Anomaly and global inconsistency matching: θ angles, $SU(3)/U(1)^2$ nonlinear sigma model, $SU(3)$ chains, and generalizations

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(Received 5 June 2018; revised manuscript received 30 August 2018; published 14 September 2018)

We discuss the $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model in 1+1D and, more broadly, its linearized counterparts. Such theories can be expressed as $U(1)\times U(1)$ gauge theories and therefore allow for two topological θ angles. These models provide a field theoretic description of the $SU(3)$ chains. We show that, for particular values of θ angles, a global symmetry group of such systems has a 't Hooft anomaly, which manifests itself as an inability to gauge the global symmetry group. By applying anomaly matching, the ground-state properties can be severely constrained. The anomaly matching is an avatar of the Lieb-Schultz-Mattis (LSM) theorem for the spin chain from which the field theory descends, and it forbids a trivially gapped ground state for particular θ angles. We generalize the statement of the LSM theorem and show that 't Hooft anomalies persist even under perturbations which break the spin-symmetry down to the discrete subgroup $\mathbb{Z}_3\times\mathbb{Z}_3\subset SU(3)/\mathbb{Z}_3$. In addition, the model can further be constrained by applying global inconsistency matching, which indicates the presence of a phase transition between different regions of θ angles. We use these constraints to give possible scenarios of the phase diagram. We also argue that at the special points of the phase diagram the anomalies are matched by the $SU(3)$ Wess-Zumino-Witten model. We generalize the discussion to the $SU(N)/U(1)^{N-1}$ nonlinear sigma models as well as the 't Hooft anomaly of the $SU(N)$ Wess-Zumino-Witten model, and show that they match. Finally, the $(2+1)$-dimensional extension is considered briefly, and we show that it has various 't Hooft anomalies leading to nontrivial consequences.

DOI: 10.1103/PhysRevB.98.115126

I. INTRODUCTION

Spin chain is an important subject of many-body physics, and has been studied extensively both in classical and quantum mechanical contexts. It also gives examples of how striking differences can arise between quantum mechanics and classical analogues. Amongst, the most studied spin chains would be the Heisenberg $SO(3)$ spin chain,[1] with the Hamiltonian of the form

$$H=J\sum_{\langle i,j\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_j,$$

where $\boldsymbol{S}_i$ is the spin vector at the lattice site i. When $J>0$, the interactions prefer anti-ferromagnetic order.

The quantum variants of such chains were conjectured by Haldane to behave radically different when spin is integer or half-integer [1,2]. In particular, by studying large-dimensional $SU(2)$ representations on each site, Haldane argued that integer and half-integer Heisenberg spin chains fall into different universality classes: the former being gapped, while the latter is gapless. The more modern perspective claims that the gapless nature of half-integer spin chains is understood as a consequence of the Lieb-Schultz-Mattis (LSM) theorem [3–6], which is a powerful theorem exploiting the fact that $SO(3)$ spin rotation acts projectively on half-integer spins. More precisely, the LSM theorem proves that either the antiferromagnetic chain is gapless or breaks translational symmetry spontaneously. Therefore the Haldane conjecture may be rephrased that as long as spin symmetry and lattice translation symmetry are good symmetries, the integer antiferromagnetic spin chains have trivial ground states, while half-integer ones are nontrivial.[2] The conjecture is confirmed explicitly by exactly solvable systems, like Bethe ansatz on spin-1/2 chain [7] and AKLT model for the spin-1 chain [8].

Generalization of $SU(2)$ chains to $SU(N)$ chains has attracted the interest in various aspects. In fact, the LSM theorem is also known for $SU(N)$ chains [4], showing a nontrivial nature of the ground states depending on the representation. Taking the large representation limit, some spin systems can again be described by nonlinear sigma models which are

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[1] We will also call this system an $SU(2)$ spin chain, to indicate that the quantum version can have states in both integer and half-integer spin representation.

[2] By a trivial ground state we mean that the system is gapped and the ground state is nondegenerate, while the nontrivial ground state is either gapless, breaks some global symmetry, or has topological degeneracy.

both asymptotically free, and have nontrivial ground states. For example, the critical nature of the $U(2N)/[U(N) \times U(N)]$ Grassmannian nonlinear sigma model was studied in Refs. [9,10]. Experimentally, there is a possibility to realize the $SU(N)$ chains via ultracold atoms [11–21], and theoretical conjectures on $SU(N)$ spin systems can be tested in the future.

Bykov [22,23] has derived the relativistic sigma models from antiferromagnetic $SU(N)$ spin chains with the p-box symmetric representation on each site. There, it is shown that the effective theory of a specific spin chain has the flag-manifold target space, $SU(N)/U(1)^{N-1}$. There, it was pointed out that such a sigma model allows $N-1$ independent topological θ terms [22], $\theta_1, \dots, \theta_{N-1}$, and they take the specific value [23] $\theta_k = 2\pi k/N$. Lajkó, Wamer, Mila, and Affleck [24] have recently analyzed the phase structure of the $SU(3)$ spin chains with the p-box symmetric representation using the $SU(3)/[U(1) \times U(1)]$ nonlinear sigma model. They showed the LSM theorem for the $SU(3)/\mathbb{Z}_3$ spin symmetry and the lattice translation for $p \neq 0$ mod 3, and thus the trivial mass gap cannot appear. They also analyzed the lattice strong-coupling limit to gain insight into the phase diagram, and performed a Monte Carlo simulation to check it using imaginary θ angles following Ref. [25].

In this paper, we shall show that the symmetry itself can constrain the possible phase diagram more strongly. For that purpose, we study the $SU(3)/[U(1) \times U(1)]$ nonlinear sigma model from the viewpoint of the 't Hooft anomaly matching and global inconsistency matching. 't Hooft anomaly is the obstruction to gauging the global symmetry. The consequence of this is that the vacuum cannot be trivially gapped [26–28] (see also Refs. [29–56] for recent developments). The 't Hooft anomaly matching provides the field-theoretic description of the LSM theorem for corresponding lattice quantum systems, and we can reproduce the same constraint on the possible low-energy physics. Global inconsistency condition is a more subtle obstruction while gauging the symmetry [40,42,49].

In our nonlinear sigma model, the spin rotation symmetry, $PSU(3) = SU(3)/\mathbb{Z}_3$, is a good symmetry for all the θ angles, but for special values of the θ angles there also exists a charge conjugation symmetry C. At all C-symmetric points, we can gauge the whole $PSU(3) \rtimes \mathsf{C}$ symmetry, so there is no 't Hooft anomaly for $PSU(3) \rtimes \mathsf{C}$. However, gauging of C requires a special choice of the discrete θ parameter of $PSU(3)$ gauge fields, and thus they can be different for different C invariant θ angles. When this occurs, we say that different regions of the parameter space have a global inconsistency [40,42,49]. A consequence of this inconsistency is that either (1) the two regions are trivially gapped, but one must encounter a phase transition in between or (2) the ground state of one of the two C-invariant regions is nontrivial. By using the matching condition for both 't Hooft anomaly and global inconsistency, we will find the constraints on the phase diagram that go beyond the LSM theorem.

We will see that the whole discussion of anomalies and global inconsistencies can be generalized to the $SU(N)/U(1)^{N-1}$ nonlinear sigma models and their linear counterparts. In particular, such models have a $PSU(N) = SU(N)/\mathbb{Z}_N$ global spin, or flavor, symmetry. They also allow $N-1$ topological θ angles. At particular values of the θ angles, they also have $\mathbb{Z}_N$ global symmetry, which we call the $\mathbb{Z}_N$ cyclic permutation symmetry.[3] The two symmetries, $PSU(N)$ and $\mathbb{Z}_N$, have a mixed 't Hooft anomaly. Moreover, the subgroup $\mathbb{Z}_N \times \mathbb{Z}_N \subset PSU(N)$ also has a mixed 't Hooft anomaly with the $\mathbb{Z}_N$ cyclic permutation symmetry. When N is even, we also show that there is a 't Hooft anomaly involving a time-reversal symmetry [54], and the phase diagram can be constrained even when the global spin symmetry is explicitly broken completely.

It is an interesting question to ask what is the possible conformal field theory if the 't Hooft anomaly is matched by the existence of gapless excitations. In order to explore it, we consider the two-dimensional $SU(N)$ Wess-Zumino-Witten (WZW) model and find the correspondence for symmetries and their 't Hooft anomaly. The anomaly can constrain the possible level number of WZW model. The computation of anomaly shall be done by gauging the symmetry of WZW models directly, and we will find the anomaly polynomial described by the $(2+1)$-dimensional symmetry-protected topological (SPT) phase. In order to elucidate why these two models have the same 't Hooft anomaly, we consider a deformation of the WZW model which reduces the $[SU(N)_L \times SU(N)_R]/\mathbb{Z}_N$ global symmetry to the $PSU(N)_V \times \mathbb{Z}_N$ symmetry. As a result, we obtain an $SU(N)/U(1)^{N-1}$ nonlinear sigma model, and hence they must contain the same 't Hooft anomaly.

