Solvable two-dimensional superconductors with l-wave pairing

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(Received 1 November 2017; revised manuscript received 5 September 2018; published 2 October 2018)

We analyze a family of two-dimensional BCS Hamiltonians with general l-wave pairing interactions, classifying the models in this family that are Bethe-ansatz solvable in the finite-size regime. We show that these solutions are characterized by nontrivial winding numbers, associated with topological phases, in some part of the corresponding phase diagrams. By means of a comparative study, we demonstrate benefits and limitations of the mean-field approximation, which is the standard approach in the limit of a large number of particles. The mean-field analysis also allows us to extend part of the results beyond integrability, clarifying the peculiarities associable with the integrability itself.

DOI: 10.1103/PhysRevB.98.144504

I. INTRODUCTION

Superconductivity, a phenomenon that is typical in condensed-matter physics but also relevant in nuclear and subnuclear physics (see, for instance, [1,2]) takes its origin from pairing between fermions. It is typically described assuming an interacting (pairing) Hamiltonian and solving it via the mean-field (MF) approximation [3], which explicitly violates particle number conservation. While this limitation has a small effect on macroscopic systems, it can lead to dramatic deviations when fluctuations are important, i.e., when dealing with a fixed small number of particles. This justifies the interest in the study of exactly solvable models that avoid any approximation at the price of assuming specific forms of the interactions, like in the so-called Richardson model [4] with s-wave pairing ($l = 0$). This model is known to be integrable, and its exact solution is known to be related to the Gaudin spin Hamiltonians [5,6]. This exact-solution approach allowed various generalizations of the Richardson-Gaudin models [7–9], relevant for condensed-matter and nuclear physics. In general, Richardson-Gaudin particle-conserving integrable models can be classified into rational, hyperbolic, and trigonometric classes. Within this classification, a realization of the hyperbolic model is the $p_x + ip_y$ model, which has been extensively studied [10–16], also in the presence of interfaces with normal conductors (see e.g. [17–19]).

These examples motivate the need for analyzing integrable models for superconductivity by elucidating the physics of some delicate aspects of strongly correlated quantum systems (see also [20]). Particularly intriguing is the possibility to include pairing interactions with higher angular momentum (a pivotal example being the d wave, i.e., $l = 2$, even chiral) in two-dimensional (2D) systems due to their direct implication for high-temperature superconductivity [21]. Among the plethora of compounds and lattice schemes belonging to this family, we report the very recent realization of high-temperature (and likely d-wave) superconductivity on twisted bilayer graphene [22]. Still on the experimental side, the p-wave ($l = 1$) pairing is present in ^{3}He [23] and in strontium ruthenates [24,25], while f-wave pairing occurs, for instance, in superfluid ^{3}He [26,27]. Moreover, new progress in the physics of ultracold Fermi gases opens up the possibility to design superconductive pairings up to the h wave ($l = 5$; see, e.g., [28–35]).

Motivated by these possibilities and by the considerable theoretical interest in the high-wave superconductivity, in the present paper we analyze a large family of 2D BCS models with arbitrary l-wave ($l_x + i\, l_y$) pairing interaction. A particular attention is focused on the phase content of these models. We first discuss (Sec. II) the cases that can be exactly solved via the Bethe ansatz in a finite-size system. Later, we describe a standard MF analysis (Sec. III), and we compare the results from the two different approaches studying the topological properties of their solutions (Sec. IV). In this way, further insight is also achieved for the cases where integrability does not hold, as well as for the role of integrability itself.

The family of superconductive models that we are going to study is described by Hamiltonians of the form

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}} - g \sum_{\mathbf{k}\mathbf{k}'} (k_x - ik_y)^l (k'_x + ik'_y)^l c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} c_{-\mathbf{k}'} c_{\mathbf{k}'}. \tag{1}$$

Here $c^\dagger_{\mathbf{k}}$ is the creation operator of 2D fermions with momentum $\mathbf{k} = (k_x, k_y)$, and g is the coupling constant, positive for an attractive interaction. Notice that the interaction term creates and annihilates pairs of fermions with opposite momenta. In order to keep the widest generality, at the beginning of our analysis we do not adopt any particular choice for the

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single-particle energy $\epsilon_{\mathbf{k}}$, assuming only that it is a function of the modulus $k \equiv |\mathbf{k}|$.

In Eq. (1), we have dropped the spin index $\{\uparrow, \downarrow\}$ in the Fermi operators, so spinless fermions are formally considered. If, instead, the Cooper pairs are spinful, the symmetry of their spin wave functions is univocally determined by the Fermi-Dirac statistics. In fact, when l is even, the Cooper pairs form a spin singlet (antisymmetric), while when l is odd, they are in the triplet sector (symmetric and polarized). In both cases, the structures of the Bethe-ansatz equations and of the spatial part of the exact Cooper wave functions (introduced in Sect. II) in the presence of integrability are the same as in the spinless model described in Eq. (1).

The familiar s-wave case corresponds to $l = 0$ and to the singlet sector of the spin wave function. This is the sole non-symmetry-breaking case under parity and time-reversal transformation. The breaking of these symmetries for $l \geqslant 1$ leads to different kinds of exact solutions, introducing non-trivial topological properties of the paired states (according to the tenfold classification of the topological insulators and superconductors; see, e.g., [36–39]).

II. EXACT SOLUTION IN THE INTEGRABLE CASES

A. General setting

In the present section we address the exact solution of the Hamiltonian in Eq. (1). We find that the precise forms of $\epsilon_{\mathbf{k}}$ and of the Cooper wave functions are constrained by requiring the integrability.

