# Asymptotic theory of bimodal quarter-wave impedance matching for full mode-converting transmission

Xiongwei Yang[1] and Yoon Young Kim[2,*]

[1]*BK21 Plus Transformative Program for Creative Mechanical & Aerospace Engineers, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, South Korea*

[2]*School of Mechanical and Aerospace Engineering and Institute of Advanced Machines and Design, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, South Korea*

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A quarter-wave impedance-matching element inserted between two dissimilar media allows full-power transmission, but this phenomenon occurs only when the mode of the incident and transmitted waves remains unaltered, either longitudinal (L) or transverse (T). Here, we report our asymptotic bimodal quarter-wave impedance-matching theory for full-power mode-converting transmission. It requires that two coupled quarter waves should be simultaneously formed inside a matching element possessing specific anisotropy for phase matching and L-to-T (T-to-L) mode conversion. Simulations using designed metamaterial matching elements show that nearly full mode-converting transmissions can be realized within a single medium or between two dissimilar media. We expect that our theory can be critically useful for medical and industrial ultrasonic applications wherever highly efficient mode conversion and high-powered transverse waves are required.

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Quarter-wave impedance matching achieves full-power transmission between two dissimilar media [1–9]. Because of the unique full transmission phenomenon, the impedance-matching concept has been widely investigated in a number of studies including some recent studies [10–17]. However, this well-known conventional impedance-matching concept is only applicable for mode-preserving transmission involving a single mode. Here, we aim to explore whether the impedance-matching concept can be generalized to achieve full-power transmission between two dissimilar modes (specifically, the longitudinal and transverse modes). Thereby, we investigate a generalized impedance-matching concept which can be used to realize full-power mode-converting transmission.

Before discussing the generalized impedance-matching phenomenon for mode-converting transmission, we start with the conventional phenomenon involving no mode conversion, as depicted in Fig. 1(a). The incident harmonic longitudinal (L) wave of frequency $f$ from an isotropic medium A having the characteristic impedance $Z_L^A$ can be fully transmitted to another isotropic medium B having the characteristic impedance $Z_L^B$ ($Z_L^A \neq Z_L^B$), provided the following well-known conditions are satisfied in the quarter-wave impedance-matching element:

$$Z_L = \sqrt{Z_L^A Z_L^B}, \tag{1a}$$

$$d = \lambda_L/4, \tag{1b}$$

where $Z_L$ and $\lambda_L$ are the characteristic impedance and wavelength of the L wave in the impedance-matching element, respectively. The conditions stated by Eqs. (1a) and (1b) suggest that the conventional impedance-matching concept is valid only when the wave mode type before and after the matching element remains unaltered. Therefore, they cannot be used directly as similar conditions for the case of mode-converting transmission that will be explored here.

Contrary to the case in Fig. 1(a), Figs. 1(b) and 1(c) describe an unusual wave phenomenon converting an incident L wave to a transmitted transverse (T) wave with full (100%) power transmission. Here, we consider acoustic waves propagating in solids because they can carry L and T modes simultaneously due to the atomic bindings [18–20]. The aim of this study is to investigate whether the impedance-matching concept that is established for the classical mode-preserving case can be generalized for the mode-converting case. If that is the case, we aim to find the corresponding conditions. In investigating this phenomenon, our postulation is that the wave phenomenon occurring in the full mode-converting transmission may be analyzed as a problem to match the characteristic impedances of the incident L (or T) and transmitted T (or L) wave modes. The mode-converting phenomenon even within a single isotropic medium may be viewed from this generalized impedance-matching concept because the T wave impedance $Z_T = \sqrt{\rho_0 c_{66}}$ ($\rho_0$: mass density; $c_{66}$: shear stiffness) always differs from the L wave impedance $Z_L = \sqrt{\rho_0 c_{11}}$ ($c_{11}$: longitudinal stiffness).

Recently, the transmodal Fabry-Perot resonance based method [21,22] was proposed for mode-converting transmission from an L to a T wave mode. Although no L wave mode is transmitted, the incident L-mode power cannot be 100% transmitted to the T mode in this method. This partial transmission is inevitable, because the constructive interference mechanism can only preserve the displacement magnitude of the incident wave without overcoming the intrinsic impedance mismatch between L and T waves. Although mode-conversion

*yykim@snu.ac.kr

 

