Superconductivity in systems exhibiting the Altshuler-Aronov anomaly

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Making use of generalized Eliashberg equations, we describe the Altshuler-Aronov (AA) effect and superconductivity on equal footing. We derive explicit expressions for the Coulomb pseudopotential in 3D, taking into account also the anomalous diffusion. We present a full numerical solution for two normal-state and two anomalous self-energies. In the normal state, we amend the known results for the purely electronic AA effect; with electron-phonon coupling turned on, we find additional anomalies in the density of states close to the phonon energy. We study how the critical temperature and density of states of strongly disordered 3D superconductors change with normal-state resistivity. We find that the type of transition from the superconducting to the insulating state depends on the strength of electron-phonon coupling: at weak coupling, there exists an intermediate normal state, whereas at strong coupling the transition is direct.

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I. INTRODUCTION

One of the central questions which have not been answered yet in the context of high-temperature superconductivity is that about the origin of the so-called pseudogap. Under the pseudogap, a complex of phenomena in the nonsuperconducting state of lightly doped cuprates is understood, among which a prominent place is taken by the suppression of the number of states in the vicinity of the Fermi level, documented by thermodynamic as well as spectroscopic methods [1].

Several candidate explanations have been proposed for the pseudogap in the cuprates, which can be classified into two large groups. In the first type of theories, the pseudogap is understood as a consequence of some symmetry-breaking phase transition leading to the formation of a "competing order." In moderate theories of this type, it is assumed that the competing order is not static, but only fluctuating. The second type of theories views the pseudogap as a consequence of the proximity of the cuprates to a Mott insulating state. It is the latter type of theories which motivates the present work.

If one assumes that the superconductor is close to being insulating, then there are again two pictures of how the pseudogap may arise, which have been widely studied also in systems other (and presumably simpler) than the cuprates [2,3]. The first (so-called bosonic) picture builds on the observation that the superfluid stiffness should be small in the vicinity of the insulating state, leading to strong phase fluctuations. Within this picture, the pseudogap appears due to the presence of Cooper pairs that have not condensed into a single macroscopically occupied state [4]. The second (so-called fermionic) picture starts from the observation that the screening of the Coulomb interactions should become progressively weaker and weaker as the insulating state is approached, and therefore the consequences of such interactions should become visible in the metallic state, irrespective of whether it ultimately becomes superconducting at low temperatures [5].

On the experimental side, pseudogap has been observed in several low-temperature superconductors which are close to being insulating. Among the first observations predating the cuprate era were those in granular aluminum [6] and in the alloy $Nb_{1-x}Si_x$ [7]. More recently, pseudogap has been observed in very diverse systems such as TiN [8], InO_x [9], NbN [10], $BaPb_{1-x}Bi_xO_3$ [11], and Cu_xTiSe_2 [12].

A mere observation of the pseudogap does not allow us to distinguish between the bosonic and fermionic scenaria, and therefore more quantitative predictions of the theory are needed. Far away from the transition, a perturbative calculation within the fermionic scenario due to Altshuler and Aronov (AA) [13] suggests that the density of states $N(\omega)$ in the vicinity of the Fermi level of a 3D metal should be suppressed according to

$$N(\omega) = N(0)(1 + \sqrt{\omega/\Delta_{\mathrm{AA}}}). \tag{1}$$

In several papers, Eq. (1) has in fact been observed experimentally [6,7,11,12]. However, as we will show, the observed magnitude of Δ_{AA} is orders of magnitude smaller than predicted by a straightforward extrapolation of the AA theory.

In this paper we will demonstrate that a generalization of the AA theory due to Anderson, Muttalib, and Ramakrishnan (AMR) [14], which takes into account the scale dependence of the diffusion constant predicted by the scaling theory of localization [15], also leads to a density of states of the form of Eq. (1), but with an energy scale Δ_{AA}, which is compatible with the experiments. A similar observation has been made also previously [16,17].

Our next goal is to apply the AA theory, as modified by AMR, to the superconducting state and to check whether the results of Refs. [6,7,11,12] can in fact be explained within the fermionic theory. To this end, we will study the

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generalized Eliashberg equations in the exact eigenstate basis. Our approach is quite similar in spirit to the pioneering work of Belitz [18], the main technical difference being that we work on the imaginary axis. Moreover, unlike Belitz, we will present the results of a full numerical solution of the Eliashberg equations. Thus we have access not only to thermodynamics, but, after analytic continuation, also to the superconducting density of states.

The outline of this paper is as follows. Following AMR, in Sec. II, we determine the energy dependence of the Coulomb pseudopotential $\mu(\varepsilon)$. We show that in addition to the logarithmic regime at intermediate ε discovered by AMR in strongly disordered systems, at the lowest ε, the pseudopotential always varies as $\sqrt{\varepsilon}$, but the relevant energy scale varies by orders of magnitude between the weakly and strongly disordered systems. In Sec. III, we write down the Eliashberg equations in the exact eigenstate basis and in Sec. IV, we show that their solution in the normal state always leads to a density of states of the form (1) in the low-energy limit. Furthermore, we show that, in the strongly disordered limit, the energy scale Δ_{AA} can become arbitrarily small, in qualitative agreement with Refs. [6,7]. We also show here that, in the presence of the AA effect, coupling to phonons leads to additional features of the density of states (in the nonsuperconducting state) close to the phonon energy. Finally, in Sec. V, we present the results of the numerical solution of the Eliashberg equations in the superconducting state.

II. COULOMB PSEUDOPOTENTIAL

Within the Eliashberg theory, the exchange contribution to the bare Coulomb pseudopotential is calculated as a Fermi-surface average of the screened Coulomb interaction. If we consider the static screening with inverse screening length k_s, one can show readily that the formula

$$\mu(\varepsilon)=\frac{e^2}{2\pi^3\epsilon_0}\int_0^{2k_F}\frac{dqq^2}{q^2+k_s^2}\frac{\hbar D_q q^2}{(\hbar D_q q^2)^2+\varepsilon^2}, \tag{2}$$

wherein we take $D_q=2v_F/(\pi q)$ with v_F the Fermi velocity, does lead—at zero energy transfer $\varepsilon=0$—to the well-known Coulomb pseudopotential $\mu_0=\frac{\alpha}{2\pi}\ln(1+\pi/\alpha)$ of a clean system with isotropic quadratic dispersion [19]. Here we have introduced the "fine structure" constant $\alpha=e^2/(4\pi\epsilon_0\hbar v_F)$ of the electron gas [20] and we have made use of the relation $2k_F/k_s=(\pi/\alpha)^{1/2}$. Note that in a typical metal $\alpha\sim 1$.

On the other hand, as shown by Refs. [13,18], Eq. (2) is applicable also to weakly disordered systems, and in this case one has to take $D_q=D_0$, where $D_0=\frac{1}{3}v_F\ell$ is the diffusion constant of the dirty system characterized by the mean free path ℓ. This result can be most simply shown in the basis of exact Hartree-Fock eigenstates of the disordered system, in which the exchange contribution to the self-energy $\chi(\varepsilon)$ of an eigenstate with bare energy ε reads as $\chi(\varepsilon)=-\int d\varepsilon'\mu(\varepsilon-\varepsilon')f(\varepsilon')$, where $f(x)$ is the Fermi function. We deliberately neglect all Hartree contributions to the self-energy, although in a more complete treatment of a disordered system they may be present [18].