The first step to proceeding is to notice that when only a single fermion occupies the level in $\mathbf{k}$ or $-\mathbf{k}$ (i.e., without its partner), it decouples from the ground-state dynamics due to the interaction in Eq. (1). So it is convenient to restrict ourselves to the dynamics of the Cooper pairs, having creation operators $b^\dagger_{\mathbf{k}} = c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}$ (see, e.g., [7]). Accordingly, the Hamiltonian in Eq. (1) takes the form

$$H = \sum_{\mathbf{k}} 2\,\epsilon_{\mathbf{k}}\, b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - g\, B^\dagger_0 B_0. \tag{2}$$

Due to the particular factorized form of the interaction in Eq. (1), H is now quadratic in terms of the new operator $B^\dagger_0 = \sum_{\mathbf{k}} z_{\mathbf{k}}\, b^\dagger_{\mathbf{k}}$, where $z_{\mathbf{k}} = (k_x - i k_y)^l$ are called pairing functions. Clearly, if the $b_{\mathbf{k}}$ operators were truly bosonic, the Hamiltonian would be directly diagonalizable. However, $b_{\mathbf{k}}$ are instead hard-core bosons obeying the following commutation relations:

$$[b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}(1 - 2\, b^\dagger_{\mathbf{k}} b_{\mathbf{k}}). \tag{3}$$

As a trial wave function for p pairs, we take the following general ansatz:

$$|\Psi_p\rangle = \prod_{\nu=1}^{p} B^\dagger_{J_\nu} |0\rangle, \quad B^\dagger_J = \sum_{\mathbf{k}} w_{\mathbf{k}}(J)\, b^\dagger_{\mathbf{k}}, \tag{4}$$

and we impose the eigenvalue equation

$$(H - \mathcal{E}_p)|\Psi_p\rangle = 0, \tag{5}$$

where the total energy $\mathcal{E}_p$ is given by the sum of the pair energies, $\mathcal{E}_p = \sum_{\nu=1}^{p} E_{J_\nu}$.

The next two sections will be devoted to the solution of Eq. (5) for one single pair and for multipair configurations. Generally, these solutions are obtained using the algebra of the pseudobosonic commutation relations to shift H in Eq. (5) through the operators $B^\dagger_{J_\nu}$ contained in $|\Psi_p\rangle$, until H acts on the vacuum $|0\rangle$, giving zero [40]. As the detailed calculation is rather cumbersome, it is presented in Appendix A.

B. One-pair case

By restricting the eigenvalue equation in Eq. (5) to one pair $|\Psi_1\rangle$ with energy E_J, we obtain the condition

$$w_{\mathbf{k}}(J) = g\, \frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J} \sum_{\mathbf{k}'} z^*_{\mathbf{k}'}\, w_{\mathbf{k}'}(J). \tag{6}$$

Multiplying both sides by $z^*_{\mathbf{k}}$ and summing in $\mathbf{k}$ (which is customary for the gap equations in the BCS theory [41,42]), unless the "order parameter" $W(J) = \sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J)$ is zero, we obtain the Richardson equation for one pair,

$$1 - g \sum_{\mathbf{k}} \frac{|z_{\mathbf{k}}|^2}{2\epsilon_{\mathbf{k}} - E_J} = 0, \tag{7}$$

as well as the expressions for the ansatz's coefficients,

$$w_{\mathbf{k}}(J) = g\, W(J)\, \frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J}, \tag{8}$$

proportional to the wave function $\frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J}$. The proportionality factors $g\, W(J)$ do not depend on $\mathbf{k}$; thus, they are irrelevant and can be neglected, as they affect only normalizations and global phases. Consequently, without any loss of generality, we can retain the wave function

$$w_{\mathbf{k}}(J) = \frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J}. \tag{9}$$

Notice that the spatial wave function (9) has the same parity of l under the transformation $\mathbf{k} \to -\mathbf{k}$. This fact has a direct consequence on the symmetry of the spin part of the wave function, as discussed in the Introduction. Moreover, if two spins $\{\uparrow, \downarrow\}$ are involved in the Cooper pair, still at fixed l, the forms of the Hamiltonian in Eq. (2) and of the commutators in Eq. (3) (as well as the consequent ones including the operators B_J; see Appendix A) remain unchanged. Therefore, the structures of the Bethe-ansatz equations and of the spatial part of the exact Cooper wave functions also do not change.

C. Many pairs

Similar to the one-pair case in the previous subsection, the ansatz in Eq. (4) for the p-pair case reads

$$|\Psi_p\rangle = \prod_{\nu=1}^{p} B^\dagger_{J_\nu} |0\rangle, \quad B^\dagger_J = \sum_{\mathbf{k}} \frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J} b^\dagger_{\mathbf{k}}, \tag{10}$$

where we have assumed the expression in Eq. (9) for the wave functions. The solution of Eq. (5), discussed in detail in Appendix A, yields the following final equations analogous to Eq. (7). These solutions can be classified into three groups, depending on the form of $z_{\mathbf{k}}$:

(1) The pairing function $z_{\mathbf{k}}$ is independent of $\mathbf{k}$. A relevant case is obtained by fixing $z_{\mathbf{k}} = 1$; therefore, from Eq. (A9) we

get the well-known Richardson equations

$$1 - g \sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}} - E_{J_\nu}} + 2g \sum_{\mu=1(\neq\nu)}^{p} \frac{1}{E_{J_\mu} - E_{J_\nu}} = 0, \quad (11)$$

whose solutions give the pair energies E_{J_ν} [7]. It is important to observe that here we have not imposed any restrictions on $\epsilon_{\mathbf{k}}$; thus, any dispersion relation (including the flat band $\epsilon_{\mathbf{k}} = 0$) allows integrability in this case.

(2) In addition to the original s-wave case $z_{\mathbf{k}} = 1$, we can also include the choice $z_{\mathbf{k}} = \exp[i\phi(\mathbf{k})]$, where $\phi(\mathbf{k})$ is a real function of momentum. Like in the previous case, the energy solutions are given by Eq. (11), and again there are no restrictions on $\epsilon_{\mathbf{k}}$. The present choice, possibly implementable in ultracold-atom setups by laser-assisted tunneling processes [28], extends the previous case, allowing for possible phases with nontrivial topology (see Appendix C).

