Continuum limits of matrix product states

Gemma De las Cuevas,[1] Norbert Schuch,[2] David Perez-Garcia,[3,4] and J. Ignacio Cirac[2]
[1]*Institut für theoretische Physik, Universität Innsbruck, Technikerstr. 21a, 6020 Innsbruck, Austria*
[2]*Max Planck Institute for Quantum Optics, Hans-Kopfermann-Str. 1, 85748 Garching, Germany*
[3]*Departamento de Análisis Matemático and IMI, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[4]*ICMAT, C/ Nicolás Cabrera, Campus de Cantoblanco, 28049 Madrid, Spain*

(Received 19 August 2017; revised manuscript received 19 October 2018; published 19 November 2018)

We determine which translationally invariant matrix product states have a continuum limit, that is, which can be considered as discretized versions of states defined in the continuum. To do this, we analyze a fine-graining renormalization procedure in real space, characterize the set of limiting states of its flow, and find that it strictly contains the set of continuous matrix product states. We also analyze which states have a continuum limit after a finite number of coarse-graining renormalization steps. We give several examples of states with and without the different kinds of continuum limits.

DOI: 10.1103/PhysRevB.98.174303

I. INTRODUCTION

The quest for continuum limits of discrete theories is a central topic in high-energy physics [1,2] and condensed-matter physics [3,4]. In many cases, the continuum limit of a theory is obtained after a renormalization process, where the lattice constant (which provides an energy cutoff) is taken to zero. This occurs, for instance, in quantum lattice models, where the continuum limit is the desired quantum field theory and the renormalization involves the redefinition of the parameters of the Hamiltonian describing the model. The question of whether a particular quantum lattice model possesses the correct continuum limit under renormalization is of central interest in several fields of quantum physics.

Tensor networks have proven to be useful tools to study strongly correlated systems in quantum lattice models [5–7]. In fact, in one spatial dimension, matrix product states (MPSs) [8,9], a special kind of tensor network states (TNSs), provide the most powerful technique to study such systems. In contrast to some traditional approaches to describe quantum many-body systems where the *Hamiltonian* (or the action) is the central object of study, the theory of tensor networks concentrates on the description of quantum many-body *states*. The reason is that they are completely characterized (for homogeneous systems) by a simple tensor, whose rank depends on the coordination number of the lattice. The fact that ground states (vacuum) and low-energy excitations of local theories are expected to have very little entanglement makes tensor networks efficient tools for describing them. Furthermore, they can be used as toy models to analyze complex phenomena associated with topology [10], symmetry protection [11,12], or even chirality [13] in relatively simple terms.

Renormalization procedures in tensor networks and, in particular, in MPSs have played an important role in the development of various methods associated with them. The renormalization of a TNS provides a coarse-grained description of the state and, in the case of MPSs, flows to a very specific family of states that can be fully characterized [14]. In fact, these fixed points of the renormalization procedure have been used to obtain a classification of the (gapped) quantum phases of spin chains in one spatial dimension [11,12].

In this work, we investigate how the same renormalization procedure can give a rigorous method to obtain the continuum limit of an MPS. That is, we consider the inverse procedure of coarse graining, i.e., fine graining, and investigate to what extent it converges and to which kind of states. Or, more boldly stated, we solve the following problem: given an MPS, when is it the coarse-grained picture of the vacuum of a quantum field theory in one spatial dimension? We will then say that such an MPS has a continuum limit (CL).

To be specific, we consider a fine-graining procedure such that the state is translationally invariant at all steps. Moreover, each fine-graining step is carried out by some isometry, which can differ from step to step. As a consequence, the finer state is, in fact, the same state as the original one, but written in a finer basis, i.e., a basis with more sites. Thus, our definition of CL is very restrictive and can be seen as a first step toward the study of CLs in more general settings.

Now, while it is clear that some states must have a CL in the sense specified below, it is also clear some others will not. For instance, a ferromagnetic state $|0,\ldots,0\rangle$ clearly has a CL, which is the vacuum of a noninteracting theory in the continuum. In contrast, a superposition of two antiferromagnetic states,

$$|\Psi_{\rm af}\rangle = \frac{1}{\sqrt{2}}(|0,1,0,1,\ldots\rangle + |1,0,1,0,\ldots\rangle), \qquad (1)$$

will not have such a limit since there exists no (translationally invariant) state such that if we coarse grain it, we obtain $|\Psi_{\rm af}\rangle$. But what about states like the Affleck, Kennedy, Lieb, and Tasaki (AKLT) [15], the cluster state [16], and other prominent states found in the field of condensed matter or quantum information theory?

On the other hand, by flipping every second spin in the z direction, $|\Psi_{\rm af}\rangle$ is mapped to a superposition of the two ferromagnetic states, $|0,0,\ldots,0\rangle + |1,1,\ldots,1\rangle$, which has a CL. While in our definition of CL we allow to apply only operations (isometries) which are the same on every site, this restriction is lifted in our second definition of the CL, called the coarse continuum limit. In the latter, we first coarse grain the state and then take the CL of the coarse-grained state. Thus, $|\Psi_{\rm af}\rangle$ has a coarse CL, but does every state have a coarse CL?