The goal of this section, which represents a generalization of the insightful AMR paper [14], is to study the evolution of the Coulomb pseudopotential $\mu(\varepsilon)$ with the amount of disorder: from the clean case, via weakly disordered systems, up to the strongly disordered (but still metallic) case.

Before proceeding, it is useful to introduce a sharp criterion which enables us to distinguish between weak and strong disorder. It is well known that the naive formula for the conductivity, $\sigma=g_0k_F^2\ell$ with $g_0=e^2/(3\pi^2\hbar)$ and what AMR call the local mean free path ℓ, should not be applied too close to the localized regime, because in that case localization corrections enter the expression for $\sigma(\ell)$. Instead, the scaling theory of localization [15] suggests to introduce a different length scale L_s such that $\sigma=g_c/L_s$, where $g_c\sim g_0$ is the critical conductance. Following AMR, we will call systems with $L_s<\ell$ weakly disordered, and those with $L_s>\ell$ strongly disordered. It turns out that it is advantageous to discuss these two cases separately.

A. Weak disorder, $L_s<\ell$

In the weakly disordered case, one can identify two qualitatively different contributions to Eq. (2). Namely, at short distances (i.e., for wave vectors $q^*<q<2k_F$), the electron motion should be ballistic and we should therefore use the clean-limit expression [21] $D_q=2v_F/(\pi q)$. On the other hand, at long length scales (i.e., for wave vectors $0<q<q^*$), the electron motion is diffusive and we should take $D_q=D_0$. Note that in doing so, we reduce the diffusion constant with respect to its ballistic value, which in turn leads to an increase of the Coulomb pseudopotential. In the spirit of AMR, we assume that the short- and long-distance forms are valid up to the crossover scale q^*. Requiring that the function D_q is continuous leads us then to the identification $q^*=6/(\pi\ell)$. Note that the inequality $q^*<2k_F$ implies that we have to require $k_F\ell>3/\pi$.

With this choice of the function D_q, the integral in Eq. (2) can be taken exactly, but it leads to a bulky formula. We find that the result can be written with good accuracy by the following expression:

$$\mu(\varepsilon)=\begin{cases}\mu_0+\frac{1}{2(k_F\ell)^2}\left[1-\delta-\sqrt{\frac{\varepsilon}{\Gamma}}\right], & \varepsilon<\varepsilon_{\max},\\ \mu_0, & \varepsilon>\varepsilon_{\max},\end{cases} \tag{3}$$

where we have introduced the energy $\Gamma=2\varepsilon_F/(3k_F\ell)$ and the dimensionless number δ,

$$\delta=\frac{\alpha(k_F\ell)^2}{6}\ln\left[1+\frac{1}{\alpha(k_F\ell)^2}\right].$$

Note that $\delta<0.17$, i.e., δ is always small. The energy scale $\varepsilon_{\max}$ can be found by requiring that $\mu(\varepsilon)$ should not drop below its value μ_0 in the clean metal.

Note that Eq. (3) looks reasonable: the Coulomb pseudopotential $\mu(\varepsilon)$ of a disordered system is larger than μ_0, at small energy transfers it exhibits the expected $\sqrt{\varepsilon}$ dependence, and in the clean limit, $k_F\ell\to\infty$, it reduces to μ_0. In Sec. IV, we will demonstrate that the well-known AA depression of the density of states at the Fermi level [13] is a simple consequence of Eq. (3).

B. Strong disorder, $\ell<L_s$

Also, in this case we will construct, following AMR, the simplest scale-dependent diffusion coefficient D_q that

is consistent with the known physical constraints. Let us start at the largest length scales, where, as noted by AMR, the macroscopic diffusion constant of a strongly disordered system with $\ell < L_s$ is reduced from its local estimate D_0 to $D_0\ell/L_s$. Therefore in the region $0 < q < L_s^{-1}$ we will assume that $D_q = D_0\ell/L_s$. At intermediate length scales, AMR have identified a region of anomalous diffusion [22], where $D_q = D_0 q\ell$ and the diffusion constant increases with decreasing length scale, ultimately approaching its local limit D_0. This functional form will therefore be assumed to be valid at momenta $L_s^{-1} < q < \ell^{-1}$. Since the diffusion constant of a dirty system can not exceed its local limit, at still shorter length scales $\ell^{-1} < q < q^*$, we have to assume that $D_q = D_0$, until ultimately at the shortest length scales $q^* < q < 2k_F$ the electrons move ballistically and therefore $D_q = 2v_F/(\pi q)$.

With the above choice of the function D_q, the integral in Eq. (2) can again be taken exactly. The result can be reasonably well described by the following function:

$$\mu(\varepsilon) = \begin{cases} \mu_0 + \frac{1}{2(k_F\ell)^2}\left[1 + \ln\left(\frac{L_s}{\ell}\right) - \delta - \sqrt{\frac{\varepsilon}{\varepsilon^*}}\right], & \varepsilon < \varepsilon^*, \\ \mu_0 + \frac{1}{2(k_F\ell)^2}\left[\frac{1}{3}\ln\left(\frac{\Gamma}{\varepsilon}\right) - \delta\right], & \varepsilon^* < \varepsilon < \varepsilon_{\max}, \\ \mu_0, & \varepsilon_{\max} < \varepsilon, \end{cases} \tag{4}$$

where $\varepsilon^* = (\ell/L_s)^3 \times \Gamma$ is a new energy scale. Note that in a strongly disordered metal $\varepsilon^* \ll \Gamma$. Requiring that $\mu(\varepsilon)$ is continuous we find $\varepsilon_{\max} = \Gamma e^{-3\delta}$.

When Eq. (4) is compared with the result (3) for the weakly disordered case, one can notice that the low-energy enhancement of the Coulomb pseudopotential is much larger in the present case. There are two reasons for this: first, the factor $k_F\ell \sim 1$ is much smaller than $k_F\ell \gg 1$ in the weakly disordered case. Second and less trivially, due to anomalous diffusion, at intermediate energy transfers, $\varepsilon^* < \varepsilon < \Gamma$, the Coulomb pseudopotential exhibits a large logarithmic increase, in qualitative agreement with the result of AMR.

It should be pointed out that at the lowest energy transfers $\varepsilon < \varepsilon^*$, which have not been considered by AMR, the Coulomb pseudopotential $\mu(\varepsilon)$ still exhibits the standard AA-type behavior, but the associated energy scale is ε^* instead of Γ, i.e., it may be much smaller than in the weakly disordered systems. This has observable consequences, as explained in Sec. IV.