(3) The pairing function is $z_{\mathbf{k}} \propto (k_x - ik_y)^l$. Since in this case $|z_{\mathbf{k}}|^2$ depends on $\mathbf{k}$ (for $l \neq 0$), we are forced to have $|z_{\mathbf{k}}|^2 \propto \epsilon_{\mathbf{k}}$ in order to guarantee integrability. As a consequence, after the substitution $|z_{\mathbf{k}}|^2 = \alpha\, \epsilon_{\mathbf{k}} = \alpha\, k^{2l}$, Eq. (A9) becomes

$$1 - \tilde{g} \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_{J_\nu}} + \tilde{g} \sum_{\mu=1(\neq\nu)}^{p} \frac{E_{J_\mu}}{E_{J_\mu} - E_{J_\nu}} = 0, \quad (12)$$

with $\tilde{g} = g\,\alpha$. For $l = 1$, our result coincides with the p-wave solution found in [11], with a massivelike dispersion $\epsilon_{\mathbf{k}} \propto k^2$. Remarkably, Eq. (12) also holds for the exact solution of the interesting d-wave case, where the relative angular momentum $l = 2$ imposes a quartic dispersion $\epsilon_{\mathbf{k}} \propto k^4$.

In [8,11] a detailed analysis was performed for case 3, with $\epsilon_{\mathbf{k}} = k^{2n}$ and $n = l = 1$, both by a MF approach in the thermodynamic limit and by comparing its results with the properties of the exact wave function from the solution of the Bethe-Ansatz equations. The topological aspects of the obtained phases were also discussed.

In the following, we generalize the latter analysis to the wider situation where $n, l \geqslant 1$, l (n) is assumed to be an integer (half integer), and n, l are allowed to be different. If $l \neq n$, integrability is broken, so that only a MF approach can be used. If, instead, $n = l$, deeper knowledge is achieved by studying again the topological properties of the exact wave functions.

We mention finally that integrability is not spoiled if an additional constant is added to the quasiparticle dispersion $\epsilon_{\mathbf{k}}$, as done in [43]. There Eqs. (11) and (12) were written in an implicit manner. Moreover, if $n \neq l$, integrability can sometimes be preserved if additional Hamiltonian terms are added; an explicit example is given in [44].

III. MEAN-FIELD ANALYSIS

A. General formalism

In this section we analyze the MF properties of the Hamiltonian in Eq. (1). Following the standard approach to MF superconductivity [41,42], we find that the MF quadratic Hamiltonian, in the thermodynamic limit and in the grand-canonical ensemble, derived from the one in Eq. (1), is

$$H = E_c + \sum_{\mathbf{k}} [\xi_k\, c^\dagger_{\mathbf{k}}\, c_{\mathbf{k}} + \Delta(k_x + ik_y)^l c_{\mathbf{k}} c_{-\mathbf{k}} + \text{H.c.}], \quad (13)$$

where E_c is the condensation energy, defined below, and $\xi_k = (\epsilon_k - \mu) = (k^{2n} - \mu)$ is the rescaled dispersion. In the chemical potential μ, the Hartree terms are also included, coming from the Wick contractions of the interaction term in the Hamiltonian of Eq. (1). According to the analysis performed in Sec. II, the integrable cases correspond to $n = l$; however, for the sake of completeness, here we do *not* fix n and l to be equal in this MF treatment.

The Hamiltonian in Eq. (13) describes potentially realistic cases if $n = 1$ and $l = 2$ (when two spins are considered) [42] and if $n = l = 1$ [23–25].

In Eq. (13) we set $\Delta = \sum_{\mathbf{k}'} g\,(k'_x + ik'_y)^l\, \langle c_{-\mathbf{k}'} c_{\mathbf{k}'} \rangle$, with $\langle c_{-\mathbf{k}'} c_{\mathbf{k}'} \rangle$ being the vacuum expectation value of the superconductive ground state. Therefore, the gap function can be written as $\Delta_{\mathbf{k}} = \Delta\,(k_x + ik_y)^l$; the quantity $(k_x + ik_y)^l$ coincides, up to a constant, with the spherical harmonic $Y_l^l(\hat{k})$ projected in the 2D plane (expected to be the more stable one in the absence of external strains or pressures; see, e.g., [42]).

The condensation energy E_{C} is given by

$$E_{\text{C}} = -4 \sum_{\mathbf{k},\mathbf{k}'>0} \frac{\Delta_{\mathbf{k}} \Delta^*_{\mathbf{k}'}}{g_{\mathbf{k}\mathbf{k}'}} = A\, \frac{M\Delta^2}{g}, \quad (14)$$

where the integer M denotes the number of states in the region of phase space considered and $g_{\mathbf{k}\mathbf{k}'}$ is the two-body potential appearing in the full Hamiltonian expressed in momentum space. In a general case, the quantity A explicitly depends on the assumed form of $g_{\mathbf{k}\mathbf{k}'}$. For the Hamiltonian in Eq. (1), this potential reads

$$g_{\mathbf{k}\mathbf{k}'} = -g(k_x - ik_y)^l (k'_x + ik'_y)^l, \quad (15)$$

so that $A = 1$. As we will check in the following, an important feature of the ground-state free energy F_{GS} is that, when expressed as a sum on the momenta via the gap equation, it does not depend on A.

The Bogoliubov spectrum corresponding to the Hamiltonian in Eq. (13) is

$$\lambda_k = \sqrt{\xi_k^2 + \Delta^2 k^{2l}} \quad (16)$$

(with k denoting again the modulus of $k_x - ik_y$). This spectrum is gapless at $\mu = 0$ and $k = 0$.

