Editors' Suggestion

Effective edge state dynamics in the fractional quantum Hall effect

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We consider the behavior of quantum Hall edges away from the Luttinger liquid fixed point that occurs in the low-energy, large-system limit. Using the close links between quantum Hall wave functions and conformal field theories, we construct effective Hamiltonians from general principles and then constrain their forms by considering the effect of bulk symmetries on the properties of the edge. In examining the effect of bulk interactions on this edge, we find remarkable simplifications to these effective theories which allow for a very accurate description of the low-energy physics of quantum Hall edges relatively far away from the Luttinger liquid fixed point, and which apply to small systems and higher energies.

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I. INTRODUCTION

The edges of quantum Hall systems are remarkable. This boundary is the paradigmatic example of a chiral quantum liquid [1,2], a one-dimensional system in which the transport is only in one direction [3–5] (the chirality being due to the breaking of time-reversal symmetry in the presence of a magnetic field). Furthermore, given that the bulk of quantum Hall systems are gapped, these edge modes constitute the only low-energy degrees of freedom in these systems, and therefore mediate the fascinating transport properties for which the quantum Hall effect is so well known [6,7].

The dynamics of these modes has been a topic of great interest since the classic papers of Wen [2,8–11]. In these works it was found that the low-energy dynamics of edge modes in the thermodynamic limit corresponds exactly to a chiral linear Luttinger liquid. Linearity means that the modes have some dispersion $\omega(k)$ which is linear in the wave number k, i.e., $\omega = vk$, and chirality implies that $k > 0$.

For generic systems, this linear picture is only true in the scaling limit of low energies and large-system sizes. However, in the presence of a set of special, model interactions, and when the confinement is finely tuned, the dispersion of the edge modes is linear regardless of system size all the way to high energies [12–14]. Away from these special cases one must consider the effect of irrelevant contributions, which introduce nonlinearities such as nontrivial dispersion or scattering processes between modes.

In this work we will consider exactly this nonideal case, which can be characterized by anharmonic confinement or interactions and will, in general, have a far richer, nonlinear edge structure. This has been discussed in numerous works such as Refs. [15–22]. To analyze these nonlinear effects, we will take the ideal case where the edge dispersion is precisely linear, with given Hamiltonian $\mathcal{H}_{\text{Parent}}$, and perturb it with $\delta\mathcal{H}$, thus moving it towards something more realistic. We then construct an effective field theory for the perturbed system. In doing so, we generate a mapping from a perturbation acting upon the whole bulk of the droplet $\delta\mathcal{H}$ onto a low-energy effective Hamiltonian which resides only on the edge.

In order to constrain the field theory, we conjecture it to be local and impose upon it the symmetries of the perturbation using a construction inspired by work from Dubail, Read, and Rezayi [23] and revisited recently in Ref. [24]. We find that this procedure is especially fruitful as it maps symmetries of the perturbations, such as rotational and translational invariance, to powerful constraints on the effective Hamiltonian's form. We illustrate this mapping from bulk interactions to their effect on the edge for the Laughlin [25] and Moore-Read [26] quantum Hall states, although the procedure could in principle be generalized to more exotic quantum Hall states, such as the Read-Rezayi states [27] or any other state which can be expressed by a conformal field theory [28,29].

We will see that this effective description of the edge dynamics is accurate for short-range interactions and confinements close to quadratic. These two conditions make our work particularly applicable to potential cold atom realizations of the quantum Hall effect where the interactions between atoms are generally short range, perhaps even hard core, the confinement can be readily tuned to a simple quadratic, and the number of particles can be quite small [30–32]. In this regime, our effective theories prove to be extremely good at capturing the effects of finite-size and nonideal interactions on the edge behavior.

We begin in Sec. II with a recap of quantum Hall edges and their construction in terms of conformal blocks. We then introduce the concept of an effective Hamiltonian in Sec. III, describe how this can be expressed as a field theory on the edge of our system, and discuss the effect of symmetries. We then use these powerful results in Sec. IV for the Laughlin and Moore-Read wave functions to propose generic theories for the edge dynamics induced by nonideal bulk interactions. Finally, we present numerics in support of these claims in Sec. V, showing the excellent agreement between finite-size exact diagonalization and our effective edge theories.

II. THEORETICAL BACKGROUND

In order to make progress, we will use the close links between quantum Hall wave functions and conformal field

theories. As such, we begin this section by introducing the concepts of parent Hamiltonians for the Laughlin state, which allows us to make a precise statement of our problem. We will then discuss the formation of the edge state wave functions for the Laughlin state in terms of conformal blocks, a construction which we shall make judicial use of going forward.

A. Quantum Hall edges

1. Trial wave functions

A quantum Hall system is one in which charged particles are confined to two dimensions in the presence of a perpendicular magnetic field [7]. The Hamiltonian of such a system containing N particles is

$$\mathcal{H} = \sum_{i=1}^{N} \frac{(\boldsymbol{p}_i - q\boldsymbol{A})^2}{2m_q} + \sum_{1\leqslant i<j\leqslant N} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|) + \sum_{i=1}^{N} U(|\boldsymbol{r}_i|), \tag{1}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$ are the positions and momenta of charge-q particles whose effective mass is m_q. These particles are acted on by a magnetic field, given by the vector potential $\boldsymbol{A}$, they interact via some generic two-body interactions $V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$, and are placed in some confining trap $U(|\boldsymbol{r}_i|)$. Note that in this work we will only consider rotationally symmetric Hamiltonians on the plane.

When the interactions are trivial, this Hamiltonian gives the integer quantum Hall effect, which is characterized by a series of *Landau levels* separated by a constant gap of $\hbar\omega_c$ where ω_c is the cyclotron frequency. The introduction of interactions introduces new gaps into this spectrum of order V, thus generating a hierarchy of fractional quantum Hall states. In both cases, the total filling of these levels is determined by the density of particles, which given a fixed number N is determined by the confinement of the particles U. We will work in the limit

$$\hbar\omega_c \gg V \gg U \tag{2}$$

and in taking this limit we will always work with the potential U as its projection into the space of the zero-energy states.

So long as the number of electrons in the system fills an integer number of Landau levels, a single-particle assumption remains reasonably accurate [33,34]. When the filling ν is fractional, however, the problem is far more difficult. Nevertheless, a variety of extremely well-educated guesses have been proposed over the years which approximate the true ground states of these fractionally filled systems remarkably well. The first of these was the Laughlin wave function [25]

$$\Psi(\boldsymbol{z}) = \prod_{1\leqslant i<j\leqslant N} (z_i - z_j)^{\beta} \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{4\ell_B^2}\right), \tag{3}$$

where $z_i \in \mathbb{C}$ are the particle positions and ℓ_B the magnetic length. This describes the fractional quantum Hall effect at fillings $\nu = 1/\beta$ for non-negative integer β. Subsequent proposals include the Moore-Read state, the Read-Rezayi series, and the generalized parafermionic states [26,27,35,36]. In each case, the wave function is a holomorphic polynomial (up to Gaussian factors) of the positions where the power attached to any z_i is the angular momentum of that ith particle.

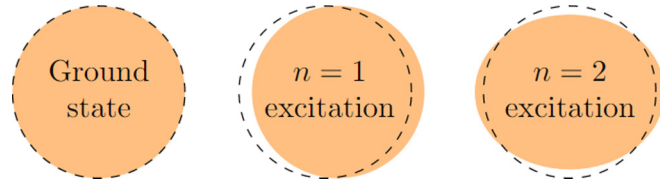

FIG. 1. A sketch of the first few edge excitations of the Laughlin droplet. These low-energy coherent excitations can be visualized as ripples propagating around the circumference of the droplet.

2. Edge states

We are interested in the low-energy states on top of these trial wave functions, which are formed by adding angular momentum to the ground state without changing the (bosonic or fermionic) symmetry of the wave function or increasing its energy too much. The specific construction in terms of holomorphic polynomials will be discussed in Sec. II B, but there exists an intuitive picture for the Laughlin state thanks to Wen [8].

In this picture one first realizes that the Laughlin state in Eq. (3) describes a circular droplet of fluid of radius $R = \ell_B\sqrt{2\beta N}$ which is both uniformly dense and incompressible. As such, the only low-energy excitations we can form are area-preserving distortions of the droplet. These are our edge modes; they are waves which encircle the droplet. The first few modes are shown in Fig. 1, with the nth edge mode, having a wavelength equal to $\frac{2\pi R}{n}$. These modes are analogous to the phonons in a lattice in that they are periodic modes of our system which distort the degrees of freedom at each point. In the same way that a system can then contain multiple phonons with the same wave number, thus making them bosonic objects, our quantum Hall system is also able to support multiple edge excitations of the same n, and so our edge modes are also bosonic.

It is then possible to derive the behavior of these modes in the presence of some confining potential in a semiclassical manner as in Wen's hydrodynamic formulation [8]. This argument proceeds by considering the classical energy of a charged fluid in the presence of an electric field and then canonically quantizes the resulting Hamiltonian. The result is the chiral linear Luttinger liquid, meaning that these waves propagate around the circumference of the droplet in one direction only. More explicitly, the statement that the Laughlin edge is a chiral linear Luttinger liquid means that the edge corresponds to a free conformal field theory of a compact boson. Note that the picture is more complicated in the Moore-Read case, where there also exists a fermionic branch of edge excitations which cannot be visualized in this way. Nevertheless, the low-energy dynamics of these modes corresponds, as in the Laughlin case, to a free conformal field theory, albeit a product of bosons *and* fermions in this case.

3. Parent Hamiltonians

Given a trial state, it is possible in many cases to work backwards and explicitly construct a parent Hamiltonian,

$\mathcal{H}_{\text{Parent}}$, for which the trial state is the exact ground state and the edge states low-energy eigenstates [37]. In the Laughlin case, this ideal Hamiltonian comprises ultra-short-range interactions consisting of only a finite number of nonzero *Haldane pseudopotentials* [12], which are all chosen to be positive (corresponding to repulsive interactions). A rapid review of these special interactions is given in Appendix A. For more exotic states, the interactions are more complicated; the $\mathbb{Z}_k$ Read-Rezayi series (where $k = 2$ corresponds to Moore-Read) are produced by a set of $(k+1)$-body interactions [27]. Within this parent Hamiltonian we also add weak quadratic confinement in the radial direction, i.e., $U(|\boldsymbol{r}|) = \frac{1}{2}U_0\boldsymbol{r}^2$, where U must be smaller than V to ensure that the system remains gapped. This specific choice to make U quadratic is made to ensure that the edge spectrum is exactly linear, matching Wen's Luttinger liquid theory from the outset.

We may now start to think more clearly about the original Hamiltonian involving generic interactions and confinement. To do so, we split it up into

$$\mathcal{H} = \mathcal{H}_{\text{Parent}} + \delta\mathcal{H}, \tag{4}$$

where $\delta\mathcal{H}$ is defined by this equation; it is the difference between the true interactions and the idealized interactions or the deviation of the true, anharmonic confinement of our system from the quadratic confinement we have imposed. Recalling then that these trial states are of interest because they approximate true systems quite accurately, we may take $\delta\mathcal{H}$ to be small and consider it as a perturbation to $\mathcal{H}_{\text{Parent}}$. The subsequent diagonalization of $\delta\mathcal{H}$ is the subject of this paper, which we will consider within the space of zero-energy states of $\mathcal{H}_{\text{Parent}}$. As such, $\delta\mathcal{H}$ will be projected into the manifold of model states which we discuss in the following section.

B. Effective descriptions

1. General construction

The trial states we will work with can be written as correlation functions of operators from a chiral conformal field theory (CFT) [23,26–28,38]. These CFTs are made up of two *sectors*, $\text{CFT}_{\text{U(1)}} \otimes \text{CFT}_\chi$, denoted the *charge sector* and the *statistics sector*, respectively. In this way, an individual particle at position z is represented by an operator $\mathcal{A}_\beta(z)$, which can be decomposed into two parts,

$$\mathcal{A}_\beta(z) =: e^{i\sqrt{\beta}\varphi(z)} : \chi(z), \tag{5}$$

where this first term is the vertex operator of a free massless Bose field $\varphi(z)$ from the U(1) charge sector and the second term $\chi(z)$ is from the statistics sector CFT_χ. Note that the notation $: X :$ refers to the normal ordering of the operator X. Furthermore, the presence of the vertex operator gives our particle a U(1) charge of $\sqrt{\beta}$.

Using these operators, one can construct a quantum Hall wave function at filling fraction $\nu = 1/\beta$ as the correlation function

$$\Psi_{\langle v|}(\boldsymbol{z}) = \langle v|c_\beta^N \mathcal{A}_\beta(z_1)\ldots\mathcal{A}_\beta(z_N)|0\rangle, \tag{6}$$

where $\boldsymbol{z} = \{z_1, \ldots, z_N\}$ are the positions of the particles and $|0\rangle$ is the vacuum of the full CFT with zero charge. The operator c_β^N is the *background charge* and is an operator with a U(1) charge of $-N\sqrt{\beta}$. Its presence gives the correlator a net U(1) charge of zero, without which the correlator must vanish.

Finally, the out state $\langle v|$ is a state in the full CFT which defines an individual edge excitation with the vacuum $\langle 0|$ corresponding to the ground state. Effectively, there exists some antilinear mapping from a CFT state to a physical wave function:

$$|v\rangle \mapsto \Psi_{\langle v|}(\boldsymbol{z}). \tag{7}$$

The CFT states have a well-defined quantum number ΔL, which, in the CFT language, is called the *conformal dimension*. The eigenoperator associated with ΔL is L_0, the 0th mode of the *Virasoro algebra*, which generates dilations and is proportional to the chiral CFT's Hamiltonian (as another example, the -1st mode of the Virasoro algebra L_{-1} generates translations). The vacuum $|0\rangle$ has a conformal dimension of $\Delta L = 0$. This quantum number also has an interpretation in the quantum Hall language as the angular momentum of a particular state relative to the vacuum state $\Psi_{\langle 0|}$.

Before continuing, it should be stressed that the mapping introduced in Eq. (7) is surjective, as it must be to reproduce the full set of quantum Hall edge states [39]. However, for finite N, the mapping is not injective as there are cases where the same edge state wave function $\Psi_{\langle v|}(\boldsymbol{z})$ can be generated by two different CFT states. We will demonstrate an example of this peculiarity for the Laughlin state in Eqs. (19) and (20) but, in the limit of $N \gg \Delta L$ which we are interested in, this complexity will not prove to be consequential. We will also see below that this mapping is not an isometry, meaning it does not preserve inner products. This has been discussed at length by Refs. [23,24] and will become important for us below [see in particular Eq. (68)].