In this paper we give an answer to these questions by determining the conditions for a state to have a CL. We also characterize which set of states of the quantum field theory is the CL of an MPS. We find that such a set contains continuous MPSs (cMPSs) [17,18], as one would expect, but it also contains some extensions that have not been encountered so far in the study of TNSs. We finally show that there exist states that do not possess a CL even if we first coarse grain any finite number of times; that is, not every state has a coarse CL. We note that different continuum limits of quantum lattice systems were considered in Ref. [19], and tensor network descriptions of quantum field theories were studied in Ref. [20].

This paper is organized as follows. In Sec. II we define and characterize the CL of MPSs. In Sec. III we define and characterize the coarse CL of MPSs, present examples of states with either kind of CL, and compare the two CLs. In Sec. IV we conclude. We leave the proof of the main result (Theorem 1) for the Appendix.

II. CONTINUUM LIMIT

In this section we present our work on the CL of an MPS. We will first explain the setting of our problem (Sec. II A), define and characterize p refining (Sec. II B), and finally define and characterize the CL of an MPS (Sec. II C).

A. The setting

Our starting point is a three-rank tensor $A = \{A^i \in \mathcal{M}_D\}_{i=1}^d$, where $\mathcal{M}_D$ denotes the set of $D \times D$ complex matrices, D is called the bond dimension, and d is the physical dimension, both of which are assumed to be fixed and finite. A generates a translationally invariant (TI) MPS,

$$|V_N(A)\rangle := \sum_{i_1,\ldots,i_N} \mathrm{Tr}(A^{i_1}A^{i_2}\cdots A^{i_N})|i_1,\ldots,i_N\rangle \quad (2)$$

for every $N \in \mathbb{N}$, as well as the family

$$\mathcal{V}(A) := \{|V_N(A)\rangle\}_{N\in\mathbb{N}}. \quad (3)$$

As the tensor A completely determines all the properties of the MPS it generates, when developing the theory of MPSs, one works directly with such a tensor.

The *transfer matrix* of $\mathcal{V}(A)$, E_A, is defined as [14]

$$E_A = \sum_{i=1}^d A^i \otimes \bar{A}^i, \quad (4)$$

where the bar indicates complex conjugation. Note that E_A is (a matrix representation of) the completely positive map (CPM) $\mathcal{E}(\cdot) = \sum_{i=1}^d A^i \cdot A^{i\dagger}$, and it is independent of any isometry applied to the physical index i. In Ref. [21] we showed that, without loss of generality, A can be taken to be in irreducible form, that is, $A^i = \oplus_j \mu_j A^i_j$, where $\mu_j > 0$, and each E_{A_j} is an irreducible CPM (i.e. a CPM with a nondegenerate eigenvalue 1 but which can have other eigenvalues of modulus 1). Moreover, E_A can be taken to be a quantum channel [i.e., a trace-preserving (TP) CPM]. We will thus indistinctively call E_A a transfer matrix or a quantum channel. If clear from the context, we will simply denote it by E.

B. Definition and characterization of *p* refining

The renormalization procedure introduced in Ref. [14] basically maps $|V_N(A)\rangle$ to

$$|V_N(B)\rangle = (W^\dagger)^{\otimes N}|V_{pN}(A)\rangle \quad \forall N, \quad (5)$$

where $p > 1$ is an integer and $W : \mathbb{C}^d \to (\mathbb{C}^d)^{\otimes p}$ is an isometry. We now introduce the inverse step.

Definition 1. We say that $\mathcal{V}(B)$ can be *p-refined* if there exists another tensor A and an isometry W such that

$$|V_{pN}(A)\rangle = W^{\otimes N}|V_N(B)\rangle \quad \forall N. \quad (6)$$

Clearly, if $\mathcal{V}(B)$ can be p-refined with the isometry W, then it can also be p-refined with the isometry $U^{\otimes p}W$, where U is a unitary. We thus call two isometries W, W' inequivalent if there is no unitary U such that $W' = U^{\otimes p}W$. Similarly, we say that $\mathcal{V}(B)$ can be p-refined in r inequivalent ways if it can be p-refined with r inequivalent isometries.

In Ref. [21] we showed that $\mathcal{V}(B)$ can be p-refined if and only if E_B is *p divisible*; that is, if there exists a quantum channel E_p such that $E_p^p = E_B$. Moreover, the number of inequivalent ways of p refining a state is precisely given by the number of pth roots of its transfer matrix which are also a transfer matrix. The divisibility of quantum channels was analyzed in Refs. [22–24] in the context of Markovian evolution of quantum systems. In particular, there exist channels that are not p divisible for any p [24]. This automatically implies that there are states that cannot be refined at all [25]; we will see two examples thereof in Examples 5 and 6. In Remark 2 we will mention examples of states that can be refined in several inequivalent ways.

C. Definition and characterization of continuum limit

One could define the CL of an MPS as the limiting point of the p-refining procedure. However, such a definition would not be satisfactory since there are states that can be refined but that should not have a CL. This can be illustrated by means of the antiferromagnetic state of Eq. (1), which can be 3-refined infinitely many times with the isometry $W = |0,1,0\rangle\langle 0| + |1,0,1\rangle\langle 1|$. However, it is clear that it cannot exist in the continuum. (This state will be more thoroughly analyzed in Example 2.)