III. ELIASHBERG EQUATIONS

In the basis of exact eigenstates of the disordered system, the Eliashberg equations can be written in the imaginary-time Nambu-Gorkov formalism in a very compact form:

$$\hat{\Sigma}(\varepsilon,\omega) = T\sum_{\omega'}\int d\varepsilon'[-\mu(\varepsilon'-\varepsilon) + g(\omega'-\omega)]\tau_3\hat{G}(\varepsilon',\omega')\tau_3,$$

where τ_3 is the Pauli matrix and the 2×2 matrices $\hat{\Sigma}(\varepsilon,\omega)$ and $\hat{G}(\varepsilon,\omega)$ are the self-energy and the Green function for a time-reversal related pair of eigenstates characterized by bare energy ε; ω is the Matsubara frequency. In what follows we do distinguish between energy and frequency; however, both will be measured in the same units, i.e., we set $\hbar = 1$. Note that in a disordered system, ε plays the same role as momentum $\mathbf{k}$ in a clean system. That is also the reason why the Coulomb pseudopotential (in a theory with static screening) is a function of transferred energy and not frequency.

The Eliashberg equations describe the contributions of self-consistent rainbow diagrams to the self-energy, where the interaction lines are either due to screened Coulomb interactions described by the Coulomb pseudopotential $\mu(\varepsilon)$ introduced in the previous section, or due to electron-electron interactions generated by the exchange of phonons and described by the function $g(\omega)$. In what follows, we will assume a simple Debye model for the phonons, and the resulting function $g(\omega)$ reads as [23]

$$g(\omega) = \lambda\left[1 - \frac{\omega^2}{\Omega^2}\ln\left(1 + \frac{\Omega^2}{\omega^2}\right)\right], \tag{5}$$

where λ is the dimensionless electron-phonon coupling and Ω is the Debye energy. Following the arguments of AMR [14,24], in what follows, we neglect the effect of disorder on λ, since we intend to concentrate on strongly disordered superconductors, where the effect of the Coulomb pseudopotential should dominate. For the same reason we keep neglecting all possible Hartree-type contributions to the self-energy.

The most general ansatz for the self-energy $\hat{\Sigma}(\varepsilon,\omega)$ can be written as

$$\hat{\Sigma}(\varepsilon,\omega) = \Sigma(\varepsilon,\omega)\tau_0 + \chi(\varepsilon,\omega)\tau_3 + \Phi(\varepsilon,\omega)\tau_1, \tag{6}$$

where τ_i are the Pauli matrices and the functions $\Sigma(\varepsilon,\omega)$ and $\Phi(\varepsilon,\omega)$ are the normal and anomalous self-energies, respectively. In clean particle-hole symmetric metals, the τ_3 component of the self-energy can be ignored, because the Coulomb pseudopotential can be taken as energy-independent. However, as explained in the previous section, in disordered systems, $\mu(\varepsilon)$ is not a constant function, and therefore in addition to $\Sigma(\varepsilon,\omega)$ and $\Phi(\varepsilon,\omega)$, also the function $\chi(\varepsilon,\omega)$ has to be determined self-consistently. Moreover, the ε dependence can not be simply ignored as in the clean case. These points have been emphasized by Belitz early in Ref. [18].

In what follows, it is useful to define also the functions $Z(\varepsilon,\omega) = 1 + \Sigma(\varepsilon,\omega)/(i\omega)$ and $R(\varepsilon,\omega) = 1 + \chi(\varepsilon,\omega)/\varepsilon$. Inserting the ansatz (6) into the Eliashberg equations and making use of the fact that the functions $\mu(\varepsilon)$ and $g(\omega)$ are even, one can show that also Z, R, and Φ can be chosen as even functions of ε,ω. Making use of this observation one finds readily that $\Sigma(\omega)$ and $Z(\omega)$ are independent of ε, and similarly $\chi(\varepsilon)$ and $R(\varepsilon)$ do not depend on ω. The Eliashberg

equations for a system at temperature T therefore simplify to the following form:

$$\Sigma(\omega) = T\sum_{\omega'}\int \frac{d\varepsilon' g(\omega'-\omega) i\omega' Z'}{(\omega' Z')^2 + (\varepsilon' R')^2 + \Phi^2(\varepsilon',\omega')}, \quad (7)$$

$$\chi(\varepsilon) = T\sum_{\omega'}\int \frac{d\varepsilon' \mu(\varepsilon'-\varepsilon)\varepsilon' R'}{(\omega' Z')^2 + (\varepsilon' R')^2 + \Phi^2(\varepsilon',\omega')}, \quad (8)$$

$$\phi(\omega) = T\sum_{\omega'}\int \frac{d\varepsilon' g(\omega'-\omega)\Phi(\varepsilon',\omega')}{(\omega' Z')^2 + (\varepsilon' R')^2 + \Phi^2(\varepsilon',\omega')}, \quad (9)$$

$$\psi(\varepsilon) = T\sum_{\omega'}\int \frac{d\varepsilon' \mu(\varepsilon'-\varepsilon)\Phi(\varepsilon',\omega')}{(\omega' Z')^2 + (\varepsilon' R')^2 + \Phi^2(\varepsilon',\omega')}, \quad (10)$$

where we have introduced the abbreviations $Z' = Z(\omega')$ and $R' = R(\varepsilon')$. We have also observed that the anomalous self-energy can be written as $\Phi(\varepsilon,\omega) = \phi(\omega) - \psi(\varepsilon)$.

In the rest of this paper, we will be concerned with the solution of Eqs. (7)–(10) with interactions given by Eqs. (3)–(5). Both in the Matsubara frequency space, as well as in the energy space, we will assume that there is a finite cutoff Λ, which restricts the studied states to the vicinity of the Fermi energy, $|\omega|, |\varepsilon| \leqslant \Lambda$. We will take Λ much larger than the Debye energy Ω, in order to have a valid description of the electron-phonon interaction.

In the special case when $\mu(\varepsilon)$ is a constant, one can easily observe that Eq. (8) implies that $\chi = 0$ and one ends up with the usual Eliashberg equations. The Coulomb pseudopotential enters only Eq. (10) in this case. Strictly speaking, we should not assume that it equals the bare value μ_0, since our cutoff Λ is much smaller than the Fermi energy (or bandwidth), and we should rather use an appropriately renormalized value. Nevertheless, since this is a minor correction, we have decided to use the bare value of μ_0 instead. On the other hand, we emphasize that the renormalization of the Coulomb interaction from the scale Λ to the phonon scale Ω is implicitly present in our self-consistent calculations.

Once the Eliashberg equations are solved, the Matsubara Green function $\hat{G}(\varepsilon,\omega)$ of the superconductor can be determined from the Dyson equation

$$\hat{G}^{-1}(\varepsilon,\omega) = i\omega\tau_0 - \varepsilon\tau_3 - \hat{\Sigma}(\varepsilon,\omega).$$

The density of states $N(\omega)$ in the superconducting state can be obtained from the textbook formula

$$N(\omega) = -\frac{1}{\pi} N_0 \int d\varepsilon \mathrm{Im} G^{\mathrm{R}}_{11}(\varepsilon,\omega), \quad (11)$$

where $G^{\mathrm{R}}_{11}(\varepsilon,\omega)$ is the upper left component of the retarded Green function $\hat{G}^{\mathrm{R}}(\varepsilon,\omega)$, and N_0 is the density of the bare levels ε.