2. Laughlin state

The Laughlin state has a trivial statistics sector, i.e., $\chi = \mathbb{1}$, making its CFT solely that of the free boson $\text{CFT}_{\text{U(1)}}$. This CFT contains only the field

$$\varphi(z) = \varphi_0 - ia_0\ln(z) + i\sum_{n\neq 0}\frac{a_n}{n}z^{-n}, \tag{8}$$

where the modes of the field satisfy the commutation relations

$$[a_n, a_{-m}] = n\delta_{n,m}, \quad [\varphi_0, a_0] = i. \tag{9}$$

All other commutators are trivial. We then define the vacuum of the theory, $|0\rangle$, as the state which is annihilated by all the positive modes:

$$a_n|0\rangle = 0 \quad \forall\, n \geqslant 0. \tag{10}$$

We define the background charge in terms of these modes as

$$c_\beta^N = e^{-iN\sqrt{\beta}\varphi_0}. \tag{11}$$

The particle operator for this case is then simply the vertex operator. Given that the operator product expansion (OPE) of two bosonic fields has the form $\varphi(z)\varphi(w) \sim -\ln(z-w)$, we find, using the Baker-Campbell-Hausdorff formula, that

$$\mathcal{A}_\beta(z)\mathcal{A}_\beta(w) = (z-w)^\beta : e^{i\sqrt{\beta}[\varphi(z)+\varphi(w)]} : . \tag{12}$$

From this it is relatively straightforward to see that the ground state, i.e., the state where $\langle v| = \langle 0|$, has the form

$$\Psi_{\langle 0|}(\boldsymbol{z}) = \prod_{i<j}(z_i - z_j)^{\beta}. \tag{13}$$

This is almost as expected for the Laughlin state but with the omission of the Gaussian factor we see in Eq. (3). This is due to our choice of background charge, which is equivalent to placing a particle with U(1) charge $-N\sqrt{\beta}$ at infinity. It is instead possible to spread this charge over the droplet, at which point the Gaussian factors are recovered, but this is slightly more complicated [26,40]. Therefore, we instead use this simpler version and include the Gaussian factors as part of the integration measure for the problem, which we define as

$$\{\Psi_{\langle v|}|\Psi_{\langle w|}\} = \int D\boldsymbol{z}\, \bar{\Psi}_{\langle v|}\Psi_{\langle w|}, \tag{14}$$

where

$$D\boldsymbol{z} = \prod_i \left[d^2 z_i \exp\left(-\frac{|z_i|^2}{2\ell_B^2}\right)\right]. \tag{15}$$

Excited states $\langle v|$ are generated by applying the positive modes a_n to the out-state vacuum, where each a_n raises the conformal dimension of the state by n. Specifically, we can define the states

$$\langle \boldsymbol{\lambda}| = \langle 0| \prod_{n\in\boldsymbol{\lambda}} a_n, \tag{16}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots\}$ is a semiordered ($\lambda_1 \geqslant \lambda_2 \geqslant \ldots$) set of positive integers. The subsequent wave function $\Psi_{\langle\boldsymbol{\lambda}|}$ is the Laughlin state multiplied by a symmetric polynomial

$$\Psi_{\langle\boldsymbol{\lambda}|}(\boldsymbol{z}) = P_{\boldsymbol{\lambda}} \prod_{i<j}(z_i - z_j)^{\beta}, \tag{17}$$

where $P_{\boldsymbol{\lambda}}$ is a product of *power sums*

$$P_{\boldsymbol{\lambda}} = \prod_{n\in\boldsymbol{\lambda}} p_n, \quad p_n = \sqrt{\beta}\sum_i z_i^n \tag{18}$$

(note that we do not normalize the z_i by factors of the radius R as is sometimes the convention). As such, we can now see how the conformal dimension of the state corresponds to the added angular momentum. If we recall that the power on any z_i is the angular momentum of that particle, we see that this polynomial adds $\sum_i \lambda_i$ units of angular momentum to the ground state, and this is exactly the conformal dimension of the state $|\boldsymbol{\lambda}\rangle$.

It is worth noting how these excitations a_n correspond to the semiclassical pictures of edge states presented in Fig. 1. We note that these a_n excitations generate states which have well-defined angular momentum and will therefore be rotationally invariant. As such, the semiclassical edge ripples of Fig. 1 correspond instead to coherent states. For example, in the $N \to \infty$ limit it can be shown that the wave function labeled by the auxiliary state $\langle\delta| = \langle 0|e^{\delta a_1}$, where δ is complex, corresponds to the $n = 1$ excitation, as it describes a droplet centered at the complex position δ. Therefore, a_1 is the generator of translations. Similarly, a_2 can be shown to be the generator of $n = 2$ distortions of the droplet's surface, and so on for subsequent a_n.

Finally, we consider a simple example of the noninjectivity of the mapping defined by Eq. (7). Taking the $N = 1$ Laughlin state then we find that

$$a_1 a_1|0\rangle \mapsto \beta z_1^2, \tag{19}$$

$$\sqrt{\beta}a_2|0\rangle \mapsto \beta z_1^2. \tag{20}$$

This equivalence these two states is due to the fact that the full space of the N-variable symmetric polynomials which make up the edge states can be fully spanned by N-independent basis polynomials. The addition of any more to this set would form an overcomplete basis. Therefore, one could similarly surmount this injectivity problem by restricting our basis of CFT states to those including the a_n for $|n| \leqslant N$. In the above example, this would exclude the a_2 mode.

3. *Moore-Read state*

The statistics sector for the Moore-Read wave function is that of a free Majorana fermion whose field $\chi(z) = \psi(z)$ has an OPE of the form

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \tag{21}$$

and admits a mode expansion of the form

$$\psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_n z^{-n-1/2}. \tag{22}$$

These modes satisfy the anticommutation relation

$$\{\psi_n, \psi_{-m}\} = \delta_{n,m} \tag{23}$$

and yield a vacuum $|0\rangle$, which is annihilated by the positive fermionic modes ψ_n for $n > 0$ (in addition to $|0\rangle$ being annihilated by the positive modes of the bosonic field; $|0\rangle = |0\rangle_{\mathrm{U}(1)} \otimes |0\rangle_{\psi}$). The background charge is unchanged from the Laughlin case.

Before we construct the trial wave functions which follow from this CFT, it is worth noting that the fermionic CFT contains a parity symmetry. This symmetry forces the correlation function of an odd number of fermionic fields to vanish, and so the construction differs slightly between odd and even particle numbers. We shall focus initially on the even case. In this case, the ground state is of the form

$$\Psi_{\langle 0|}(\boldsymbol{z}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^{\beta}, \tag{24}$$

where this extra term, Pf(...), arising from the contraction of the fermionic fields is called the *Pfaffian*, and is an antisymmetrized sum over all products of the fractions $\frac{1}{z_i - z_j}$:

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathbb{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\right), \tag{25}$$

where $\mathbb{A}$ refers to the antisymmetrization over all the indices $1, \ldots, N$. Once again, the form of the wave function (24) omits the necessity Gaussian factors which we once again place within the integration measure, whose form is exactly equivalent to Eq. (14).

Consistent with the Laughlin state, we excite edge modes by applying the positive modes of the fields in our CFT on the

vacuum. In this case, we have two branches of excitations, one of which is an exact replication of the bosonic excitations seen for the Laughlin and another from the fermionic field. This second branch is more restricted than for the bosons given that the states must obey parity symmetry and possess a fermionic exclusion principle. As such, general states have the form

$$\langle \boldsymbol{\lambda} \,;\, \boldsymbol{\mu}| = \langle 0| \prod_{n\in\boldsymbol{\lambda}} a_n \prod_{l\in\boldsymbol{\mu}} \psi_l, \tag{26}$$

where $\boldsymbol{\lambda}$ is once again a semiordered set of positive integers while $\boldsymbol{\mu}$ is an ordered ($\mu_1 > \mu_2 > \cdots$) set of positive half-integers ($\mu_i \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$). Parity symmetry then forces the number of elements within the set to be even. As the ψ_l anticommute, we must also enforce an ordering on this product and we choose to order the ψ_l with the smallest l at the leftmost position, i.e.,

$$\left\langle \emptyset \,;\, \tfrac{3}{2}, \tfrac{1}{2}\right| = \langle 0|\psi_{\frac{1}{2}}\psi_{\frac{3}{2}}. \tag{27}$$

Despite the extra complexity of these states, they once again raise the angular momentum by an amount equal to the conformal dimension of the state, which in this case is

$$\Delta L = \sum_i \lambda_i + \sum_j \mu_j. \tag{28}$$

They can also be expressed in first quantized notation as

$$\Psi_{\langle\boldsymbol{\lambda};\boldsymbol{\mu}|} = P_{\boldsymbol{\lambda}} \frac{\left\langle \left(\prod_{l\in\boldsymbol{\mu}} \psi_l\right)\psi(z_1)\ldots\psi(z_N)\right\rangle}{\langle\psi(z_1)\ldots\psi(z_N)\rangle}\Psi_{\langle 0|}, \tag{29}$$

where these correlation functions can be evaluated by Wick's theorem to give similar Pfaffians to Eq. (24) (see Ref. [23] for an explicit example).

Finally, we consider the case when N is odd. The picture is almost identical but, due to parity symmetry, we must be careful to ensure that the total number of fermionic operators within any given correlator is even. As such, the edge states from Eq. (26) are identical except for the condition that $\boldsymbol{\mu}$ must be a set containing an *odd* number of elements. This even holds for the ground state, which is instead defined by the CFT state $\langle 0|\psi_{\frac{1}{2}}$. Therefore, note also that the angular momentum added by state $\langle v|$ is now the conformal dimension of $\langle v|$ less the conformal dimension of this ground state (whose conformal dimension is $\frac{1}{2}$).

III. EFFECTIVE HAMILTONIANS

A. Effective Hamiltonian

We are now in a position to use the powerful language of CFT to generate effective low-energy theories for quantum Hall edges. As we have already discussed, we do so by considering the problem as one of degenerate perturbation theory. In the construction we have just introduced our physical states are labeled by auxiliary states in the CFT $\Psi_{\langle v|}$. The state $\langle v|$ is such that it describes a state with ΔL units of angular momentum with respect to the ground state. The parent Hamiltonian, with its parabolic confinement, is then such that the energies of these states are linear in this added angular momentum $E_v \propto \Delta L$.

However, these subspaces at a given ΔL will be degenerate. For example, in the Laughlin case at $\Delta L = 2$ there are two states $\Psi_{\langle 2|}$ and $\Psi_{\langle 1,1|}$. Once we impose our perturbation $\delta\mathcal{H}$ on the system, this degeneracy will in general break, mixing the two states,

$$\mathcal{P}\,\delta\mathcal{H}\,\mathcal{P}\,\Psi_{\langle v|} = \sum_w H_{v,w}\Psi_{\langle w|}, \tag{30}$$

where $\mathcal{P}$ is the aforementioned projection to the zero-energy space of model states. However, given the linearity of the description of these wave functions in terms of CFT (i.e., $\alpha\Psi_{\langle v|} + \beta\Psi_{\langle w|} = \Psi_{\langle v|\alpha+\langle w|\beta}$), we in fact have that

$$\delta\mathcal{H}\Psi_{\langle v|} = \Psi_{\langle v'|} \quad \text{where} \quad \langle v'| = \sum_w \langle w|H_{v,w}. \tag{31}$$

As such, there is an operator H which is the image of $\delta\mathcal{H}$ under a linear mapping from the physical space of states to the CFT which reproduces the mixing of the real states in the CFT language, i.e.,

$$\delta\mathcal{H}\Psi_{\langle v|} = \Psi_{\langle v|H}. \tag{32}$$

This equation defines H. However, recall that for finite N the mapping from CFT states to edge state wave functions is not injective and therefore the operator H is only uniquely defined in the limit $N \to \infty$. Nevertheless, this nonuniqueness will not alter the calculated spectrum even for finite N.

In what follows, we will consider the constraints imposed upon H by the symmetries of $\delta\mathcal{H}$. As usual in quantum mechanics, the symmetries of the Hamiltonian will be expressed as a vanishing commutation relation with some operator $\mathcal{B}$, which encodes the particular symmetry. Consider then that this operator also has a mapping to the CFT:

$$\mathcal{B}\Psi_{\langle v|} = \Psi_{\langle v|B}. \tag{33}$$

Note that there is an implicit assumption here that we are considering an operator $\mathcal{B}$, which has already been projected to the zero-energy subspace of states. In this way, the symmetry of $\delta\mathcal{H}$ (which we also assume to be implicitly projected down) can be simply mapped to a symmetry of H:

$$[\mathcal{B}, \delta\mathcal{H}] = 0 \quad \mapsto \quad [H, B] = 0. \tag{34}$$

This procedure allows us to impose strong constraints on the form of H. However, we once again stress that the commutation relation of $\mathcal{B}$ and $\delta\mathcal{H}$ must hold for the projected operators, which may be different from the bare operators (for example, the position operators x and y commute but their projections to the lowest Landau level do not).

It is worth noting a key consequence of the fact that the mapping is linear. Consider, for example, that we perturb the trial wave function by two perturbations $\delta\mathcal{H} = \delta\mathcal{H}_1 + \delta\mathcal{H}_2$. Given that our mapping to the CFT language is linear, each of these perturbations admits its own effective description and the two simply add:

$$\delta\mathcal{H} = \delta\mathcal{H}_1 + \delta\mathcal{H}_2 \quad \mapsto \quad H = H_1 + H_2. \tag{35}$$

Now, consider that $\delta\mathcal{H}$ commutes with a set of operators $\{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ but that $\delta\mathcal{H}_2$ also has one extra symmetry $\mathcal{B}_{n+1}$. The former statement implies that H commutes with each of B_1 to B_n. However, $\delta\mathcal{H}_2$ also commutes with $\mathcal{B}_{n+1}$ and, given that the mapping to the CFT language is linear, it must also be the case that $[H_2, B_{n+1}] = 0$. Therefore, the

individual effective Hamiltonian satisfies symmetries that the whole might not.

This may seem like an obvious point, but it is of crucial importance. Consider, for example, that $\delta\mathcal{H}_1$ is some confinement imposed on the system and $\delta\mathcal{H}_2$ corresponds to an interaction. In this case, $\delta\mathcal{H}_2$ possesses an extra symmetry to $\delta\mathcal{H}_1$, that of translational invariance, and this imposes extra constraints on the form of H_2. However, because the mapping $\delta\mathcal{H} \to H$ is linear, the only part of the effective Hamiltonian that knows anything about the form of the interactions is H_2. Therefore, even when the generic perturbation to the system as a whole, $\delta\mathcal{H}$, does not possess translational symmetry, the fact that our mapping is linear means that we are still able to make strong statements about all the contributions to H arising from interactions.