The ground-state free energy $F_{\text{GS}} = E_{\text{GS}} + \mu\, N$, $N = 2p$, corresponding to the spectrum in Eq. (16), is

$$F_{\text{GS}} = \sum_{\mathbf{k}>0} (\xi_k - \lambda_k) + \frac{M\Delta^2}{g} + \mu\, N, \quad (17)$$

independent of A, as anticipated. The Bogoliubov coefficients are

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2 k^{2l}}}\right), \quad |v_k|^2 = 1 - |u_k|^2, \quad (18)$$

so that the MF wave function results:

$$w^{(\text{MF})}_{\mathbf{k}} = \frac{v_k}{u_k} = \frac{\lambda_k - \xi_k}{\Delta\,(k_x + i\,k_y)^l}. \quad (19)$$

The equations for Δ and μ are as follows:

$$\frac{\partial F_{\rm GS}}{\partial \Delta} = 0 \rightarrow \frac{M}{g} = \frac{1}{2} \sum_{\mathbf{k}>0} \frac{k^{2l}}{\lambda_k}, \tag{20}$$

$$\frac{\partial F_{\rm GS}}{\partial \mu} = 0 \rightarrow N = \sum_{\mathbf{k}>0} \left(1 - \frac{\xi_k}{\lambda_k}\right). \tag{21}$$

The last equation can also be written as

$$\mu \sum_{\mathbf{k}>0} \frac{1}{\lambda_k} = N + \sum_{\mathbf{k}>0} \frac{k^{2n}}{\lambda_k} - \frac{M}{2}, \tag{22}$$

which, in the case of $n = l$, becomes, from Eq. (20),

$$\mu \sum_{\mathbf{k}>0} \frac{1}{\lambda_k} = N + 2\frac{M}{g} - \frac{M}{2}. \tag{23}$$

Using Eq. (20), the ground-state free energy is written as

$$F_{\rm GS} = \sum_{\mathbf{k}>0} \left(\xi_k - \lambda_k + \frac{\Delta^2}{2}\frac{k^{2l}}{\lambda_k}\right) + \mu N \tag{24}$$

and, exploiting Eq. (22), also as

$$F_{\rm GS} = \sum_{\mathbf{k}>0} k^{2n} \left(1 - \frac{2k^{2n} - 2\mu + \Delta^2 k^{2(l-n)}}{2\lambda_k}\right). \tag{25}$$

If $n = l$, the latter expression shows a duality between different MF solutions, in that two solutions (labeled 1 and 2) are related by the equations $\mu_1 = -\mu_2$ and $\Delta_1^2 - 2\mu_1 = \Delta_2^2 - 2\mu_2$, such that the corresponding free energies coincide: $F_{\rm GS}^{(1)} = F_{\rm GS}^{(2)}$. If $n = l = 1$, this duality is justified by the exact solution of the Richardson equations (11).

Once one considers working in a lattice, as opposed to the continuum, the above analysis can be extended straightforwardly. Some spin models are, indeed, quadratic in Fermi operators in momentum space with pair creation [45]. For sufficiently small interaction strength $\propto g$, we expect that superconductivity involves only quasiparticles with momenta within a small range $\delta k \approx g^{\frac{1}{n}}$ around the Fermi momentum k_F. Here the lattice dispersion, with discretized momenta, can be expanded in powers of k, such that it ends up in a power-law dispersion. At that point, the MF analysis proceeds as described before.

B. Mean-field phase diagram

Using the derived expressions for the ground-state free energy, for the wave functions of the Bogoliubov excitations, and for the self-consistency equations, it is interesting to characterize the phase diagram of the Hamiltonian in Eq. (13) as a function of g and of the (average) filling $N/M \equiv x$.

Various transition lines, between different quantum phases, can be identified. A notable transition occurs at $\mu = 0$, where the spectrum in Eq. (16) is gapless at $k = 0$. There the MF wave function behaves as

$$w_{\mathbf{k}}^{(\rm MF)} \approx \begin{cases} (k_x - ik_y)^l\, k^{2(n-l)} & \text{if } \mu < 0 \text{ and } n \geqslant l, \\ (k_x - ik_y)^l & \text{if } \mu < 0 \text{ and } n < l, \\ \frac{1}{(k_x + ik_y)^l} & \text{if } \mu > 0. \end{cases} \tag{26}$$

This transition has a nature similar to the Read-Green one described in the case $n = l = 1$ [8,11,14,46] (and found to be a third-order transition in [14]); for this reason in the following the same name will be adopted for it. The condition $\mu = 0$ translates, from Eq. (23), to the relation

$$x = \frac{1}{2}\left(1 - \frac{4}{g}\right). \tag{27}$$

The line identified by this equation does not depend on the distribution of the momenta, thus is topologically protected against every perturbation changing it and possibly breaking the integrability of the Hamiltonian in Eq. (1).

Another notable line, denoted as the (generalized) Moore-Read line [11,47], is found for every $n = l$, parametrized by the relation $\mu = \frac{\Delta^2}{4}$; along this line the condition $F_{\rm GS} = 0$ holds: the same free energy of the vacuum, intended as the absence of fermions ($x = 0$), is obtained for the superconductive ground state. Notice that, in order to obtain this result, the positiveness of μ is crucial. The condition $\mu = \frac{\Delta^2}{4}$ is fulfilled on the line

$$x = \left(1 - \frac{4}{g}\right), \tag{28}$$

a result found by exploiting Eq. (21). There the mass gap does not vanish, but the ground-state free energy is discontinuous in the thermodynamic limit.

As for the case $n = l = 1$ [11,47], the duality mentioned in the previous section holds, at least at the MF level, between a point (g, x_w) in the weak-pairing regime ($\mu > 0$) and a point (g, x_s) in the strong-pairing regime ($\mu < 0$); these points are related to each other by the relation

$$x_w + x_s = \left(1 - \frac{4}{g}\right), \tag{29}$$

which is still obtained directly from Eqs. (21) and (23). Therefore, the Read-Green line is self-dual, while the MR state is dual to the vacuum, where $x = 0$.

The Read-Green and Moore-Read lines meet at the point $g = 4$, where the limit $x = 0$ is achieved.