To deal with this problem, we notice that if we had a CL, it would be reasonable to demand that the limit should not depend on whether we block a few spins when we are close to that limit. Differently speaking, introducing an intermediate coarse-graining step should not affect the form of the CL. This, e.g., rules out the antiferromagnetic state: In Eq. (1), if we 3-refine many times with the isometry $W = |0,1,0\rangle\langle 0| + |1,0,1\rangle\langle 1|$ and then block two spins, with the isometry $W' = |0,1\rangle\langle 0| + |1,0\rangle\langle 1|$ we obtain a Greenberger-

Horne-Zeilinger-like state [26], $|0,0,\ldots,0\rangle + |1,1,\ldots,1\rangle$, which is very different from the fixed point if we had not blocked. This motivates the following definition.

Definition 2. We say that $\mathcal{V}(B)$ *has a continuum limit* if there is a $p > 1$ such that the procedure of p-refining ℓ times followed by the blocking of $n_\ell \in \mathbb{N}$ of the resulting spins converges in ℓ, as long as $(n_\ell/p^\ell)_\ell \to 0$ as $\ell \to \infty$.

Note that $(n_\ell/p^\ell)_\ell$ denotes the infinite sequence whose elements are n_ℓ/p^ℓ with $\ell \in \mathbb{N}$. We now want to characterize which states have a CL in terms of the divisibility properties of its transfer matrix. The requirement that the state be p-refinable infinitely many times translates to the requirement that its transfer matrix E be *p-infinitely divisible*. This means that E is p^ℓ-divisible for any $\ell \in \mathbb{N}$, that is, that for any $\ell \in \mathbb{N}$ there is a quantum channel E_{p^ℓ} such that $E_{p^\ell}^{p^\ell} = E$. Note that a quantum channel E is called *infinitely divisible* if it is n-divisible for any n, i.e., $E = E_n^n$ for all $n \in \mathbb{N}$ [24].

We also need to characterize the condition of stability of the limiting procedure under blocking (see Definition 2). To this end, we introduce the following function (see, e.g., Ref. [27]). Let E be a p-infinitely divisible quantum channel, and let $\{E_{p^\ell}\}_{\ell\in\mathbb{N}}$ be a set of roots which are quantum channels themselves. We define the function $f_{p,E}$ as

$$f_{p,E}(n,\ell) = E_{p^\ell}^n, \tag{7}$$

where $n,\ell \in \mathbb{N}$. Now, we say that $f_{p,E}$ is *continuous at zero* if there exists a set $\{E_{p^\ell}\}_{\ell\in\mathbb{N}}$ and a matrix Q, such that for all sequences $\{n_k,\ell_k\}_{k=1}^\infty$ fulfilling $\lim_{k\to\infty} n_k/p^{\ell_k} = 0$, it holds that $\lim_{k\to\infty} f_{p,E}(n_k,\ell_k) = Q$. Thus, the existence of a CL is equivalent to the existence of a $p > 1$ such that E_B is p-infinitely divisible and an f_{p,E_B} which is continuous at zero. With this, we can characterize the set of MPSs with a CL.

Theorem 1 (Main result). Given $\mathcal{V}(B)$ with B in irreducible form, the following statements are equivalent:

(1) $\mathcal{V}(B)$ has a CL.

(2) E_B is infinitely divisible.

(3) There is a quantum channel P and a Liouvillian of Lindblad form L such that $E_B = Pe^L$, $P^2 = P$, and $PLP = PL$.

The proof is given in the Appendix.

Note that the last item fully characterizes all possible CLs. If $P = \mathbb{1}$, the corresponding transfer matrix e^L coincides with that of a TI cMPS. Thus, as expected, all TI cMPSs can be limits of TI MPSs. However, for $P \neq \mathbb{1}$, states other than cMPS appear as possible CLs. Note also that one can easily see from condition (2) of Theorem 1 that the limit is smooth, as $\lim_{t\to 0} E^t = \lim_{t\to 0} Pe^{tL} = P$. Finally, note that from Theorem 1 and the results of [21] it follows that if $\mathcal{V}(B)$ has a CL, then $\mathcal{V}(B)$ can be p-refined for any $p > 1$.

III. COARSE CONTINUUM LIMIT

We now present a more relaxed definition of a CL of an MPS, which we call the coarse CL. We will first define and characterize it (Sec. III A), give several examples of states with or without a coarse CL (Sec. III B), and finally use these examples to compare the two notions of CLs (Sec. III C).

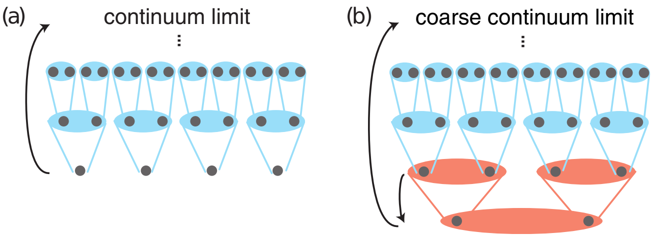

FIG. 1. Sketch of (a) the continuum limit and (b) the coarse continuum limit.

A. Definition and characterization

We have seen that to obtain a meaningful definition of a CL we have to impose that we can block towards the end of the refinement and still obtain the same limit. We can thus ask what happens if we allow for blocking before the refinement. For example, by blocking two sites of the antiferromagnetic state [Eq. (1)], we obtain the ferromagnetic state, which has a trivial CL. This motivates the following definition (see Fig. 1).