IV. NORMAL STATE

In this section, we will investigate the implications of the Eliashberg equations for the normal-state properties of disordered metals. Since in the normal state $\Phi = 0$, we have to solve Eqs. (7) and (8) for the self-energies $\chi(\varepsilon)$ and $\Sigma(\omega)$. We will be especially interested in the tunneling density of states. We will start by discussing the case when the electron-phonon coupling is turned off and we will show that the AA anomaly exhibits novel features in the limit of strong disorder. Next, we will show how switching on a finite electron-phonon interaction leads to additional structure in the density of states.

A. Systems without electron-phonon coupling

In this case, the self-energy due to phonons vanishes, $\Sigma(\omega) = 0$ and $Z = 1$. Assuming a sufficiently large cutoff Λ, the sum over the Matsubara frequencies in Eq. (8) can be performed explicitly and we find a self-consistent equation for the self-energy $\chi(\varepsilon)$,

$$\chi(\varepsilon) = \frac{1}{2}\int_{-\Lambda}^{\Lambda} d\varepsilon' \mu(\varepsilon'-\varepsilon)\tanh\frac{\varepsilon' + \chi(\varepsilon')}{2T}.$$

In order to proceed, let us take into account that the Coulomb pseudopotential can be written as $\mu(\varepsilon) = \mu_0 + \delta\mu(\varepsilon)$, where the function $\delta\mu(\varepsilon)$ vanishes for $\varepsilon > \varepsilon_{\max}$. For the sake of simplicity, let us specialize to the case of $T = 0$. A simple calculation shows that in this case

$$\chi(\varepsilon) = \int_0^{\varepsilon} dE \delta\mu(E),$$

a result which is valid for $\varepsilon < \varepsilon_{\max}$. On the other hand, for $\varepsilon > \varepsilon_{\max}$ we find that $\chi(\varepsilon) = \chi(\varepsilon_{\max})$ is a constant.

From Eq. (11), it follows that the density of states $N(\omega)$ of an interacting disordered system is given by

$$N(\omega) = N_0 \int dE \delta[E + \chi(E) - \omega] = \frac{N_0}{1 + \delta\mu(E_0)},$$

where E_0 is the solution of the equation $E_0 + \chi(E_0) = \omega$.

In the weakly disordered regime where Eq. (3) applies, we thus find that the density of states in the low-frequency limit $\omega \ll \Gamma$ can be described (to order $\sqrt{\omega/\Gamma}$) by Eq. (1) with

$$N(0) = \frac{N_0}{1 + \frac{1-\delta}{2(k_F\ell)^2}}, \qquad \Delta_{\mathrm{AA}} = \frac{8}{3}\left(k_F\ell + \frac{1-\delta}{2k_F\ell}\right)^3 \varepsilon_F,$$

a well-known result due to Altshuler and Aronov [13]. However, since throughout the weakly disordered regime we have $1 \lesssim k_F\ell$, the AA energy scale Δ_{AA} is at least of order ε_F, and therefore not directly observable on the meV scale of typical tunneling experiments. This suggests that the experimental findings of Refs. [6,7,11,12] can not be explained by a straightforward application of Altshuler-Aronov physics.

On the other hand, in the strongly disordered regime, the energy scale Δ_{AA} can be reduced substantially. In fact, for $\omega \ll \varepsilon^*$, the density of states can be again written in the form of Eq. (1), and from Eq. (3) it follows that

$$N(0) = \frac{N_0}{1 + \frac{1-\delta+\ln(L_s/\ell)}{2(k_F\ell)^2}},$$

$$\Delta_{\mathrm{AA}} = \frac{8}{3}\left[k_F\ell + \frac{1-\delta+\ln(L_s/\ell)}{2k_F\ell}\right]^3 \left(\frac{\ell}{L_s}\right)^3 \varepsilon_F.$$

Note that in the strongly disordered regime, $k_F\ell \sim 1$. Since we can write $L_s/\ell = \rho/\rho_c$ where $\rho_c = \ell/g_c$, in the immediate vicinity of the metal-insulator transition (where the resistivity ρ blows up), the AA energy scale Δ_{AA} can become arbitrarily small, $\Delta_{\mathrm{AA}} \propto [\ln(\rho/\rho_c)/(\rho/\rho_c)]^3$, and this scaling is not inconsistent with the scaling found experimentally in

Refs. [6,7]. A similar result for the energy scale Δ_{AA}, except for the logarithmic correction, has been found previously [16,17].

As regards the density of states right at the Fermi energy, $N(0)$, in perturbative calculations it is typically identified with the bare value N_0 [17]. Also in our self-consistent calculation, $N(0)$ differs only weakly from the bare value N_0, if the system is weakly disordered. However, in the strongly disordered regime, we find that $N(0)$ becomes heavily suppressed when $\rho \to \infty$ and the insulating state is approached, and it varies ultimately as $N(0) \sim N_0/\ln(\rho/\rho_c)$. It is worth pointing out that the ratio $N(0)/N_0$ is measurable and experiments with 2D systems [25] do find that $N(0)/N_0 < 1$.

Finally, for the sake of completeness let us note that in the limit $L_s/\ell \to \infty$ the density of states exhibits a logarithmic correction in the limit of small frequencies:

$$N(\omega) = \frac{N_0}{1 + \frac{1}{6(k_F\ell)^2} \ln \frac{\Gamma}{6(k_F\ell)^2\omega}}.$$

Logarithmic scaling of the density of states in the critical regime has been found also earlier [16,17].

B. Finite electron-phonon coupling

For a finite coupling between the electrons and phonons, an analytic solution can be found for a constant Coulomb pseudopotential, since in this case $\chi = 0$ and $R = 1$. If we furthermore assume that $T = 0$ and $\omega \ll \Lambda$, a standard calculation shows that the real part of the retarded wave-function renormalization is

$$\mathrm{Re}Z(\omega) - 1 = \frac{\lambda}{3}\left(1 + \frac{\Omega}{\omega}\ln\left|\frac{\omega+\Omega}{\omega-\Omega}\right| + \frac{\omega^2}{\Omega^2}\ln\left|1 - \frac{\Omega^2}{\omega^2}\right|\right),$$

which, inter alia, implies the well-known result for the mass enhancement $Z(0) = 1 + \lambda$.

In presence of both, a finite electron-phonon coupling λ and an energy-dependent Coulomb pseudopotential $\mu(\varepsilon)$, we have solved the coupled Eqs. (7) and (8) numerically. The analytic continuation from the Matsubara frequencies to the real axis has been carried out by means of the Padé approximation [26].

As a typical example of the results which we find, in Fig. 1, we present the density of states calculated using Eq. (11) for a strongly disordered metal with $k_F\ell = 1.8$ (close to the critical value) and $L_s/\ell = 3$. For the fine-structure constant, we take $\alpha = 1.3$, implying that the bare Coulomb pseudopotential of the clean system is $\mu_0 \approx 0.25$. For this choice of parameters, we find that the dimensionless number $\delta \approx 0.15$.