By a direct numerical analysis of the MF free energy in Eq. (25), performed on various cases with $n \neq l$, we have found strong indications that the Moore-Read line does not persist out of the integrability [48], as $F_{\rm GS} \neq 0$.

If $n = l$, the minimum $E_{\rm GAP}$ of λ_k, Eq. (16), is

$$E_{\rm GAP} = \begin{cases} |\mu| & \text{if } \mu < \frac{\Delta^2}{2}, \\ \Delta\sqrt{\mu - \frac{\Delta^2}{4}} & \text{if } \mu > \frac{\Delta^2}{2}. \end{cases} \tag{30}$$

The condition $\mu = \frac{\Delta^2}{2}$ defines a third notable transition line, the so-called Volovik line [8,11]. Along it a first-order quantum phase transition, reminiscent of the Higgs transition, occurs [23]. The same line depends on the distribution of the momenta; thus, it is *not* topologically protected (and its presence must be verified beyond the MF approach, adopted in the following). Setting $\mu = \frac{\Delta^2}{2}$ and exploiting Eqs. (20) and (23), we find that, if $n = l$, the Volovik line reads explicitly as

$$x = \frac{1}{2}\left(1 - \frac{1}{M}\sum_{\mathbf{k}>0} \frac{2k^{2l} - \Delta^2}{\lambda_k}\right). \tag{31}$$

From a numerical study of λ_k in Eq. (16), we conclude that the Volovik line does not survive if $n < l$ since $E_{\rm GAP}$ always arises at $k \neq 0$. On the contrary, if $n > l$, $E_{\rm GAP}$ is located at $k = 0$ for some values of Δ and μ, so that a Volovik line can still be identified [the defining equation, similar to (31), is not easily writable as a closed formula].

IV. TOPOLOGICAL PROPERTIES

In this section we give a deeper characterization of the MF phase diagram, sketched in the previous section, studying the topology of the various identified phases. Focusing first on the case $n = l$, we start by taking the MF Cooper wave function $w_{\mathbf{k}}^{(\rm MF)}$ in Eq. (19) to calculate the topological invariant [8]:

$$I_{\rm MF} = \frac{1}{4\pi}\int_{S^2} d\mathbf{k}\; w_{\mathbf{k}}, \tag{32}$$

where S^2 is the sphere of radius $|\mathbf{k}| = 1$ obtained from the plane R^2 by the inverse of the stereographic projection [49,50]. We obtain $I_{\rm MF} = l$ if $\mu > 0$ and $I_{\rm MF} = 0$ if $\mu < 0$. This result matches the previously found values $I_{\rm MF} = 1$ for the p-wave case [8,11,46] and $I_{\rm MF} = 2$ for the d-wave case [46]. As generally expected (see, e.g., [38]), $I_{\rm MF}$ is sensitive to the vanishing of the energy for the Bogoliubov quasiparticles, occurring at $\mu = 0$. Finally, it is worth noticing that, although the location of the Read-Green line is independent of the momentum distribution and of the Bogoliubov dispersion law λ_k, the (topological) phases bounded by it depend on l. This index can affect the topology since it induces global (on the entire set of allowed momenta) and not smooth (l is discrete) modifications on λ_k.

The topological content of the phase diagram can be inferred not only from the MF wave function of a single Cooper pair, Eq. (19), but also from the MF ground-state wave function, following a procedure common in the study of topological insulators and superconductors [49]. In particular, denoting by $|u_{\mathbf{k}}\rangle$ the positive-energy eigenvector of the quadratic Hamiltonian in Eq. (13), $I_{\rm MF}$ is expressed as the integral on the momentum space of the Berry curvature:

$$I_{\rm MF} = \frac{1}{4\pi}\int_{S^2} d\mathbf{k}\; \nabla_{\mathbf{k}} \times \langle u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}}\rangle. \tag{33}$$

The equivalence between the two MF calculations for $I_{\rm MF}$ stems directly from the fact that $|u_{\mathbf{k}}\rangle$ is an excited state obtained by breaking a Cooper pair. In turn, expression (33) is also equivalent to the spin-texture one [46,49,51],

$$I_{\rm MF} = \frac{1}{8\pi}\int_{S^2} d\mathbf{k}\; \epsilon_{abc}\epsilon_{ij}\, \hat{d}_a(\mathbf{k})\, \partial_{k_i}\hat{d}_b(\mathbf{k})\, \partial_{k_j}\hat{d}_c(\mathbf{k}) \tag{34}$$

$[(i, j) = \{x, y\}$ and $(a, b, c) = \{1, 2, 3\}]$, obtained expressing the Hamiltonian (13) in terms of the Pauli matrices in the basis $(c_{\mathbf{k}}, c_{-\mathbf{k}})^T$: $H = \sum_{\mathbf{k}} \hat{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$. Direct numerical calculation of both expressions (33) and (34) confirmed the result $I_{\rm MF} = l$ if $\mu > 0$.

The content in topology obtained using the MF wave functions can also be probed calculating the same quantity as in Eq. (32) in terms of the exact wave function $w_{\mathbf{k}}$ of a single Cooper pair and then considering again the limit $x = 0$. We implicitly assume that fluctuations beyond MF do not change the MF phase diagram significantly; thus, the solution of the Bethe-ansatz equations essentially leads to the same phase diagram. This hypothesis will not be contradicted in the following. The exact wave function, derived in Sec. II, reads, up to an unimportant multiplicative constant,