The linear sigma model description is also discussed, and it provides a useful consistency check of the phase diagram when all the matter fields are very massive. In that limit, the theory becomes a gauge theory of free photons, and we clarify the concrete consequence of the 't Hooft anomaly and global inconsistency using that example. We also propose the circle compactification of the model so that the 't Hooft anomaly discussed in this paper persists for any size of the compactification radius. Since the model has asymptotic freedom, this provides an opportunity to study the $SU(3)/[U(1) \times U(1)]$ sigma model semiclassically.

We also discuss the $(2+1)$-dimensional version of our model very briefly. While it does not have the θ terms, it contains a $U(1) \times U(1)$ topological symmetry. We show that the model has various 't Hooft anomalies involving the topological symmetry, indicating that the model cannot be trivially gapped.

The paper is organized as follows. In Sec. II, we explain details about $SU(3)/[U(1) \times U(1)]$ nonlinear sigma model and its symmetries. We discuss their 't Hooft anomaly and global inconsistency in Sec. III, and their implication on the phase diagram is also discussed there. Section IV is devoted to the generalization of our analysis to $SU(N)/U(1)^{N-1}$ nonlinear sigma models, and an anomaly involving time-reversal is found for even N. We discuss the 't Hooft anomaly of the $SU(N)$ Wess-Zumino-Witten model in Sec. V. In Sec. VI, we construct the linear sigma model having the same 't Hooft anomaly and global inconsistency, and perform the analytic computation of the partition function in certain cases.

[3]The name comes from the fact that the relevant models involve N copies of fields, which can be mapped cyclically into each other, as we shall see.

In Sec. VII, we discuss the a small-circle compactification of the nonlinear sigma model, whose phase structure can be adiabatically connected to the large circle limit from the viewpoint of anomaly. We discuss the $(2+1)$-dimensional version of the model in Sec. VIII. We make conclusions in Sec. IX.

II. $SU(3)/[U(1)\times U(1)]$ SIGMA MODEL AND SYMMETRY

An $SU(3)$ spin chain with the p-box symmetric representation on each site can be described by a nonlinear sigma model whose target space is the flag manifold $SU(3)/[U(1)\times U(1)]$ with the specific θ terms $\theta=2\pi p/3$ in the large-p limit [22–24]. We first explain the nonlinear sigma model in Sec. II A, and discuss its symmetries in Sec. II B. To be self-contained, we briefly review its connection with the corresponding lattice spin Hamiltonian in Sec. II C following Refs. [22–24].

A. $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model

We consider the nonlinear sigma model with the target space $SU(3)/[U(1)\times U(1)]$. The Lagrangian is given by

$$S=\sum_{\ell=1}^{3}\int_{M_2}\left[-\frac{1}{2g}|(\mathrm{d}+\mathrm{i}a_\ell)\boldsymbol{\phi}_\ell|^2+\frac{\mathrm{i}\theta_\ell}{2\pi}\mathrm{d}a_\ell\right.$$
$$\left.+\frac{\lambda}{2\pi}(\overline{\boldsymbol{\phi}}_{\ell+1}\cdot\mathrm{d}\boldsymbol{\phi}_\ell)\wedge(\boldsymbol{\phi}_{\ell+1}\cdot\mathrm{d}\overline{\boldsymbol{\phi}}_\ell)\right],\tag{1}$$

where $\boldsymbol{\phi}_\ell=(\phi_{1,\ell},\phi_{2,\ell},\phi_{3,\ell}):M_2\to\mathbb{C}^3$ are three-component complex scalar fields with the constraint,

$$\overline{\boldsymbol{\phi}}_\ell\cdot\boldsymbol{\phi}_{\ell'}=\delta_{\ell\ell'},\tag{2}$$

$$\varepsilon_{abc}\phi_{a,1}\phi_{b,2}\phi_{c,3}=1,\tag{3}$$

and a_i are $U(1)$ gauge fields.[4] The constraint claims that the 3×3 matrix, $\mathcal{U}=[\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\boldsymbol{\phi}_3]$, is special unitary,

$$\mathcal{U}^\dagger\mathcal{U}=\mathbf{1}_3,\quad \det[\mathcal{U}]=1.\tag{4}$$

As we shall see soon later, the gauge fields obey the constraint $a_1+a_2+a_3=0$ as a consequence of the equation of motion, so the target space is divided by $U(1)\times U(1)$ by gauge invariance and becomes the flag manifold $SU(3)/[U(1)\times U(1)]$. The first term is the usual kinetic term of the nonlinear sigma model, and the second one is the topological θ term of the 2d $U(1)$ gauge theory. The last term is the new feature of this nonlinear sigma model, called the λ term in Ref. [24]. It is linear both in space and time derivatives, but not topologically quantized to integers unlike the θ terms. It will not be important for our discussion, as it will not be important for the 't Hooft anomaly matching. Furthermore, it is not a universal term of the underlying spin model and is also perturbatively irrelevant [24].

[4] We take the convention that the dynamical gauge fields are denoted by lowercases a_i, and that the background ones by uppercase $A,B,C,\ldots$, unless stated explicitly.

The λ-term may not look gauge-invariant at the first sight, so let us confirm it explicitly. Consider the $U(1)$ gauge transformation, $\boldsymbol{\phi}_\ell\mapsto g_\ell\boldsymbol{\phi}_\ell$ with $g_\ell:M_2\to U(1)$, then

$$\begin{aligned}\overline{\boldsymbol{\phi}}_{\ell+1}\cdot\mathrm{d}\boldsymbol{\phi}_\ell &\mapsto g_{\ell+1}^{-1}g_\ell\overline{\boldsymbol{\phi}}_{\ell+1}\cdot\mathrm{d}\boldsymbol{\phi}_\ell+\overline{\boldsymbol{\phi}}_{\ell+1}\cdot\boldsymbol{\phi}_\ell g_{\ell+1}^{-1}\mathrm{d}g_\ell\\ &= g_{\ell+1}^{-1}g_\ell\overline{\boldsymbol{\phi}}_{\ell+1}\cdot\mathrm{d}\boldsymbol{\phi}_\ell,\end{aligned}\tag{5}$$

and here we use the orthogonality condition. This proves the $U(1)$ gauge invariance of the λ term.

When $\lambda=0$, the Lagrangian looks like three independent copies of $\mathbb{C}P^2$ nonlinear sigma models, but they are still coupled via the orthonormality constraint. Consequence of the constraint on the topological charges is very important. Solving the equation of motion of a_ℓ, we find that

$$a_\ell=\frac{\mathrm{i}}{2}(\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_\ell-\mathrm{d}\overline{\boldsymbol{\phi}}_\ell\cdot\boldsymbol{\phi}_\ell)=\mathrm{i}\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_\ell.\tag{6}$$

As a result of orthonormality, we shall find that

$$Q_1+Q_2+Q_3=0,\tag{7}$$

where $Q_\ell=\frac{1}{2\pi}\int\mathrm{d}a_\ell$ are topological charges $\in\mathbb{Z}$, and thus the Lagrangian contain only two independent $U(1)$ topological charges. Therefore we can always set one of the θ angles equal to zero without loss of generality, and we will set $\theta_2=0$ following Ref. [24] in this and the next sections.

Let us derive this constraint on the topological charge. We can solve the constraints (2) and (3) for $\boldsymbol{\phi}_3$ uniquely using $\boldsymbol{\phi}_{1,2}$:

$$\phi_{3a}=\varepsilon_{abc}\overline{\phi}_{b,1}\overline{\phi}_{c,2}.\tag{8}$$

Using this expression, a_3 becomes

$$\begin{aligned}a_3&=\mathrm{i}\overline{\phi}_{a,3}\mathrm{d}\phi_{a,3}\\ &=\mathrm{i}\varepsilon_{abc}\phi_{b,1}\phi_{c,2}\mathrm{d}(\varepsilon_{ab'c'}\overline{\phi}_{b',1}\overline{\phi}_{c',2})\\ &=\mathrm{i}(-\overline{\boldsymbol{\phi}}_1\cdot\mathrm{d}\boldsymbol{\phi}_1-\overline{\boldsymbol{\phi}}_2\cdot\mathrm{d}\boldsymbol{\phi}_2).\end{aligned}\tag{9}$$

As a result, we find that

$$a_1+a_2+a_3=0.\tag{10}$$

Physical meaning of this constraint is that the sum of $U(1)$ charges of $\boldsymbol{\phi}_\ell'$s must be equal to zero, and this is indeed necessary for the condition (3) having gauge invariance. In particular, we obtain the constraint (7) on the topological charges by taking derivatives. Since the Lagrangian is quadratic in $U(1)$ gauge fields a_ℓ, this constraint obtained by the equation of motion holds at the quantum level.

B. Global symmetries

Next, we discuss the global symmetry of the model. There are four symmetries of this system: (1) $SU(3)/\mathbb{Z}_3$ flavor symmetry, (2) time reversal T, and (3) $\mathbb{Z}_3$ permutation symmetry (for special θ's), and (4) charge conjugations C (for different special θ's). We shall explain these symmetry.

