and (6), one can seek a solution which is a combination of exponential functions of  $x$  and impose the boundary conditions. This readily leads to the dispersion relation.

One may also specialize our general solution, Eqs. (22), (19), and (13), to the Fetter approximation by plugging in the kernel  $\bar{L}_0$ . In this case the Riemann-Hilbert problem whose solution defines  $\bar{X}_\pm$  [see Eq. (13)] involves only rational functions and can be easily solved as

$$\bar{X}_\pm(k) = \left[ \frac{(k \pm k_{H1})(k \pm k_{H2})(k \pm i\sqrt{2}|q|)}{(k \pm z_1)(k \pm z_2)(k \pm z_3)} \right]^{\pm 1}, \quad (\text{A5})$$

where  $z_{1,2,3}$  are the roots in the upper complex  $k$  plane of the a sextic polynomial equation,

$$(k^2 + 2q^2)\{qs_\xi^2 \omega_s(k)(k^2 + q^2) + q\omega[\omega_H^2(k) - \omega_s^2(k)]\} + 2\Omega_q^2 \omega_s(k)(k^2 + q^2) = 0. \quad (\text{A6})$$

We have verified numerically that the Wiener-Hopf dispersion, Eq. (22), indeed agrees with the direct solution outlined above in this case.

## APPENDIX B: SUFFICIENCY OF EQUATION (21)

In this appendix we will verify that, for the no-slip boundary conditions, besides Eq. (21), there are no additional equations to be satisfied in order for our solution to have the required analyticity properties and boundary conditions.

First, let us check that  $\bar{\psi}_+$  is analytic in the upper complex plane. Performing the integral in Eq. (15) gives

$$\bar{\psi}_+(k) = \frac{1}{\bar{X}_+(k)} \frac{1}{k^2 + q^2} \left[ Ak + iqB + (iqA + kB) \frac{\omega_H}{\omega_s} \right] + \frac{1}{2|q|} \left( \frac{C_-}{k + i|q|} - \frac{C_+}{k - i|q|} \right) + \frac{1}{2k_s} \frac{\omega_H(k_s)}{\tilde{\omega}} \left( \frac{D_+}{k - k_s} - \frac{D_-}{k + k_s} \right). \quad (\text{B1})$$

This expression has two potential poles in the upper complex  $k$  plane at  $k = i|q|$  and  $k = k_s$ . The corresponding residues are

$$\text{Res} \bar{\psi}_+(i|q|) = \frac{1}{\bar{X}_+(i|q|)} \frac{1}{2i|q|} \left[ Ai|q| + iqB + (iqA + i|q|B) \frac{\omega_H(i|q|)}{\omega_s(i|q|)} \right] - \frac{1}{2|q|} C_+, \quad (\text{B2a})$$

$$\text{Res} \bar{\psi}_+(k_s) = \frac{1}{\bar{X}_+(k_s)} \frac{1}{k_s^2 + q^2} (iqA + k_s B) \frac{\omega_H(k_s)}{2i\gamma_s k_s} + \frac{1}{2k_s} \frac{\omega_H(k_s)}{\tilde{\omega}} D_+. \quad (\text{B2b})$$

Both can be shown to vanish when substituting the definitions of  $C_+$  and  $D_+$ . This, together with the corresponding behavior of  $\bar{\psi}_-$ , is sufficient to guarantee the correct analyticity properties of the Fourier-transformed potential  $\bar{\phi}_\pm$ .

Let us now turn to the Fourier-transformed velocities. From Eqs. (11) we obtain

$$\bar{v}_x^+ = \frac{1}{k} \left( \omega \frac{\bar{\delta n}_+}{n_0} - q \bar{v}_y^+ \right), \quad (\text{B3a})$$

$$\bar{v}_y^+ = \frac{1}{q\omega_H - ik\omega_s} \left[ iqk \frac{e}{m} \bar{\phi}_+ - ikB + (\omega\omega_H - iqk s_\zeta^2) \frac{\bar{\delta n}_+}{n_0} \right]. \quad (\text{B3b})$$

The correct behavior of  $\bar{v}_x^+$  is thus determined by that of  $\bar{v}_y^+$ . For the latter the residue at  $\omega_H = ik\omega_s/q$  remains to be checked. Using Eq. (18) for  $\bar{\delta n}_+/n_0$ , it can be shown to vanish.