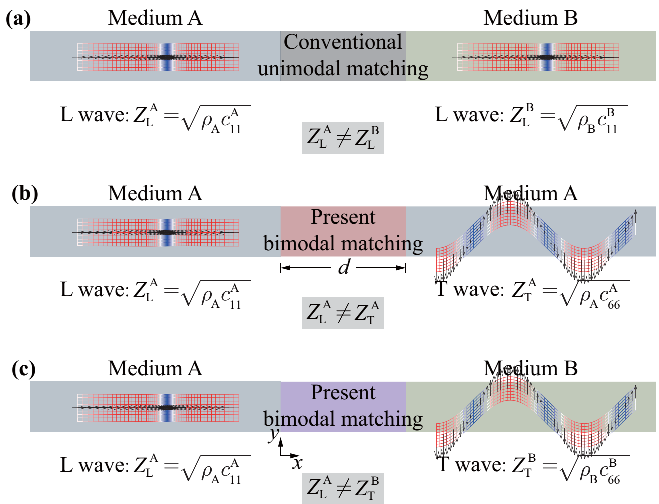


FIG. 1. Illustration of possible quarter-wave impedance matching for full-power transmission. (a) The classical case of mode-preserving transmission involving a single mode with a conventional quarter-wave impedance-matching element; (b), (c) generalized cases of mode-converting transmission involving two different wave modes within the same medium and between two dissimilar media, respectively. The arrows in the sketches indicate the directions of polarization.

phenomena have been much explored (see, e.g., [23–26]), the converting efficiency through designing mode convertors based on Snell's critical angle is very low [18]. If such mode convertors were made of double- or triple-negative metamaterials [27–29], complete mode conversion might be theoretically possible, but such negative metamaterials are nearly impossible to fabricate because very restrictive conditions must be satisfied between the metamaterial and the background medium.

Now, we examine the possibility to achieve the full mode-converting transmission from the perspective of impedance matching. Because the impedance matching for the mode-preserving case requires the formation of a quarter wave inside the matching element, we postulate that a similar phase-matching mechanism could be possible even in the mode-converting case involving two different modes. With this postulation, we first consider the case described in Fig. 1(b). Here, we assume that the matching element is made of a general anisotropic medium with its stiffness coefficients $C_{11}$, $C_{66}$, and $C_{16}$ (longitudinal, shear, and longitudinal-shear coupling stiffness coefficients, respectively) and density $\rho$. Note that no mode conversion is possible for plane-wave incidence if $C_{16} = 0$. The material properties of the background isotropic medium are denoted by $c_{11}$, $c_{66}$, and $\rho_0$ (or Young's modulus $E_0$ and Poisson's ratio $\nu_0$). In this work, we will mainly consider the mode-converting transmission under an incident L wave, but the obtained results are equally applicable for the case of an incident T wave.

For a harmonic wave excited at frequency $\omega = 2\pi f$, the following two wave numbers, $k = \alpha$ and $k = \beta$, in an anisotropic medium can be expressed in terms of $\omega$ and the material properties by using the Christoffel equation (dispersion equation):

$$\alpha = \sqrt{\frac{\rho\omega^2(C_{11}+C_{66}) - \rho\omega^2\sqrt{(C_{11}-C_{66})^2+4C_{16}^2}}{2(C_{11}C_{66}-C_{16}^2)}},$$
$$\beta = \sqrt{\frac{\rho\omega^2(C_{11}+C_{66}) + \rho\omega^2\sqrt{(C_{11}-C_{66})^2+4C_{16}^2}}{2(C_{11}C_{66}-C_{16}^2)}}. \quad (2)$$

The analysis of the polarization vectors ($P_x^k$, $P_y^k$) ($k = \alpha, \beta$) shows that both waves with $k = \alpha$ and $k = \beta$ exhibit skew polarizations, implying that they are neither purely longitudinal nor purely transverse:

$$P_x^k = \frac{X_k}{\sqrt{1+|X_k|^2}}, \quad P_y^k = \frac{1}{\sqrt{1+|X_k|^2}}, \quad (3a)$$

where

$$X_k = -\frac{C_{16}k^2}{C_{11}k_x^2 - \rho\omega^2} = -\frac{C_{66}k^2 - \rho\omega^2}{C_{16}k^2}(k = \alpha, \beta). \quad (3b)$$

Therefore, each mode has nonzero horizontal (longitudinal) $u_x$ and vertical (transverse) $u_y$ displacement components. This skew polarization field pattern is inevitable for mode conversion. Because $\alpha < \beta$, the corresponding modes will be referred to as the fast-skew and slow-skew modes, respectively.

The L-to-T ($T_T$) and L-to-L ($T_L$) transmission power ratios and the L-to-T ($R_T$) and L-to-L ($R_L$) reflection power ratios with respect to the input L wave power are given by (under the plane strain assumption):

$$T_T = \xi|C_{LT}^T|^2, \quad T_L = |C_{LL}^T|^2,$$
$$R_T = \xi|C_{LT}^R|^2, \quad R_L = |C_{LL}^R|^2, \quad (4)$$

where $C_{LL}^R$, $C_{LT}^R$, $C_{LL}^T$, and $C_{LT}^T$ denote the reflection and transmission coefficients, and $\xi \triangleq (\beta_0 c_{66})/(\alpha_0 c_{11})$(see the Supplemental Material [30] for more detailed derivations). The symbols $\alpha_0$ and $\beta_0$ refers to the Land T wave numbers in the background isotropic medium. If the full mode-converting transmission is possible, the following conditions must be identically satisfied:

$$T_T = 1, \quad T_L = R_T = R_L = 0, \quad (5)$$

and, equivalently,

$$|C_{LT}^T| = \sqrt{1/\xi}, \quad C_{LL}^T = C_{LT}^R = C_{LL}^R = 0. \quad (6)$$