B. Preliminary example: Harmonic case

We begin with a simple example where the mapping $\delta\mathcal{H} \mapsto H$ can be performed exactly. In general, this will not be possible but this example provides a taste of the machinery involved in the subsequent calculations. The perturbation we will consider is a harmonic confinement imposed upon the droplet. In the first quantized language, this perturbation has the form

$$\delta\mathcal{H} = U_0 \sum_{i=1}^{N} \left|\frac{z_i}{R}\right|^2. \tag{36}$$

We now wish to consider the effect of $\delta\mathcal{H}$ acting upon our wave function $\Psi_{\langle v|}$. At first glance, this does not appear possible as the wave function $\Psi_{\langle v|}$ can only be holomorphic but the interaction contains $\bar{z}_i$ terms. We are saved in this instance by a procedure known as *projection to the lowest Landau level*. Effectively, we need only consider the matrix elements of $\delta\mathcal{H}$ within our subspace of states given by Eq. (6) and where the integration measure is given by Eq. (14). In this case, we can replace any $\bar{z}_i$ in the integral with the differential operator $-2\ell_B^2\partial_i$ acting on the exponential factors inside the integration measure. Thus, following integration by parts we find that

$$\bar{z}_i \mapsto 2\ell_B^2\partial_i. \tag{37}$$

In this way, we can reformulate our perturbation as a differential operator. Recalling that $R = \ell_B\sqrt{2\beta N}$ we find

$$\delta\mathcal{H} = \frac{U_0}{\beta} + \frac{U_0}{\beta N}\sum_{i=1}^{N} z_i\partial_i \tag{38}$$

(where the constant term arises from the action of ∂_i on z_i itself). Thus, we find a constant energy shift plus an extra term which we can map into the CFT using a *Ward identity* [28]. Using the identity for the Laughlin state

$$[L_0, \mathcal{A}_\beta(z)] = \left(z\partial + \frac{\beta}{2}\right)\mathcal{A}_\beta(z), \tag{39}$$

we are able to map $\delta\mathcal{H}$ into the CFT as

$$\delta\mathcal{H}\langle v|c_\beta^N\mathcal{A}_\beta(z_1)\ldots|0\rangle = \langle v|c_\beta^N\tilde{H}\mathcal{A}_\beta(z_1)\ldots|0\rangle, \tag{40}$$

where this operator $\tilde{H}$ is

$$\tilde{H} = \frac{U_0}{\beta} + \frac{U_0}{\beta N}\left(L_0 - \frac{\beta N}{2}\right). \tag{41}$$

Therefore, to find the Hamiltonian we simply need to commute $\tilde{H}$ through the background charge, which takes $a_0 \to a_0 + N\sqrt{\beta}$ and therefore gives

$$H = \frac{U_0}{\beta N}(L_0 + a_0 N\sqrt{\beta}) + U_0\frac{\beta(N-1)+2}{2\beta}. \tag{42}$$

Thus, we have our first effective Hamiltonian. In the space of neutral edge states (i.e., $a_0|v\rangle = 0$) this simply reduces to a linear model

$$H = \frac{U_0}{\beta N}L_0 + \text{const}, \tag{43}$$

where L_0 simply counts the conformal dimension ΔL of the state $|v\rangle$, and so

$$E = \frac{U_0}{\beta N}\Delta L + \text{const}. \tag{44}$$

This also proves that any perturbing confinement to the quantum Hall system which is quadratic gives only a linear contribution to the spectrum.

Note that a similar mapping is also possible for a quadratic interaction

$$\delta\mathcal{H} = V_0\sum_{i\neq j}\left|\frac{z_i - z_j}{2\ell_B}\right|^2. \tag{45}$$

We find that

$$H = NV_0\left(L_0 - a_{-1}a_1 - \frac{1}{N\sqrt{\beta}}L_{-1}a_1\right) + \text{const} \tag{46}$$

in the sector where the states have zero U(1) charge (i.e., $a_0 = 0$). The derivation of this result is presented in Appendix C.

C. Symmetries of the effective Hamiltonian

A priori, we know nothing about the form of H but we shall attempt to constrain it using a simple symmetry analysis. In what follows, we shall use a local field theory description for H and then map the symmetries of the perturbation $\delta\mathcal{H}$ into the CFT language in order to find symmetry constraints on the terms in H.

1. A local field theory

We begin by noting that our effective Hamiltonian operator H is a CFT operator which acts only within the space of edge states and so is supported along the edge. We then conjecture this Hamiltonian should be local, following from the work of Dubail, Read, and Rezayi [23] who emphasized the importance of local field theories in the description of quantum Hall states. To motivate this idea, we appeal to the generalized screening hypothesis, which states that correlations within the bulk of quantum Hall droplets decay exponentially [7,25]. For the Laughlin state, this can be understood in terms of the plasma analogy, which reimagines the probability distribution of particles in the ground state as the partition function of a two-dimensional plasma of charge-β particles. Numerical

work [41] suggests that this plasma is in a screening phase when $\beta < \beta_c$ for a critical value of $\beta_c \simeq 65$. Hence, the interactions between particles in this plasma decay exponentially and one expects the relevant physics to be local. Generalized screening arguments also exist for the Moore-Read and $k = 3$ Read-Rezayi states which, while much more in depth, also conclude that bulk correlations decay exponentially [42,43].

Thus, we are conjecturing that our effective Hamiltonian, which by its definition is some operator supported along the edge of our droplet, should be local. This means that we take H to be of the form

$$H = \int dx \sum_a \tilde{h}_a(x, \delta\mathcal{H})\Phi_a(x), \tag{47}$$

where x is the one-dimensional coordinate encircling the surface of the droplet and $\Phi_a(x)$ are local operators made up of the fields $\varphi(x)$ and $\chi(x)$ and which have associated coupling constants $\tilde{h}_a$ that depend *a priori* on the position x and the perturbation $\delta\mathcal{H}$. A nonlocal form would be an integral over multiple edge coordinates with local Hamiltonian densities depending in some nontrivial manner on each coordinate, thus coupling well-separated regions of space along our edge.

The mapping between this edge coordinate x and a complex planar coordinate z is given by $z = re^{ix/R}$ for any r (but physically we consider $r \simeq R$) and where R is the droplet's radius. Working with this planar coordinate proves to be a large simplification. If we make this change of variables, we find that

$$H = \sum_a \oint \frac{dz}{2\pi i} h_a(z, \delta\mathcal{H})\left(\frac{z}{R}\right)^{d_a - 1}\Phi_a(z), \tag{48}$$

where d_a is the scaling dimension of the field Φ_a. Note, however, that we do not wish to restrict ourselves to Φ_a which are primary fields and so the general transformation of Φ_a will be

$$\Phi_a(x) \to \left(\frac{z}{R}\right)^{d_a}\Phi_a(z) + \text{subleading contributions}, \tag{49}$$

where these subleading contributions will also scale as R^{-d_a}. Therefore, we have also redefined the coupling coefficients $\tilde{h}_a \to h_a$ within Eq. (48) to work more readily with this basis of fields in the planar coordinate.

Given that we are interested in cases where the number of particles is finite but still large, we see that the effective Hamiltonian is an expansion in $\frac{1}{R}$ where R is large. As such, we may restrict ourselves to considering only contributions with a small scaling dimension d_a, and still hope to gain an accurate picture for relatively large system sizes. Furthermore, in the scaling limit, for which $R \to \infty$, the behavior is given simply by the term with the lowest scaling dimension d_a.

It should be stressed that this locality conjecture is not a rigorous constraint on H. We have motivated it here on the idea that the bulk physics is local though exactly how this should transfer to the form of H is not fully understood (though we provide some further evidence in the integer quantum Hall effect in a future publication [44]). Therefore, we provide supporting numerical evidence in Sec. V which further substantiates that this local description is very accurate for at least short-range interactions. Nevertheless, the locality conjecture may incur some loss of generality when the perturbation we add to the Hamiltonian describes some long-range interactions, in which case one might no longer expect a local field theory to provide an ample description of the dynamics. However, without the powerful simplification that this conjecture imposes on the theory, it would be extremely difficult to make significant progress.

2. Number conservation

We now move onto rigorous constraints on our Hamiltonian, of which we shall consider three. The first is the conservation of total number of particles in the system. This is conservation of U(1) charge, as each particle possesses a charge of $\sqrt{\beta}$. The operator which counts this charge is a_0 and, therefore, the particle number in the CFT language is $\hat{N} = a_0/\sqrt{\beta}$. As such, the Hamiltonian must commute with a_0:

$$[H, a_0] = 0. \tag{50}$$

The consequence of this is relatively simple and means that our Hamiltonian must obey the same underlying U(1) symmetry of the free boson CFT. For the field this symmetry manifests itself as a shift to the field under the action of the U(1) generator

$$\varphi(z) \to \varphi(z) + \delta\varphi_0. \tag{51}$$

As such, the individual operators $\Phi_a(z)$ can only involve $\varphi(z)$ as a derivative, i.e., $\partial^n\varphi(z)$ for $n > 0$. Note that there is no constraint on the statistics sector due to this conservation law.

3. Rotational invariance

We will also consider perturbations $\delta\mathcal{H}$ which are rotationally invariant. Once again, this is a very reasonable constraint for interactions, though perhaps more restrictive for confining potentials. On a mathematical level, it is relatively simple to derive the concomitant commutation relation for this symmetry by noting that the Hamiltonian should leave the total amount of angular momentum in the system invariant. Recalling that the angular momentum relative to the ground state is equal to the conformal dimension of the state $\langle v|$, and this is measured by the operator L_0, we therefore have that

$$[H, L_0] = 0. \tag{52}$$

However, once again, it is perhaps simpler to consider this constraint from a more physical perspective. Our droplet is a disk, and this system is invariant under rotations. Our edge is the circle at the edge of this disk, and the rotation of the bulk coordinate corresponds to a translation of the edge coordinate. Therefore, our edge Hamiltonian must be translationally invariant. This has a simple consequence that the coupling coefficients h_a which we introduced in Eq. (47) must be independent of x.

4. Translational invariance

The final bulk symmetry we may consider is two-dimensional translational invariance. Of course, this symmetry is not applicable to confining potentials but it does provide very strong constraints on the forms of field theories which describe interaction perturbations. Such perturbations will be

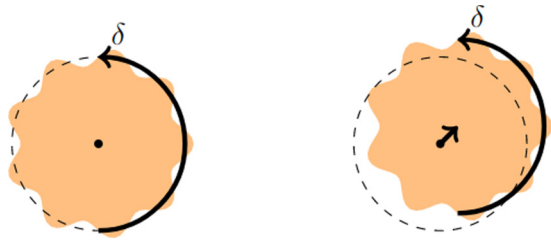

FIG. 2. On the left we show a droplet with some edge excitation which is evolved by our translationally invariant perturbation $\delta\mathcal{H}$. Given that $\delta\mathcal{H}$ is translationally invariant, this evolution will be about the center of mass, and thus should be the same for the droplet on the right, whose center of mass is slightly shifted. Crucially, $\delta\mathcal{H}$ cannot move this center of mass.

translationally invariant within the bulk, meaning that the perturbation commutes with the generator of translations

$$\left[\delta\mathcal{H}\ ,\ \sum_i \partial_i\right] = 0, \tag{53}$$

where $\partial_i = \frac{\partial}{\partial z_i}$. Thus, if we can map this generator into our conformal field theory, i.e.,

$$\sum_i \partial_i \Psi_{\langle v|} = \Psi_{\langle v|\mathcal{D}}, \tag{54}$$

then we know that our effective Hamiltonian H must commute with $\mathcal{D}$.

Nevertheless, there is once again a simpler picture to keep in mind. Consider that our perturbation $\delta\mathcal{H}$ will induce dynamics on the edge states of our system. This is shown in Fig. 2. On the left of this figure the droplet is centered and the arrow indicates the dynamics induced by $\delta\mathcal{H}$ (there may be additional dynamics due to the parabolic confinement in $\mathcal{H}_{\text{parent}}$ which we ignore here). Now, consider the right picture, where we shift the position of our droplet. The dynamics induced by $\delta\mathcal{H}$, assuming that it is a translationally invariant perturbation, should be identical, but now about a new origin. Therefore, there must be some decoupling of the center-of-mass mode from the underlying field theory. If we reconsider our pictures for the edge modes in Fig. 1, then we see that the a_n edge mode creates n equally sized lobes of charge around the surface of the droplet. For $n \geqslant 2$ these cancel out, leading to no net shift of the center of mass. The a_1 mode, however, is equivalent to a shift of the droplet. As such, the field theory H cannot induce any dynamics on this edge mode. Therefore, a state $\langle v|$ will have the same energy under $\delta\mathcal{H}$ as a shifted state $\langle v|a_1$ and so

$$[H, a_1] = 0. \tag{55}$$

However, this argument is far from rigorous and is too much of a simplistic treatment which does not capture the full complexity of the situation. Therefore, we perform the explicit mapping of the symmetry in Eq. (53) directly into the CFT language in two different ways, producing two (closely related) consequences for our effective Hamiltonian.

Constraint 1. For the first consequence, we simply map the generator of translations into the edge mode language. This can be done quite simply given the L_{-1} Virasoro mode, which is the generator of translations in the CFT language. This operator has the properties that

$$[L_{-1}, \mathcal{A}_\beta(z)] = \partial\mathcal{A}_\beta(z), \quad L_{-1}|0\rangle = 0. \tag{56}$$

As such, it is simple to reproduce the action of our translation operator on the correlation function via

$$\sum_i \partial_i \Psi_{\langle v|} = \langle v|c_\beta^N L_{-1}\mathcal{A}_\beta(z_1)\ldots\mathcal{A}_\beta(z_N)|0\rangle. \tag{57}$$

This is one step away from finding the CFT operator $\mathcal{D}$ as defined in Eq. (54). We now simply need to move L_{-1} through the background charge.