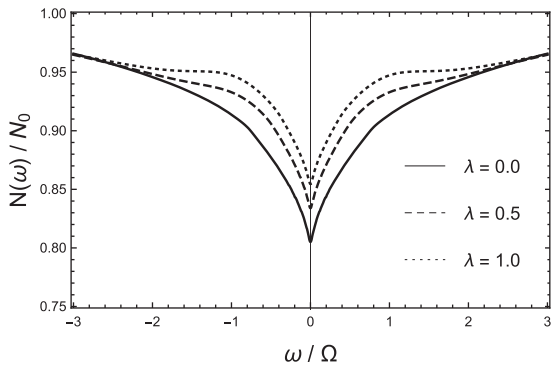

FIG. 1. Density of states of a strongly disordered AA metal with $k_F\ell = 1.8$ and $L_s/\ell = 3$ in the normal state at temperature $T = 0.01\Omega$ for three values of the electron-phonon coupling constant λ.

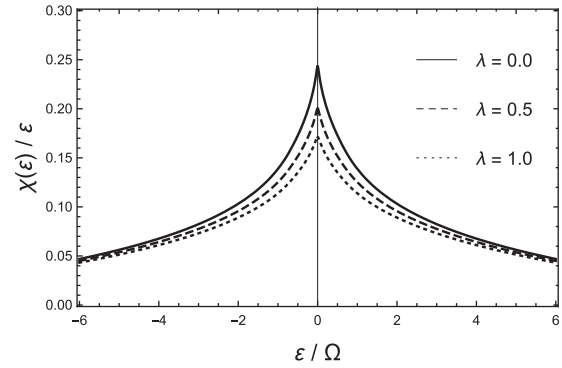

FIG. 2. Normalized self-energy $\chi(\varepsilon)/\varepsilon$ for the same set of parameters as used in Fig. 1.

Throughout this paper, energies will be measured in units of the Debye energy Ω. For the Fermi energy and the cutoff we take $\varepsilon_F = 50\Omega$ and $\Lambda = 10\Omega$, respectively, so that the set of inequalities $\Omega \ll \Lambda \ll \varepsilon_F$ is well satisfied. For our choice of parameters we have $\Gamma = 2\varepsilon_F/(3k_F\ell) \approx 18.5\Omega$ and $\varepsilon_{\max} = \Gamma e^{-3\delta} \approx 11.8\Omega$. This implies that essentially the whole anomalous part of the Coulomb pseudopotential is within the cutoff, except for a small tail which can be neglected.

Figure 1 shows that, without coupling to phonons (i.e., for $\lambda = 0$), the density of states exhibits a strong AA-type singularity at low frequencies, as well as a feature close to the energy scale $\varepsilon^* \approx 0.69\Omega$, as should have been expected. When a finite λ is turned on, two new effects become apparent.

First, for frequencies close to the Debye energy, $\omega \approx \Omega$, an additional feature of the density of states starts to develop, and its strength grows with the magnitude of λ. This is very similar to the phonon features in the density of states of strong-coupling superconductors. We emphasize, however, that our theory predicts that the typical frequencies of phonons coupled to electrons can be measured already in the normal, nonsuperconducting state of a strongly disordered metal.

Second, when λ increases, the dip in the density of states at the Fermi level weakens. This effect is due to an anticorrelation between the effects of the Coulomb pseudopotential and of the electron-phonon coupling: increasing λ diminishes the self-energy $\chi(\varepsilon)$, see Fig. 2, while increasing $\mu(\varepsilon)$ diminishes the self-energy $\Sigma(\omega)$, see Fig. 6. Looking at the Eliashberg equations Eqs. (7) and (8), the origin of the anticorrelation can be traced back to the simultaneous presence of both $\Sigma(\omega)$ and $\chi(\varepsilon)$ in the denominators of the right-hand sides of both equations. Both Figs. 2 and 6 show, however, that the anticorrelation is relatively weak and to a first approximation it can be neglected.

V. SUPERCONDUCTING STATE

Finally, we address the main subject of this paper, namely superconductors with a sizable AA anomaly in their normal

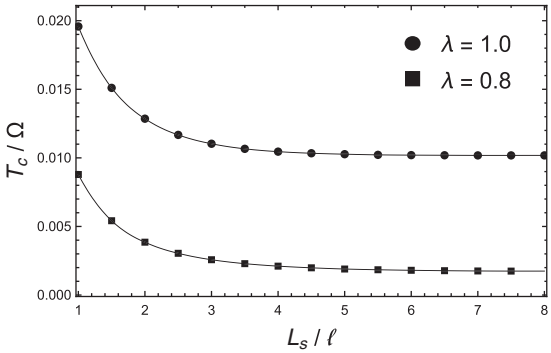

FIG. 3. Critical temperature T_c of strongly disordered AA superconductors with two different electron-phonon coupling constants λ as a function of $L_s/\ell = \rho/\rho_c$, where $\rho_c = \ell/g_c$. In both cases, we take the same parameters $\alpha = 1.3$, $\varepsilon_F = 50\Omega$, and $\Lambda = 10\Omega$. The lines are guides to the eye.

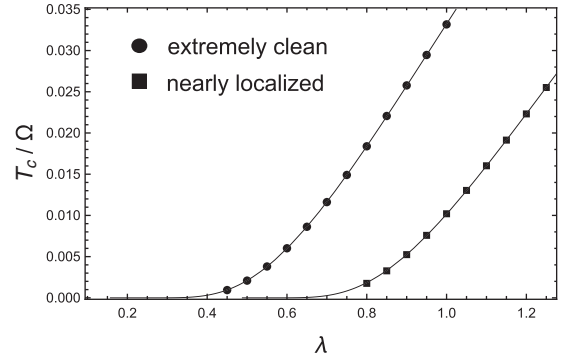

FIG. 4. Critical temperature T_c as a function of the electron-phonon coupling λ for extremely clean and nearly localized metals. The lines are fits described in the text.

state. As explained in the previous section, the requirement of experimental observability of AA-type anomalies forces us to concentrate on the strongly disordered regime with the Coulomb pseudopotential described by Eq. (4). Unless stated otherwise, in our numerical calculations, we assume the same set of parameters as in the previous section: $k_F\ell = 1.8$, the fine-structure constant $\alpha = 1.3$, the Fermi energy $\varepsilon_F = 50\Omega$, and the cutoff $\Lambda = 10\Omega$. For the electron-phonon coupling we take $\lambda = 1.0$, and the length scale $L_s \geqslant \ell$ is taken as a free parameter corresponding to the sample resistivity ρ via $L_s = g_c\rho$.