Flavor symmetry $SU(3)/\mathbb{Z}_3$. The flavor symmetry acts on $\boldsymbol{\phi}_\ell$ as $\boldsymbol{\phi}_\ell\mapsto U\boldsymbol{\phi}_\ell$ for $U\in SU(3)$. U must be the same for ϕ_1, ϕ_2, and ϕ_3, in order to maintain the orthonormality (2) and also the λ term, and its determinant must be unity in order to maintain (3). $SU(3)$ acts faithfully on $\boldsymbol{\phi}_i$, but these operators are not $U(1)$ gauge invariant. The center $\mathbb{Z}_3\subset$

$SU(3)$ acts trivially on the local gauge-invariant operators, such as $\overline{\phi}_{a,\ell}\phi_{b,\ell}$, and thus the correct global symmetry is $PSU(3) = SU(3)/\mathbb{Z}_3$.

Time-reversal T. Time-reversal symmetry acts as

$$\begin{aligned}\mathsf{T}: \boldsymbol{\phi}_\ell(x,t) &\mapsto \overline{\boldsymbol{\phi}}_\ell(x,-t),\\ a_{\ell 0}(x,t) &\mapsto a_{\ell 0}(x,-t),\\ a_{\ell 1}(x,t) &\mapsto -a_{\ell 1}(x,-t).\end{aligned} \tag{11}$$

The kinetic term is invariant trivially. Under this definition of the time reversal, $\int \mathrm{d}a_\ell$ is invariant under the orientation flip of M_2, and thus the θ terms are time-reversal invariant at any θ angles. The λ term is also invariant as follows. Notice that $\overline{\boldsymbol{\phi}}_{\ell+1} \cdot \mathrm{d}\boldsymbol{\phi}_\ell(x,t) \mapsto \boldsymbol{\phi}_{\ell+1} \cdot \mathrm{d}\overline{\boldsymbol{\phi}}_\ell(x,-t)$. Since the wedge product anti-commutes, we get one negative sign for the λ term, but the linear time derivative gives another negative sign, so the action becomes invariant in total.

$\mathbb{Z}_3$ permutation. $\mathbb{Z}_3$ symmetry is the symmetry by the cyclic permutation of the fields

$$\boldsymbol{\phi}_i \mapsto \boldsymbol{\phi}_{\ell+1}, \quad a_i \mapsto a_{\ell+1}, \tag{12}$$

where the label ℓ should be identified mod 3. Under this transformation for $\theta_2 = 0$, the θ term changes as

$$\theta_1 Q_1 + \theta_3 Q_3 \mapsto \theta_1 Q_2 + \theta_3 Q_1 = (\theta_3 - \theta_1)Q_1 - \theta_1 Q_3. \tag{13}$$

In order for the $\mathbb{Z}_3$ permutation to be a symmetry, the θ angles must satisfy

$$2\theta_1 = \theta_3, \quad \theta_1 + \theta_3 = 0 \mod 2\pi. \tag{14}$$

As a result, the $\mathbb{Z}_3$ invariant points are

$$(\theta_1, \theta_3) = (0,0), (\pm 2\pi/3, \mp 2\pi/3) \mod 2\pi\mathbb{Z} \times 2\pi\mathbb{Z}. \tag{15}$$

Charge conjugations C. We again take the convention $\theta_2 = 0$. Let us define three different charge conjugation operators

$$\mathsf{C}_k : \boldsymbol{\phi}_\ell \mapsto -\overline{\boldsymbol{\phi}}_{-\ell-k}, a_\ell \mapsto -a_{-\ell-k}. \tag{16}$$

For example, $\mathsf{C}_2 : \boldsymbol{\phi}_{1(3)} \mapsto -\overline{\boldsymbol{\phi}}_{3(1)}$ and $\boldsymbol{\phi}_2 \mapsto -\overline{\boldsymbol{\phi}}_2$, so C_i acts on $\boldsymbol{\phi}_i$ as a complex conjugation, but other two fields are exchanged in addition to the complex conjugation. The negative sign on $\boldsymbol{\phi}$ fields is necessary for consistency with the constraint (3). Note that the three charge-conjugations differ by a $\mathbb{Z}_3$ symmetry, so at $\mathbb{Z}_3$ symmetric point, they really correspond to the same charge conjugation.

The kinetic and λ terms are invariant under this transformation, and the above reordering $\ell \mapsto -\ell - k$ mod 3 for some $k = 1, 2, 3$ is necessary for invariance of the λ term. The θ terms change nontrivially, and they are symmetry only for special θ angles.

For C_2, $Q_{1(3)} \mapsto -Q_{3(1)}$ and $Q_2 \mapsto -Q_2$, and then C_2 is the symmetry only if

$$\theta_1 Q_1 + \theta_3 Q_3 = -\theta_1 Q_3 - \theta_3 Q_1 \mod 2\pi, \tag{17}$$

for all $Q_{1,3} \in \mathbb{Z}$. This is solved as

$$\theta_1 = -\theta_3 \mod 2\pi, \tag{18}$$

and C_2-invariant points form parallel lines.

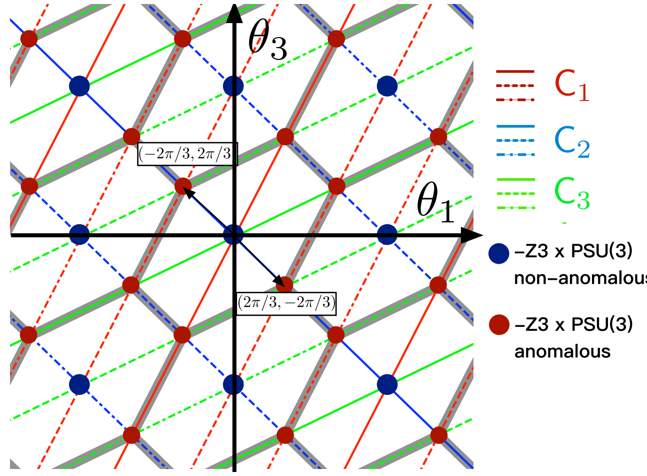

FIG. 1. The plot of the phase diagram of $SU(3)/[U(1) \times U(1)]$ nonlinear sigma model. The $\mathbb{Z}_3$-symmetric points are shown with blobs, and the blue blobs show that there is no 't Hooft anomaly for $PSU(3) \times \mathbb{Z}_3$ while the red ones show that there is the 't Hooft anomaly. The global inconsistency lines in the (θ_1, θ_3) plane for the symmetries $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$ are sketched as red, blue, and green lines. The solid, dashed, and dot-dashed lines indicate that different counterterms are needed to restore the corresponding C-symmetries when the $SU(3)/\mathbb{Z}_3$ symmetry is gauged, indicating that there is a global inconsistency between different-type lines (e.g., between solid and dashed). The inconsistency can be saturated either by at least one of these lines having a nontrivial ground state, or that they are separated by a phase transition.

For C_1, $Q_{2(3)} \mapsto -Q_{3(2)}$ and $Q_1 \mapsto -Q_1$, and it is a symmetry only if

$$\begin{aligned}\theta_1 Q_1 + \theta_3 Q_3 &= -\theta_1 Q_1 - \theta_3 Q_2\\ &= (\theta_3 - \theta_1)Q_1 + \theta_3 Q_3 \mod 2\pi,\end{aligned} \tag{19}$$

for all $Q_1, Q_3 \in \mathbb{Z}$. That is, the C_1-invariant points are

$$\theta_3 = 2\theta_1 \mod 2\pi, \tag{20}$$

and they form parallel lines. Similarly, the C_3-invariant points are

$$\theta_1 = 2\theta_3 \mod 2\pi. \tag{21}$$

These C_k invariant lines will be shown later in Fig. 1. In particular, we should notice that all C_k are symmetries at the $\mathbb{Z}_3$-invariant points. If we define the parity as the Euclid π rotation of the $\mathsf{C}_k\mathsf{T}$ transformation, then there are also three distinct parity transformations P_k,

$$\mathsf{P}_k : \boldsymbol{\phi}_\ell(x,t) \mapsto -\boldsymbol{\phi}_{-\ell-k}(-x,t), \tag{22}$$

and they are symmetries only for above special θ angles but not for general θ's. By construction, $\mathsf{C}_k\mathsf{P}_k\mathsf{T}$ is always a symmetry, as is required by the CPT theorem for relativistic field theories.