To do so, we need to better understand the exact form of L_{-1}. This operator is a sum of the Virasoro modes from each CFT:

$$L_{-1} = L_{-1}^{\mathrm{U}(1)} + L_{-1}^{\chi}, \tag{58}$$

where

$$L_{-1}^{\mathrm{U}(1)} = \frac{1}{2}\oint \frac{dz}{2\pi i} : (i\partial\varphi(z))^2 : \tag{59}$$

and L_{-1}^{χ} is the -1 mode of the stress energy tensor $T(z)$ of CFT_χ. Therefore, L_{-1}^{χ} moves through the background charge (which depends only on φ_0) without consequence, whereas $L_{-1}^{\mathrm{U}(1)}$ acquires an extra term. Specifically, when the field $\varphi(z)$ is conjugated by the background charge it becomes

$$c_\beta^N \varphi(z) c_\beta^{-N} = \varphi(z) - iN\sqrt{\beta}\ln(z). \tag{60}$$

As such, one can conjugate the full L_{-1} operator by the background charge to find that

$$\mathcal{D} = L_{-1} + N\sqrt{\beta}a_{-1}, \tag{61}$$

which implies that our effective Hamiltonian has the symmetry

$$\left[H, a_{-1} + \frac{1}{N\sqrt{\beta}}L_{-1}\right] = 0. \tag{62}$$

Constraint 2. The second consequence of translational invariance is exactly that in Eq. (55), that $[H, a_1] = 0$. To prove this, we will consider the Hermitian conjugate of the derivation which produced the result in Eq. (62). In that derivation we began with the condition $[\delta\mathcal{H}, \sum_i \partial_i] = 0$. When considering the matrix elements of this condition,

$$\{\Psi_{\langle v|}|\left[\delta\mathcal{H}, \sum_i \partial_i\right]|\Psi_{\langle w|}\} = 0, \tag{63}$$

we mapped both $\sum_i \partial_i$ and $\delta\mathcal{H}$ into the CFT to find that

$$\{\Psi_{\langle v|}|\left[\delta\mathcal{H}, \sum_i \partial_i\right]|\Psi_{\langle w|}\} = \{\Psi_{\langle v|}|\Psi_{\langle w|[\mathcal{D},H]}\} = 0. \tag{64}$$

Therefore, we claimed that, for this to hold for any states $\langle v|$ and $\langle w|$, it must be that $\Psi_{\langle w|[\mathcal{D},H]} = 0$ which implies that $[\mathcal{D}, H] = 0$.

We now run through the Hermitian conjugate of this argument. First, the perturbation $\delta\mathcal{H}$ is Hermitian and therefore can act backwards on the bra-state

$$\{\Psi_{\langle v|}|\delta\mathcal{H}|\Psi_{\langle w|}\} = \{\Psi_{\langle v|H}\ |\ \Psi_{\langle w|}\}. \tag{65}$$

Therefore, if we can also find a mapping of $\sum_i \partial_i$ on the bra-state, i.e., some $\mathcal{K}$ such that

$$\{\Psi_{\langle v|}| \sum_i \partial_i |\Psi_{\langle w|}\} = \{\Psi_{\langle v|\mathcal{K}}|\Psi_{\langle w|}\}, \tag{66}$$

then we have another condition on H, namely, $[H, \mathcal{K}] = 0$.

To find $\mathcal{K}$ we need to find a way to convert the translation operator $\sum_i \partial_i$, which acts on holomorphic wave functions, into some operator which acts only on antiholomorphic wave functions, i.e., the bra-state $\bar{\Psi}_{\langle v|}$, which is a function of the $\bar{z}_i$ instead of the z_i. Recall then the procedure of projection to the lowest Landau level [Eq. (37)], the result of which is that $\bar{z}_i \equiv 2\ell_B^2 \partial_i$ within the lowest Landau level [that $\partial_i \equiv \bar{z}_i/2\ell_B^2$ can be seen by considering the matrix element of ∂_i in integral form and integrating by parts, taking care to remember the exponential factors in the integration measure, Eq. (14)]. Therefore, we find that

$$\{\Psi_{\langle v|}| \sum_i \partial_i |\Psi_{\langle w|}\} = \{\Psi_{\langle v|}| \sum_i \frac{\bar{z}_i}{2\ell_B^2} |\Psi_{\langle w|}\}. \tag{67}$$

This can be readily mapped to an operator acting on the bra-state's auxiliary state when we remember that the edge mode a_1 is simply the polynomial $\sqrt{\beta}\sum_i \bar{z}_i$. Therefore, $\mathcal{K} = a_1/2\ell_B^2\sqrt{\beta}$, thus confirming that $[H, a_1] = 0$ as claimed.

One may well worry about the apparent contradiction between the two constraints in Eqs. (55) and (62). If one takes a Hermitian conjugate of the former, using the fact that $(a_1)^\dagger = a_{-1}$, one does not reproduce the latter if one assumes that the effective Hamiltonian is Hermitian. This is because the effective Hamiltonian is not necessarily Hermitian with respect to the inner product of the CFT. As we have defined it, it is an operator which reproduces the action of a Hermitian perturbation on some auxiliary space. Therefore, it must have only real eigenvalues. In fact, in the case of quadratically increasing repulsive interactions which we are able to map exactly into the CFT (see Appendix C), the resulting Hamiltonian was indeed non-Hermitian with respect to this CFT inner product. The cause of this non-Hermiticity is that the inner product of two quantum Hall states labeled by different $\langle v|$ are not necessarily orthogonal. In fact,

$$\{\Psi_{\langle v|}|\Psi_{\langle w|}\} = \langle w|G_N|v\rangle, \tag{68}$$

where G_N is some operator close, but not equal, to the identity [23,24]. Therefore, while H need not be Hermitian in the CFT, it must be Hermitian with respect to the inner product of the model states (which then implies that HG_N be Hermitian with respect to the CFT inner product).

5. *Confinement vs interactions*

To summarize the results of the preceding section, we have found the mapping of various symmetries of a general perturbation $\delta\mathcal{H}$ into our CFT language. They are as follows:

(1) Number conservation implies that

$$[H, a_0] = 0. \tag{69}$$

(2) Rotational invariance implies that

$$[H, L_0] = 0. \tag{70}$$

(3) Translational invariance of $\delta\mathcal{H}$ implies that

$$[H, a_1] = 0, \tag{71}$$

$$\left[H, a_{-1} + \frac{1}{N\sqrt{\beta}} L_{-1}\right] = 0. \tag{72}$$

Note that these are general symmetries which apply to H *regardless of our conjecture of locality*.

Now, consider some generic perturbation $\delta\mathcal{H}$, which we split up into a confining part $\delta\mathcal{H}_U$, and an interaction term $\delta\mathcal{H}_V$. As discussed previously, these individual perturbations will have mappings into the CFT of the form H_U and H_V which simply add. In the rotationally symmetric cases we will consider, H_U will then satisfy the first two symmetries, of number conservation and rotational invariance. However, we will find that we can constrain the form of H_V significantly more given that it must also satisfy the third symmetry, corresponding to bulk translational invariance.

IV. RESULTS

A. Confinement

We begin by considering the effect of number conservation and rotational symmetry on the form of effective Hamiltonians. This is the situation which corresponds to the part of our field theory which describes the effects of confinement on the droplet. We recall that number conservation forces H to commute with a_0, thus forcing $\varphi(z)$ to appear in H only as a derivative and that rotational invariance equates to one-dimensional translations along the edge, removing the possibility for our coupling coefficients h_a to depend on position z.

Let us consider the effect of these symmetries for the Laughlin and Moore-Read wave functions. In the Laughlin case the statistics sector is trivial ($\chi = \mathbb{1}$) leaving only a theory made from the bosonic field. In general, the fields $\Phi_a(z)$ have the form

$$\Phi_a(z) = (i\partial\varphi(z))^{m_1}(i\partial^2\varphi(z))^{m_2}\ldots \tag{73}$$

for non-negative integers $m_1, m_2, \ldots$. Each term has a scaling dimension $d_a = m_1 + 2m_2 + \cdots$ and it is always assumed that they are normal ordered. Therefore, we can consider the first few most relevant terms (up to total derivatives) to be

$$H = \oint \frac{dz}{2\pi i}\left(\frac{v}{2} z(i\partial\varphi(z))^2 + g z^2 (i\partial\varphi(z))^3\right) + O(R^{-3}). \tag{74}$$

The first of these terms is the usual chiral linear Luttinger liquid term, and by itself would lead to a dispersion $E = v\Delta L$. The second term is then a scattering term which, for example, might take the $n = 2$ mode (recall Fig. 1) and scatter this into two $n = 1$ modes.

We may also consider the consequences of these simple symmetries on the generic effective Hamiltonian describing a Moore-Read edge. In this case, the statistics sector is a free fermion CFT with $\chi(z) = \psi(z)$. Therefore, our general fields have the form

$$\Phi_a(z) = (i\partial\varphi(z))^{m_1}\cdots\times(\psi(z))^{k_0}(\partial\psi(z))^{k_1}\ldots, \tag{75}$$

where the m_i are once again any non-negative integer but $k_i \in \{0, 1\}$ due to fermionic exclusion. In this case, the scaling dimension of a given term is $d_a = \sum_n n m_n + \sum_l (l + \frac{1}{2}) k_l$. As such, the first few most relevant terms will be

$$H = \oint \frac{dz}{2\pi i} \Big(-\frac{v_1}{2} z(\psi \partial \psi(z)) + \frac{v_2}{2} z(i\partial\varphi(z))^2 + g_1 z^2 (i\partial\varphi(z))^3 + g_2 z^2 (i\partial\varphi \psi \partial\psi(z)) \Big) + O(R^{-3}). \tag{76}$$

In this Hamiltonian, the first two terms are once again linear edge velocities which by themselves would simply give us the spectrum $E = v_1 \Delta L_\psi + v_2 \Delta L_\varphi$, giving the fermionic modes a velocity v_1 and the bosonic modes a velocity v_2. We then have two scattering terms, one exactly equivalent to the Laughlin scattering term, which scatters bosonic modes, and one coupling term which scatters a single bosonic mode into two fermions and vice versa.

B. Interacting Laughlin

We now consider the addition of two-dimensional translational symmetry to the effective Hamiltonian, which will allow us to describe the effects of interactions on the edge dynamics. We will find that this symmetry is very restrictive, removing the majority of those terms which were present in those effective Hamiltonians describing confinement. Given these restrictions, we will need to go to rather high order (large d_a) to find the major contributing terms. In doing so, we need to find a good basis of operators with which to work and then use those to construct linearly independent Hamiltonians H_a such that

$$H = \sum_a h_a H_a, \tag{77}$$

where the H_a are individual blocks which satisfy the translational symmetry commutation relations and $h_a(\delta\mathcal{H})$ are coefficients which scale as $h_a \sim R^{1-d_a}$ where d_a is the scaling dimension of the leading term (the term with the lowest scaling dimension) in H_a.

1. A basis of fields

To consider this problem generally, it is useful to generate a basis of terms which can arise within the field theory. We label these terms with partitions $\mathbf{\Gamma} = \{\gamma_1, \gamma_2, \ldots\}$ of integers $\gamma_i > 0$. These integers refer to the derivatives we have of the field $\varphi(z)$, as defined by terms

$$T_{\mathbf{\Gamma}} = \oint \frac{dz}{2\pi i} z^{|\mathbf{\Gamma}|-1} \prod_{\gamma_i \in \mathbf{\Gamma}} i\partial^{\gamma_i} \varphi(z), \tag{78}$$

where the factor of $z^{|\mathbf{\Gamma}|-1}$ for $|\mathbf{\Gamma}| = \sum_i \gamma_i$ is required by the rotational invariance condition. These terms have a scaling dimension $d_{\mathbf{\Gamma}} = |\mathbf{\Gamma}|$. In this way, we may generate any general contribution H_a as some linear combination of these $T_{\mathbf{\Gamma}}$ with $d_{\mathbf{\Gamma}} = d_a$.

Note that the order of γ_i in $\mathbf{\Gamma}$ is unimportant in the definition of $T_{\mathbf{\Gamma}}$ as the fields are bosonic and so we take the ordering $\gamma_1 \geqslant \gamma_2 \geqslant \cdots$. Note further that the basis is overcomplete. To see this, consider T_{21}, which we can integrate by parts to give

$$T_{21} = \oint \frac{dz}{2\pi i} z^2 \frac{1}{2} \partial((i\partial\varphi(z))^2) = -T_{11} = -2L_0. \tag{79}$$

Therefore, one must be careful to use a basis of $T_{\mathbf{\Gamma}}$ which includes only "unique" $\mathbf{\Gamma}$ (i.e., linearly independent $T_{\mathbf{\Gamma}}$). At low orders it is sufficient to find these unique $\mathbf{\Gamma}$ simply by the integration by parts procedure described above, though for higher orders it may be necessary to decompose the subsequent terms into bosonic modes and analyze whether the individual matrices are linearly independent.

It is worth noting an apparent oddity in Eq. (79). On the left-hand side we have T_{21} with scaling dimension 3 and on the right-hand side T_{11}, whose scaling dimension is 2. The source of this contradiction is really just notation. When we construct the Hamiltonian in Eq. (47) as $H = \int dx \sum_a h_a \Phi_a(x)$ we do not expect $(\partial\varphi)^2$ to appear at scaling dimension $d_a = 3$. We only expect $\partial^2\varphi\partial\varphi$ at this order [and also $(\partial\varphi)^3$ and $\partial^3\varphi$]. However, we are searching for a convenient basis to use and so it makes little sense to keep careful track of both T_{21} and T_{11} in the effective Hamiltonian if they are simply the same operator. Therefore, it is more convenient to simply throw away T_{21} and allow T_{11} to appear at both scaling dimensions 2 and 3. A similar reasoning applies at higher orders and this means that any $T_{\mathbf{\Gamma}}$ in the final basis can appear at scaling dimension $d_{\mathbf{\Gamma}}$ *and any scaling dimension higher*. This further implies that the coefficients will vary as $h_a \sim cR^{1-d_a} + c'R^{-d_a} + \cdots$ for some constants c, c', and so on.

Once we rid ourselves of these extraneous terms, we are left with a linearly independent basis up to scaling dimension 7 (or R^{-6}) given in Table I. Note that we do not consider the scaling dimension $d_{\mathbf{\Gamma}} = 1$ as the sole term here is $T_1 = a_0$ and we work with states $|\boldsymbol{\lambda}\rangle$ [recall the definition in Eq. (16)] which have U(1) charge of zero (i.e., $a_0|\boldsymbol{\lambda}\rangle = 0$ for all $|\boldsymbol{\lambda}\rangle$). Therefore, this term and others like it are trivial.

2. Constraining H

We will now restrict to interactions, considering $\delta\mathcal{H}$ to be translationally invariant, thus implying that each of our effective Hamiltonians H_a must satisfy the commutation

TABLE I. A table of some of the first few linearly independent terms from which H can be constructed. Note that the choices made for $\mathbf{\Gamma}$ are not unique. For example, as we have already seen, T_{11} and T_{21} are exactly equivalent up to an overall sign and so to keep notation simpler we replace T_{21} with T_{11}, although must then allow part of the coefficient of T_{11} to vary as $R^{1-d_{21}}$.