Definition 3. We say that $\mathcal{V}(A)$ has a *coarse CL* if there is a $\mathcal{V}(B)$ and an $n \in \mathbb{N}$ such that $\mathcal{V}(A)$ is the n-refinement of $\mathcal{V}(B)$ and $\mathcal{V}(B)$ has a CL.

Note that every state $\mathcal{V}(A)$ is the p-refinement of some other state $\mathcal{V}(B)$; that is, given A and p, there is always an isometry W and a tensor B that satisfies Eq. (5). Moreover, the process of "coarse-graining" p sites (the opposite of p-refining) is essentially unique; more precisely, different isometries will give rise to tensor B's which are related by a unitary matrix in the physical index, as shown in Ref. [14]. This is again best understood at the level of the transfer matrix: coarse-graining p sites corresponds to taking the pth power of the transfer matrix, which gives a unique result and which always corresponds to a valid transfer matrix. This is to be contrasted with p-refining, which is possible only if there is at least one pth root of E which is a valid transfer matrix, and in case there is, there may be multiple such roots.

The following characterization is immediate from the above results.

Corollary 1. $\mathcal{V}(A)$ has a coarse CL if and only if there exists an $n \in \mathbb{N}$ such that E_A^n is infinitely divisible.

Remark 1. Computational complexity. What is the computational complexity of deciding whether a state has a (coarse) CL? Concerning the CL, deciding infinite divisibility is at least as hard as deciding Markovianity since the latter amounts to deciding the former together with being full rank (see condition (3) of Theorem 1), and being full rank can be decided efficiently. Deciding Markovianity has been formulated as an integer semidefinite program for fixed input dimension [25] and shown to be NP-hard as a function of the bond dimension [28]. Concerning the coarse CL, to the best of our knowledge, the computational complexity of determining whether, given a channel E, there is some $n \in \mathbb{N}$ such that E^n is infinitely divisible is not known.

B. Examples

We now present several examples of states with either kind of CL which illustrate Theorem 1 and Corollary 1.

Example 1. The ferromagnet. Let us start with an equal superposition of m ferromagnetic states,

$$|V_N(B)\rangle = \sum_{i=0}^{m-1} |i, i \ldots, i\rangle, \tag{8}$$

which is given by the tensor $B = \{B^i \in \mathcal{M}_D\}_{i=0}^{m-1}$, where $B^i = |i\rangle\langle i|$ for $i = 0, 1, \ldots, m-1$. $\mathcal{V}(B)$ can be p-refined into p copies of itself for any p with $W = \sum_{i=0}^{m-1} |i, i, \ldots\rangle\langle i|$, and this is also true after the blocking of an arbitrary number of spins. Equivalently (see Theorem 1), the transfer matrix

$$E_{\mathrm{f}} = \sum_{i=0}^{m-1} |i, i\rangle\langle i, i| \tag{9}$$

is a projector; thus, it is infinitely divisible, and thus, the state has a CL. Recall that the transfer matrix [see (4)] acts on the auxiliary space, whereas $|V_N(B)\rangle$ is a state living in the physical space.

Example 2. The antiferromagnet. Consider an equal superposition of m antiferromagnetic states,

$$|V_m(B)\rangle = \sum_{i=0}^{m-1} |i, i+1, \ldots, i+m-1\rangle, \tag{10}$$

where the sum is modulo m (and similarly for an N multiple of m and $|V_N(B)\rangle = 0$ otherwise), which is given by $B^i = |i\rangle\langle i+1|$ for $i = 0, 1, \ldots, m-1$. $\mathcal{V}(B)$ can be p-refined into p copies of itself, with $p = m+1$, with the isometry

$$W = \sum_{i=0}^{m-1} |i, i+1, \ldots, i+m-1, i\rangle\langle i|. \tag{11}$$

However, as we have discussed, this state does not have a CL since the limit of this refinement is not stable under blocking. Equivalently (see Theorem 1), the transfer matrix E_{af} is p-infinitely divisible with $p = m+1$ since

$$E_{\mathrm{af}} = \sum_{i=0}^{m-1} |i, i\rangle\langle i+1, i+1| = E_{\mathrm{af}}^{m+1}, \tag{12}$$

but it is not infinitely divisible since it does not have, e.g., an mth root which is a quantum channel. To see the latter, note that the nonzero part of the spectrum of E_{af} is $\{e^{2\pi i r/m}\}_{r=0}^{m-1}$ and thus for its mth root $\{e^{2\pi i \ell_r/m^2}\}_{r=0}^{m-1}$ (with, e.g., ℓ_1 coprime to m^2), whereas the set of eigenvalues of modulus 1 of a quantum channel needs to be of the form $\{e^{2\pi i r/n}\}_{r=0}^{n-1}$ for some n [29]. On the other hand, $\mathcal{V}(B)$ has a coarse CL since after blocking m sites we obtain the ferromagnet of Example 1.