C. Lattice $SU(3)$ chains

D. Bykov [22,23] and also M. Lajkó, K. Wamer, F. Mila, and I. Affleck [24] have shown that the $SU(3)/[U(1) \times U(1)]$

nonlinear sigma model (1) is the field theoretic description of a certain $SU(3)$ spin chains in the large representation limit. The Hamiltonian in Ref. [24] is given by the antiferromagnetic nearest and next-to-nearest and ferromagnetic next-to-next-to-nearest Heisenberg interaction,

$$H = \sum_{j\in\mathbb{Z}} \big[J_1 S^\alpha_\beta(j) S^\beta_\alpha(j+1) + J_2 S^\alpha_\beta(j) S^\beta_\alpha(j+2) - J_3 S^\alpha_\beta(j) S^\beta_\alpha(j+3)\big], \tag{23}$$

where $S(j)$ is the $SU(3)$ spin operator of the p-box symmetric representation at the site j. It is interesting to argue that the coupling J_2, J_3 of order of $1/p$ are generated by the quantum fluctuation of the nearest-neighbor coupling J_1, so the results of the nonlinear sigma model are expected to apply for the nearest-neighbor Hamiltonian [24].

Since the symmetric representation can be constructed by the symmetric tensor product of the defining representation, the coherent state of $S(j)$ can be written by $\mathbb{C}P^2(= SU(3)/[SU(2)\times U(1)])$ field,[5] and we denote the corresponding unit vector field $\Phi(\ell,\tau)\in SU(3)/SU(2)\subset\mathbb{C}^3$. To discuss the low-energy physics of this lattice Hamiltonian, it is convenient to consider the three-site unit cell since it contains up to next-to-next-to-nearest-neighbor interaction, and decompose the fluctuation into the slow field among unit cells and fast field inside each unit cell. To that end, we decompose the 3×3 complex matrix field for the unit cell into the transverse fluctuation L and slow rotation $\mathcal{U} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3]$ given by the $SU(3)$ matrix:

$$[\Phi(3j,\tau), \Phi(3j+1,\tau), \Phi(3j+2,\tau)] = L(j,\tau)\cdot[\boldsymbol{\phi}_1(j,\tau), \boldsymbol{\phi}_2(j,\tau), \boldsymbol{\phi}_3(j,\tau)]. \tag{24}$$

In the large p limit, the fluctuation of L is of order $\mathcal{O}(1/p)$, and can be integrated out. As a result, the $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model is obtained with $1/g = p\sqrt{(J_1J_2+2J_3J_1+2J_3J_2)}/(J_1+J_2)$, $\lambda = p2\pi(2J_2-J_1)/(3(J_1+J_2))$, and $\theta=\theta_1=-\theta_3 = 2\pi p/3$. That is, the theory lies on the $\mathbb{Z}_3$-invariant point. For details, see Ref. [24].

This clarifies the origin of the discrete symmetries, $\mathbb{Z}_3$ permutation. The $\mathbb{Z}_3$ permutation originates from the lattice translational symmetry by one lattice unit. Since the low-energy description treats the three consecutive sites as a single unit cell, the translation symmetry act as the $\mathbb{Z}_3$ internal symmetry.

[5]For more general representations involving antisymmetric products, we need to prepare the bosonic field for each fundamental representation satisfying the orthogonality constraint, and the coherent state for each spin becomes $SU(3)/[U(1)\times U(1)]$ in certain representations (see, e.g., Ref. [23]). In the present case, the coherent state of each spin is described by $\mathbb{C}P^2$ because the representation is a totally symmetric tensor, and the flag manifold $SU(3)/[U(1)\times U(1)]$ for nonlinear sigma model comes out from the nature of the antiferromagnetic interaction. The similar thing occurs also for (2+1)-dimensional $SU(3)$ spin magnets [57].

III. ANOMALY, GLOBAL INCONSISTENCY, AND PHASE STRUCTURE

In this section, we compute the 't Hooft anomaly and the global inconsistency for the effective theory of $SU(3)$ chains. The 't Hooft anomaly is the manifestation of the Lieb-Schultz-Mattis theorem of the lattice model in the continuum low-energy description, and rules out the trivially gapped ground state.

Global inconsistency is a more subtle obstruction for gauging the symmetry. When considering the parameter space of the theory, we have enhancement of symmetry at different values of the θ angles, e.g., when $(\theta_1,\theta_3)=(0,0),(\pi,0)$ in $SU(3)/[U(1)\times U(1)]$ model. At each point, the symmetry can be gauged, but their gauging is in a sense inconsistent. In such a case, the ground state of of these points in the phase diagram can be trivially gapped, but they cannot *both* be trivially gapped, and hence still give information about the structure of the phase diagram. We will now see how the 't Hooft anomaly and the global inconsistencies arise.

A. Gauging $SU(3)/\mathbb{Z}_3$ flavor symmetry

In order to detect the mixed 't Hooft anomaly and global inconsistency, we first gauge the $SU(3)/\mathbb{Z}_3$ flavor symmetry. Naively we would promote the covariant derivatives to include a non-Abelian gauge field. Indeed, we would have to replace

$$(\mathrm{d}+\mathrm{i}a_\ell)\boldsymbol{\phi}_\ell \to (\mathrm{d}+\mathrm{i}a_\ell+\mathrm{i}A)\boldsymbol{\phi}_\ell, \tag{25}$$

where A is the $SU(3)$ gauge field. Seemingly, nothing dramatic happens by this promotion. However, we shall see that this is not true when we gauge $SU(3)/\mathbb{Z}_3$, i.e., the a_ℓ gauge fields above can no longer be properly quantized gauge fields.

To see that a_1, a_2 are not properly quantized gauge fields, consider a gauge transformation, which takes

$$\boldsymbol{\phi}_\ell \to U_\ell\boldsymbol{\phi}_\ell = e^{i\varphi_\ell}U\boldsymbol{\phi}_\ell,\quad U\in SU(3),\quad \varphi_3 = -\varphi_1-\varphi_2. \tag{26}$$

Now for this to be a gauge transformation on a compact manifold, e.g., a two-torus $\mathbb{T}^2$, we must have that the gauge transformation U_ℓ is single valued on the torus to ensure the single-valuedness of $\boldsymbol{\phi}_\ell$. Since $U\in SU(3)$ should be regarded as the lift of the $PSU(3)$ matrix, it is required to be periodic up to a center $e^{-\mathrm{i}\frac{2\pi}{3}}\mathbf{1}_3$, which is mapped to unity in the $PSU(3)$ group. However, U_ℓ must be single valued, which requires φ_1 and φ_2 to be periodic only up to $2\pi/3$ (so that φ_3 is periodic up to $-4\pi/3 = 2\pi/3 \bmod 2\pi$). Since the gauge transformation affects the gauge fields $a_\ell\to a_\ell - \mathrm{d}\varphi_\ell$, we see that the a_ℓ are no longer properly quantized $U(1)$ gauge fields. In particular, holonomies $e^{i\int a_\ell}$ are no longer gauge-invariant operators. Rather the gauge field $3a_\ell$ are properly normalized $U(1)$ gauge fields. This in turn implies that the fluxes of a_ℓ will be quantized as multiples of $2\pi/3$.

In fact, this deviation of the lack of 2π quantization is related to a topological invariant of the $PSU(3)=SU(3)/\mathbb{Z}_3$ gauge bundle, which is a member of the second cohomology $B\in H^2(M_2,\pi_1(PSU(3)))$ (see, e.g., Ref. [58]). This topological invariant can also be thought of as the two-form $\mathbb{Z}_3$ gauge field [59–61], which is necessary to convert the gauge group $SU(3)$ to $PSU(3)$.

We therefore introduce the background $SU(3)/\mathbb{Z}_3$ gauge fields, which consist of two ingredients: (1) A field, a $SU(3)$ one-form gauge field; (2) B field, a $\mathbb{Z}_3$ two-form gauge field.

We realize the $\mathbb{Z}_3$ two-form gauge field as a pair of the $U(1)$ two-form gauge field B and $U(1)$ one-form gauge field C satisfying the constraint $3B = \mathrm{d}C$. To see how the B gauge field arises in the $SU(3)/\mathbb{Z}_3$ gauge theory, we first embed the $SU(3)/\mathbb{Z}_3$ gauge field into the $U(3)$ gauge field [59,60],

$$\widetilde{A} = A + \tfrac{1}{3}C\mathbf{1}_3, \tag{27}$$

where A is traceless, and $C = \mathrm{tr}\widetilde{A}$.