$d_{\mathbf{\Gamma}}$	Unique terms
2	T_{11}
3	T_{111}
4	T_{22}, T_{1111}
5	T_{221}, T_{11111}
6	$T_{33}, T_{222}, T_{2211}, T_{111111}$
7	$T_{331}, T_{2221}, T_{22111}, T_{1111111}$
⋮	⋮

relations (55) and (62), which we recall to have the form

$$[H_a, a_1] = 0, \tag{80}$$

$$\left[H_a, a_{-1} + \frac{1}{N\sqrt{\beta}} L_{-1}\right] = 0. \tag{81}$$

By inspecting Eq. (81) we note that we will need to expand H_a in powers of $\frac{1}{N\sqrt{\beta}}$ of the form

$$H_a = H_a^{(0)} + \frac{H_a^{(1)}}{N\sqrt{\beta}} + \frac{H_a^{(2)}}{(N\sqrt{\beta})^2} + \cdots. \tag{82}$$

Note here that the expansion in powers of $1/N$ can be compared to our overall expansion of the effective Hamiltonian in powers of $1/R$. To relate the two, we recall that the radius varies as the square root of the particle number $R = \ell_B\sqrt{2\beta N}$. Therefore, if $H_a^{(0)}$ is made up from terms of scaling dimension d_a this tells us that $H_a^{(1)}$ can be made only from terms with a scaling dimension $d_a + 2$ or smaller. In general, $H_a^{(n)}$ is a combination of terms with scaling dimension $d_a + 2n$ or smaller. Furthermore, recalling that h_a scales as R^{1-d_a} and given that $R \sim \sqrt{N}$ we note that the overall coefficient in front of any $H_a^{(n)}$ must vary as $\sqrt{N}^{1-d_a-2n}$.

Using this expansion, we find from Eqs. (80) and (81) that the leading-order terms in any H_a are those which satisfy the simple relations

$$\left[H_a^{(0)}, a_1\right] = \left[H_a^{(0)}, a_{-1}\right] = 0. \tag{83}$$

For example, of the terms listed in Table I those which satisfy both of these constraints are T_{22} and T_{33}. A summary of these leading terms up to a scaling dimension of 9 is provided in Table II. The majority of these are simple bilinear terms, which will predominantly modify the dispersion of the edge modes (see below), although we will also find at $d_a = 9$ that there is a three-body scattering term.

Nevertheless, the leading terms do not tell the whole story and we must generate the subleading terms via Eq. (81), which necessarily generates interesting nonlinear scattering terms. This ladder of subleading corrections is generated by the constraints

$$\left[H_a^{(n)}, a_1\right] = 0 \quad \forall\, n, \tag{84}$$

$$\left[H_a^{(n)}, a_{-1}\right] = \left[L_{-1}, H_a^{(n-1)}\right] \tag{85}$$

TABLE II. The first few leading contributions to the effective Hamiltonian for the Laughlin case which commute with both a_1 and a_{-1} and whose effects are suppressed by factors of R^{1-d_a}. We note that the first three are bilinears in the bosonic modes, which means that their primary effect is on the dispersion of edge modes. The final example at $d_a = 9$ is a trilinear, which corresponds to some three-body scattering of bosonic modes.

d_a	$H_a^{(0)}$
4	T_{22}
6	T_{33}
8	T_{44}
9	$T_{333} + T_{332} - \frac{1}{3}T_{222}$
⋮	⋮

with the former equation from Eq. (80) later arising from Eq. (81).

3. Leading contributions

Let us consider this in greater detail for the least irrelevant term with leading contribution T_{22}. To see that this satisfies the leading-order requirements for a translationally invariant effective Hamiltonian (83), consider that in terms of bosonic modes this term has the form

$$T_{22} = \oint \frac{dz}{2\pi i} z^3 (i\partial^2\varphi)^2 = -2\sum_{n>0}(n^2-1)a_{-n}a_n. \tag{86}$$

Therefore, it is clear that the $n = \pm 1$ modes are absent from this term. We then proceed to apply Eqs. (84) and (85) to produce the subleading terms $H_a^{(n)}$. We first consider Eq. (84) in the context of the individual fields:

$$[i\partial^n\varphi(z), a_1] = -\delta_{n,1}. \tag{87}$$

Therefore, in order to satisfy this condition, we must have that $n > 1$, and so we only consider $T_{\mathbf{\Gamma}}$ where $\mathbf{\Gamma}$ contains integers greater than or equal to 2 such as T_{222}, T_{33}, T_{2222}, and so on.

We then consider Eq. (85), recalling that L_{-1} acts to differentiate the fields of the CFT [28], and so this constraint becomes

$$\left[H_{22}^{(1)}, a_{-1}\right] = \oint \frac{dz}{2\pi i} z^3 \partial((i\partial^2\varphi)^2). \tag{88}$$

We must now consider which terms might appear in $H_{22}^{(1)}$. Recall that by scaling arguments, the pool of terms which might appear is restricted to those with a scaling dimension $d_{22} + 2 = 6$ or smaller. This therefore includes T_{22}, T_{33}, or T_{222}. *A priori*, any of these might appear in $H_{22}^{(1)}$ but, given that T_{22} and T_{33} commute with a_{-1} their coefficients are unconstrained by this equation. Initially, this appears worrisome as it suggests that

$$H_{22}^{(1)} = \alpha T_{22} + \beta T_{33} + \gamma T_{222}, \tag{89}$$

where we are unable to say anything about α and β. However, in practice this is not a problem as T_{22} and T_{33} are also permissible $H_a^{(0)}$ terms, exactly because they commute with a_{-1}. Therefore, whatever the values of α and β, these coefficients can be combined with h_{22} and h_{33}, thus yielding no new coefficients. This argument is indicative of a more general principle. When considering what terms might appear in any $H_a^{(n)}$ where $n > 0$, one should first remove any terms which commute with a_{-1} and therefore produce their own $H_{a'}^{(0)}$ to avoid adding unnecessary extra coefficients.

Therefore, returning to our example, we note that $H_{22}^{(1)}$ should be made up of only T_{222}. To find how much of this term is present, consider once again the commutation of the mode with the field, which in this case is

$$[i\partial^n\varphi(z), a_{-1}] = \partial^{n-1}(z^{-2}). \tag{90}$$

Therefore, if we state that $H_{22}^{(1)} = \alpha T_{222}$, then we find by evaluating Eq. (88) that

$$\alpha = \tfrac{1}{2}. \tag{91}$$

Thus,

$$H_{22} = T_{22} + \frac{1}{2N\sqrt{\beta}} T_{222} + \cdots . \tag{92}$$

In fact, this procedure can be continued easily to all orders, with the result that

$$\boxed{H_{22} = T_{22} + \sum_{n=1}^{\infty} \frac{8}{(N\sqrt{\beta})^n} \frac{(2n+1)!!}{(2n+4)!!} T_{2^n},} \tag{93}$$

where 2^n refers to the partition containing n copies of 2.

We can also consider this picture for the H_a whose leading contribution is T_{33}. For this root, the first subleading correction as calculated using Eq. (85) has the form

$$H_{33} = T_{33} + \frac{5}{2N\sqrt{\beta}} T_{332} - \frac{15}{2N\sqrt{\beta}} T_{222} + \cdots . \tag{94}$$

In general, this contribution is made from the terms $T_{332^{n-2}}$ and T_{2^n} of the form

$$\boxed{H_{33} = T_{33} + \sum_{n=1}^{\infty} \left(\frac{\alpha_n}{(N\sqrt{\beta})^n} T_{332^n} + \frac{\beta_n}{(N\sqrt{\beta})^n} T_{2^{n+2}} \right),} \tag{95}$$

where the coefficients α_n and β_n satisfy the recursive relation

$$\alpha_n = \frac{2n+3}{2n} \alpha_n, \tag{96}$$

$$\beta_n = \frac{2n+1}{2n+4} \beta_{n-1} - \frac{6(2n+1)}{(n+1)(n+2)} \alpha_n, \tag{97}$$

for all $n > 0$ and where $(\alpha_0, \beta_0) = (1, 0)$. A similar procedure can be repeated to fix all the H_a, leaving only h_a as free parameters which depend on the bulk interactions and cannot be fixed by symmetry only.

C. Interacting Moore-Read state

Effective Hamiltonians for the Moore-Read state follow an extremely similar pattern to that we have seen for the Laughlin state. Once again, the imposition of bulk translational symmetry on the effective Hamiltonian prompts us to search for individual, independent contributions $H_a = H_a^{(0)} + \frac{1}{N\sqrt{\beta}} H_a^{(1)} + \cdots$, which satisfy the commutation relations in Eqs. (84) and (85). The sole difference is that the individual terms $H_a^{(n)}$ for the expansions in $\frac{1}{N\sqrt{\beta}}$ are some terms expressed in terms of both the bosonic field $\varphi(z)$ and the fermionic field $\psi(z)$.

1. A basis of fields

We proceed exactly as before, beginning with a definition of the basis we can use to express any local Hamiltonian constructed from these two fields $\varphi(z)$ and $\psi(z)$, which are relevant to the problem. These will now be labeled by a pair of partitions $\mathbf{\Gamma} = \{\gamma_1, \gamma_2, \ldots\}$ and also $\mathbf{\Xi} = \{\xi_1, \xi_2, \ldots\}$, one for the bosonic sector and one for the fermionic sector:

$$T_{\mathbf{\Gamma},\mathbf{\Xi}} = \oint \frac{dz}{2\pi i} z^{|\mathbf{\Gamma}|+|\mathbf{\Xi}|+\frac{l(\mathbf{\Xi})}{2}-1} \prod_{\gamma_i \in \mathbf{\Gamma}} i\partial^{\gamma_i} \varphi(z) \prod_{\xi_i \in \mathbf{\Xi}} \partial^{\xi_i} \psi(z). \tag{98}$$

There are a few things to note about this definition. First, $\mathbf{\Gamma}$, relating to the bosonic fields, is as before; it is a partition of positive integers which we order such that $\gamma_1 \geqslant \gamma_2 \geqslant \cdots$. The new, fermionic partition $\mathbf{\Xi}$ is also a set of integers although it cannot contain any integer twice (due to fermionic exclusion) and it must have an even number of elements (due to parity symmetry). We denote the number of elements by $l(\mathbf{\Xi})$ so, for example, $l(\{0, 1\}) = 2$ and $l(\{0, 1, 2, 3\}) = 4$. Furthermore, it can also contain 0 as an entry ($i\partial^0\varphi$ is excluded due to number conservation though no such symmetry prevents $\partial^0\psi$). To further distinguish this partition from $\mathbf{\Gamma}$, we will also order it in reverse, with $\xi_1 < \xi_2 < \cdots$. Finally, the product over the elements ξ_i must have a specific ordering, as permutations may bring about an overall sign change, so we define this product such that

$$\prod_{\xi_i \in \mathbf{\Xi}} \partial^{\xi_i} \psi = \partial^{\xi_1} \psi \partial^{\xi_2} \psi \ldots . \tag{99}$$

Note that in both the bosonic and fermionic cases the empty set, $\emptyset$, refers to no contributions from that sector. So, for example, $T_{\mathbf{\Gamma},\emptyset}$ are all the purely bosonic terms which we found when considering the Laughlin state and include no fermionic fields.

Once again, there is a large degeneracy in this definition of basis elements. For example,

$$T_{\emptyset,02} = \oint \frac{dz}{2\pi i} z^2 \psi \partial^2 \psi = -\oint \frac{dz}{2\pi i} \partial(z^2\psi)\partial\psi = -2T_{\emptyset,01}. \tag{100}$$

Therefore, we must once again weed out these duplicate entries, and so we provide a particular choice of a linearly independent basis of terms in Table III for the first few scaling dimensions, where the scaling dimension of a given term is $d_{\mathbf{\Gamma},\mathbf{\Xi}} = |\mathbf{\Gamma}| + |\mathbf{\Xi}| + \frac{l(\mathbf{\Xi})}{2}$.

2. Leading contributions

We will now restrict the wide array of possible terms in Table III to those which satisfy the symmetries of our Hamiltonian. Recall that to first order this entails searching for $H_a^{(0)}$ which commute with both a_1 and a_{-1}. In the Laughlin case, this was extremely restrictive. However, in this Moore-Read case the constraint is somewhat less powerful due to the fact that the center-of-mass mode commutes with terms of purely fermionic nature. As such, there are more contributions than previously, as provided in Table IV.

TABLE III. A table of some of the first few linearly independent terms from which H can be constructed in the Moore-Read case. The number of possibilities grows much quicker with the scaling dimension than it did in the Laughlin case.

$d_{\mathbf{\Gamma},\mathbf{\Xi}}$	Unique terms
2	$T_{11,\emptyset}$, $T_{\emptyset,01}$
3	$T_{111,\emptyset}$, $T_{1,01}$
4	$T_{22,\emptyset}$, $T_{1111,\emptyset}$, $T_{2,01}$, $T_{11,01}$, $T_{\emptyset,12}$
5	$T_{221,\emptyset}$, $T_{11111,\emptyset}$, $T_{3,01}$, $T_{21,01}$, $T_{111,01}$, $T_{1,12}$
⋮	⋮

TABLE IV. The leading contributions to the effective Hamiltonian in the Moore-Read case. These contributions occur at order R^{1-d_a} and are mostly diagonal in the basis defined by Eq. (26), and so primarily affect the dispersion. However, there is a scattering term at order R^{-4} which couples the fermionic edge states with the bosonic modes.

d_a	$H_a^{(0)}$
2	$T_{\emptyset,01}$
3	
4	$T_{22,\emptyset}$ $T_{\emptyset,12}$
5	$T_{3,01} + 3T_{2,01}$
6	$T_{33,\emptyset}$ $T_{\emptyset,23}$
⋮	⋮

Finding all the subleading contributions is then a case of applying the commutation relation in Eq. (85) to generate the ladder of terms $H_a^{(1)}$, $H_a^{(2)}$, and so on. As before, the situation is relatively simple, with L_{-1} acting as a derivative on the fields within each $T_{\Gamma,\Xi}$ on the right and a_{-1} seeing only bosonic fields on the left. Applying this procedure, one then finds that the three leading terms for Moore-Read Hamiltonians have the form

$$H_{\emptyset,01} = T_{\emptyset,01} + \sum_{n=1}^{\infty} \frac{1}{(N\sqrt{\beta})^n} \frac{(2n-1)!!}{(2n)!!} T_{2^n,01}, \tag{101}$$

$$H_{22,\emptyset} = T_{22,\emptyset} + \sum_{n=1}^{\infty} \frac{8}{(N\sqrt{\beta})^n} \frac{(2n+1)!!}{(2n+4)!!} T_{2^{n+2},\emptyset}, \tag{102}$$

$$H_{\emptyset,12} = T_{\emptyset,12} + \sum_{n=1}^{\infty} \frac{1}{(N\sqrt{\beta})^n} \frac{(2n+1)!!}{(2n)!!} T_{2^n,12}. \tag{103}$$

Therefore, each of the terms which, to leading order, simply change the dispersion of the fermionic modes (i.e., those of the form $H_{\emptyset,\Xi}$) necessarily have subleading contributions which couple the bosonic and fermionic excitations, suppressed by a factor of $\frac{1}{N\sqrt{\beta}}$.