Example 3. A deformed antiferromagnet. We consider the tensor $B(\alpha)$ (with $0 < \alpha < 1$)

$$B^0(\alpha) = \sqrt{\alpha}\, |0\rangle\langle 1| + \sqrt{1-\alpha}\, |1\rangle\langle 0|, \tag{13}$$

$$B^1(\alpha) = B^0(\alpha)^t, \tag{14}$$

where t denotes transpose. The corresponding state has periodicity 2, as for even N we have that

$$|V_N(B(\alpha))\rangle = |\mu_0, \mu_1, \mu_0, \mu_1, \ldots\rangle + |\mu_1, \mu_0, \mu_1, \mu_0, \ldots\rangle, \tag{15}$$

where $|\mu_i\rangle$ is shorthand for $|\mu_i(\alpha)\rangle$ and

$$|\mu_i(\alpha)\rangle = \sqrt{\alpha}|i\rangle + \sqrt{1-\alpha}|i+1\rangle \tag{16}$$

for $i = 0, 1$, where the sum on i is mod 2. Now, let

$$g_\pm(\alpha) = \tfrac{1}{2}\{1 \pm \sqrt{1 - [4\alpha(1-\alpha)]^{1/3}}\}. \tag{17}$$

Then $\mathcal{V}(B(\alpha))$ can be 3-refined into $\mathcal{V}(B(g_+(\alpha)))$ or $\mathcal{V}(B(g_-(\alpha)))$. The corresponding isometries are given by

$$W_\pm = \frac{1}{1-\lambda(\alpha)^2}(|\nu_0^\pm\rangle\langle\mu_0| + |\nu_1^\pm\rangle\langle\mu_1| - \lambda(\alpha)|\nu_0^\pm\rangle\langle\mu_1| - \lambda(\alpha)|\nu_1^\pm\rangle\langle\mu_0|), \tag{18}$$

where $|\nu_i^\pm\rangle = |\mu_i(g_\pm(\alpha)), \mu_{i+1}(g_\pm(\alpha)), \mu_i(g_\pm(\alpha))\rangle$ for $i = 0, 1$, where the sum on i is modulo 2, and

$$\lambda(\alpha) = 2\sqrt{\alpha(1-\alpha)}. \tag{19}$$

However, this refinement is not stable under the blocking of two spins since that would give rise to a state without periodicity. Equivalently (see Theorem 1), the transfer matrix $E_{B(\alpha)}$ is 3-infinitely divisible but not infinitely divisible. To see this, note that in the Pauli basis (which is defined as usual, namely, $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$, $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$) we have that

$$E_{B(\alpha)} = \mathrm{diag}(1, \lambda(\alpha), -\lambda(\alpha), -1). \tag{20}$$

Therefore, $E_{B(\alpha)} = E^{3^\ell}_{B(g_\pm^\ell(\alpha))}$ for all natural ℓ, where $E_{B(g_\pm(\alpha))} = \mathrm{diag}(1, \lambda(g_\pm(\alpha)), -\lambda(g_\pm(\alpha)), -1)$, where we choose either g_+ or g_- for both eigenvalues and $g_\pm^\ell$ denotes the ℓ-fold application of the map $g_\pm$. Yet $E_{B(\alpha)}$ does not have, e.g., a square root which is a quantum channel since the spectrum of a channel needs to be closed under complex conjugation, which is impossible given (20). Thus, this state does not have a CL. However, after blocking two sites we obtain a Markovian transfer matrix, namely, $E_{B(\alpha)}^2 = e^L$, with $\mathcal{L}(\rho) = -\ln[\lambda(\alpha)](Z\rho Z - \rho)$. Thus, this state has a coarse CL.

Example 4. The cluster state. Consider the one-dimensional (1D) cluster state $\mathcal{V}(A)$ [16], which is obtained with the tensor

$$A^1 = |1\rangle\langle +|, \quad A^2 = |0\rangle\langle -|, \tag{21}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ [9]. The transfer matrix

$$E_A = |0, 0\rangle\langle -, -| + |1, 1\rangle\langle +, +| \tag{22}$$

has eigenvalues (1,0,0,0), but the eigenvalue 0 is associated with a nontrivial Jordan block. This block does not have a pth root for any p (see Definition 1.2 of Ref. [30]), and thus, $\mathcal{V}(A)$ cannot be p-refined for any p. However, $E^2 = (1/2)(|0, 0\rangle + |1, 1\rangle)(\langle 0, 0| + \langle 1, 1|)$ is a projector and hence has a trivial CL. Thus, the 1D cluster state has a coarse CL.

Example 5. The Holevo-Werner channel. Consider the Holevo-Werner channel for qubits, $\mathcal{E}(\rho) = \frac{1}{3}[\rho^t + \mathrm{Tr}(\rho)\mathbb{1}]$, where ρ^t denotes its transpose. The corresponding state is given by the tensor

$$A^1 = \sqrt{\frac{2}{3}}|0\rangle\langle 0|, \quad A^2 = \sqrt{\frac{2}{3}}|1\rangle\langle 1|, \quad A^3 = \frac{1}{\sqrt{3}}X. \tag{23}$$

In the Pauli basis, $E = \mathrm{diag}(1, 1/3, -1/3, 1/3)$. This channel cannot be expressed as a nontrivial composition of two quantum channels (even if these two are different) [24], and thus, $\mathcal{V}(A)$ cannot be p-refined for any p. However, E^2 is Markovian, namely, $\mathcal{E}^2 = e^{\mathcal{L}_\gamma}$, with

$$\mathcal{L}_\gamma(\rho) = \gamma(X\rho X + Y\rho Y + Z\rho Z - 3\rho), \quad \gamma = \ln(9)/4. \tag{24}$$

Thus, this state has a coarse CL. More generally, note that every odd power of E is not infinitely divisible, $\det(E^n) < 0$ for odd n (see Proposition 15 of [24]), and every even power of E is Markovian.