The Eliashberg equations Eqs. (7)–(10) have been solved numerically. In a clean system with a constant Coulomb pseudopotential $\mu(\varepsilon) = \mu_0$, our choice of parameters leads to a reasonable critical temperature $T_{c0} \approx 0.033\Omega$. With increasing disorder, T_c drops and, when entering the strongly disordered regime, $T_c \approx 0.02\Omega$. Further decrease of T_c as a function of L_s/ℓ in the strongly disordered regime is shown in Fig. 3. In the same figure, we also plot T_c for a somewhat smaller electron-phonon coupling constant, $\lambda = 0.8$.

An unexpected observation is that, although the insulating state is approached as $L_s \to \infty$, the critical temperature does not drop to zero in this limit and it stays constant. Of course, the mean-field Eliashberg equations can not be quantitatively correct for $L_s \to \infty$, since fluctuation effects should be large close to the insulating phase. Nevertheless, our fermionic theory is certainly consistent with a direct superconductor-insulator transition in 3D, without any intervening metallic phase.

In order to understand Fig. 3, we have calculated the critical temperature for a set of metals with fixed electronic parameters and varying electron-phonon coupling λ. We have considered two opposite limits for each λ: the metal was assumed to be either extremely clean, $k_F\ell = 10^8$, or nearly localized, $k_F\ell = 1.8$ and $L_s/\ell = 7$. The results are plotted in Fig. 4. As expected, the critical temperature grows with λ. Moreover, in both limits, T_c seems to be finite only for λ larger than a critical coupling strength λ_c. This was also to be expected, since the phonon-mediated attraction has to overcome the Coulomb repulsion.

Surprisingly, both data sets in Fig. 4 can be fitted well by the simple formula $T_c = a\exp(-b/(\lambda - \lambda_c))$. From these fits we estimate that in the extremely clean case the critical coupling strength $\lambda_{c1} \approx 0.15$, whereas in the nearly localized case $\lambda_{c2} \approx 0.46$. Note that $\lambda_{c2} > \lambda_{c1}$, since the Coulomb repulsion is obviously stronger in the nearly localized case. It follows that three scenaria for the metal-insulator transition are possible: (i) for $\lambda < \lambda_{c1}$ the metal never becomes superconducting. (ii) For $\lambda_{c1} < \lambda < \lambda_{c2}$, the metal can be superconducting, provided it is sufficiently clean. Upon increasing disorder, superconductivity disappears before entering the insulating state [27]. (iii) For $\lambda > \lambda_{c2}$, all metallic states become superconducting at low temperatures.

In the rest of this paper, we concentrate on the physical properties of strongly disordered AA superconductors. In Fig. 5, we plot the spectral gap $\Delta = \Phi(0,0)/Z(0)$ as a function of temperature for an AA superconductor with $L_s/\ell = 5$. We find that the numerical data can be fitted well by the formula $\Delta(T) = \Delta(0)\tanh[\alpha(T_c/T - 1)^{1/2}]$, compatible with simple BCS theory. From the fit we obtain $\alpha \approx 1.77$, $\Delta(0) \approx 0.0190\Omega$, and $T_c \approx 0.0107\Omega$, implying that the ratio $2\Delta(0)/T_c \approx 3.55$, slightly smaller than the clean-limit value for which we find 3.79.

In what follows, we will compare the properties of two AA superconductors, one with $L_s/\ell = 1$, i.e., on the border

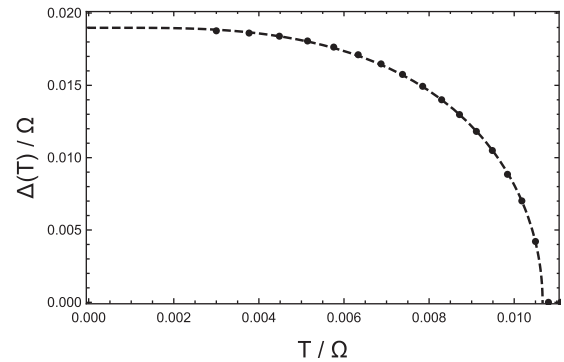

FIG. 5. Temperature dependence of the spectral gap $\Delta = \Phi(0,0)/Z(0)$ for an AA superconductor with $L_s/\ell = 5$. The dots are numerical data and the dashed line is a fit described in the text.

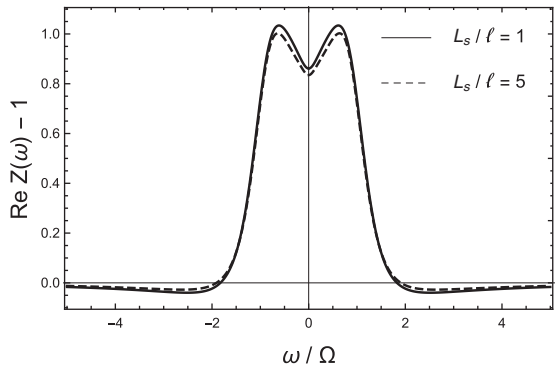

FIG. 6. The real part of the retarded wave-function renormalization $Z(\omega)$ for AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$ at temperature $T = 0.003\Omega$.

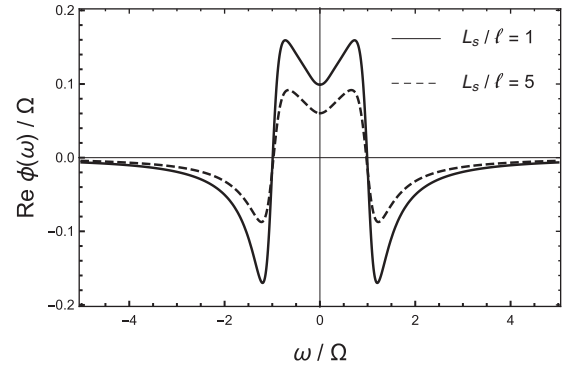

FIG. 8. The real part of the frequency-dependent anomalous self-energy $\phi(\omega)$ on the real frequency axis for AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$ at temperature $T = 0.003\Omega$.

between weak and strong disorder, and another one with $L_s/\ell = 5$, i.e., deeply within the strongly disordered regime.

In Fig. 6, we plot the real part of the retarded wave-function renormalization $Z(\omega)$, obtained by analytic continuation from the imaginary axis. The overall shape of $Z(\omega)$ is in good agreement with the phonon-only analytic result. One can observe that the phonon-related function $Z(\omega)$ exhibits only small changes with L_s/ℓ. This is an example of the weak anticorrelation between $Z(\omega)$ and the Coulomb pseudopotential described in the previous section.

On the other hand, as shown in Fig. 7, the Coulomb pseudopotential-related self-energy $\chi(\varepsilon)$ strongly increases with increasing L_s/ℓ. This was, of course, to be expected. As explained in Sec. IV, larger values of $\chi(\varepsilon)$ imply stronger depression of the density of states at the Fermi level in the hypothetical normal (nonsuperconducting) state, see also Figs. 10 and 11. It is also worth pointing out that, as usual at moderate coupling, the self-energies $Z(\omega)$ and $\chi(\varepsilon)$ change only little between the normal and superconducting states.