V. NUMERICS

We once again stress that the claim of locality which was so instrumental in calculating these allowed contributions to our Hamiltonian was a conjecture based primarily on the understanding of the Laughlin state as a plasma in its screening phase (for $\frac{1}{\nu} \lesssim 65$). As such, we present thorough numerical evidence to further motivate this claim by comparing the results of these local effective Hamiltonians, which we hope will provide a very accurate description of the low-energy physics, with exact numerical results [45].

To assess our effective Hamiltonians, we will take H to be simply a sum of the first few, least irrelevant terms from the field theoretic considerations we have made, including their coupling coefficients h_a, which we cannot fix by symmetry alone. These Hamiltonians are then simple to diagonalize, being phrased in terms of second quantized operators. We then perform the exact numerics by generating the full edge states in a basis of single-particle states, the monomial basis, by using their expression in terms of Jack polynomials [46–49]. We then exactly diagonalize the interaction within reduced subspaces of edge states at fixed angular momentum to find the eigenstates and eigenvalues, and compare to the effective Hamiltonians $H(h_a)$ by fitting the coupling coefficients to the data.

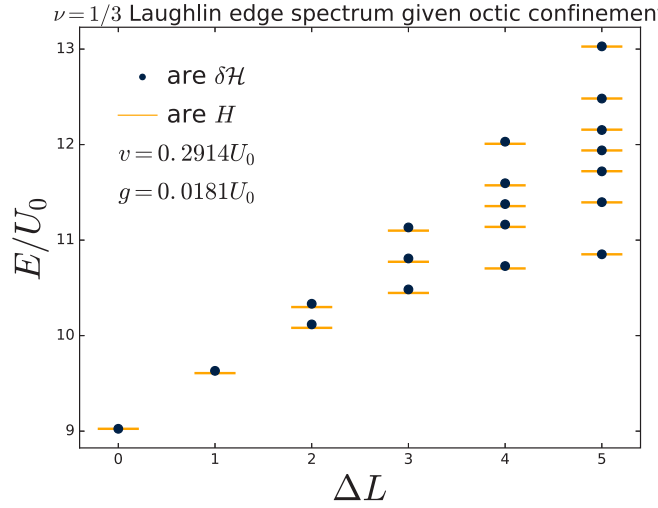

FIG. 3. A comparison of our effective Hamiltonian with fit parameters g and v with the exact spectrum for an $N = 10$ Laughlin state at $\nu = \frac{1}{3}$ confined purely by a weak octic confinement $U = U_0 \sum_i (r_i/R)^8$ (i.e., there is no additional quadratic confinement).

A. Confinement

We will initially consider the simpler case of confinement where the perturbation is simply a single-body term in the Hamiltonian. Recall then that we cannot use translational invariance to significantly constrain the final form of the effective Hamiltonian, so we simply take the first few least irrelevant contributions

$$H = \oint \frac{dz}{2\pi i} \left(\frac{v}{2} z (i\partial\varphi(z))^2 + g z^2 (i\partial\varphi(z))^3 \right) + O(R^{-3}). \tag{104}$$

We will use this to consider a relatively steep edge confinement of $U = U_0(r/R)^8$ and fit the coefficients v and g by simply matching the exact and effective spectra. We find the results in Fig. 3 for the $\nu = \frac{1}{3}$ Laughlin state containing $N = 10$ particles, and find a very good match.

We also consider the Moore-Read case. Once again, without translational invariance the resulting effective Hamiltonian cannot be significantly simplified, which leaves us with the following least irrelevant terms:

$$H = \oint \frac{dz}{2\pi i} \left(-\frac{v_1}{2} z(\psi\partial\psi(z)) + \frac{v_2}{2} z(i\partial\varphi(z))^2 + g_1 z^2 (i\partial\varphi(z))^3 + g_2 z^2 (i\partial\varphi\psi\partial\psi(z)) \right) + O(R^{-3}). \tag{105}$$

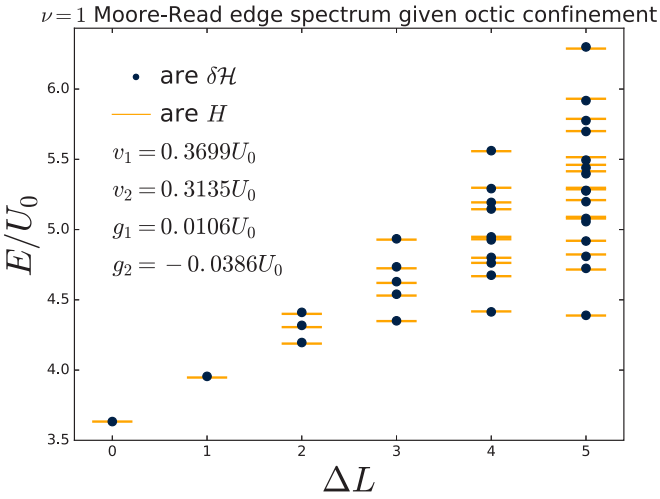

FIG. 4. A comparison of our effective Hamiltonian with fit parameters v_1, v_2, g_1, and g_2 with the exact spectrum for a $\nu = 1$ Moore-Read state confined by a weak octic confinement $U = U_0 \sum_i (r_i/R)^8$ when $N = 14$. Once again, the quadratic confinement was set to zero for these results.

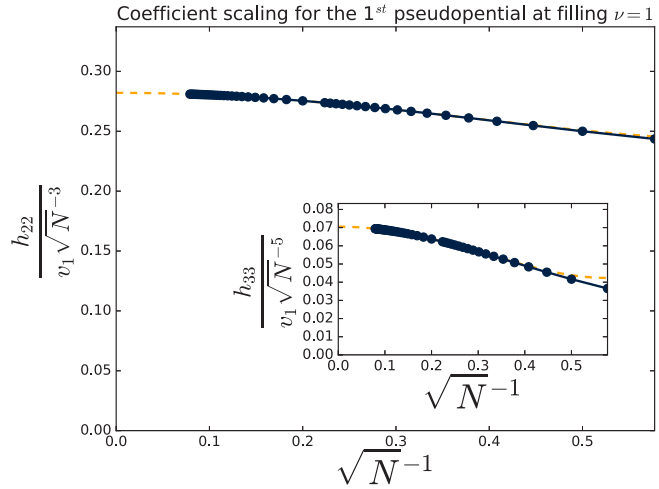

FIG. 5. Perturbing the integer quantum Hall effect at $\nu = 1$ with a first pseudopotential V_1, we find the above scaling of the coefficients h_{22} and h_{33} in the effective Hamiltonian. In both cases, the coefficients appear to obey the scaling hypothesis well, varying as $h_a \sim \sqrt{N}^{1-d_a}$ where $d_{22} = 4$ and $d_{33} = 6$.

Once again, we check this against exact numerical results for a Moore-Read droplet confined by radial potential of the form $U_0(r/R)^8$. The results are given in Fig. 4 and show good agreement once again.

B. Interacting Laughlin state

For the Laughlin state perturbed by nontrivial interactions we shall employ a simple two-parameter effective Hamiltonian, including only the two least irrelevant contributions:

$$H = h_{22}H_{22} + h_{33}H_{33}. \tag{106}$$

This should replicate the matrix elements of the Hamiltonian as calculated numerically up to corrections of order R^{-7}.

We then note that, given this form for the effective Hamiltonian, certain matrix elements will constrain the coefficients h_{22} and h_{33} exactly. Specifically, consider the element $\langle 2|H|2\rangle$ where $|2\rangle = a_{-2}|0\rangle$ as defined in Eq. (16). The only nonzero contribution to this comes from T_{22} contained in H_{22}. Not only is this true for the truncated expansion, but we expect it to hold true at every order. Furthermore, the only local terms which we expect can contribute to the matrix element $\langle 3|H|3\rangle$, even in an infinite-order expansion, are T_{22} and T_{33}. Therefore, these two matrix elements should determine h_{22} and h_{33} exactly, and these are what we use to fit these coefficients.

Note that to find the effective Hamiltonian H from the data requires one to also calculate the overlaps of the wave functions. To see this, recall that the states in the quantum Hall language are not orthogonal, i.e.,

$$\{\Psi_{\langle v|}|\Psi_{\langle w|}\} = \langle w|G_N|v\rangle. \tag{107}$$

The quadratic form G_N was discussed in length in Refs. [23,24]. Furthermore,

$$\{\Psi_{\langle v|}|\delta\mathcal{H}|\Psi_{\langle w|}\} = \langle w|HG_N|v\rangle. \tag{108}$$

Therefore, we can calculate the matrices G_N and (HG_N), and hence we find the effective Hamiltonian by

$$H = (HG_N)G_N^{-1}. \tag{109}$$

1. Coefficient scaling

Upon fitting the coefficients, it is important to check that the scaling arguments made previously hold. These claimed that the coefficients h_a should scale as R^{1-d_a}. Then, given that the radius scales as the square root of particle number N means

$$h_a \sim \sqrt{N}^{1-d_a}. \tag{110}$$

To check that this scaling is borne out in the data, we fit the parameters for a variety of system sizes for fillings $\nu = 1$ and $\frac{1}{2}$ and plot the results in Figs. 5 and 6. The exact results will depend upon the interaction and filling, and in these examples we take the interaction to be the first Haldane pseudopotential for $\nu = 1$ and the second Haldane pseudopotential at $\nu = \frac{1}{2}$. We see that the results appear to be consistent with the scaling hypothesis, with h_{22} varying in the large-N limit as $h_{22} \sim \sqrt{N}^{-3}$, exactly as expected. h_{33} is less clear. For the integer case it also appears to vary as expected, $h_{33} \sim \sqrt{N}^{-5}$, but at $\nu = \frac{1}{2}$ it appears that subleading corrections to this coefficient are large enough that the value does not converge for the range of N we can reach with exact methods. Nevertheless, it does not appear to fall off slower than $\sqrt{N}^{-5}$, so this also appears consistent with scaling arguments.

2. Effective Hamiltonian spectra

Perhaps the most crucial check that our effective Hamiltonians describe the true behavior of quantum Hall edges is that they faithfully reproduce the spectrum of edge states. Therefore, we take the Hamiltonian in Eq. (106) and fit values for h_{22} and h_{33} based on the method described above. We then plot the agreement for the case of exponential repulsion

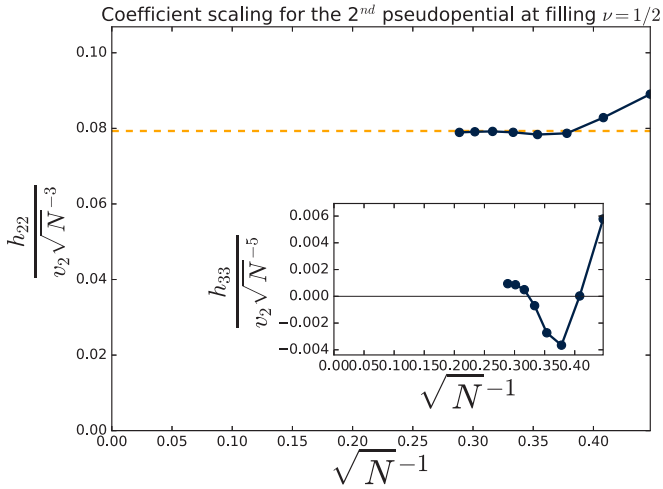

FIG. 6. We perturb the Laughlin wave function at $\nu = \frac{1}{2}$ by the second pseudopotential. Plotted here is the value we fit for h_{22} (which we expect to scale as $v_2\sqrt{N}^{-3}$) in the effective Hamiltonian as N is varied, which appears to converge very quickly to some constant$/\sqrt{N}^{3}$. Unfortunately, h_{33}, which is expected to vary as $\sqrt{N}^{-5}$, does not converge for this range of system sizes but does appear to decay at least as quickly as $\sqrt{N}^{-5}$, if not faster, as required by the scaling hypothesis.

between particles,

$$V = w_0 \sum_{i<j} \exp\left(-\left|\frac{z_i - z_j}{2\ell_B^2}\right|^2\right), \tag{111}$$

at a variety of filling fractions in Figs. 7–9. In each case, we find that this two-parameter fit is very good.

Note that in these plots the linear slope is a free parameter of the parent Hamiltonian derived from the confinement, which we always take as quadratic. Therefore, in each plot we add an arbitrary harmonic potential (specified in each figure) which produces the accompanying linear slope. The repulsive interactions then cause the energies to decrease with respect to this linear edge as angular momentum is increased. This is because these excited states allow the particles to avoid each other more successfully, increasing their average separations. The states which then lie exactly on the Luttinger liquid line are those edge states which correspond simply to translations of the original circular droplet as a whole, i.e., the states $a_{-1}^{\Delta L}|0\rangle$.

3. Constraints on nonlocal terms

Throughout the preceding text we have assumed that the only terms which contribute to H are local. However, this is a conjecture based on the Laughlin state being in a screening phase for $\beta \lesssim 65$, which makes correlations short range. However, for sufficiently long-range interactions we expect nonlocal terms to also contribute. One such example is the harmonic interaction, for which we can find the effective Hamiltonian exactly as presented in Appendix C and which contains nonlocal contributions.