The conditions given by Eq. (6) imply that the scattering matrix **S** must have [30]

$$S_{41} = 0, \quad (7a)$$
$$S_{21} = 0, \quad (7b)$$
$$S_{22} = 0. \quad (7c)$$

Because the impedance matching for the classical mode-preserving transmission requires the formation of a quarter wave inside the matching element [$\sin kd = \pm 1$, i.e., Eq. (1b)], we postulate that a similar phase-matching mechanism would be valid in the mode-converting case. Since there are two wave modes, the fast-skew and slow-skew modes, the phase-matching conditions should be simultaneously satisfied for both of them:

$$\sin(\alpha d) = \pm 1, \quad \sin(\beta d) = \mp 1. \tag{8}$$

Using Eq. (8), one can derive the following condition, which we call the "bimodal" quarter-wave phase-matching condition (Supplemental Material [30]):

$$d = m n_{\mathrm{FS}} \frac{\lambda_{\mathrm{FS}}}{4}, \tag{9a}$$

$$d = m n_{\mathrm{SS}} \frac{\lambda_{\mathrm{SS}}}{4}, \tag{9b}$$

where $m = 1, 3, 5, \cdots$; $\lambda_{\mathrm{FS}}$ and $\lambda_{\mathrm{SS}}$ are the wavelengths of the fast-skew and slow-skew modes, respectively; and the coprime integers $n_{\mathrm{FS}}$ and $n_{\mathrm{SS}}$ must satisfy

$$n_{\mathrm{SS}}/2 - n_{\mathrm{FS}}/2 = \mathrm{odd}. \tag{9c}$$

If the fundamental bimodal quarter-wave frequency (with $m = 1$) is $f_{\mathrm{BQW}}$, Eq. (9) states that the same phenomenon would repeatedly occur at $f = f_{\mathrm{BQW}}, 3f_{\mathrm{BQW}}, 5f_{\mathrm{BQW}}, \cdots$. By solving Eqs. (9) and (2), one can derive the relations between the material properties ($C_{ij}$, $\rho$) of the anisotropic slab for the bimodal phase-matching condition:

$$C_{11} + C_{66} = 16\rho f_{\mathrm{BQW}}^2 d^2 \left( \frac{1}{n_{\mathrm{FS}}^2} + \frac{1}{n_{\mathrm{SS}}^2} \right), \tag{10a}$$

$$\sqrt{C_{11}C_{66} - C_{16}^2} = \frac{16\rho f_{\mathrm{BQW}}^2 d^2}{n_{\mathrm{FS}} n_{\mathrm{SS}}}. \tag{10b}$$

Based on Eq. (8) or Eq. (9), we can show that the condition given by Eq. (7a) is equivalent to the following condition:

$$\rho\sqrt{C_{11}C_{66} - C_{16}^2} = \rho_0\sqrt{c_{11}c_{66}}. \tag{11}$$

The physical meaning of Eq. (11) can be clearly extracted if we introduce the so-called "bimodal" impedances $\tilde{Z}$ and $\tilde{Z}_0$ defined as

$$\tilde{Z} = \rho\sqrt{C_{11}C_{66} - C_{16}^2}, \tag{12a}$$

$$\tilde{Z}_0 = \rho_0\sqrt{c_{11}c_{66}} = \sqrt{(\rho_0 c_{11})(\rho_0 c_{66})}. \tag{12b}$$

In terms of $\tilde{Z}$ and $\tilde{Z}_0$, Eq. (11) is written in compact form as

$$\tilde{Z} = \tilde{Z}_0. \tag{12c}$$

Therefore, the condition given by Eq. (11) or (12c) can be called the bimodal impedance-matching condition. Clearly, the condition (12c) can be viewed as a generalized version of the conventional impedance-matching condition in Eq. (1a) that is only valid for unimodal transmission.

Using Eqs. (7b) and (7c), one can obtain the following relation:

$$P_y^\alpha P_x^\beta = -\frac{\beta C_{66} + \alpha C_{11}}{(\alpha + \beta)(C_{11} + C_{66})}. \tag{13}$$

The use of Eq. (3) yields another expression for $P_y^\alpha P_x^\beta$:

$$P_y^\alpha P_x^\beta = \frac{C_{11} - C_{66}}{2\sqrt{(C_{11} - C_{66})^2 + 4C_{16}^2}} - \frac{1}{2}. \tag{14}$$

Equating Eqs. (13) and (14) yields

$$C_{11} = C_{66}, \tag{15a}$$

$$C_{16} = 0. \tag{15b}$$

By using Eqs. (3) and (15a), one can find that

$$P_x^\alpha = P_y^\alpha = \frac{1}{\sqrt{2}}, \quad P_x^\beta = -P_y^\beta = \frac{1}{\sqrt{2}}. \tag{16}$$

Therefore, Eq. (15a) physically means that the fast-skew and slow-skew modes inside the anisotropic slab are polarized by 45° and −45°, respectively. Therefore, the condition in Eq. (15a) can be called the polarization condition.