The anomalous self-energy $\Phi(\varepsilon, \omega) = \phi(\omega) - \psi(\varepsilon)$ is given by the difference between an ω-dependent part $\phi(\omega)$ due to the phonons, and an ε-dependent part $\psi(\varepsilon)$ due to the Coulomb pseudopotential. As shown in Fig. 8, the function $\phi(\omega)$ exhibits the standard shape expected for boson exchange. When L_s/ℓ grows, the overall scale of $\phi(\omega)$ decreases, but its shape remains roughly intact. This can again be interpreted as an anticorrelation effect. In fact, the dominant effect of increasing L_s/ℓ is that $R(\varepsilon) = 1 + \chi(\varepsilon)/\varepsilon$ grows, but from Eq. (9) it therefore follows that $\phi(\omega)$ has to decrease.

The function $\psi(\varepsilon)$ is the superconducting analog of the normal-state self-energy $\chi(\varepsilon)$. However, there is an important difference between the two functions: for an energy-independent Coulomb pseudopotential, we have $\chi = 0$, but ψ is a nonzero constant even in this case. If $\mu(\varepsilon)$ is not constant, then $\psi(\varepsilon)$ acquires a finite energy dependence, as demonstrated in Fig. 9. Note that with increasing L_s/ℓ, the energy dependence of $\psi(\varepsilon)$ becomes more prominent.

A very rough estimate of the magnitude of ψ can be obtained from Bogoliubov's two-gap model: let us assume that $\phi(\omega)$ is a finite constant for $\omega < \Omega$ and zero outside this interval, and let $\psi(\varepsilon)$ be a constant up to the cutoff Λ. Moreover, let us assume the presence of featureless electron-phonon and Coulomb couplings λ and μ, where μ is an appropriately taken average of $\mu(\varepsilon)$. Then we find that $\psi \sim (\mu^*/\lambda)\phi$, where $\mu^* = \mu/[1 + \mu \ln(\Lambda/\Omega)]$ is the renormalized Coulomb pseudopotential. The data in Fig. 9 are roughly consistent with this estimate.

Finally, the superconducting density of states of AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$, calculated using Eq. (11), is shown in Figs. 10 and 11. As expected, the pseudogap grows with increasing L_s/ℓ. Figure 11 shows that the phonon-related peak at $\omega \approx \Omega$, visible already in the

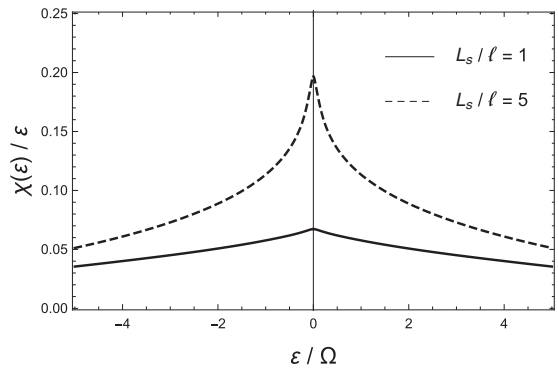

FIG. 7. Normalized self-energy $\chi(\varepsilon)/\varepsilon$ for AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$ at temperature $T = 0.003\Omega$.

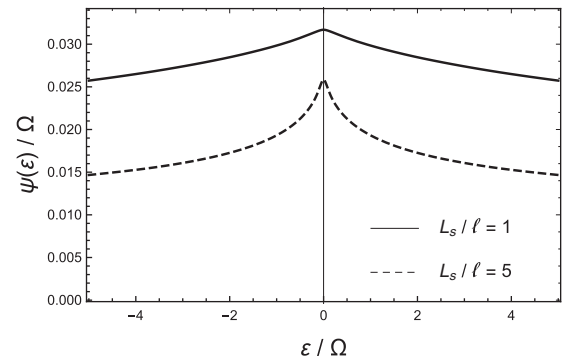

FIG. 9. The energy-dependent anomalous self-energy $\psi(\varepsilon)$ for AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$ at temperature $T = 0.003\Omega$.

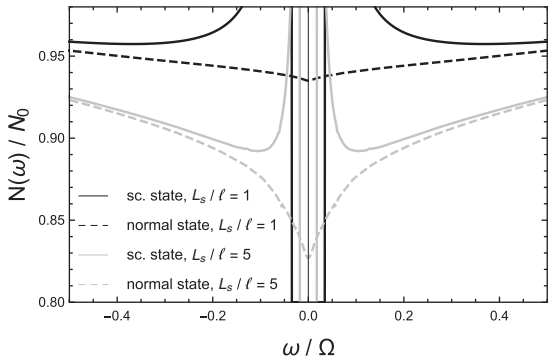

FIG. 10. Superconducting and (hypothetical) normal density of states for AA superconductors with $L_s/\ell = 1$ and $L_s/\ell = 5$ at temperature $T = 0.003\Omega$.

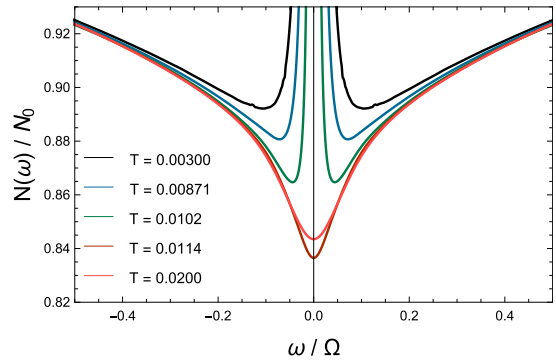

FIG. 12. Temperature dependence of the density of states of a strongly disordered AA superconductor with $L_s/\ell = 5$. At temperatures below T_c, $N(\omega)$ is shown only for $|\omega| > \Delta$ for the sake of clarity.

normal state, acquires additional structure in the superconducting state, in complete analogy with what is observed in superconductors with a constant Coulomb pseudopotential.

In Fig. 12, we show the temperature dependence of the density of states of a strongly disordered AA superconductor with $L_s/\ell = 5$. A pure $\sqrt{\omega}$ behavior takes place only for $T \lesssim \omega \lesssim \varepsilon^*$, and at low energies the normal-state singularity at $\omega = 0$ is either cut off by the finite temperature T (above T_c), or completely masked by the superconducting gap below T_c. Thus weak-coupling superconductors with $T_c \ll \Omega$ offer the most favorable conditions to simultaneously observe both, the AA effect and the superconducting gap in $N(\omega)$.

Let us comment on the relation of our theory to experiments. In this work we have studied 3D superconductors and we have assumed that the only effect of disorder is to introduce additional electron scattering. This means, however, that our theory can not be directly compared with Refs. [6,7,11,12]. In fact, in order to interpret Refs. [7,11,12], at the very least it would be necessary to take into account the changes of the electron density, but this is by no means straightforward, since not only the fine-structure constant α, but also ε_F, N_0, and even λ will change in that case. Similar complications are to be expected also when varying the grain size of granular aluminum [6]. Moreover, large phase fluctuations typical of the bosonic scenario are expected in the latter case.