Finally, for any function  $f(x)$  defined for  $x < 0$ , with a corresponding Fourier transform  $\bar{f}_+(k)$  analytic in the upper complex plain and vanishing as  $|k| \rightarrow \infty$ , its real-space boundary values are given by  $f(0) = -i \lim_{|k| \rightarrow \infty} k \bar{f}_+(k)$ , and  $\partial_x f(0) = \lim_{|k| \rightarrow \infty} k^2 \bar{f}_+(k)$  if  $f(0) = 0$ . Since the Fourier functions are analytic in the upper complex plane, the boundary conditions for the velocity are obeyed automatically. Using this, it is straightforward to verify that all the boundary conditions are obeyed by our solution. For example, from the leading behavior of  $\bar{v}_y^+$  at large  $|k|$ ,

$$\bar{v}_y^+(k) \sim -\frac{1}{\gamma_s k^3} \left[ ikB + \omega\gamma_H \frac{i\gamma_s \frac{e}{m} k^2 \bar{\phi}_+(\infty) - k\gamma_s A + ik\gamma_H B}{i\gamma_s s_\zeta^2 + \omega(\gamma_H^2 + \gamma_s^2)} \right],$$

one can verify that  $v_y(0) = 0$  and  $\partial_y v_y(0)$  equals the value encoded in  $A$  and  $B$  [see Eq. (17)]. Similar relations hold for all the other physical quantities, without imposing any additional constraints. The only exception is the potential for the no-stress boundary conditions, as discussed in Sec. III B.

- 
- [1] R. N. Gurzhi, *Usp. Fiz. Nauk* **94**, 689 (1968) [*Sov. Phys. Usp.* **11**, 255 (1968)].
- [2] M. J. M. de Jong and L. W. Molenkamp, *Phys. Rev. B* **51**, 13389 (1995).
- [3] M. Müller, J. Schmalian, and L. Fritz, *Phys. Rev. Lett.* **103**, 025301 (2009).
- [4] A. V. Andreev, S. A. Kivelson, and B. Spivak, *Phys. Rev. Lett.* **106**, 256804 (2011).
- [5] M. Titov, R. V. Gorbachev, B. N. Narozhny, T. Tudorovskiy, M. Schütt, P. M. Ostrovsky, I. V. Gornyi, A. D. Mirlin, M. I. Katsnelson, K. S. Novoselov, A. K. Geim, and L. A. Ponomarenko, *Phys. Rev. Lett.* **111**, 166601 (2013).
- [6] R. A. Davison, K. Schalm, and J. Zaanen, *Phys. Rev. B* **89**, 245116 (2014).
- [7] U. Briskot, M. Schütt, I. V. Gornyi, M. Titov, B. N. Narozhny, and A. D. Mirlin, *Phys. Rev. B* **92**, 115426 (2015).
- [8] I. Torre, A. Tomadin, A. K. Geim, and M. Polini, *Phys. Rev. B* **92**, 165433 (2015).
- [9] F. Ghahari, H.-Y. Xie, T. Taniguchi, K. Watanabe, M. S. Foster, and P. Kim, *Phys. Rev. Lett.* **116**, 136802 (2016).
- [10] P. S. Alekseev, *Phys. Rev. Lett.* **117**, 166601 (2016).
- [11] L. S. Levitov and G. Falkovich, *Nat. Phys.* **12**, 672 (2016).
- [12] D. A. Bandurin, I. Torre, R. K. Kumar, M. Ben Shalom, A. Tomadin, A. Principi, G. H. Auton, E. Khestanova, K. S. Novoselov, I. V. Grigorieva, L. A. Ponomarenko, A. K. Geim, and M. Polini, *Science* **351**, 1055 (2016).
- [13] A. Lucas, R. A. Davison, and S. Sachdev, *Proc. Natl. Acad. Sci. USA* **113**, 9463 (2016).
- [14] P. J. W. Moll, P. Kushwaha, N. Nandi, B. Schmidt, and A. P. Mackenzie, *Science* **351**, 1061 (2016).
- [15] J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas, S. Sachdev, P. Kim, T. Taniguchi, K. Watanabe,