The analysis above shows that unlike the conventional unimodal transmission, the full mode-converting transmission requires that the matching element possess specific anisotropy as stated by Eqs. (10a), (10b), (12c), (15a), and (15b). However, there is no possible combination of ($C_{11}, C_{66}, C_{16}, \rho$) that can satisfy these five equations simultaneously. Accordingly, one condition must be relaxed to obtain nontrivial material properties and thus we propose to relax Eq. (15b). Relaxing the $C_{16} = 0$ condition to the condition of a small nonzero $C_{16}$ would result in weak longitudinal-shear coupling.

Solving Eqs. (10a), (10b), (12c), and (15a) for ($C_{11}$, $C_{66}$, $C_{16}$, $\rho$) yields

$$C_{11} = C_{66} = \frac{\tilde{Z}_0}{2\rho}\left(\frac{n_{\mathrm{SS}}}{n_{\mathrm{FS}}} + \frac{n_{\mathrm{FS}}}{n_{\mathrm{SS}}}\right), \quad C_{16} = \frac{\tilde{Z}_0}{2\rho}\left(\frac{n_{\mathrm{SS}}}{n_{\mathrm{FS}}} - \frac{n_{\mathrm{FS}}}{n_{\mathrm{SS}}}\right), \tag{17a}$$

$$\rho = \frac{\sqrt{\tilde{Z}_0 n_{\mathrm{FS}} n_{\mathrm{SS}}}}{4 f_{\mathrm{BQW}} d}. \tag{17b}$$

The amount of the relaxation may be quantified in terms of a small parameter $\varepsilon$ defined as

$$\varepsilon = \frac{2C_{16}}{C_{11} + C_{66}} = \frac{n_{\mathrm{SS}}^2 - n_{\mathrm{FS}}^2}{n_{\mathrm{SS}}^2 + n_{\mathrm{FS}}^2}. \tag{18}$$

Clearly, Eq. (15b) can be asymptotically satisfied as $n_{\mathrm{FS}}$ and $n_{\mathrm{SS}} \to \infty$ because $\varepsilon \to 0$.

At this point, it is worth summarizing the conditions that have been derived for the full mode-converting transmission:

(a) the bimodal quarter-wave phase-matching condition, Eq. (9);

(b) the bimodal impedance-matching condition, Eq. (12);

(c) the polarization condition, Eq. (15a);

(d) the weak mode-coupling condition with a sufficiently small $\varepsilon$.

To interpret the physical significance of these conditions, we note that the bimodal quarter-wave phase-matching and polarization conditions enable the incident longitudinal wave to become a transverse wave at the existing side of the anisotropic quarter-wave converter. On the other hand, the bimodal impedance-matching condition ensures the full-power transmission through the converter involving mode conversion (Supplemental Material [30]).

Using the conditions (a–c) above, we can write the transmission and reflection coefficients explicitly in terms of $\varepsilon$ as

$$C_{\mathrm{LL}}^{\mathrm{R}}(f_{\mathrm{BQW}}) = \frac{(\xi^2-1)(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon})^2}{(\xi-1)^2(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon})^2+4\xi(\sqrt{1+\varepsilon}+\sqrt{1-\varepsilon})^2},$$
$$C_{\mathrm{LT}}^{\mathrm{R}}(f_{\mathrm{BQW}}) = \frac{2(\xi+1)\varepsilon}{(\xi-1)^2(1-\sqrt{1-\varepsilon^2})+4\xi(1+\sqrt{1-\varepsilon^2})},$$
$$C_{\mathrm{LL}}^{\mathrm{T}}(f_{\mathrm{BQW}}) = S_{11}+S_{12}C_{\mathrm{LL}}^{\mathrm{R}},$$
$$C_{\mathrm{LL}}^{\mathrm{T}}(f_{\mathrm{BQW}}) = S_{31}+S_{34}C_{\mathrm{LT}}^{\mathrm{R}}, \quad (19a)$$

where

$$S_{11}(f_{\mathrm{BQW}}) = \frac{-j}{4}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{-\frac{1}{4}}-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}}\right](\xi^{\frac{1}{2}}-\xi^{-\frac{1}{2}}), \quad S_{12}(f_{\mathrm{BQW}}) = \frac{-j}{4}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{-\frac{1}{4}}-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}}\right](\xi^{\frac{1}{2}}+\xi^{-\frac{1}{2}}),$$
$$S_{31}(f_{\mathrm{BQW}}) = \frac{-j}{2}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}}+\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{-\frac{1}{4}}\right]\xi^{-\frac{1}{2}}, \quad S_{34}(f_{\mathrm{BQW}}) = \frac{-j}{4}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}}-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{-\frac{1}{4}}\right](\xi^{\frac{1}{2}}+\xi^{-\frac{1}{2}}). \quad (19b)$$