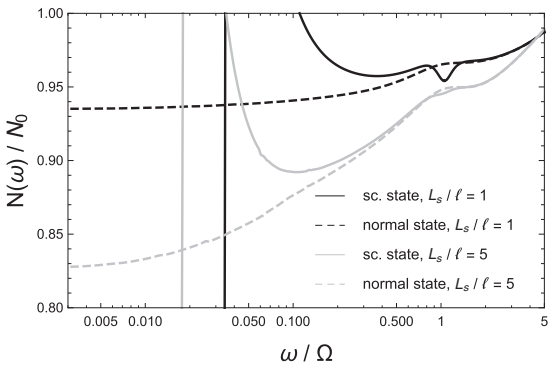

FIG. 11. The same as Fig. 10, but in a logarithmic scale of ω.

It seems that the best example of AA superconductivity might be provided by materials in which radiation damage causes large resistivity enhancements, such as the $A15$ superconductors [28,29]. In order to check whether our picture for the suppression of T_c is valid in this group of materials, one should start by looking for an AA anomaly in the normal state of the high-resistivity samples. Tunneling data on the $A15$ compounds are in fact available [30,31], but unfortunately the authors concentrate on the McMillan-Rowell inversion and do not report the normal-state data [32].

In order to estimate the order of magnitude of the changes in $N(\omega)$ to be expected in the tunneling experiments, let us take for the Debye frequency a typical value of $\Omega = 40$ meV, which for our choice of parameters implies $\varepsilon_F = 2$ eV and $T_{c0} \approx 15$ K. From Fig. 12, it then follows that in a strongly disordered AA superconductor with $L_s/\ell = 5$ the density of states at a temperature $T \approx T_c \approx 5$ K can change by $r \approx 12\%$ between $\omega = 0$ and $\omega = 20$ meV, and for the AA energy scale we get $\Delta_{\mathrm{AA}} \approx 650$ meV. These estimates are quite similar to $r \approx 14\%$ and $\Delta_{\mathrm{AA}} \approx 520$ meV, which have been measured for the $x = 0.02$ sample of Ref. [12]. Therefore such changes of $N(\omega)$ should be observable.

VI. CONCLUSIONS

Building on the pioneering work of Belitz [18], in this paper we have developed a formalism that can deal with both, the AA effect and superconductivity, on equal footing. In particular, this enables us to study the superconducting instability of systems with a pseudogap caused by the AA effect in their normal state.

The set of generalized Eliashberg equations (7)–(10) has been complemented by the simplest but physically well motivated explicit expressions for the Coulomb pseudopotential in 3D, Eqs. (3) and (4), and for the phonon-mediated electron-electron interaction, Eq. (5). Following AMR [14], we distinguish between weakly and strongly disordered conductors. In both cases, our expressions for the Coulomb pseudopotential $\mu(\varepsilon)$ take into account the $\sqrt{\varepsilon}$-type enhancement at the lowest energy transfers ε. In the strongly disordered case, $\mu(\varepsilon)$ in addition exhibits a logarithmic regime at intermediate

energies, predicted by AMR as a consequence of the anomalous diffusion [14].

A complete numerical solution of the imaginary-time Eliashberg equations has been presented, with emphasis on the parameter values representative of strongly disordered AA superconductors. This point is crucially different from the approach of Belitz [18,33], who postulates a simple functional form for the self-energy $\chi(\varepsilon)$ that does not allow for the AA anomaly, and solves the Eliashberg equations in the simple two-square-well approximation.

Keeping the full energy dependence of the self-energy $\chi(\varepsilon)$ allows us to show that the low-frequency behavior of the density of states in the normal (nonsuperconducting) state is well described by Eq. (1). We have also calculated the energy scale Δ_{AA} and the density of states at the Fermi level $N(0)$ in both, the weakly and strongly disordered regimes. In agreement with earlier work [16,17], we find that the AA anomalies are best observable in the strongly disordered regime. When the electron-phonon coupling is turned on, we find (still in the normal state!) additional anomalies in the density of states at $\omega \approx \Omega$, where Ω is the Debye energy.

Numerical solution of the Eliashberg equations suggests there are two possible scenaria for disorder-controlled superconductor-insulator transition. If the electron-phonon coupling is weak, then the transition proceeds via an intermediate metallic phase, in agreement with the approximate theory of AMR [14]. On the other hand, for sufficiently strong electron-phonon coupling, the transition occurs without any intermediate phases. It should be pointed out, however, that the superconducting state in the vicinity of the insulator is presumably fragile, and sufficiently strong *extrinsic* pair breaking [34] may result in stabilization of an intermediate metallic phase. If this happens, then the transition again proceeds via an intermediate metallic phase.

A straightforward comparison of our results to experimental data is not possible at the moment, since in all available experiments on 3D disordered superconductors [6,7,11,12] introduction of disorder led, in addition to increased scattering, also to a change of other relevant electronic parameters, such as the Fermi velocity. In order to circumvent such difficulties, we have instead proposed to search for the AA effect by tunneling spectroscopy of radiation-damaged $A15$ superconductors.

Our assumption that the electron-phonon coupling λ does not change with disorder is by no means obvious. The most complete discussion of disorder-induced renormalization of the electron-phonon coupling is due to Keck and Schmid [24]. These authors study interaction between electrons and long-wavelength acoustic modes and find that the coupling to longitudinal (transverse) modes decreases (increases) with increasing disorder strength. Making use of these results, Belitz argues that the total electron-phonon coupling strength increases with disorder [18]. There are, however, several caveats in this line of reasoning. First, when treating the transverse phonons, Keck and Schmid consider only the so-called collision-drag mechanism, and they neglect the electromagnetic mechanism [35] with a different dependence on disorder. Moreover, the effect of disorder on the phonons is not taken into account. Second, to the best of our knowledge, disorder-dependence of the coupling to the optical phonons has not been studied yet. Since the electron-optical phonon interaction is essentially due to electrostatics as for the longitudinal acoustic modes, we expect a decrease of the coupling strength with increasing disorder also in this case. Whether the total λ increases or decreases with disorder should therefore depend on the relative contribution of the acoustic and optical phonons to the electron-phonon coupling strength. Third, no systematic treatment of the electron-phonon coupling taking into account the anomalous diffusion of electrons is available at present. Weak-localization effects have been treated in the literature, but very different conclusions have been reached: within a wave function-based approach, it was argued that λ decreases [36], whereas the σ-model renormalization-group framework predicts an enhancement of the interaction matrix elements [37]. Further work is therefore clearly needed to arrive at definitive conclusions about the disorder dependence of the electron-phonon coupling function $g(\omega)$. In any case, the formalism developed in the present paper will allow for a simple accommodation of the results of such studies in a unified description of the Altshuler-Aronov effect and superconductivity.

Further extensions of our theory are possible in several ways: a procedure analogous to the McMillan-Rowell inversion, but taking into account the AA effect, should be worked out. 2D systems should be studied, since in 2D one can make use of surface disorder, which should be free of the unwanted side effects; moreover, high-quality data are available in this case [8,9]. Finally and most ambitiously, it remains to be seen whether the AA effect proper or some analogous effect play any role in the physics of the cuprates.

ACKNOWLEDGMENTS

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