- T. A. Ohki, and K. C. Fong, *Science* **351**, 1058 (2016).
- [16] R. Krishna Kumar, D. A. Bandurin, F. M. D. Pellegrino, Y. Cao, A. Principi, H. Guo, G. Auton, M. Ben Shalom, L. A. Ponomarenko, G. Falkovich, K. Watanabe, T. Taniguchi, I. Grigorieva, L. S. Levitov, M. Polini, and A. Geim, *Nat. Phys.* **13**, 1182 (2017).
- [17] T. Scaffidi, N. Nandi, B. Schmidt, A. P. Mackenzie, and J. E. Moore, *Phys. Rev. Lett.* **118**, 226601 (2017).
- [18] J. Gooth, F. Menges, C. Shekhar, V. Süß, N. Kumar, Y. Sun, U. Drechsler, R. Zierold, C. Felser, and B. Gotsmann, [arXiv:1706.05925](https://arxiv.org/abs/1706.05925).
- [19] B. N. Narozhny, I. V. Gornyi, A. D. Mirlin, and J. Schmalian, *Ann. Phys. (Berlin, Ger.)* **529**, 1700043 (2017).
- [20] O. Kashuba, B. Trauzettel, and L. W. Molenkamp, *Phys. Rev. B* **97**, 205129 (2018).
- [21] A. Lucas and K. C. Fong, *J. Phys.: Condens. Matter* **30**, 053001 (2018).
- [22] A. D. Levin, G. M. Gusev, E. V. Levinson, Z. D. Kvon, and A. K. Bakarov, *Phys. Rev. B* **97**, 245308 (2018).
- [23] E. I. Kiselev and J. Schmalian, [arXiv:1806.03933](https://arxiv.org/abs/1806.03933).
- [24] A. I. Berdyugin, S. G. Xu, F. M. D. Pellegrino, R. Krishna Kumar, A. Principi, I. Torre, M. Ben Shalom, T. Taniguchi, K. Watanabe, I. V. Grigorieva, M. Polini, A. K. Geim, and D. A. Bandurin, [arXiv:1806.01606](https://arxiv.org/abs/1806.01606).
- [25] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, 1987).
- [26] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, New York, 1953).
- [27] W. Marshall, Kinetic Theory of an Ionized Gas, AERE T/R 2419, Harwell, Berkshire, UK, 1958.
- [28] M. S. Steinberg, *Phys. Rev.* **109**, 1486 (1958).
- [29] A. N. Kaufman, *Phys. Fluids* **3**, 610 (1960).
- [30] W. B. Thompson, *Rep. Prog. Phys.* **24**, 363 (1961).
- [31] S. I. Braginskii, in *Transport Processes in Plasma*, edited by M. Leontovich (Gosatomizdat, Moscow, 1963), Vol. 1, p. 183.
- [32] J. E. Avron, R. Seiler, and P. G. Zograf, *Phys. Rev. Lett.* **75**, 697 (1995).
- [33] J. E. Avron, *J. Stat. Phys.* **92**, 543 (1998).
- [34] P. Lévy, *J. Math. Phys.* **36**, 2792 (1995).
- [35] I. V. Tokatly and G. Vignale, *Phys. Rev. B* **76**, 161305 (2007).
- [36] I. V. Tokatly and G. Vignale, *J. Phys.: Condens. Matter* **21**, 275603 (2009).
- [37] N. Read, *Phys. Rev. B* **79**, 045308 (2009).
- [38] F. D. M. Haldane, [arXiv:0906.1854](https://arxiv.org/abs/0906.1854).
- [39] N. Read and E. H. Rezayi, *Phys. Rev. B* **84**, 085316 (2011).
- [40] T. L. Hughes, R. G. Leigh, and E. Fradkin, *Phys. Rev. Lett.* **107**, 075502 (2011).
- [41] M. Barkeshli, S. B. Chung, and X.-L. Qi, *Phys. Rev. B* **85**, 245107 (2012).
- [42] C. Hoyos and D. T. Son, *Phys. Rev. Lett.* **108**, 066805 (2012).
- [43] B. Bradlyn, M. Goldstein, and N. Read, *Phys. Rev. B* **86**, 245309 (2012).
- [44] P. B. Wiegmann, *Phys. Rev. B* **88**, 241305 (2013).
- [45] T. Can, M. Laskin, and P. Wiegmann, *Phys. Rev. Lett.* **113**, 046803 (2014).
- [46] A. G. Abanov, *J. Phys. A* **46**, 292001 (2013).
- [47] A. Gromov and A. G. Abanov, *Phys. Rev. Lett.* **113**, 266802 (2014).