With the weak mode-coupling condition (d), one can show that

$$\lim_{\varepsilon\to 0} C_{\mathrm{LL}}^{\mathrm{R}}(f_{\mathrm{BQW}}) = 0, \quad \lim_{\varepsilon\to 0} C_{\mathrm{LT}}^{\mathrm{R}}(f_{\mathrm{BQW}}) = 0,$$
$$\lim_{\varepsilon\to 0} C_{\mathrm{LL}}^{\mathrm{T}}(f_{\mathrm{BQW}}) = 0, \quad \lim_{\varepsilon\to 0} C_{\mathrm{LT}}^{\mathrm{T}}(f_{\mathrm{BQW}}) = \sqrt{1/\xi}. \quad (20)$$

Therefore, the conditions given by Eq. (6) are shown to be asymptotically satisfied. According to Eq. (4),

$$\lim_{\varepsilon\to 0} R_{\mathrm{L}}(f_{\mathrm{BQW}}) = 0, \quad \lim_{\varepsilon\to 0} R_{\mathrm{T}}(f_{\mathrm{BQW}}) = 0,$$
$$\lim_{\varepsilon\to 0} T_{\mathrm{L}}(f_{\mathrm{BQW}}) = 0, \quad \lim_{\varepsilon\to 0} T_{\mathrm{T}}(f_{\mathrm{BQW}}) = 1. \quad (21)$$

The asymptotic behavior of $T_{\mathrm{T}}$ and $T_{\mathrm{L}}$ discussed above is demonstrated in Fig. 2 as $\varepsilon$ goes to zero. Two different cases are presented for the background media of aluminum ($\nu_0 = \nu_{\mathrm{Al}} = 0.33$) and silicone rubber ($\nu_0 = \nu_{\mathrm{SR}} = 0.49$). For both cases, $T_{\mathrm{L}}$ and $T_{\mathrm{T}}$ approach zero and 1, respectively, at $f = f_{\mathrm{BQW}}$ as $\varepsilon \to 0$.

| $(n_{\mathrm{FS}}, n_{\mathrm{SS}})$ | (1,3) | (3,5) | (5,7) | (7,9) | ... | (25,27) | ... |
|---|---|---|---|---|---|---|---|
| $\epsilon$ | 0.8 | 0.4706 | 0.3243 | 0.2462 | ... | 0.0768 | ... |

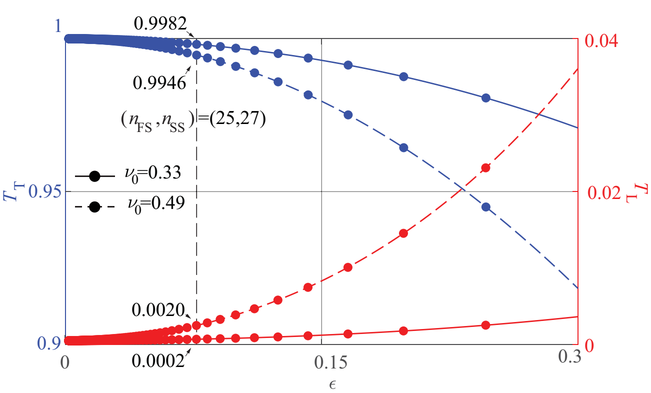


FIG. 2. The asymptotic behavior of $T_{\mathrm{T}}$ and $T_{\mathrm{L}}$ as a function of $\varepsilon = (n_{\mathrm{SS}}^2 - n_{\mathrm{FS}}^2)/(n_{\mathrm{SS}}^2 + n_{\mathrm{FS}}^2)$ at $f = f_{\mathrm{BQW}}, 3f_{\mathrm{BQW}}, \cdots$. Here, we varied $n_{\mathrm{FS}}$ only while $n_{\mathrm{SS}} = n_{\mathrm{FS}} + 2$ to facilitate plotting.

In that no natural material can satisfy the extreme properties stated by Eq. (17), the bimodal matching element must be designed using metamaterials. Here, we try to design a nonresonant metamaterial unit cell by cutting two elaborately perpendicular elliptical slits [see Fig. 3(a)]. We assume that the unit cell size $l$ is much smaller than the matching element width $d$.

Figure 3 depicts the realized full mode-converting transmission phenomenon and the metamaterial of the bimodal matching element inserted in an aluminum background medium. The single-phased unit cell is made in a steel base medium with the geometric parameters listed in Fig. 3(a). The design procedure with the selected parameters of ($n_{\mathrm{FS}}$, $n_{\mathrm{SS}}$) = (25, 27) is presented in the Supplemental Material [30]. With the effective stiffness coefficients calculated by the homogenization method [31], one can find that the bimodal impedance-matching condition is nearly satisfied with $\tilde{Z}/\tilde{Z}_0 = 0.9992$, as required by Eq. (12c). The bimodal quarter-wave phase-matching condition at frequency $f_{\mathrm{BQW}}$ is also satisfied with $(4d/\lambda_{\mathrm{FS}}, 4d/\lambda_{\mathrm{SS}}) = (25.0029, 27.0020)$.