However, $PSU(3)$ gauge theory and $U(3)$ gauge theory are different in two ways: (1) $U(3)$ gauge field has an extra $U(1)$ photon C and (2) the 't Hooft flux of $PSU(3)$ bundle is in $H^2(M_2,\mathbb{Z}_3)$, while that of $U(3)$ bundle is in $H^2(M_2,\mathbb{Z})$. These differences can be resolved simultaneously [59] by postulating the $U(1)$ one-form gauge invariance of B. In fact, what we want to do is to allow the extra $U(1)$ photon to be absorbed by the already existing a_1, a_2 photons (recall that $a_3 = -a_1 - a_2$ due to the constraint). To that end, let us replace the covariant derivatives $(\mathrm{d} + \mathrm{i}a_\ell)\boldsymbol{\phi}_\ell$ by

$$(\mathrm{d} + \mathrm{i}a_\ell + \mathrm{i}\widetilde{A})\boldsymbol{\phi}_\ell, \tag{28}$$

If we vary $B \mapsto B + \mathrm{d}\xi$ and

$$a_\ell \mapsto a_\ell - \xi, \quad C \mapsto C + 3\xi, \quad \widetilde{A} \mapsto \widetilde{A} + \xi\mathbf{1}_3, \tag{29}$$

the above action will be invariant. However, notice that the above transformation is not consistent with the constraint $a_1 + a_2 + a_3 = 0$. To fix it, let us promote this constraint to

$$a_1 + a_2 + a_3 + C = 0, \tag{30}$$

which is still manifestly $\mathbb{Z}_3$ invariant.[6]

We emphasize that the gauge-variation parameter ξ is a properly normalized $U(1)$ gauge field, so that we call this gauge symmetry a $U(1)$, one-form gauge symmetry. This effectively gauges the $U(1)$ center of $U(3)$ gauge bundle, and reduces it to the $SU(3)/\mathbb{Z}_3$ bundle. Locally, the C field can therefore be gauged away, by choosing $\xi = -C/3$, so that there is no photon associated with C. However, since both ξ and C are properly normalized $U(1)$ gauge fields, the equation $\xi = -C/3$ cannot be satisfied globally. Namely, the flux $dC/3$ is gauge invariant mod 2π. Indeed, $B = \mathrm{d}C/3$ is the $\mathbb{Z}_3$ gauge field of the $PSU(3)$ gauge bundle, which was advertised above.

Further, to maintain this gauge invariance in the θ terms, we must replace

$$\mathrm{d}a_\ell \to \mathrm{d}a_\ell + B. \tag{31}$$

Since they are quantized by $2\pi/3$, this forces a 6π periodicity in θ angles. Since the $\mathbb{Z}_3$ exchange symmetry crucially depended on the 2π periodicity of θ angles, the $\mathbb{Z}_3$ symmetry is explicitly broken by the presence of the B field. This is the source of the 't Hooft anomaly which we will examine more closely in the next section. We now obtain the fully gauged action,

$$\begin{aligned} S_{\text{gauged}} &= \sum_{\ell=1}^{3}\int_{M_2}\bigg[-\frac{1}{2g}|(\mathrm{d}+\mathrm{i}a_\ell+\mathrm{i}\widetilde{A})\boldsymbol{\phi}_\ell|^2 + \frac{\mathrm{i}\theta_\ell}{2\pi}(\mathrm{d}a_\ell + B) \\ &\quad + \frac{\lambda}{2\pi}\{\overline{\boldsymbol{\phi}}_{\ell+1}\cdot(\mathrm{d}+\mathrm{i}\widetilde{A})\boldsymbol{\phi}_\ell\}\wedge\{\boldsymbol{\phi}_{\ell+1}\cdot(\mathrm{d}+\mathrm{i}\widetilde{A})\overline{\boldsymbol{\phi}}_\ell\}\bigg]. \end{aligned} \tag{32}$$

We should notice that the λ term is invariant under $U(1)$ one-form gauge transformations because of the orthogonality constraint. Performing the path integral,

$$Z[(A,B)] = \int \mathcal{D}a\mathcal{D}\overline{\boldsymbol{\phi}}\mathcal{D}\boldsymbol{\phi}\,\exp(S_{\text{gauged}}), \tag{33}$$

we obtain the partition function $Z[(A,B)]$ under the background $SU(3)/\mathbb{Z}_3$ background gauge field.

B. $SU(3)/\mathbb{Z}_3$-$\mathbb{Z}_3$ anomaly

Now we turn the mixed 't Hooft anomaly between the $SU(3)/\mathbb{Z}_3$ flavor symmetry and the $\mathbb{Z}_3$ permutation symmetry. To see it, we show that the partition function under the $SU(3)/\mathbb{Z}_3$ gauge field, $Z[(A,B)]$, is not invariant under the $\mathbb{Z}_3$ permutation at a $\mathbb{Z}_3$-invariant point $(\theta_1,\theta_2,\theta_3) = (2\pi/3, 0, -2\pi/3)$.

In the presence of the $SU(3)/\mathbb{Z}_3$ background gauge field, the constraint on the topological charges becomes

$$\mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 + \mathrm{d}C = 0, \tag{34}$$

or, equivalently,

$$(\mathrm{d}a_1 + B) + (\mathrm{d}a_2 + B) + (\mathrm{d}a_3 + B) = 0. \tag{35}$$

After gauging $SU(3)/\mathbb{Z}_3$, the action is given by (32), with the above θ-terms. The kinetic and λ terms are evidently invariant under $\mathbb{Z}_3$ permutation $\boldsymbol{\phi}_\ell \mapsto \boldsymbol{\phi}_{\ell+1}$, so we compute the topological term only. At $(\theta_1,\theta_2,\theta_3) = (2\pi/3, 0, -2\pi/3)$,

$$\begin{aligned} S_{\text{top}} &= \mathrm{i}\int\left(\frac{1}{3}(\mathrm{d}a_1 + B) - \frac{1}{3}(\mathrm{d}a_3 + B)\right) \\ &\mapsto \mathrm{i}\int\left(\frac{1}{3}(\mathrm{d}a_2 + B) - \frac{1}{3}(\mathrm{d}a_1 + B)\right) \\ &= S_{\text{top}} - \mathrm{i}\int(\mathrm{d}a_1 + B). \end{aligned} \tag{36}$$

Since $\int \mathrm{d}a_1 \in 2\pi\mathbb{Z}$, this term drops off in the path integral. However, the B-term $\int B \in \frac{2\pi}{3}\mathbb{Z}$ contributes a phase, so we have

$$Z[(A,B)] \mapsto Z[(A,B)]\exp\left(-\mathrm{i}\int B\right) \tag{37}$$

under $\mathbb{Z}_3$ permutation. This is the mixed 't Hooft anomaly between $SU(3)/\mathbb{Z}_3$ and $\mathbb{Z}_3$, implying the generalization of the Haldane conjecture to $SU(3)$ chains. There is no local counterterm that can eliminate the generation of the B term under the $\mathbb{Z}_3$ exchange symmetry. Indeed, the only countert-

[6] We could have also chosen to maintain the constraint $a_1 + a_2 + a_3 = 0$, but then we would have had add an extra term $-\mathrm{i}C$ in the definition of the covariant derivative (28) for, say, $\ell = 3$. This would have made the $\mathbb{Z}_3$ symmetry slightly less manifest, but the discussion remains unchanged.

erms allowed are

$$\mathrm{i}p\int B, \tag{38}$$

where $p\in\mathbb{Z}$ mod 3, and these are is invariant under the $\mathbb{Z}_3$ symmetry.

By anomaly matching argument, the ground state at the $\mathbb{Z}_3$ invariant point, $(\theta_1,\theta_2,\theta_3)=(2\pi/3,0,-2\pi/3)$, cannot be trivially gapped, i.e., the system must have either (1) spontaneous symmetry breaking (SSB), (2) topological order, or (3) conformal behavior. In $1+1$ dimension, the intrinsic topological order is ruled out by Ref. [62], so the system must either have the spontaneous symmetry breaking or the conformal behavior in the low-energy limit. The same statement is obtained by the Lieb-Schultz-Mattis theorem for the lattice $SU(N)$ chain [4,24], and we here provided the field-theoretic counterpart.

C. $SU(3)/\mathbb{Z}_3$-C global inconsistency

Here, we will discuss a constraint which arises from the $SU(3)/\mathbb{Z}_3$ symmetry and the charge conjugation C. As we discussed for generic values of the θ angles, the charge conjugation is not a symmetry. We always set one of the θ angles to be zero without the loss of generality, and here we work with θ_1 and θ_3 angles only.

As we already discussed, there are three distinct ways that we can define the charge conjugation symmetry, and we labeled them by C_k, $k=1,2,3$, given by (16). The three ways differ by the $\mathbb{Z}_3$ exchange symmetry, and so when $\mathbb{Z}_3$ is a symmetry (i.e., when $\theta_1=-\theta_3=2\pi/3s$, $s=0,1,2$) any one of them can be used. Here we will discuss the values of (θ_1,θ_3), where $\mathbb{Z}_3$ permutation symmetry is not necessarily present but there is a sensible C_k-symmetry for some $k=1,2,3$. As we have discussed around Eqs. (17)–(21), we get that, under C_k, the θ angles are mapped as

$$\mathsf{C}_1:(\theta_1,\theta_3)\to(\theta_3-\theta_1,\theta_3), \tag{39}$$

$$\mathsf{C}_2:(\theta_1,\theta_3)\to(-\theta_3,-\theta_1,), \tag{40}$$

$$\mathsf{C}_3:(\theta_1,\theta_3)\to(\theta_1,\theta_1-\theta_3,). \tag{41}$$

Therefore, for C_1 to be a symmetry, we must have $\theta_3=2\theta_1$ mod 2π, for C_2, we must have $\theta_3=-\theta_1$ mod 2π and for C_3, $\theta_1=2\theta_3$ mod 2π.