- [48] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, *Phys. Rev. Lett.* **114**, 016805 (2015).
- [49] Z. E. Ezawa, *Quantum Hall Effects*, 2nd ed. (World Scientific, Singapore, 2008).
- [50] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [51] C. Hoyos, B. S. Kim, and Y. Oz, *J. High Energy Phys.* **2014**, 54 (2014).
- [52] S. Moroz, C. Hoyos, and L. Radzihovsky, *Phys. Rev. B* **91**, 195409 (2015).
- [53] J. Höller and N. Read, *Phys. Rev. B* **93**, 197401 (2016).
- [54] M. Sherafati, A. Principi, and G. Vignale, *Phys. Rev. B* **94**, 125427 (2016).
- [55] F. M. D. Pellegrino, I. Torre, and M. Polini, *Phys. Rev. B* **96**, 195401 (2017).
- [56] S. Ganeshan and A. G. Abanov, *Phys. Rev. Fluids* **2**, 094101 (2017).
- [57] L. V. Delacrétaz and A. Gromov, *Phys. Rev. Lett.* **119**, 226602 (2017).
- [58] D. C. Glatli, E. Y. Andrei, G. Deville, J. Poitrenaud, and F. I. B. Williams, *Phys. Rev. Lett.* **54**, 1710 (1985).
- [59] A. L. Fetter, *Phys. Rev. B* **32**, 7676 (1985).
- [60] V. A. Volkov and S. A. Mikhailov, *Pis'ma Zh. Eksp. Teor. Fiz.* **42**, 450 (1985) [*Sov. Phys. JETP Lett.* **42**, 556 (1986)].
- [61] V. A. Volkov, D. V. Galchenkov, L. A. Galchenkov, I. M. Grodnenskii, O. R. Matov, and S. A. Mikhailov, *Pis'ma Zh. Eksp. Teor. Fiz.* **44**, 510 (1986) [*Sov. Phys. JETP Lett.* **44**, 655 (1987)].
- [62] A. L. Fetter, *Phys. Rev. B* **33**, 5221 (1986).
- [63] A. L. Fetter, *Phys. Rev. B* **33**, 3717 (1986).
- [64] S. S. Nazin and V. B. Shikin, *Zh. Eksp. Teor. Fiz.* **94**, 133 (1988) [*Sov. Phys. JETP* **67**, 288 (1988)].
- [65] V. A. Volkov and S. A. Mikhailov, *Zh. Eksp. Teor. Fiz.* **94**, 217 (1988) [*Sov. Phys. JETP* **67**, 1639 (1988)].
- [66] E. Y. Andrei, D. C. Glatli, F. I. B. Williams, and M. Heiblum, *Surf. Sci.* **196**, 501 (1988).
- [67] I. Grodnensky and A. Kamaev, *Surf. Sci.* **229**, 522 (1990).
- [68] M. Wassermeier, J. Oshinowo, J. P. Kotthaus, A. H. MacDonald, C. T. Foxon, and J. J. Harris, *Phys. Rev. B* **41**, 10287 (1990).
- [69] R. C. Ashoori, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. West, *Phys. Rev. B* **45**, 3894 (1992).
- [70] V. I. Talyanskii, J. E. F. Frost, M. Pepper, D. A. Ritchie, M. Grimshaw, and G. A. C. Jones, *J. Phys.: Condens. Matter* **5**, 7643 (1993).
- [71] N. B. Zhitenev, R. J. Haug, K. v. Klitzing, and K. Eberl, *Phys. Rev. Lett.* **71**, 2292 (1993).
- [72] I. L. Aleiner and L. I. Glazman, *Phys. Rev. Lett.* **72**, 2935 (1994).
- [73] I. L. Aleiner, D. Yue, and L. I. Glazman, *Phys. Rev. B* **51**, 13467 (1995).
- [74] G. Sukhodub, F. Hohls, and R. J. Haug, *Phys. Rev. Lett.* **93**, 196801 (2004).
- [75] H. Kamata, T. Ota, K. Muraki, and T. Fujisawa, *Phys. Rev. B* **81**, 085329 (2010).
- [76] N. Kumada, H. Kamata, and T. Fujisawa, *Phys. Rev. B* **84**, 045314 (2011).
- [77] I. Petković, F. I. B. Williams, K. Bennaceur, F. Portier, P. Roche, and D. C. Glatli, *Phys. Rev. Lett.* **110**, 016801 (2013).
- [78] K.-i. Sasaki, S. Murakami, Y. Tokura, and H. Yamamoto, *Phys. Rev. B* **93**, 125402 (2016).