However, this is a special case of an interaction which actually grows with separation. In a more general setting, one

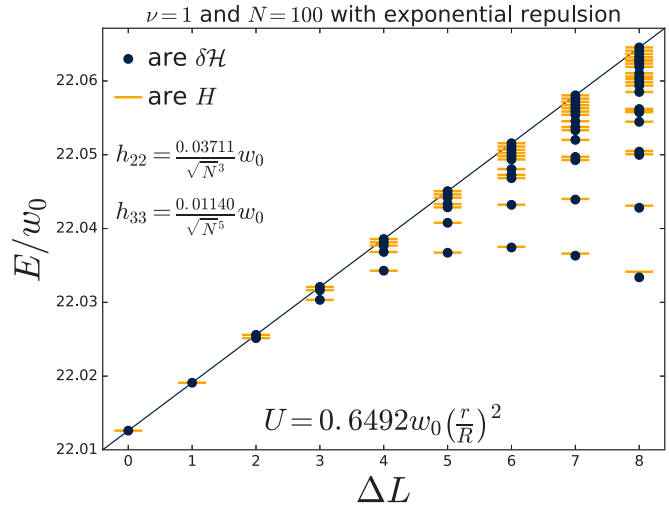

FIG. 7. A comparison of our two-parameter effective Hamiltonian with the exact edge spectrum for a $\nu = 1$ quantum Hall state perturbed by exponential repulsive interactions in Eq. (111) in the limit where w_0 is small. Recall that, because we expect the only terms in H which contribute to the matrix elements we use to fit h_{22} and h_{33} are H_{22} and H_{33}, we expect that the fits to these coefficients are exact for these systems at these particle numbers. We find that the subsequent agreement is extremely good, capturing the distinct characteristics of the spectrum and matching most points extremely closely.

of the most well-known nonlocal terms we might expect to contribute is the Benjamin-Ono term [21,22,50], which has the form

$$T_{\mathrm{BO}} = \oint \frac{dz}{2\pi i} \oint_{|z|>|w|} \frac{dw}{2\pi i} \frac{zw}{(z-w)^2} : i\partial\varphi(z) i\partial\varphi(w) :$$
$$= \sum_{n>0} n a_{-n} a_n \tag{112}$$

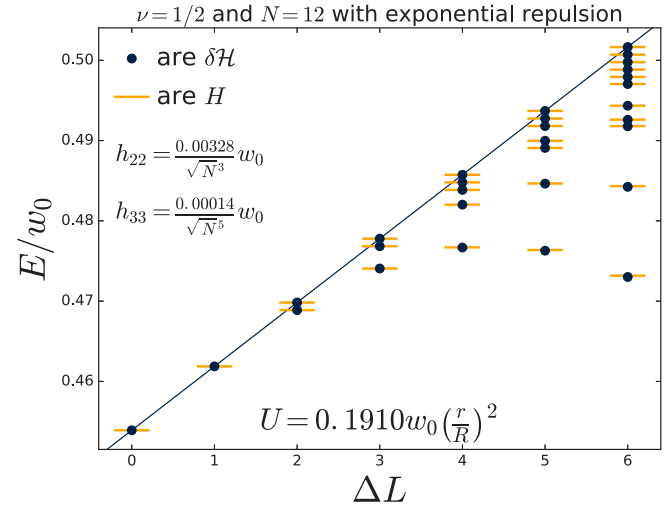

FIG. 8. A comparison of numerical results for a $\nu = \frac{1}{2}$ Laughlin state perturbed by exponential interactions with our two-parameter effective Hamiltonians. As with the integer quantum Hall case, the agreement is excellent.

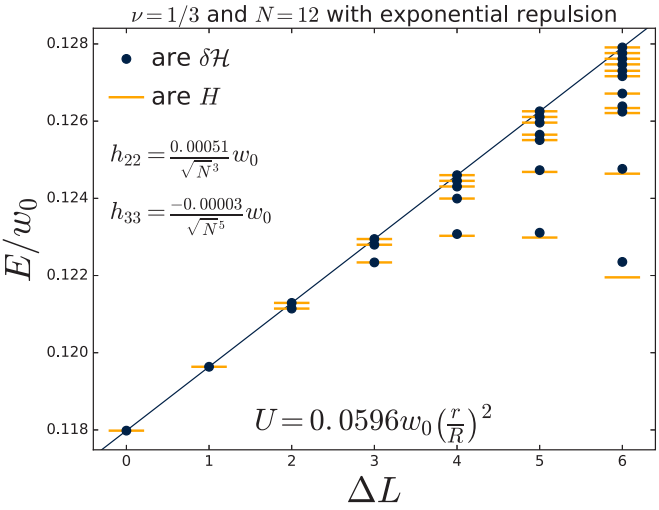

FIG. 9. A final comparison of numerical results with our effective description for the $\nu = \frac{1}{3}$ Laughlin state, which once again shows excellent agreement.

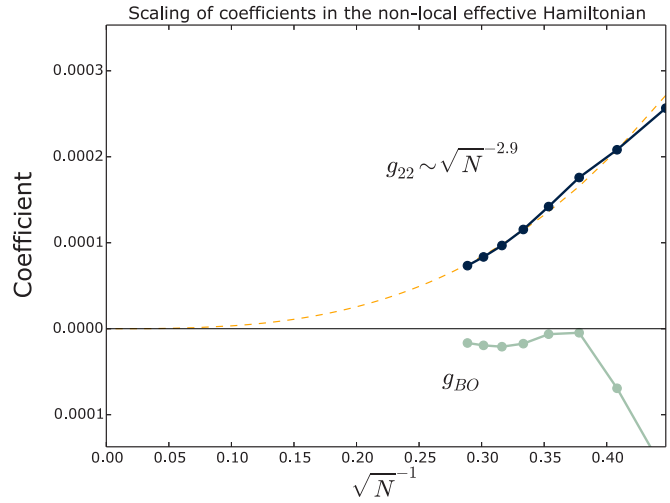

FIG. 10. We assume for the moment that the Hamiltonian might contain nonlocal terms and fit them taking the form in Eq. (113) for H and taking exponential interactions between the particles at filling $\nu = \frac{1}{2}$. We see that the Benjamin-Ono coefficient is consistently smaller than g_{22} despite scaling arguments suggesting that it should be $\sim\sqrt{N}$ times larger. As such, any contribution this term makes to the dynamics is much smaller than expected. Clearly, this does not prevent any nonlocal term being present at any order in the expansion, but it does provide evidence against this most likely contribution.

and has the lowest scaling dimension possible for such a double-integral term. Therefore, to ascertain the likelihood that a nonlocal term might appear, we insert it into the Hamiltonian and attempt to fit its coefficient. We will then analyze the scaling of the resulting coefficient.

Therefore, we take the ultrasimple Hamiltonian

$$H = g_{\mathrm{BO}}\left(T_{\mathrm{BO}} - \tfrac{1}{2}T_{11}\right) + g_{22}T_{22} + \cdots \tag{113}$$

and fit these two coefficients. Note that the inclusion of the T_{11} here ensures that the overall Benjamin-Ono term which we have inserted here obeys the leading condition of translational symmetry, i.e.,

$$\left[a_1, H_a^{(0)}\right] = \left[a_{-1}, H_a^{(0)}\right] = 0. \tag{114}$$

The terms that we throw away in Eq. (113) are then either off diagonal (terms which are generated by this Benjamin-Ono term to satisfy translational invariance at all orders) or of order $N^{-5/2}$. We then fit the coefficients g_{BO} and g_{22} with the first two nontrivial diagonal elements $\langle 2|H|2\rangle$ and $\langle 3|H|3\rangle$.

The fits for the coefficients are plotted in Fig. 10 for the case of an exponential interaction at $\nu = \frac{1}{2}$. We see that the scaling of g_{22} is very close to the expected $\sqrt{N}^{-3}$, whereas the Benjamin-Ono coefficient does not scale in a manner which can even be approximated by a power law. Nevertheless, we see that it is always much smaller than g_{22} despite the fact that it is expected to be roughly $\sqrt{N}$ times larger. Therefore, it is unlikely that the Benjamin-Ono term contributes to our effective Hamiltonian, or at least its effects are much smaller than expected by simple scaling arguments.

4. Couplings for pseudopotentials

We now look at what the values for the couplings h_{22} and h_{33} look like for the first few Haldane pseudopotentials. In theory, each pseudopotential has associated with it an effective Hamiltonian,

$$V_k \Psi_{\langle \lambda|} = \Psi_{\langle \lambda|H_k}, \tag{115}$$

and, therefore, given that any interaction can be formed from a sum of pseudopotentials, i.e., $V = \sum_k v_k V_k$, the effective Hamiltonian of such an interaction is simply

$$H = \sum_k v_k H_k. \tag{116}$$

Recall that the parent Hamiltonian at $\nu = 1/m$ is constructed from all the pseudopotentials V_k with $k < m$ and so these annihilate all our wave functions at these filling fractions. Therefore, knowledge of the coupling coefficients within the H_k of Eq. (116) for $k \geqslant m$ is all we require to be able to build the effective Hamiltonian for any interaction.

Unfortunately, as we have already seen in Fig. 6, for example, fitting some of these coefficients can be difficult or simply unreliable for the system sizes we are able to compute exactly. Therefore, we fit only the leading-order contribution h_{22} for each and give the values in Table V for the pseudopotentials contributing to the interactions at $\nu = \frac{1}{2}$.

TABLE V. The leading coupling coefficient for the effective Hamiltonian of various pseudopotentials at fractional fillings $\nu = \frac{1}{2}$. For the case at half-filling the convergence to a constant appears robust and the errors are very small. Similar extrapolations could not be made reliably for the case at $\nu = \frac{1}{3}$ given the inferior convergence in this case.

Pseudopotential	$h_{22}^{\nu=1/2}$
V_2	0.079 ± 0.007
V_4	0.074 ± 0.020
V_6	0.087 ± 0.036

TABLE VI. The coupling coefficients for the first few contributing pseudopotentials at filling $\nu = 1$. We note that the higher-order coefficients are larger relative to the lower-order coefficients for the higher pseudopotentials (which are the pseudopotentials that are more important for less local interactions), i.e., $|h_{22}^{(3)}/h_{22}^{(5)}|$ is smaller for V_k where k is larger. Furthermore, we note that the coefficient of h_{22} at order $\sqrt{N}^{-4}$ is very small relative to the coefficients at fractional powers of N. This is in good agreement with our calculation for the asymptotic behavior of the exponential potential shown in a future publication [44], which predicts vanishing coefficients at even order (i.e., at order $\sqrt{N}^{-2n}$).

Pseudopotential	$h_{22}^{(3)}$	$h_{22}^{(4)}$	$h_{22}^{(5)}$	$h_{33}^{(5)}$
V_1	0.282 ± 0.001	0.000 ± 0.002	-0.179 ± 0.016	0.071 ± 0.001
V_3	0.423 ± 0.001	0.001 ± 0.017	-0.633 ± 0.187	0.247 ± 0.001
V_5	0.529 ± 0.001	0.004 ± 0.014	-1.268 ± 0.156	0.485 ± 0.001
V_7	0.617 ± 0.001	0.010 ± 0.010	-2.088 ± 0.135	0.771 ± 0.005
V_9	0.694 ± 0.006	0.014 ± 0.278	-2.982 ± 5.472	1.103 ± 0.189

However, while our data for fractional fillings remain too small to form a reliable conclusion about the subleading corrections to the effective Hamiltonian, we can perform this analysis extremely reliably for $\nu = 1$, where we consider N as high as 160. Thus, we fit the effective Hamiltonian to fifth order, expanding the leading coupling coefficient as

$$h_{22} = \frac{h_{22}^{(3)}}{\sqrt{N}^3} + \frac{h_{22}^{(4)}}{\sqrt{N}^4} + \frac{h_{22}^{(5)}}{\sqrt{N}^5} + O(\sqrt{N}^{-6}) \tag{117}$$

and considering only the leading order for $h_{33} = h_{33}^{(5)}/\sqrt{N}^5 + O(\sqrt{N}^{-6})$. Each of these coefficients is shown in Table VI.

C. Interacting Moore-Read state

The spectra for the Moore-Read state are more complex than those for the Laughlin state. For example, consider the subspace of states with $\Delta L = 5$ units of angular momentum added with respect to the ground state. In the Laughlin case, this is a seven-dimensional subspace but in the Moore-Read case there are 16 states. As such, the matrix $\delta\mathcal{H}$, which our effective Hamiltonian H must reproduce, includes over four times as many matrix elements. Nevertheless, as we shall see, an effective Hamiltonian containing only three terms (and thus three fit parameters) can still provide an extremely good description of the resulting behavior. Thus, in the resulting discussion we will take the effective Hamiltonian to include the three least irrelevant terms

$$H = h_{\emptyset,01} H_{\emptyset,01} + h_{22,\emptyset} H_{22,\emptyset} + h_{\emptyset,12} H_{\emptyset,12}. \tag{118}$$

Note that to leading order, each term of this Hamiltonian is purely fermionic or bosonic, and hence the modes appear to decouple. However, one should recall that to first order each of these fermionic terms [as given in Eqs. (101) and (103)] will couple the bosonic and fermionic edge channels.

As in the Laughlin case, we will fit the coupling coefficients in Eq. (118) by comparison with particular matrix elements. Specifically, $\langle 2;\emptyset|H|2;\emptyset\rangle$ will fit the coefficient $h_{22,\emptyset}$. Additionally, we use $\langle\emptyset;\frac{3}{2}\frac{1}{2}|H|\emptyset;\frac{3}{2}\frac{1}{2}\rangle$ and $\langle\emptyset;\frac{5}{2}\frac{1}{2}|H|\emptyset;\frac{5}{2}\frac{1}{2}\rangle$ together to fit the coefficients $h_{\emptyset,01}$ and $h_{\emptyset,12}$. We present a scaling analysis of the resulting fit coefficients for the bosonic $\nu = 1$ Moore-Read state perturbed by exponential interactions in Fig. 11. As these plots show, the scaling hypothesis appears to work well even for these relatively small system sizes. The convergence for the least irrelevant contribution $h_{\emptyset,01}$ is very good and is shown to vary as $\sqrt{N}^{-1}$ as expected by scaling arguments. Less clear are the forms of scaling for $h_{22,\emptyset}$ and $h_{\emptyset,12}$ which do not converge so convincingly over the system sizes we are able to access though still appear to fall off no faster than the $\sqrt{N}^{-3}$ required.