Example 6. AKLT state. Consider the AKLT state [15], which is described in terms of the tensor

$$A^1 = \frac{1}{\sqrt{3}} Z, \quad A^2 = \sqrt{\frac{2}{3}}|1\rangle\langle 0|, \quad A^3 = -\sqrt{\frac{2}{3}}|0\rangle\langle 1|. \tag{25}$$

In the Pauli basis, $E = \mathrm{diag}(1, -1/3, -1/3, -1/3)$. We thus have that $\det(E) = -1/27$, and the channel cannot be expressed as a nontrivial composition of two quantum channels [24]. Thus, the AKLT state cannot be p-refined for any p. However, $\mathcal{E}^2 = e^{\mathcal{L}_\gamma}$, with $\mathcal{L}_\gamma$ given by (24). More specifically, $E^2 = \sum_{i=1}^4 B^i \otimes \bar{B}^i$, with

$$B^1 = \sqrt{q} I, \quad B^2 = \sqrt{\frac{1-q}{3}} X,$$
$$B^3 = \sqrt{\frac{1-q}{3}} Y, \quad B^4 = \sqrt{\frac{1-q}{3}} Z, \tag{26}$$

with $q = 1/3$. This state can be p-refined for any p into a state with the same matrices, but with q replaced by $q_p = (1 + 3^{(p-2)/p})/4$. Thus, the AKLT state has a coarse CL.

Remark 2. Multiple roots of the transfer matrix. Examples 1 and 2 illustrate that the transfer matrix of the ferromagnet with m states [Eq. (9)] has two $p = m + 1$ roots which correspond to a transfer matrix, namely, itself and the transfer matrix of the antiferromagnet [Eq. (12)]. These correspond to the two inequivalent ways of p-refining the state.

Similarly, Examples 5 and 6 illustrate that the depolarizing channel $E = e^{L_\gamma}$ with L_γ given in (24) has three square roots which are valid quantum channels: the Markovian one ($e^{L_\gamma/2}$), the Holevo-Werner channel, and the transfer matrix corresponding to the AKLT state. Only the Markovian root can be further refined, and thus, the state corresponding to $E = e^{L_\gamma}$ has a CL.

Finally, we give an example of a state without a coarse CL.

Example 7. A state without a coarse CL. Consider the family of qubit channels of the form $E = 1 \oplus \Delta$ in the Pauli basis, with Δ being positive definite and with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$. We claim that if $0 < \lambda_3 < \lambda_1\lambda_2$, then E^n is not infinitely divisible for any finite n. To see this, note that by Theorem 24 in Ref. [24] E is not infinitesimal divisible, and this is preserved under powers. Since infinitely divisible channels are a subset of infinitesimal divisible channels [24], it follows that the state corresponding to this transfer matrix does not have a coarse CL.

Take, for example, diagonal Δ and $\lambda_1 = \lambda_2 = a$, $\lambda_3 = a^2/2$ (with $0 < a \leqslant 2 - \sqrt{2}$; see the proof of Proposition 2).

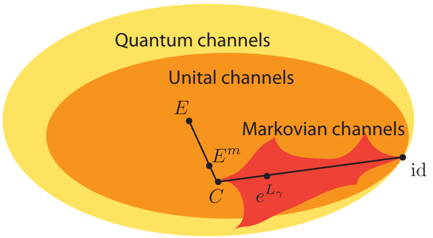

FIG. 2. Sketch of part of the geometry of qubit channels. The volume of the sets is drawn arbitrarily.

The corresponding tensor is given by

$$A^1 = \sqrt{\tfrac{2+4a+a^2}{8}}\mathbb{1}, \quad A^2 = -\sqrt{\tfrac{2-4a+a^2}{8}} Z,$$
$$A^3 = \sqrt{\tfrac{2-a^2}{8}}|1\rangle\langle 0|, \quad A^4 = \sqrt{\tfrac{2-a^2}{8}}|0\rangle\langle 1|. \tag{27}$$

Note that $\lim_{n\to\infty} E^n = C$, where C is the completely depolarizing channel, $C(\rho) = \mathrm{Tr}(\rho)\mathbb{1}/2$. The latter is in the closure of the set of Markovian channels [e.g., $C = \lim_{\gamma\to\infty} e^{\mathcal{L}_\gamma}$, with $\mathcal{L}_\gamma$ given in (24); see Fig. 2].

C. Comparison between the two continuum limits

The previous examples allow us to compare the two CLs. Let C_D and $C_D^{\rm coarse}$ denote the set of families of states $\mathcal{V}(A)$ of bond dimension D with a CL and a coarse CL, respectively.

Proposition 1. For every bond dimension D, (1) C_D is strictly included in $C_D^{\rm coarse}$, and (2) there are states not in $C_D^{\rm coarse}$.

Proof. That C_D is included in $C_D^{\rm coarse}$ is trivial from the definition, and for $D = 2$, that the inclusion is strict follows, e.g., from Example 5. For $D = 2$, the second claim is proven by Example 7. In both cases, the extension to higher D follows trivially by embedding $\mathcal{M}_2$ into $\mathcal{M}_D$, for example, as $\mathcal{M}_D = \mathcal{M}_2 \oplus 0_{D-2}$, where 0_{D-2} is the zero matrix. ■

We also gain the following insight from Example 7.