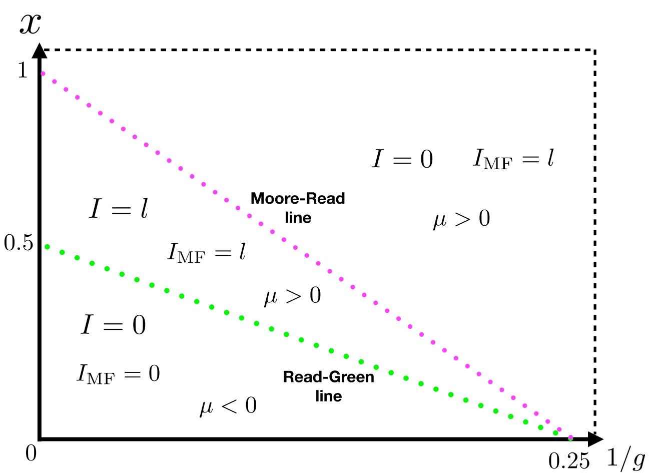

FIG. 1. Mean-field phase diagram for $n = l$ as a function of x and g. The topological invariant $I_{\rm MF}$, relative to a single Cooper pair, is reported, as well as the invariant I from the exact wave function in Eq. (35). Notice the difference between MF and exact invariants in the phases above the Moore-Read line. The different length scale for the axes is chosen for sake of clarity of the picture. The Moore-Read line disappears in general out of integrability if $n \neq l$.

$$w_{\mathbf{k}} = \frac{(k_x - ik_y)^l}{2\epsilon_{\mathbf{k}} - E}, \tag{35}$$

where E is the pair energy (complex in general [7]), derived from the solution of the Richardson equations. The integral in Eq. (32) can be recast as follows:

$$I = l^2 \int_0^{\infty} du\, \frac{u^{(3l-1)} - E\bar{E}\, u^{(l-1)}}{[u^l + (u^l - E)(u^l - \bar{E})]^2}, \tag{36}$$

with $u = k^2$. The result of Eq. (36) is

$$\begin{aligned} I &= l \;\text{ if }\; E = 0,\\ I &= 0 \;\text{ if }\; E \neq 0. \end{aligned} \tag{37}$$

An alternative derivation of the winding number I is discussed in Appendix B; this turns out to be useful also for the pure-phase case in Appendix C. Moreover, it would also be interesting to extend the calculation of I to multipair states, e.g., following the approaches in [44,52].

Referring to the MF diagram in Fig. 1, the condition $E = 0$ in (37) is fulfilled if $x = 0$ at the intersection with the Moore-Read line, where $g = 4$. This fact indicates that $I = l$ in the region between the Read-Green and Moore-Read lines, while $I = 0$ in the other phases. Therefore, I matches the MF phase diagram opposite to $I_{\rm MF}$: indeed, $I_{\rm MF}$ is nonvanishing also in the region to the right of the Moore-Read line, and thus $I_{\rm MF}$ does not detect this line. The described mismatch is indeed interesting since it can indicate a general inability of the topological invariants from the MF wave functions to correctly detect some phases of (topological) insulators or superconductors. In our case, the mismatch occurs since the mass gap does not vanish on the MR line. It remains an open question whether the origin of the puzzle is due to integrability of the full model in Eq. (1). However, such interpretation is suggested by the fact that from the MF analysis the MR line seems generally absent for $n \neq l$, where integrability is broken (and no divergencies occur in the spectrum, a situation found instead in the presence of long-range Hamiltonian couplings; see [53] and references therein [54]).

We note finally that in [14] it has been suggested, for the case $n = l = 1$, that the Moore-Read line does not identify a genuine quantum phase transition, a possibility partly solving the mismatch mentioned above. However, the same result for I (different from zero only at $E = 0$) from the exact pair wave function in [8] and in the present paper seems to rule out this scenario.

V. DISCUSSION AND CONCLUSIONS

In this paper we have analyzed the physical features of a large set of superconductive models for which an exact solution is available, composed of two-dimensional systems with a factorized form for the momentum-dependent interaction. Besides the known cases of the s-wave pairing, solved by Richardson [4], and p-wave pairing, discussed by Ibañez *et al.* [11], we have found that, in general, l-wave pairing is exactly solvable on a finite-size system, provided that the single-particle dispersion is proportional to k^{2n}, with $n = l$.

Analyzing the integrable case, we also found that the topological invariants calculated in the framework of the mean-field approach cannot reproduce correctly the phase diagrams of the considered integrable models, in contrast to the corresponding invariants obtained from the exact (Bethe-ansatz) solutions. This discussion has shown the potential inadequacy of the mean-field topological invariants to predict the correct phase diagram of (topological) insulators and superconductors, at least in peculiar situations. In our case, the origin of this problem seems to be the (possible) presence of quantum phase transitions without vanishing of the mass gap, a feature possibly related to integrability. We notice that quite recently a change in topology without mass gap closing, in the presence of large interaction, was found numerically in [55].

In the nonintegrable cases $n \neq l$ [as well as for other perturbed models where interactions do not assume the special form of Eq. (1)], exact wave functions analogous to Eq. (9) cannot be derived because the Bethe ansatz is not applicable; therefore, only the mean-field approach can be exploited. The reliability of this approach out of the integrable regime is suggested by its prediction about the general absence of quantum phase transitions with a nonvanishing mass gap.

ACKNOWLEDGMENT

The authors are pleased to thank M. Ibanez Berganza, S. Paganelli, and G. Sierra for useful discussions.