1. Effective Hamiltonian spectra

Once again, the most crucial check of our effective theory is that they are able to reproduce the spectra of the corresponding systems. As such, we calculate the spectrum numerically for Moore-Read states containing $N = 16$ particles at filling $\nu = 1$. The data when the interactions we perturb with are exponentially repulsive [i.e., the same as in Eq. (111)] are shown in Fig. 12 alongside a comparison to the effective Hamiltonian

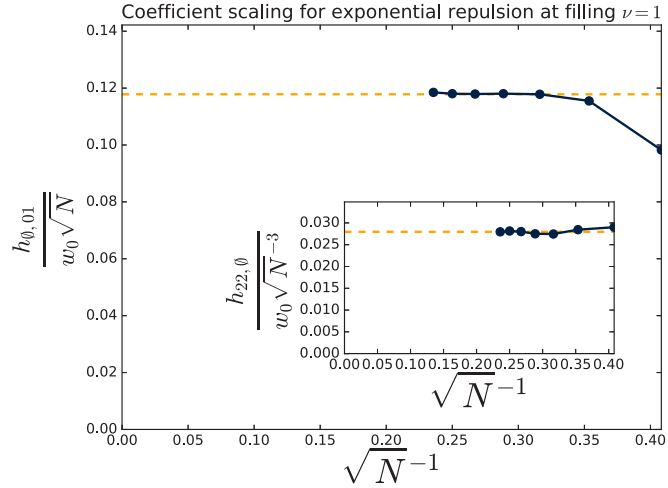

FIG. 11. We consider the Moore-Read state at $\nu = 1$ perturbed by an exponential repulsive interaction [Eq. (111)] and fit the coupling coefficients $h_{\emptyset,01}$, $h_{22,\emptyset}$, and $h_{\emptyset,12}$ at a variety of system sizes N. We then plot the variation of $h_{\emptyset,01}$ and $h_{22,\emptyset}$ with N, which are expected to vary as $\sqrt{N}^{-1}$ and $\sqrt{N}^{-3}$, respectively, based on scaling arguments. The convergence to this scaling behavior is quite good for both $h_{\emptyset,01}$ and $h_{22,\emptyset}$. However, the convergence is unclear for the coefficient $h_{\emptyset,12}$, whose scaling behavior we do not show here, as we cannot reach large enough N with these exact methods for the value to converge. Nevertheless, the scaling hypothesis does not appear to be incompatible with any of these data.

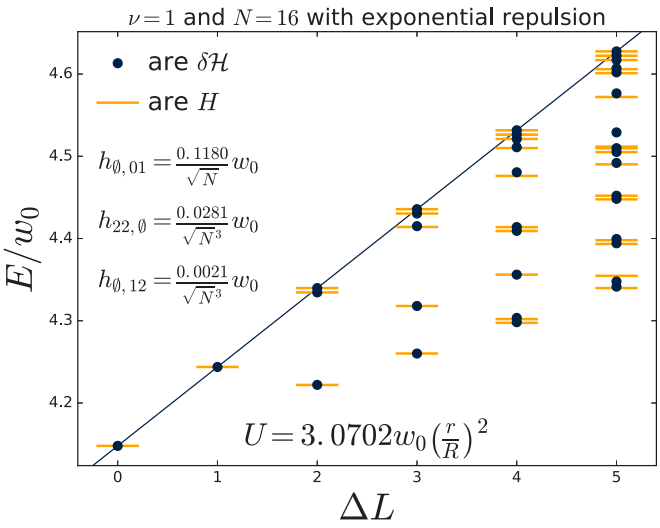

FIG. 12. We show the spectrum for the Moore-Read state containing $N = 16$ particles at filling $\nu = 1$ perturbed by an exponential repulsive interaction [Eq. (111)]. The three coupling coefficients in the effective Hamiltonian H are fit using the process described above and provide a very accurate fit to the numerically calculated spectrum (the dots correspond to the spectrum of $\delta\mathcal{H}$ while the orange lines are the spectra of H). As in the Laughlin case, the gradient of this linear Luttinger liquid slope (the blue line corresponding to an unperturbed droplet) is a free parameter of the parent Hamiltonian arising from the assumption of quadratic confinement.

$H(h_{\emptyset,01}, h_{22,\emptyset}, h_{\emptyset,12})$, where these coupling coefficients are fit using the procedure described above.

This comparison shows very good agreement between the exact numerical data and our low-energy effective theories. Notably, the renormalization of the velocity of the fermionic modes is indeed borne out by the data, with the lowest-energy modes in Fig. 12 corresponding to cases where all of the angular momentum goes into the excitation of fermionic edge modes. Their velocity is reduced by the presence of interactions.

However, we show here data only for the $\nu = 1$ Moore-Read state as smaller filling fractions do not converge sufficiently to be described by our effective theories at the system sizes we can reach with these exact methods. We suspect that this is due to a larger correlation length in these systems [43], which therefore requires higher-order terms to be included in H to provide an adequate description of the data at small system sizes. This point is also true for the Laughlin state, whose agreement with our low-energy theories also worsens as the filling fraction decreases. We present an illustration of this point in Appendix B.

VI. CONCLUSION

We have found the behavior of quantum Hall edge states in anharmonic traps and in the presence of short-range interactions and shown that these theories are extremely accurate descriptions of the low-energy structure of both the Laughlin and Moore-Read states. These effective theories show that the addition of bulk translational symmetry causes surprising simplifications in the nonlinear Luttinger liquid expansion. Furthermore, we find that these local descriptions of the edge behavior are at odds with alternative proposals for the edge state behavior relying on nonlocal models such as the quantum Benjamin-Ono equation [22].

However, the present analysis does not cover the full behavior of the resulting theories. It would be extremely interesting to analyze the consequences of the remaining terms in greater details, considering for example the implications on the hydrodynamics of the systems, perhaps along similar lines to previous works [16,18,21] which have, among other things, considered the potential for shock waves along the edge. Furthermore, the line of reasoning we use in this paper is readily applicable to any other quantum Hall wave function which can be expressed in terms of conformal blocks, which might indicate further interesting results in, for example, the Read-Rezayi states. Finally, given that these results depend intimately on our conjecture of locality and that this is not fully understood, it would be worth exploring exactly how and why the Hamiltonian can be claimed to be local, something which we partially consider for the integer quantum Hall effect in a future publication [44].

In compliance with EPSRC policy framework on research data, this publication is theoretical work that does not require supporting research data.

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APPENDIX A: HALDANE PSEUDOPOTENTIALS

We provide a rapid review of the two-body problem and discuss the concept of Haldane pseudopotentials. These pseudopotentials are a convenient way to think about potentials and generate parent Hamiltonians for the Laughlin state. They are reviewed in [7] but we summarize the points pertinent to our discussion here.

The lowest Landau level of the single-particle quantum Hall effect is a set of degenerate states gapped by $\hbar\omega_c$ where ω_c is the cyclotron frequency. As is customary [51], we use the Bargmann-Fock space of holomorphic polynomials as our Hilbert space with an inner product of the form

$$\{f|g\} = \int \frac{d^2z}{\pi} e^{-\frac{|z|^2}{2\ell_B^2}} \overline{g(z)} f(z), \tag{A1}$$

where ℓ_B is the magnetic length and $z = x + iy$ is the particle position. Note that we use curly kets to distinguish states in the physical space from those in the conformal field theory, which we introduce later. Within this space, the wave function $\{z \mid \psi_l\} = z^l$ describes a particle with angular momentum l about the origin.

We perturb the single-particle picture with a two-body interaction $V(|z_1 - z_2|) \ll \hbar\omega_c$. This perturbation is rotationally invariant and so is diagonal in a basis of states with well-defined relative and total angular momentum

$$\{z_1, z_2|m, M\} = z_r^m Z^M, \tag{A2}$$

where $z_r = z_1 - z_2$ and $Z = \frac{z_1+z_2}{2}$ are the relative and center-of-mass coordinates. The interaction is also translationally invariant so the eigenvalues of V are independent of M.

This leads to the concept of pseudopotentials. We define the mth pseudopotential of V as

$$v_m[V] = \frac{\{m, M|V|m, M\}}{\{m, M|m, M\}}. \tag{A3}$$

Therefore, given some set of functions $V_k(|z_r|)$ for which $v_m[V_k] = \delta_{m,k}$, we can represent any generic interaction potential in this Hilbert space as

$$V(|z_r|) = \sum_{k=0}^{\infty} v_k[V] V_k(|z_r|). \tag{A4}$$

These $V_k(|z|)$ may then be expressed as the derivative of a delta function

$$V_k(|z_r|) = L_k\left(-\ell_B^2 \nabla_r^2\right)\left[4\pi \ell_B^2 \delta^{(2)}(z_r)\right], \tag{A5}$$

where L_k is the kth Laguerre polynomial.

APPENDIX B: SMALLER FILLINGS

We present a series of plots in Fig. 13 to demonstrate the agreement between our effective Hamiltonians and numerics for a variety of filling fractions in the Laughlin state. In each case, we take only the least irrelevant term in the Hamiltonian, neglecting all other terms,

$$H = g_{22} T_{22} + O(R^{-5}). \tag{B1}$$

Then, as in the main text, we fit the coupling coefficient g_{22} using the matrix element $\langle 0|a_2 H a_{-2}|0\rangle$ which, in theory, should depend only on g_{22} to all orders. Plotted are the subsequent agreements between this simple fit and the numerics in the case where the interactions we take in each case are the first contributing pseudopotentials for fillings $\nu = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, each at a system size of $N = 8$ (which constitutes an approximate limit for our exact numerical methods in the $\nu = \frac{1}{4}$ state). They show that the agreement becomes worse for smaller filling fractions, which also corresponds to larger correlation lengths and therefore one might expect to require a larger number of higher-order terms to achieve an adequate agreement with the data.

APPENDIX C: HARMONIC INTERACTIONS

Consider the Harmonic interaction, which we take to have the form

$$\delta\mathcal{H} = V_0 \sum_{i \neq j} \left| \frac{z_i - z_j}{2\ell_B} \right|^2. \tag{C1}$$

This interaction is clearly very nonlocal, coupling particles with larger separations more than those which are close. In order to find the mapping of this operator onto the CFT, we must first convert it into a differential operator using projection to the lowest Landau level $\bar{z}_i \to 2\ell_B^2 \partial_i$. The result of this is that

$$\delta\mathcal{H} = N V_0 \sum_i z_i \partial_i - V_0 \sum_i z_i \sum_j \partial_j + V_0 N(N-1). \tag{C2}$$

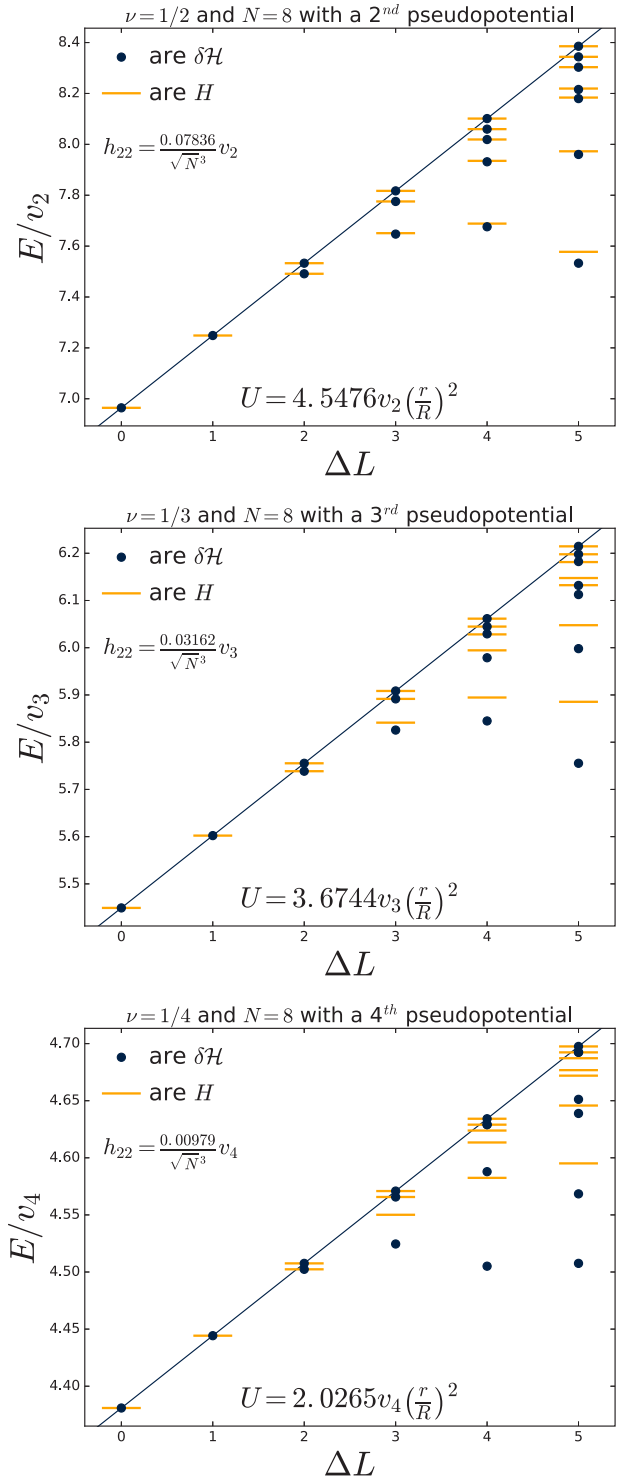

FIG. 13. A trio of plots showing the spectrum for the quantum Hall edge when the bulk interactions are the first nonvanishing pseudopotentials at filling fractions $\nu = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ (i.e., at filling $1/m$ we add the pseudopotential V_k) and the system size is $N = 8$. The effective Hamiltonian we fit is simply $H = g_{22} T_{22}$, with the coefficient fit using only the data at $\Delta L = 2$. The blue points are the numerical data and the orange levels are the result of diagonalizing the effective Hamiltonian. These data show that the agreement becomes steadily worse as the filling fraction decreases, with the agreement very poor for $\nu = \frac{1}{4}$. To describe this case well, one would require higher-order terms in the effective Hamiltonian.

Therefore, we simply need the mapping of these operators into the CFT. These are all derived in the main text with

$$\sum_i z_i \to \frac{1}{\sqrt{\beta}} a_1, \tag{C3}$$

$$\sum_i z_i \partial_i \to L_0 + \frac{\beta N(N-1)}{2}, \tag{C4}$$

$$\sum_j \partial_j \to L_{-1} + N\sqrt{\beta} a_{-1} \tag{C5}$$

in the sector with zero charge (i.e., $a_0 = 0$). Collating these results we therefore find that

$$H = NV_0\left(L_0 - a_{-1}a_1 - \frac{1}{N\sqrt{\beta}} L_{-1}a_1\right) + \frac{1}{2}V_0 N(N-1)(N\beta+2). \tag{C6}$$

Note that this effective Hamiltonian is clearly nonlocal with the $L_{-1}a_1$ term being of the form

$$L_{-1}a_1 = \frac{1}{2}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i} : (i\partial\varphi(w))^2 : zi\partial\varphi(z). \tag{C7}$$

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