Proposition 2. There are states that can be p-refined only a finite number of times.

Proof. Consider the family of channels whose Lorentz normal form [24] is given by $E(a, \eta) := \mathrm{diag}(1, a, a, \eta a^2)$, with $a \in (0, 1]$ and $\eta \in (0, 1)$. It is easy to see that $E(a, \eta)$ is completely positive if and only if $a \leqslant \frac{1}{\eta}(1 - \sqrt{1-\eta}) =: g(\eta)$(this can be seen by applying Eq. (9) of Ref. [31] to our case). Denoting by $\ell_{\rm sol}$ the solution to the equation $a^{-\ell} = g(\eta^{-\ell})$, we see that $E(a, \eta)$ is $\lfloor \ell_{\rm sol} \rfloor$-divisible but not $(\lfloor \ell_{\rm sol} \rfloor + 1)$-divisible. Correspondingly, the state can be n-refined only $\lfloor \log_p \lfloor \ell_{\rm sol} \rfloor \rfloor$ times. For example, for $a = 0.1$ and $\eta = 0.9$, we have that the state can be 2-refined only five times. ■

IV. CONCLUSIONS AND OUTLOOK

In summary, we have investigated which TI MPSs have a CL, which is defined as the infinite iteration of the inverse of a renormalization procedure, together with a regularity condition in the limit. We have found that a TI MPS has a CL if and only if its transfer matrix is infinitely divisible. We have then defined the coarse CL as the CL of some of the coarser descriptions of the state and have characterized the states with a coarse CL using the divisibility properties of their transfer matrices. We have shown that various well-studied states (such as the AKLT state, the 1D cluster state, and the antiferromagnet) have a coarse CL but that not all states have one.

This work raises several questions. One concerns the representation of the states obtained in the limit as matrix products, which would require a generalization of the class of cMPSs. This would also allow us to study the uniqueness of the CL. It also remains to be seen whether there is a meaningful definition of CL such that all TI MPSs have a limit of this sort. A further possibility is to consider the renormalization procedure determined by the multiscale entanglement renormalization ansatz (MERA) [32], for which the class of continuous MERAs was defined in [33], and study continuum limits in that setting.

ACKNOWLEDGMENTS

G.D.L.C. thanks T. J. Osborne for discussions. G.D.L.C. acknowledges support from the Elise Richter Fellowship of the FWF. This work was supported in part by the Perimeter Institute of Theoretical Physics. Research at Perimeter Institute is supported by the government of Canada through Industry Canada and by the province of Ontario through the Ministry of Economic Development and Innovation. N.S. acknowledges support by the European Union through the ERC-StG WASCOSYS (Grant No. 636201). D.P.-G. acknowledges support from MINECO (Grant No. MTM2014-54240-P), Comunidad de Madrid (Grant No. QUITEMAD+-CM, ref. S2013/ICE-2801), and Severo Ochoa project SEV-2015-556. This work was made possible through the support of Grant No. 48322 from the John Templeton Foundation. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 648913). J.I.C. acknowledges support from the DFG through the NIM (Nano Initiative Munich).

APPENDIX: PROOF OF THEOREM 1

Here we prove Theorem 1, which we state again.

Theorem 1. Given $\mathcal{V}(B)$ with B in irreducible form, the following statements are equivalent:

(1) $\mathcal{V}(B)$ has a CL.

(2) E_B is infinitely divisible.

(3) There is a TPCPM P and a Liouvillian of Lindblad form L such that $E_B = Pe^L$, $P^2 = P$, and $PLP = PL$.

Proof. That items 2 and 3 are equivalent was proven by Holevo [22] and Denisov [23].

By Definition 2 and the subsequent discussion, $\mathcal{V}(B)$ has a CL if there is $p > 1$ such that E_B is p-infinitely divisible and f_{p,E_B} is continuous at zero. It is thus immediate to see that item 2 implies item 1 since being p-infinitely divisible is a particular case of being infinitely divisible, and using item 3, we have that $f_{p,E_B}(n/p^\ell) = Pe^{Ln/p^\ell}$ is continuous at zero.

Finally, to see that item 1 implies item 2, assume that E_B is p-infinitely divisible and that f_{p,E_B} is continuous at zero. We will construct the nth root of $E \equiv E_B$ by using the expansion of $1/n$ in terms of $1/p^\ell$. So for an arbitrary $n \in \mathbb{N}$, we have that

$$\frac{1}{n} = \frac{1}{p^\ell}\left(\left\lfloor \frac{p^\ell}{n} \right\rfloor + \frac{r_\ell}{n}\right), \tag{A1}$$

where $\lfloor \frac{p^\ell}{n} \rfloor$ is the largest integer which is, at most, that number (floor) and $0 \leqslant r_k < n$ is the residue of the division.

Let us consider

$$\left(E_{p^\ell}^{\lfloor p^\ell/n \rfloor}\right)_\ell. \tag{A2}$$

Since this is a sequence in a compact space, there must exist a subsequence that converges to a limit which we call E_n,

$$\left(E_{p^{l_k}}^{\lfloor p^{l_k}/n \rfloor} =: T_k\right)_k \to E_n. \tag{A3}$$

By completeness, E_n is a quantum channel. In the rest of the proof we will show that E_n is an nth root of E, i.e., $E_n^n = E$.