APPENDIX A: BETHE-ANSATZ SOLUTION OF EQUATION (5)

The eigenvalue equation in (5) can be written as

$$\left(\left[H, \prod_{\nu=1}^{p} B^{\dagger}_{J_\nu}\right] - \mathcal{E}_p \prod_{\nu=1}^{p} B^{\dagger}_{J_\nu}\right)|0\rangle = 0, \tag{A1}$$

where the commutator on the left side expands as

$$\sum_{\nu=1}^{p}\left\{\left(\prod_{\eta=1}^{\nu-1} B^{\dagger}_{J_\eta}\right)\left[H, B^{\dagger}_{J_\nu}\right]\left(\prod_{\mu=\nu+1}^{p} B^{\dagger}_{J_\mu}\right)\right\}. \tag{A2}$$

Using the relations

$$\begin{aligned}[b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}, B^{\dagger}_J] &= w_{\mathbf{k}}(J) b^{\dagger}_{\mathbf{k}},\\ [B_0, B^{\dagger}_J] &= \sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J)(1 - 2b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}),\end{aligned} \tag{A3}$$

we find the expression for every single commutator appearing in Eq. (A2):

$$\begin{aligned}[H, B^{\dagger}_J] = {}& E_J B^{\dagger}_J + \sum_{\mathbf{k}} (2\epsilon_{\mathbf{k}} - E_J) w_{\mathbf{k}}(J) b^{\dagger}_{\mathbf{k}}\\ &- g B^{\dagger}_0 \sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J)(1 - 2b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}).\end{aligned} \tag{A4}$$

Putting Eq. (A4) in Eq. (A2) and using the basic relation $H|0\rangle = 0$, we find

$$\begin{aligned}H|\Psi_p\rangle = {}& \mathcal{E}_p|\Psi_p\rangle + \sum_{\nu=1}^{p}\left[\left(\sum_{\mathbf{k}}(2\epsilon_{\mathbf{k}} - E_{J_\nu}) w_{\mathbf{k}}(J_\nu) b^{\dagger}_{\mathbf{k}} - g B^{\dagger}_0 \sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J_\nu)\right)\left(\prod_{\substack{\eta=1\\ \eta\neq\nu}}^{p} B^{\dagger}_{J_\eta}\right)\right]|0\rangle\\ &+ \sum_{\nu=1}^{p}\left\{\left(\prod_{\eta=1}^{\nu-1} B^{\dagger}_{J_\eta}\right) 2g B^{\dagger}_0 \left(\sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J_\nu) b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}\right)\left(\prod_{\mu=\nu+1}^{p} B^{\dagger}_{J_\mu}\right)\right\}|0\rangle.\end{aligned} \tag{A5}$$

In the last term of Eq. (A5), we want to commute the operator $b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}$ to the extreme right, where it annihilates the vacuum $|0\rangle$. To this aim, we write this term as

$$\sum_{\nu=1}^{p}\left[2g B^{\dagger}_0\left(\prod_{\eta=1}^{\nu-1} B^{\dagger}_{J_\eta}\right)\sum_{\mu=\nu+1}^{p}\left\{\left(\prod_{\eta'=\nu+1}^{\mu-1} B^{\dagger}_{J_{\eta'}}\right)\left[\sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J_\nu) b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}, B^{\dagger}_{J_\mu}\right]\left(\prod_{\mu'=\mu+1}^{p} B^{\dagger}_{J_{\mu'}}\right)\right\}\right]|0\rangle. \tag{A6}$$

At this point, it is crucial to use the following manageable form for the commutator in Eq. (A6):

$$\left[\sum_{\mathbf{k}} z^*_{\mathbf{k}}\, w_{\mathbf{k}}(J_\nu)\, b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}}, B^{\dagger}_{J_\mu}\right] = \sum_{\mathbf{k}} z^*_{\mathbf{k}}\, w_{\mathbf{k}}(J_\nu) w_{\mathbf{k}}(J_\mu)\, b^{\dagger}_{\mathbf{k}}. \tag{A7}$$

In general, for every μ and ν, we want to express Eq. (A7) in the form $C_{\mu,\nu}B^\dagger_{J_\nu} + D_{\mu,\nu}B^\dagger_{J_\mu}$, where $C_{\mu,\nu}$ and $D_{\mu,\nu}$ are some coefficients. For this reason, we impose the condition

$$\sum_{\mathbf{k}} z^*_{\mathbf{k}} w_{\mathbf{k}}(J_\nu) w_{\mathbf{k}}(J_\mu) b^\dagger_{\mathbf{k}} = C_{\mu,\nu}B^\dagger_{J_\nu} + C_{\nu,\mu}B^\dagger_{J_\mu}, \tag{A8}$$

where we have used the symmetry under the exchange $\nu \leftrightarrow \mu$. Assuming that Eq. (A8) is correct, then we find that the eigenvalue equation (A5) holds, provided that

$$1 - g\sum_{\mathbf{k}} \frac{|z_{\mathbf{k}}|^2}{2\epsilon_{\mathbf{k}} - E_{J_\nu}} + 2g \sum_{\mu=1(\neq\nu)}^{p} C_{\nu,\mu} = 0, \tag{A9}$$

where we have used the expression for the wave function $w_{\mathbf{k}}(J) = \frac{z_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_J}$. Equation (A8) gives

$$(2\epsilon_{\mathbf{k}} - E_{J_\mu})C_{\mu,\nu} + (2\epsilon_{\mathbf{k}} - E_{J_\nu})C_{\nu,\mu} = |z_{\mathbf{k}}|^2$$

with two different kinds of solutions:

(1) In the s-wave case, $|z_{\mathbf{k}}|^2 = 1$ and $C_{\mu,\nu} = -C_{\nu,\mu} = (E_{J_\nu} - E_{J_\mu})^{-1}$. Thus, from (A9) we get the well-known Richardson equation (11), with no restrictions on $\epsilon_{\mathbf{k}}$. Notice that the condition $|z_{\mathbf{k}}|^2 = 1$ is more general than the s-wave case $z_{\mathbf{k}} = 1$.

(2) In the l-wave case, $z_{\mathbf{k}} = (k_x - ik_y)^l$ depends on $\mathbf{k}$ (for $l \neq 0$), and the coefficients are given by

$$C_{\mu,\nu} = \frac{|z_{\mathbf{k}}|^2}{2\epsilon_{\mathbf{k}}} \frac{E_{J_\nu}}{E_{J_\nu} - E_{J_\mu}}, \tag{A10}$$

but we must have $|z_{\mathbf{k}}|^2 \propto \epsilon_{\mathbf{k}}$ to have a $C_{\mu,\nu}$ independent of $\mathbf{k}$. As a consequence, after the substitution $|z_{\mathbf{k}}|^2 = \alpha\,\epsilon_{\mathbf{k}}$, Eq. (A9) becomes Eq. (12).