Since C_k invariance is trivially true for kinetic and λ terms even after gauging $SU(3)/\mathbb{Z}_3$, all we have to discuss is the effect of topological θ terms. Now let us set $\theta=\theta_1=-\theta_3+\alpha$. If $\alpha=0$ mod 2π the C_2 is the symmetry. Upon gauging the $SU(3)/\mathbb{Z}_3$ symmetry, we have that the θ terms become

$$S_{\text{top}}=\frac{\mathrm{i}}{2\pi}\int\{\theta(\mathrm{d}a_1+B)+(\alpha-\theta)(\mathrm{d}a_3+B)\}. \tag{42}$$

When $\alpha=0$, the C_2 transformation is clearly a symmetry, if we define that $\mathsf{C}_2:B\mapsto -B$. However, if we now dial $\alpha=2\pi k$, $k\in\mathbb{Z}$, we will get that under the transformation,

$$\begin{aligned}S_{\text{top}}&=\frac{\mathrm{i}}{2\pi}\int\{\theta(\mathrm{d}a_1+B)+(2\pi k-\theta)(\mathrm{d}a_3+B)\}\\&\mapsto\frac{\mathrm{i}}{2\pi}\int\{-\theta(\mathrm{d}a_3+B)-(2\pi k-\theta)(\mathrm{d}a_1+B)\}\\&=S_{\text{top}}-2\mathrm{i}k\int B\mod 2\pi.\end{aligned} \tag{43}$$

Therefore the partition function $Z[(A,B)]$ at $(\theta_1,\theta_3)=(\theta,-\theta+2k\pi)$ changes under C_2 as

$$\mathsf{C}_2:Z[(A,B)]\mapsto Z[(A,B)]\exp\left(-2\mathrm{i}k\int B\right). \tag{44}$$

This, however, is not necessarily a ’t Hooft anomaly, because when gauging $SU(3)/\mathbb{Z}_3$, we have the freedom to add a local gauge-invariant term of the background field. We can define

$$Z_n[(A,B)]=Z[(A,B)]\exp\left(\mathrm{i}n\int B\right), \tag{45}$$

where n is called the discrete θ parameter, and $n\in\mathbb{Z}_3$. This gauged partition functions obeys

$$\mathsf{C}_2:Z_n[(A,B)]\mapsto Z_n[(A,B)]\exp\left(-2\mathrm{i}(n+k)\int B\right), \tag{46}$$

and thus it becomes C_2 invariant if

$$n=2k\mod 3. \tag{47}$$

In this manner, we can always write a local counterterm that restores the symmetry on every C_2-invariant line. Therefore there is no ’t Hooft anomaly between $SU(3)/\mathbb{Z}_3$ and C_2.

However, the local counterterm $\int B$ does not allow continuous parameters for its coefficient n in order to satisfy the $U(1)$ one-form gauge invariance. In such a case, we can apply the global inconsistency condition; when interpolating adiabatically from $\alpha=2\pi k_1$ to $\alpha=2\pi k_2$, if the local counterterms of $SU(3)/\mathbb{Z}_3$ added for C_2-invariant gauged partition functions at those points are different, then it is called a global inconsistency [40,42,49] or a secondary anomaly [43]. The conjectured matching condition [42,49] states that (1) both are trivially gapped, but they are distinct as the symmetry-protected topological (SPT) phases protected by $SU(3)/\mathbb{Z}_3$, or (2) one of them has nontrivial ground states as in the case of ’t Hooft anomaly matching. This consequence obtained by global inconsistency does not have the Lieb-Schultz-Mattis type counterpart. We can obtain the same conclusion for C_1 and C_3 by setting $\theta_3=2\theta+\alpha$, $\theta_1=\theta$ and $\theta_1=2\theta+\alpha$, $\theta_3=\theta$, and obtain that there is a global inconsistency between $\alpha=0,2\pi,4\pi$ mod 6π lines. This situations are sketched in Fig. 1.

D. Anomaly involving the $\mathbb{Z}_3\times\mathbb{Z}_3\subset PSU(3)$ subgroup

So far, we have discussed gauging the full $SU(3)/\mathbb{Z}_3=PSU(3)$ flavor symmetry and established that there exists an anomaly between it and the $\mathbb{Z}_3$ cyclic permutation symmetry at the appropriate points in the (θ_1,θ_3) phase diagram. Moreover, we have also seen that when (θ_1,θ_3) are chosen such that the $\mathbb{Z}_3$ cyclic permutation symmetry is broken, but that a form of charge conjugation is preserved, there exists a global inconsistency between certain regions of the phase diagram, constraining the system significantly more than the LSM theorem.

Here, we wish to make a remark that a lot of our discussion applies even to the case of the subgroup $\mathbb{Z}_3\times\mathbb{Z}_3\subset PSU(3)$. As we shall see, there is a mixed ’t Hooft anomaly between this $\mathbb{Z}_3\times\mathbb{Z}_3\subset PSU(3)$ flavor symmetry and the $\mathbb{Z}_3$ cyclic permutation symmetry. Moreover, this anomaly immediately implies the anomaly involving the full $PSU(3)$, so it is more general. We will also use the opportunity to complement the discussion so far, by introducing a slightly different, but equivalent, perspective on the anomaly.

The practical consequence of using the $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup of $PSU(3)$ is that the system may be allowed to break the spin-$PSU(3)$ symmetry all the way down to $\mathbb{Z}_3 \times \mathbb{Z}_3$, keeping the nontrivial constraints of the anomaly. Further, such deformations of the theory will have a richer structure, as spontaneous breaking of the discrete symmetries are not forbidden by the Mermin-Wagner-Coleman theorem. Spontaneous breaking of the discrete symmetries also gives rise to domain walls. These too will be constrained by the anomaly as we shall see, and are interesting in their own right.

Before we can argue that there is an anomaly involving the subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset PSU(3)$, let us first discuss how this group acts on the fields of the theory. To that end, consider the lift of the $PSU(3)$ to $SU(3)$. $SU(3)$ group contains two matrices,

$$M_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}2\pi/3} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}4\pi/3} \end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{48}$$

which are dubbed *clock* and *shift* matrices. They satisfy the algebra

$$M_C M_S = e^{\frac{2\pi \mathrm{i}}{3}} M_S M_C, \tag{49}$$

i.e., they differ by a center element of $SU(3)$. Further, a homomorphism $H : SU(3) \to SU(3)/\mathbb{Z}_3$ map has the center as the kernel, so the two group elements $H(M_C), H(M_S) \in SU(3)/\mathbb{Z}_3$ commute. Since they also have the property that $M_C^3 = M_S^3 = \mathbf{1}_3$, they generate a group $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset SU(3)/\mathbb{Z}_3$.

Now we want to promote the global symmetry $\mathbb{Z}_3 \times \mathbb{Z}_3$ to a gauge symmetry, i.e., we wish to promote the $U(1)^2$ gauge bundle to $\mathbb{Z}_3^2 \times U(1)^2$. Notice, however, that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ [just like $SU(3)/\mathbb{Z}_3$] acts projectively on the fields $\boldsymbol{\phi}_\ell$.

Before continuing, let us first gauge the $PSU(3)$ global symmetry. Before we do, recall that the $PSU(3)$ gauge bundle contains a topological class $H^2(M_2, \mathbb{Z}_3)$ with $\mathbb{Z}_3$ coefficients. In fact, we have already seen the representative of this topological class. It is the B-field used extensively in the discussion so far. This topological class is an obstruction to the lifting of the $PSU(3)$ bundle to $SU(3)$ bundle. To see this, let T_{ij}, T_{jk}, T_{ki} be the transition functions between the three local coordinate charts U_i, U_j, U_k of M_2. The cocycle condition on the triple overlap $U_i \cap U_j \cap U_k$ demands

$$T_{ij} T_{jk} T_{ki} = \mathbb{I}. \tag{50}$$

Now, let $\tilde{T}_{ij}, \tilde{T}_{jk}, \tilde{T}_{ki}$ be the lifts of T_{ij}, T_{jk}, T_{ki} from $PSU(3)$ to $SU(3)$. The cocycle condition translates into

$$\tilde{T}_{ij} \tilde{T}_{jk} \tilde{T}_{ki} = z\mathbb{I}, \quad z \in \mathbb{Z}_3. \tag{51}$$

In other words, the obstructions which cause the cocycle condition of $PSU(3)$ bundle can fail to satisfy the cocycle condition of the $SU(3)$ bundle are classified by the center element $z \in \mathbb{Z}_3$ of $SU(3)$.