To see this, observe that

$$\left|\left|E_n^n - E\right|\right| \leqslant \left|\left|E_n^n - T_k^n\right|\right| + \left|\left|T_k^n - E\right|\right|, \tag{A4}$$

where for a superoperator L we use the norm $||L|| = \sup_X ||L(X)||_1/||X||_1$, where $||X||_1$ denotes the Schatten 1-norm. The first term of (A4) vanishes as $k \to \infty$ since

$$\left|\left|E_n^n - T_k^n\right|\right| \leqslant n||E_n - T_k|| \leqslant n\varepsilon, \tag{A5}$$

where the first inequality follows from the identity $T_k^n - E_n^n = (T_k^{n-1} + T_k^{n-2}E_n + \cdots + E_n^{n-1})(T_k - E_n)$ and the fact that $||T_k^{n-j}E_n^{j-1}|| = 1$ for all $j = 1, \ldots, n$ and the second follows from (A3).

To show that the second term of (A4) vanishes, we use that

$$\begin{aligned}\left\|T_k^n - E\right\| &\overset{(A1)}{\leqslant} \left\|E_{p^{\ell_k}}^{\lfloor p^{\ell_k}/n \rfloor n} - E_{p^{\ell_k}}^{\lfloor p^{\ell_k}/n \rfloor n + r_k}\right\| \\ &\leqslant \left\|E_{p^{\ell_k}}^{\lfloor p^{\ell_k}/n \rfloor n - 1}\right\| \left\|E_{p^{\ell_k}} - E_{p^{\ell_k}}^{r_k+1}\right\| \\ &\leqslant \left\|E_{p^{\ell_k}} - E_{p^{\ell_k}}^{r_k+1}\right\|,\end{aligned}$$

where we have used that $\|E_{p^{\ell_k}}\| = 1$. Since $r_k + 1 \leqslant n$, we have that both $1/p^{\ell_k}$ and $(r_k+1)/p^{\ell_k} \to 0$, and thus, continuity of $f_{p,E_B}(n, p^{\ell_k}) = E_{p^{\ell_k}}^n$ at zero implies that $\|E_{p^{\ell_k}} - E_{p^{\ell_k}}^{r_k+1}\| \to 0$. ■

[1] S. Weinberg, *The Quantum Theory of Fields, Vol. 1, Foundations* (Cambridge University Press, Cambridge, 1995).

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, 1995).

[3] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).

[4] E. Fradkin, *Field Theories in Condensed Matter Physics*, 2nd ed. (Cambridge University Press, Cambridge, 2013).

[5] R. Orús, Ann. Phys. (N.Y.) **349**, 117 (2014).

[6] U. Schollwöck, Rev. Mod. Phys. **77**, 259 (2005).

[7] F. Verstraete, V. Murg, and J. I. Cirac, Adv. Phys. **57**, 143 (2008).

[8] M. Fannes, B. Nachtergaele, and R. F. Werner, Commun. Math. Phys. **144**, 443 (1992).
[9] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum Inf. Comput. **7**, 401 (2007).
[10] N. Schuch, D. Perez-Garcia, and J. I. Cirac, Ann. Phys. (N.Y.) **325**, 2153 (2010).
[11] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[12] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).
[13] T. B. Wahl, S. T. Haßler, H.-H. Tu, J. I. Cirac, and N. Schuch, Phys. Rev. B **90**, 115133 (2014).
[14] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Phys. Rev. Lett. **94**, 140601 (2005).
[15] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).
[16] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
[17] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **104**, 190405 (2010).
[18] J. Haegeman, J. I. Cirac, T. J. Osborne, and F. Verstraete, Phys. Rev. B **88**, 085118 (2013).
[19] T. J. Osborne, Continuous limits of quantum lattice systems (2015), https://github.com/tobiasosborne/Continuous-Limits-of-Quantum-Lattice-Systems.
[20] D. Jennings, C. Brockt, J. Haegeman, T. J. Osborne, and F. Verstraete, New J. Phys. **17**, 063039 (2015).
[21] G. De las Cuevas, J. I. Cirac, N. Schuch, and D. Perez-Garcia, J. Math. Phys. **58**, 121901 (2017).
[22] A. S. Kholevo, Theory Probab. Appl. (Engl. Transl.) **31**, 493 (1987).
[23] L. V. Denisov, Theory Probab. Appl. (Engl. Transl.) **33**, 392 (1989).
[24] M. M. Wolf and J. I. Cirac, Commun. Math. Phys. **279**, 147 (2008).
[25] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008).
[26] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), pp. 69–72.
[27] J. Yuan, Pac. J. Math. **65**, 285 (1976).
[28] T. S. Cubitt, J. Eisert, and M. M. Wolf, Commun. Math. Phys. **310**, 383 (2012).
[29] M. M. Wolf, Quantum channels & operations. Guided tour (2012), https://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf.
[30] N. J. Higham, *Functions of Matrices* (Society for Industrial and Applied Mathematics, Philadelphia, 2008).
[31] M. B. Ruskai, S. Szarek, and E. Werner, Linear Algebra Appl. **347**, 159 (2002).
[32] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007).
[33] J. Haegeman, T. J. Osborne, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. **110**, 100402 (2013).