Figure 3(b) plots a snapshot of the displacement field at $f_{\mathrm{BQW}}$ under a time-harmonic L wave incidence prescribed by $u_x = \sin(2\pi f_{\mathrm{BWQ}} t)$ at $x = -d/2$ ($t$ is time). It clearly shows that the incident L wave is nearly completely mode converted. The calculated vertical displacement amplitude $|u_y|$ after transmission is 1.4070, which is sufficiently close to the theoretical value $|C_{\mathrm{LT}}^{\mathrm{T}}| = \sqrt{1/\xi} = 1.4080$ [see Eq. (6)]. Figure 3(c) shows the power transmission curves as a function of frequency $f$. At $f = f_{\mathrm{BQW}}$ and $3f_{\mathrm{BQW}}$, the ratio $T_{\mathrm{T}}$ reaches almost unity while $T_{\mathrm{L}}$ is nearly zero. We also carried out finite element simulations by modeling the matching element with the detailed microstructures shown in Fig. 3(a) (with $l = d/300$). The phenomenon occurring at $f = 2f_{\mathrm{BQW}}$, exhibiting $T_{\mathrm{L}} \approx 1$ and $T_{\mathrm{T}} \approx 0$, is explained in the Supplemental Material [30].

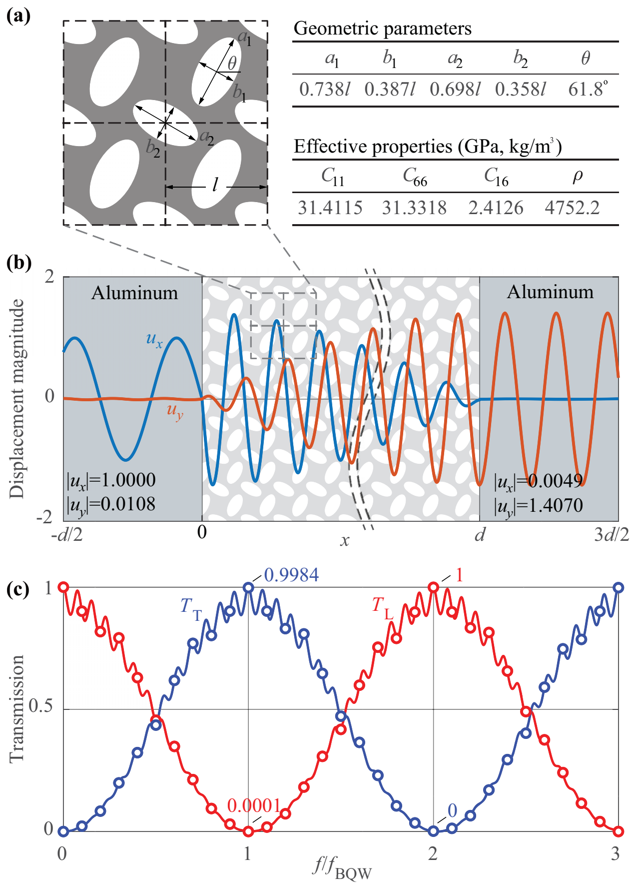


FIG. 3. The realized full mode-converting transmission phenomenon and the metamaterial unit cell of the bimodal quarter-wave impedance-matching element inserted in an aluminum (Al) background medium. (a) The unit cell geometry and effective material properties of the designed anisotropic metamaterial. The base material is steel (Fe). (b) A snapshot of the displacement field under an L wave of frequency $f_{\mathrm{BQW}}$ incidence at $x = -d/2$. (c) Transmission power ratios through the designed matching element as a function of frequency $f$; lines: analytical results using effective material properties; dots: numerical results using detailed microstructures. $E_{\mathrm{Al}} = 71\,\mathrm{GPa}$, $\nu_{\mathrm{Al}} = 0.33$, and $\rho_{\mathrm{Al}} = 2700\,\mathrm{kg/m^3}$; $E_{\mathrm{Fe}} = 213\,\mathrm{GPa}$, $\nu_{\mathrm{Fe}} = 0.286$, and $\rho_{\mathrm{Fe}} = 7900\,\mathrm{kg/m^3}$. $f_{\mathrm{BQW}} = 169.91\,\mathrm{kHz}$, $d = 10\,\mathrm{cm}$.

It is remarked that even in a nearly incompressible medium such as silicone rubber, the proposed theory achieves full mode-converting transmission [30], but the transmodal Fabry-Perot resonance [22] can only realize a low-efficiency mode-converting transmission because of the significant intermodal impedance mismatch.