Now consider the cocycle condition with a nontrivial element $z \in \mathbb{Z}_3$ and how it affects the fields $\boldsymbol{\phi}_\ell$ of our theory. The $PSU(3)$ transition functions act as $SU(3)$ matrices on them, so in order to have $\boldsymbol{\phi}_\ell$ to be well defined in the triple intersection, we must compensate the change of phase $z \in \mathbb{Z}_3$ by an equivalent change of phase in the $U(1)^2$ transition functions. Namely we must have that

$$\exp\left(\mathrm{i}\varphi_\ell^{ij} + \mathrm{i}\varphi_\ell^{jk} + \mathrm{i}\varphi_\ell^{ki}\right) = \bar{z}, \tag{52}$$

where $t_{ij}^\ell = \mathrm{e}^{i\varphi_\ell^{ij}}$ is the transition function for the $U(1)$ gauge bundle acting on the field $\boldsymbol{\phi}_\ell$ (notice that we have a constraint $\sum_{\ell=1,2,3} \varphi_\ell^{ij} = 0$). In turn this means that the gauge fields associated with the $U(1)$ gauge bundles are no longer properly quantized, and their fluxes are no longer quantized in multiples of 2π. However, their deviation from the quantization is correlated with the value of $B \in H^2(M_2, \mathbb{Z}_3)$. In other words,

$$\int F_\ell = \int B \mod 2\pi. \tag{53}$$

The failure for the Abelian fluxes to be properly quantized is reflected in the loss of the $\mathbb{Z}_3$ cyclic permutation symmetry, exactly by a value of the $B \in H^2(M_2, \mathbb{Z}_3)$.

Now we see that nothing will change when we break $PSU(3)$ down to $\mathbb{Z}_3 \times \mathbb{Z}_3$, defined as above. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ still acts projectively on the $\boldsymbol{\phi}_\ell$ fields, and the $\mathbb{Z}_3 \times \mathbb{Z}_3$ bundle is classified by the obstruction to the lifts by $\mathbb{Z}_3$ central extensions, which we will still call B.

Let us see the same thing by another way. We can think of gauging the $\mathbb{Z}_3 \times \mathbb{Z}_3$ as putting the twisted boundary condition on the 2D manifold. We do this by using the clock and shift matrices M_C and M_S defined above, and twisting the ϕ fields with the clock and shift matrices. In other words, let us take that

$$\phi_{\ell,a}(L, t) = \mathrm{e}^{\mathrm{i}\varphi_\ell(t)} (M_C)_a{}^b \phi_{\ell,b}(0, t), \tag{54}$$

$$\phi_{\ell,a}(x, \beta) = \mathrm{e}^{\mathrm{i}\tilde{\varphi}_\ell(x)} (M_S)_a{}^b \phi_{\ell,b}(x, 0), \tag{55}$$

$$a_\ell(L, t) = a_\ell(0, t) - \mathrm{d}\varphi_\ell(t), \tag{56}$$

$$a_\ell(x, \beta) = a_\ell(x, 0) - \mathrm{d}\tilde{\varphi}_\ell(x), \tag{57}$$

where φ_ℓ and $\tilde{\varphi}_\ell$ at the moment undetermined phases, with the constraint that $\varphi_3 = -\varphi_1 - \varphi_2$ and $\tilde{\varphi}_3 = -\tilde{\varphi}_1 - \tilde{\varphi}_2$. By setting $t = \beta$ in the first equation and $x = L$ in the second, we have

$$\begin{aligned} \phi_{\ell,a}(L, \beta) &= \mathrm{e}^{\mathrm{i}\varphi_\ell(\beta)} (M_C)_a{}^b \phi_{\ell,b}(0, \beta) \\ &= \mathrm{e}^{\mathrm{i}\varphi_\ell(\beta) + \mathrm{i}\tilde{\varphi}_\ell(0)} (M_C M_S)_a{}^b \phi_{\ell,b}(0, 0), \end{aligned} \tag{58}$$

$$\begin{aligned} \phi_{\ell,a}(L, \beta) &= \mathrm{e}^{\mathrm{i}\tilde{\varphi}_\ell(L)} (M_S)_a{}^b \phi_{\ell,b}(L, 0) \\ &= \mathrm{e}^{\mathrm{i}\tilde{\varphi}_\ell(L) + \mathrm{i}\varphi_\ell(0)} (M_S M_C)_a{}^b \phi_{\ell,b}(0, 0). \end{aligned} \tag{59}$$

Since the LHS of the two lines above are equal, we must have that

$$\varphi_\ell(\beta) + \tilde{\varphi}_\ell(0) = \tilde{\varphi}_\ell(L) + \varphi_\ell(0) + \frac{2\pi}{3} \mod 2\pi, \tag{60}$$

from which it follows that

$$\int \mathrm{d}a_\ell = [\varphi_\ell(\beta) - \varphi_\ell(0)] - [\tilde{\varphi}_\ell(L) - \tilde{\varphi}_\ell(0)] = \frac{2\pi}{3} \mod 2\pi. \tag{61}$$

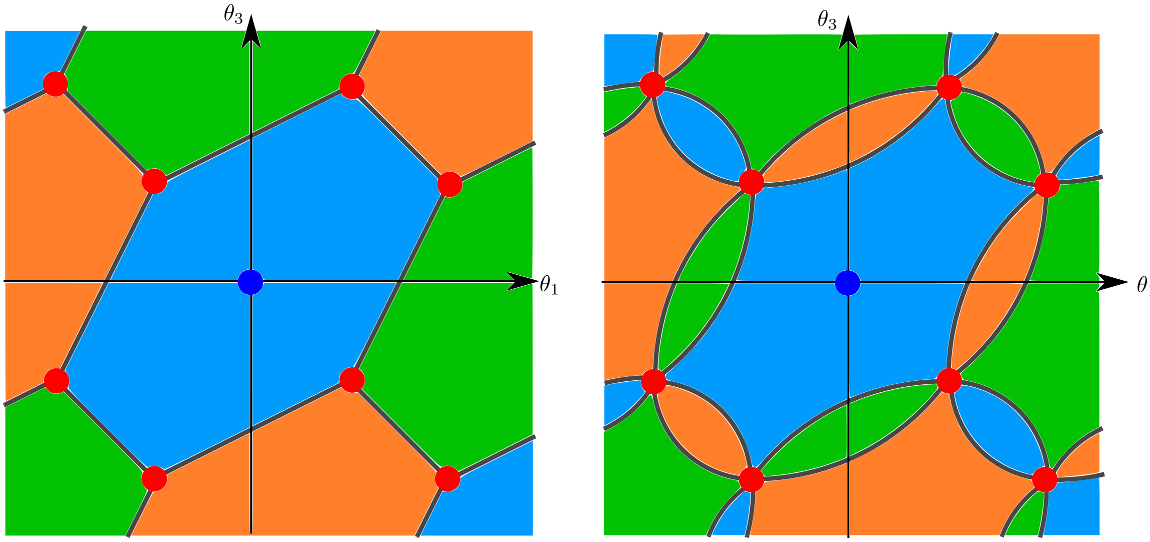

FIG. 2. Possible scenarios consistent with global inconsistency. The red blobs are $\mathbb{Z}_3$ symmetric points with $PSU(3)\times\mathbb{Z}_3$ 't Hooft anomaly, and the origin (blue blob) is the $\mathbb{Z}_3$ symmetric point without anomaly. Blank regions painted with different colors (light blue, orange, green) all correspond to trivially gapped phases, but they are different as SPT phases protected by $PSU(3)$ symmetry. (Left) The global inconsistency is matched by the spontaneous breaking of C on thick gray lines. (Right) The global inconsistency is matched by the phase transitions lines (gray curves) separating distinct C-symmetric trivial vacua.

The deviation from the 2π quantization can be seen as the cup product between the $\mathbb{Z}_3$ gauge fields for the two generators of $\mathbb{Z}_3$, which means that we can identify

$$B=\frac{3}{2\pi}A^1\wedge A^2, \tag{62}$$

where A^1 and A^2 are the $\mathbb{Z}_3$ gauge fields for the two generators of $\mathbb{Z}_3\times\mathbb{Z}_3$. Indeed, if we think of A^1 and A^2 as embedded in the $U(1)$ gauge group, the above term is gauge invariant under $A^{1,2}\to A^{1,2}+\mathrm{d}\varphi^{1,2}$.

E. The phase structure

In this section, we discuss details how the 't Hooft anomaly and the global inconsistency constrain the phase diagram of the $SU(3)/[U(1)\times U(1)]$ sigma model. Figure 1 shows one of the possible phase diagrams consistent with the matching condition when the nonvanishing mass gap is assumed everywhere. The red and blue blobs indicate the $\mathbb{Z}_3$-invariant points, red being the points with a 't Hooft anomaly. The thin lines (blue, green, red, and solid, dashed dotted) indicate that the system has a charge-conjugation symmetry (i.e., either C_1, C_2, or C_3 symmetry. The thick gray lines show the first-order phase transitions, on which the charge conjugation is spontaneously broken. While this is a minimal way to saturate the global inconsistency, it is not the only way. Indeed, we will soon discuss a more exotic scenario of the phase diagram (see Fig. 2). Let us first discuss the standard scenario depicted in Fig. 1.