- [79] S. Bosco and D. P. DiVincenzo, *Phys. Rev. B* **95**, 195317 (2017).
- [80] A. Endo, K. Koike, S. Katsumoto, and Y. Iye, *J. Phys. Soc. Jpn.* **87**, 064709 (2018).
- [81] D. Jin, Y. Xia, T. Christensen, S. Wang, K. Y. Fong, M. Freeman, G. C. Gardner, S. Fallahi, Q. Hu, Y. Wang, L. Engel, M. J. Manfra, N. X. Fang, and X. Zhang, [arXiv:1803.02913](https://arxiv.org/abs/1803.02913).
- [82] J. C. W. Song and M. S. Rudner, *Proc. Natl. Acad. Sci.* **113**, 4658 (2016).
- [83] D. Jin, L. Lu, Z. Wang, C. Fang, J. D. Joannopoulos, M. Soljačić, L. Fu, and N. X. Fang, *Nat. Commun.* **7**, 13486 (2016).
- [84] A. Principi, M. I. Katsnelson, and G. Vignale, *Phys. Rev. Lett.* **117**, 196803 (2016).
- [85] A. Principi, M. I. Katsnelson, and A. Levchenko, *Phys. Rev. Lett.* **118**, 036802 (2017).
- [86] Y. Zhang, F. Zhai, B. Guo, L. Yi, and W. Jiang, *Phys. Rev. B* **96**, 045104 (2017).
- [87] X. Jia, S. Zhang, R. Sankar, F.-C. Chou, W. Wang, K. Kempa, E. W. Plummer, J. Zhang, X. Zhu, and J. Guo, *Phys. Rev. Lett.* **119**, 136805 (2017).
- [88] A. C. Mahoney, J. I. Colless, L. Peeters, S. J. Pauka, E. J. Fox, X. Kou, L. Pan, K. L. Wang, D. Goldhaber-Gordon, and D. J. Reilly, *Nat. Commun.* **8**, 1836 (2017).
- [89] Steven S.-L. Zhang and G. Vignale, *Phys. Rev. B* **97**, 224408 (2018).
- [90] A. G. Abanov, T. Can, and S. Ganeshan, *SciPost Phys.* **5**, 10 (2018).
- [91] A. Souslov, K. Dasbiswas, S. Vaikuntanathan, and V. Vitelli, [arXiv:1802.09649](https://arxiv.org/abs/1802.09649).
- [92] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge University Press, Cambridge, 2009).
- [93] N. R. Cooper, B. I. Halperin, and I. M. Ruzin, *Phys. Rev. B* **55**, 2344 (1997).
- [94] E. Batke, D. Heitmann, J. P. Kotthaus, and K. Ploog, *Phys. Rev. Lett.* **54**, 2367 (1985).
- [95] L. L. Sohn, A. Pinczuk, B. S. Dennis, L. N. Pfeiffer, K. W. West, and L. Brey, *Solid State Commun.* **93**, 897 (1995).
- [96] I. V. Kukushkin, J. H. Smet, V. W. Scarola, V. Umansky, and K. von Klitzing, *Science* **324**, 1044 (2009).
- [97] D. A. Bandurin, D. Svintsov, I. Gayduchenko, S. G. Xu, A. Principi, M. Moskotin, I. Tretyakov, D. Yagodkin, S. Zhukov, T. Taniguchi, K. Watanabe, I. V. Grigorieva, M. Polini, G. Goltsman, A. K. Geim, and G. Fedorov, [arXiv:1807.04703](https://arxiv.org/abs/1807.04703).
- [98] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic, New York, 2007).