Let us now consider the case in Fig. 1(c), where the materials of the incident and transmitted sides are dissimilar. For the full mode-converting transmission from medium A carrying the L wave to medium B carrying the T wave, the (asymptotic) theory of bimodal quarter-wave impedance matching established above remains valid except for the bimodal impedance-matching condition. Namely, Eq. (12b) should be replaced by a more general formula:

$$\tilde{Z}_0 = \tilde{Z}_0^{\mathrm{A_L\text{-}B_T}} = \sqrt{\rho_0^{\mathrm{A}} c_{11}^{\mathrm{A}} \rho_0^{\mathrm{B}} c_{66}^{\mathrm{B}}}. \tag{22}$$

Here, we introduced the superscript $\mathrm{A_L}$-$\mathrm{B_T}$ to indicate that the L wave incident from medium A is to be converted to the T mode to medium B. Therefore, $\tilde{Z}_0^{\mathrm{B_L\text{-}A_T}} = \sqrt{\rho_0^{\mathrm{B}} c_{11}^{B} \rho_0^{\mathrm{A}} c_{66}^{\mathrm{A}}} \neq \tilde{Z}_0^{\mathrm{A_L\text{-}B_T}}$, implying that the bimodal matching element designed for the $\mathrm{A_L}$-$\mathrm{B_T}$ transmission cannot realize the $\mathrm{B_L}$-$\mathrm{A_T}$ transmission. However, for the $\mathrm{B_T}$-$\mathrm{A_L}$ transmission, the required bimodal impedance $\tilde{Z}_0^{\mathrm{B_T\text{-}A_L}}$ is equally given by Eq. (22), owing to the time-reversal symmetry. In addition to the redefinition of the bimodal impedance, the parameter $\xi$ is redefined as $\xi = \sqrt{(\rho_0^{\mathrm{B}} c_{66}^{\mathrm{B}})/(\rho_0^{\mathrm{A}} c_{11}^{\mathrm{A}})}$.

Figures 4(a) and 4(b) present the results for transmission from aluminum (Al) to lead (Pb) and from Pb to Al, respectively. The metamaterial unit cells have the same configurations as illustrated in Fig. 3(a) while their detailed geometric parameters are listed in Fig. 4. The base medium of the metamaterials is steel and the chosen values of $(n_{\mathrm{FS}}, n_{\mathrm{SS}})$ are (25, 27). At $f = f_{\mathrm{BQW}}$, the impedance ratios are found to be $\tilde{Z}^{\mathrm{Al_L\text{-}Pb_T}}/\tilde{Z}_0^{\mathrm{Al_L\text{-}Pb_T}} = 0.9967$ and $\tilde{Z}^{\mathrm{Pb_L\text{-}Al_T}}/\tilde{Z}_0^{\mathrm{Pb_L\text{-}Al_T}} = 0.9996$, which are nearly equal to 1. The bimodal quarter-wave phase-matching condition at frequency $f_{\mathrm{BQW}}$ is satisfied with $(4d/\lambda_{\mathrm{FS}}, 4d/\lambda_{\mathrm{SS}}) = (24.9947, 27.0034)$, and (24.9979, 27.0043), respectively, for the $\mathrm{Al_L}$-$\mathrm{Pb_T}$ and $\mathrm{Pb_L}$-$\mathrm{Al_T}$ cases.

The displacement fields and the L-to-T power transmission ratios plotted in Fig. 4 support the (nearly) full mode-converting transmission phenomenon. The vertical displacement amplitude after transmission $|u_y|$ calculated from the numerical results is nearly equal to the ideal theoretical result $|C_{\mathrm{LT}}^{\mathrm{T}}| = \sqrt{1/\xi}$ for an incident L wave of unit magnitude:

$\mathrm{Al_L}$-$\mathrm{Pb_T}$ case: 1.4543 (simulation), 1.4572 (theory);
$\mathrm{Pb_L}$-$\mathrm{Al_T}$ case: 1.6893 (simulation), 1.6901 (theory).

The bottom plots in Fig. 4 clearly show the repeated peaks at $f = f_{\mathrm{BQW}}, 3f_{\mathrm{BQW}}, \cdots$, with $T_{\mathrm{T}} = 0.9987$ for the $\mathrm{Al_L}$-$\mathrm{Pb_T}$ case and $T_{\mathrm{T}} = 0.9975$ for the $\mathrm{Pb_L}$-$\mathrm{Al_T}$ case, which confirm the realization of the (nearly) full mode-converting transmission. We can also show that $T_{\mathrm{L}}$ at $f = 2f_{\mathrm{BQW}}, 4f_{\mathrm{BQW}}, \cdots$, is $T_L = 1 - R_L = 4\zeta/(1+\zeta)^2$ with $\zeta = \sqrt{(\rho_0^{\mathrm{B}} c_{11}^{\mathrm{B}})/(\rho_0^{\mathrm{A}} c_{11}^{\mathrm{A}})}$, which becomes 0.9676 for both the cases considered here.

In this study, an asymptotic theory of bimodal quarter-wave impedance matching was established for full mode-converting transmission from longitudinal (transverse) to transverse (longitudinal) waves. The derived impedance-matching condition requires that $\tilde{Z} = \rho\sqrt{C_{11}C_{66} - C_{16}^2}$ of the matching element should be equal to $\tilde{Z}_0 = \tilde{Z}_0^{\mathrm{A_L\text{-}B_T}} = \sqrt{\rho_0^{\mathrm{A}} c_{11}^{\mathrm{A}} \rho_0^{\mathrm{B}} c_{66}^{\mathrm{B}}}$ of the background media A and B carrying L and T modes, respectively. This condition degenerates to the classical matching condition for the case of unimodal transmission. The phase-matching conditions for two L-T coupled modes state that the matching element width should be multiples of their quarter waves, but the multiplicities cannot be arbitrary. It also shows that 100% mode-converting transmission is only asymptotically possible when $C_{16}$ of the matching element approaches zero. However, the matching element satisfying the asymptotic bimodal matching conditions is shown to achieve nearly full mode-converting transmission.