APPENDIX B: ALTERNATIVE CALCULATION OF I

In this appendix we discuss an alternative derivation of the winding number I, which is also useful for the pure phase case in Appendix C, that can be performed by analyzing directly the map $\omega_{\mathbf{k}}$ in the case of real E. In order to do that, we first separate Eq. (35) as

$$\omega_{\mathbf{k}} = [f_-(k) + f_+(k)]\, e^{i\phi_k l}, \tag{B1}$$

with $f_-(k) = \frac{k^l}{k^{2l}-E}$, $k < E^{1/2l}$, and $f_+(k) = \frac{k^l}{k^{2l}-E}$, $k > E^{1/2l}$. The part $f_+(k)\, e^{i\phi_k l}$ gives a contribution $I_+ = l$ to I since $f_+(k)$ is monotonic in k and assumes values $[0,\infty)$, so that $f_+(k)\, e^{i\phi_k l}$ covers l times (because of the phase $l\,\phi_k$) the entire plane $R^2 \sim S^2$ (the identification relying again on the stereographic projection).

Assuming now that $E \neq 0$, we put $k = 1/p$ in $f_-(k)$, obtaining $f_-(p) = -\frac{1}{E}\,\frac{p^l}{p^{2l}-E} = -f_+(p)$, with $p > E^{1/2l}$. Apart from the unimportant multiplicative factor E^{-1}, we can write (renaming $p \equiv k$)

$$\omega_{\mathbf{k}} = [f_-(k) - f_-(k)]\, e^{i\phi_k l} = 0, \tag{B2}$$

showing that $I = 0$ if $E \neq 0$. The minus sign in $f_-(p)$, responsible for the vanishing result for I, is related to the fact that, for k varying, $f_+(k)$ and $f_-(k)$ span the space $R^2 \sim S^2$ in the opposite sense.

The situation is different if $E = 0$: in this case we get only

$$\omega_{\mathbf{k}} = f_+(k)\, e^{i\phi_k l} \tag{B3}$$

and $I = I_+ = l$.

APPENDIX C: PURE PHASE GAP

We can also calculate the topological index I in the case when $\Delta(k) = e^{i\phi_{\bar{k}} l}$. In this case, we have shown in Sec. II that we have integrability no matter what the particular single-particle dispersion $\epsilon_{\mathbf{k}}$ is; therefore, we assume again $\xi_{(l)}(k) = k^{2l}$. The exact wave function reads, in momentum space and up to an unimportant multiplicative constant,

$$\omega_{\mathbf{k}} = \frac{(k_x - ik_y)^l}{k^l\,(2\epsilon_{\mathbf{k}} - E)}. \tag{C1}$$

In this case we obtain

$$I = 2l^2 \int_0^\infty dk\, \frac{k^{(2l-1)}[2k^{2l} - (E + \bar{E})]}{[1 + (k^{2l} - E)(k^{2l} - \bar{E})]^2}. \tag{C2}$$

This integral yields $I = \frac{l}{|E|^2+1}$, a pretty unexpected result since, in general, an integer winding number should be expected. However, this result can be explained quite naturally by analyzing the map (C1) directly. This map can be expressed as

$$\omega_{\mathbf{k}} = \frac{1}{k^{2l} - E}\, e^{i\phi_k l}. \tag{C3}$$

As for (35), we can write again

$$\omega_{\mathbf{k}} = [f_-(k) + f_+(k)]\, e^{i\phi_k l}, \tag{C4}$$

with $f_-(k) = \frac{1}{k^{2l}-E}$, $k < E^{1/2l}$, and $f_+(k) = \frac{1}{k^{2l}-E}$, $k > E^{1/2l}$. We notice that $f_-(k)\, e^{i\phi_k l}$ is homotopic to a constant map $\tilde{f}_-(k) = c$ since $f_-(k) = (-\infty, -\frac{1}{E}]$ (the minus sign is reabsorbable in the phase ϕ_k) and not every point of the target stereographic plane R^2 is covered by $f_-(k)\, e^{i\phi_k l}$. Then we can write

$$\omega_{\mathbf{k}} = [f_-(k) + f_+(k)]\, e^{i\phi_k l} \sim f_-(k)\, e^{i\phi_k l} \tag{C5}$$

(here the symbol $\sim$ means here "continuously deformable to"). Since, again, $f_+(k) = [0,\infty)$ and is monotonic, it yields a contribution $I_+ = l$ to I for every value of E. However, $f_-(k)$ gives a nonvanishing contribution to I, covering a part of the sphere with area

$$I_- = -\frac{1}{\pi}\int_0^{\frac{1}{E}} dk\, \frac{2\pi\,k}{(1+k^2)^2} = -\frac{E^2}{E^2+1}, \tag{C6}$$

where the minus sign appears since $|f_-(k \to \infty)| \to \infty$. This contribution sums up to I_+, giving the result (C2):

$$I = I_+ + I_- = l - l\,\frac{E^2}{E^2+1} = l\,\frac{E^2}{E^2+1}. \tag{C7}$$

In spite of the value of I, the real winding number related to (C1) is $\tilde{I} = I_+ = l$ since we know that $f_-(k)$ is homotopic to a constant map, a fact also resulting in a value of $|I_-|$ smaller than 1.

This result matches the fact that the BCS case and the (C3) case are linked by the transformation in the gap $\Delta \to \Delta(k) = \Delta\, e^{i\phi_k l}$. However, this map is continuous but not invertible, wrapping l times: this is the reason $\tilde{I} = l$.

In conclusion, the case (C3) describes a phase with winding number $I = l\frac{N}{2}$ (with $\frac{N}{2}$ being the number of Cooper pairs in the ground state). However, the energy of Bogoliubov quasiparticles is the same as in the BCS case and always gapped; thus, no phase transitions arise, and the system is always in a phase with nontrivial topology.

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