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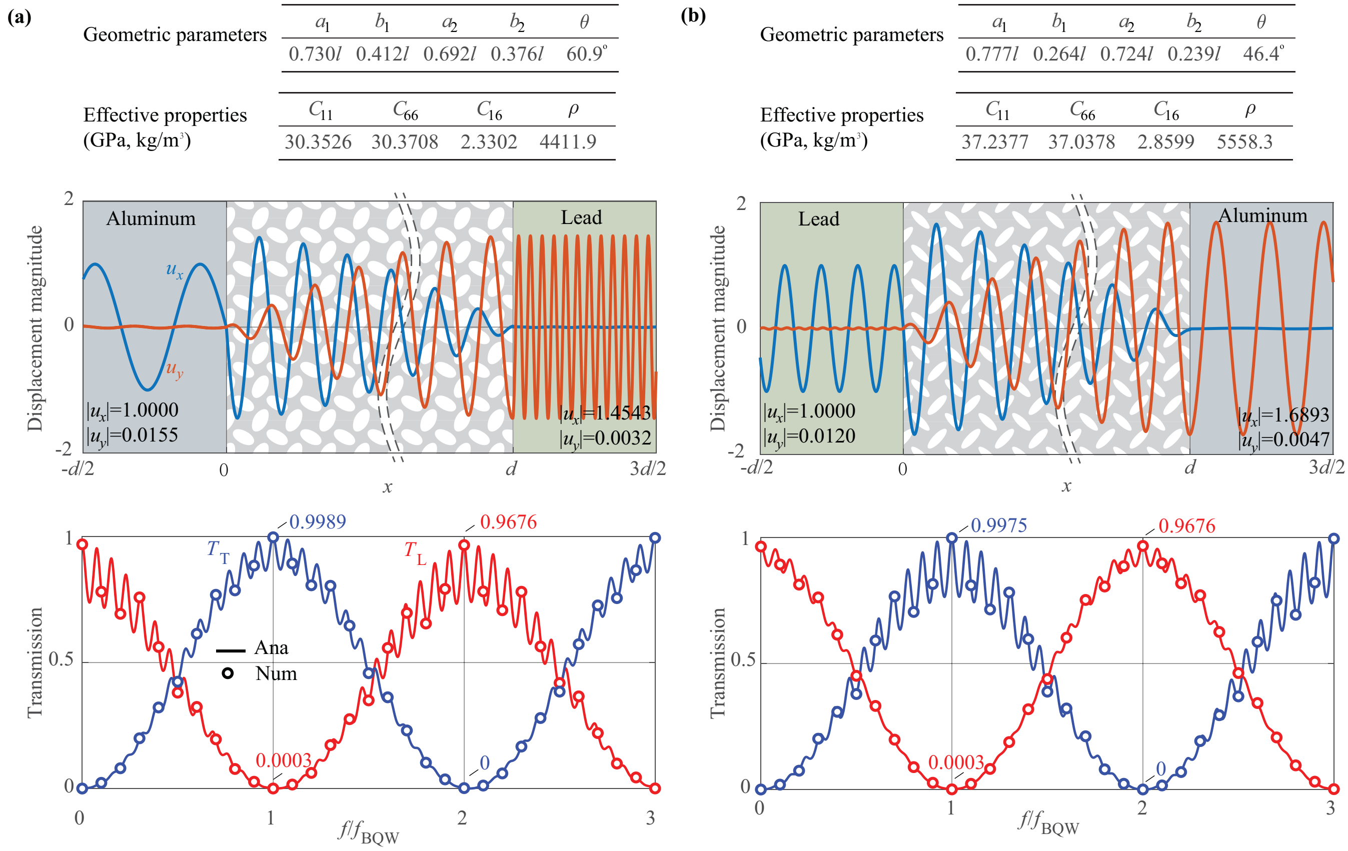


FIG. 4. The realized full mode-converting transmission phenomenon through bimodal impedance-matching elements between two dissimilar isotropic media of lead (Pb) and aluminum (Al). (a) $Al_L$-$Pb_T$ case with $f_{BQW} = 170.165$ kHz, and (b) $Pb_L$-$Al_T$ case with $f_{BQW} = 167.648$ kHz. In the transmission plots, lines: analytical results using effective material properties; dots: numerical results using detailed microstructures. $d = 10$ cm; $E_{Pb} = 16$ GPa, $\nu_{Pb} = 0.44$, and $\rho_{Pb} = 11\,340$ kg/m$^3$.

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