The red $\mathbb{Z}_3$-invariant points have a mixed 't Hooft anomaly between $SU(3)/\mathbb{Z}_3$ and $\mathbb{Z}_3$ permutation, requiring matching with a nontrivial vacuum. The $PSU(3)$ symmetry cannot be spontaneously broken due to the Coleman-Mermin-Wagner theorem [63,64], the possible choice of the low-energy theory is SSB of $\mathbb{Z}_3$ or the conformal field theory (CFT). The $SU(3)$ spin chain discussed in Ref. [22] has the trimerized phase and the $\mathbb{Z}_3$ symmetry is spontaneously broken. Also, the strong-coupling analysis of (23), given in Ref. [24], shows that the anomaly is matched by SSB of $\mathbb{Z}_3$ permutation. (This should be compared with a free photon theory of the linear version of the sigma model, discussed in Sec. VI B.) Reference [24] also performed a Monte Carlo simulation at imaginary θ angles, and extrapolated the mass gap with the ansatz $(c_1+c_2\theta^2)/(1+c_3\theta^2)$ indicated by Ref. [25]. It claims that the gapless excitation appears for $g<g_c\simeq 2.55$ [24], and then the 't Hooft anomaly is matched by some CFT if this were really the case. To get a more conclusive remark, it would be quite appealing if the result with the real θ angles is directly obtained via the lattice dualization [65–68].

The nature of the conformal field theory is one of the open questions and still under debate. Numerical results of exact diagonalization [69] suggest that it is $SU(3)_2$ Wess-Zumino-Witten (WZW) model for $p=2$, but the authors also mention that the crossover towards the gapped or $SU(3)_1$ WZW phase may occur as the system size becomes larger. Reference [24] also argues that it is the $SU(3)_1$ WZW model for all $p\neq 0$ mod 3. To constrain the possible CFT from anomaly matching, we have to compute the 't Hooft anomaly of the $SU(3)$ WZW model. We will give a detailed analysis on the anomaly of $SU(N)$ WZW models in Sec. V, and we claim that the anomaly matching condition also admits the crossover from the $SU(3)_2$ WZW model toward the $SU(3)_1$ WZW model.

Now consider deviations from the $\mathbb{Z}_3$ symmetric points. In Ref. [24], it was argued that the deformation along one of the C-invariant lines, depicted in Fig. 1, will result in a flow away from a CFT, because, in the absence of the $\mathbb{Z}_3$ symmetry, $SU(3)_1$ WZW theory has relevant perturbations driving it away from conformality. Using the strong-coupling analysis, they have argued that the lines connecting the red $\mathbb{Z}_3$ invariant points along C-invariant lines are phase separating lines which break the corresponding C-symmetry spontaneously. These phase-separating lines are depicted as thick gray lines on Fig. 1.

We now argue that the global inconsistency between C-invariant lines makes this picture robust. To argue this, we will assume that the system at $\theta_1 = \theta_3 = 0$ has a trivial mass gap, which is consistent with some previous studies [22,24,70–72] although some others show no indication of the mass gap [69]. We believe that this is a reasonable assumption since the nonlinear sigma model is asymptotically free and has no imaginary terms in the action at $\theta = 0$ (up to irrelevant and nonuniversal λ terms), which are typically trivially gapped in 1+1D.

When the trivial mass gap is assumed at blue points of Fig. 1, we must have that as we move from such a point all the way to the thick gray lines, we must either encounter a phase transition on the way or have a nontrivial ground state on the gray lines, matched by breaking $PSU(N)$ symmetry, C symmetry, or a CFT. The Mermin-Wagner-Coleman theorem prevents the first, while there is no obvious candidate for the last option, leaving the breaking of the C-symmetry as an obvious choice. This is consistent with the picture of Ref. [24]. We argue that a similar discussion on the phase diagram using the global inconsistency can be found in previous study of the 4d gauge theories [40,42] and also of the quantum mechanics on a circle [49].

We should make a comment that while this is a natural way to saturate the global inconsistency it is not the only way. We could imagine that the thick gray lines of Fig. 1 splits into two phase-separating lines as one goes from one nontrivial $\mathbb{Z}_3$ (red points) to another, causing the vacuum on the C-invariant line to be trivial. We illustrate this behavior in Fig. 2. The left figure shows the conventional scenario explained above, while the right one gives an exotic one. This scenario, however, seems contrived to us for the model at hand, but it should be possible to achieve by some deformations of the linear sigma model where more tunable parameters are allowed.

When the charge-conjugation C is spontaneously broken, we can consider the domain wall connecting two vacua. Since the partition functions of these vacua are different by $\mathrm{i}\int B$ under the $SU(3)/\mathbb{Z}_3$ background gauge field, the difference must be compensated by a nontrivial domain wall [44,73,74]. In fact, in this case, the domain wall is an $SU(3)$-spin triplet. In other words, these two vacua are trivially gapped[7] but distinct as SPT phases protected by $SU(3)/\mathbb{Z}_3$, so the fundamental representation of $SU(3)$ is excited on the domain wall without any energy cost. In Fig. 2, we paint different colors for distinct SPT phases protected by $PSU(3)$ symmetry. Phase transition lines required by global inconsistency must exist to describe these different SPT phases.

Finally, we recall that the anomalies and global inconsistencies remain even when the global $PSU(3)$ symmetry group is reduced down to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The theory, however, will not show conformal behavior at anomalous $\mathbb{Z}_3$ cyclic permutation symmetry invariant points, but will instead be saturated by a breaking either the $\mathbb{Z}_3 \times \mathbb{Z}_3$ (a Néel phase) or the $\mathbb{Z}_3$ cyclic permutation symmetry (the VBS phase). What makes this scenario interesting is that the system will support domain walls, all of which will have anomaly inflow and therefore carry nontrivial (i.e., degenerate) particle excitations (note that domain walls are particles in 1+1D).

IV. GENERALIZATION TO $SU(N)/U(1)^{N-1}$ NONLINEAR SIGMA MODEL

In this section, we will show that the whole analysis on anomalies and global inconsistencies in the previous section, Sec. III, can be extended to the $SU(N)/U(1)^{N-1}$ nonlinear sigma model. We take the same form of the Lagrangian

$$S = \sum_{\ell=1}^{N} \int_{M_2} \left[-\frac{1}{2g} |(\mathrm{d} + \mathrm{i}a_\ell)\boldsymbol{\phi}_\ell|^2 + \frac{\mathrm{i}\theta_\ell}{2\pi} \mathrm{d}a_\ell + \frac{\lambda}{2\pi} (\overline{\boldsymbol{\phi}}_{\ell+1} \cdot \mathrm{d}\boldsymbol{\phi}_\ell) \wedge (\boldsymbol{\phi}_{\ell+1} \cdot \mathrm{d}\overline{\boldsymbol{\phi}}_\ell) \right], \tag{63}$$

where $\boldsymbol{\phi}_\ell : M_2 \to \mathbb{C}^N$ satisfies the constraint,

$$\overline{\boldsymbol{\phi}}_\ell \cdot \boldsymbol{\phi}_{\ell'} = \delta_{\ell\ell'}, \tag{64}$$

$$\varepsilon_{f_1 f_2 \dots f_N} \phi_{f_1 1} \phi_{f_2 2} \cdots \phi_{f_N N} = 1. \tag{65}$$

The equation of motion of a_ℓ gives the constraint on the gauge field,

$$\sum_{\ell=1}^{N} a_\ell = 0, \tag{66}$$

and this is necessary for gauge invariance of (65). Since the sum of topological charges vanishes, we can put one of the θ parameters equal to 0. We take the convention $\theta_N = 0$. The kinetic and λ terms are invariant under the following symmetries: (1) $SU(N)/\mathbb{Z}_N$ flavor symmetry, (2) time-reversal T : $\boldsymbol{\phi}_\ell(x,t) \mapsto \overline{\boldsymbol{\phi}}_\ell(x,-t)$, (3) $\mathbb{Z}_N$ permutation symmetry, $\boldsymbol{\phi}_\ell \mapsto e^{2\pi \mathrm{i}(\ell+1)/N} \boldsymbol{\phi}_{\ell+1}$ and $a_\ell \mapsto a_{\ell+1}$, and (4) charge conjugations, C_k: $\boldsymbol{\phi}_i \mapsto (-1)^N \overline{\boldsymbol{\phi}}_{-i-k}$ and $a_\ell \mapsto -a_{-\ell-k}$. The first two symmetries $SU(N)/\mathbb{Z}_N$ and T are symmetries at any θ angles, while the last two, $\mathbb{Z}_N$ permutation and C_k, are symmetries only for special θ angles.

For $\mathbb{Z}_N$ permutation and charge conjugations, an appropriate phase factor must be multiplied so that those transformations become consistent with the constraint on the determinant, (65). Although it does not affect the following anomaly and global inconsistency argument,[8] let us make a

[7]Let us clarify terminologies to avoid possible confusions. A phase is called trivially gapped if it has a mass gap with a unique ground state on any closed spatial manifolds (i.e., no spontaneous symmetry breaking nor topological order). Trivially gapped states can be nontrivial as symmetry-protected topological phases, which can be used to detect the degeneracy of boundary states on open spatial manifolds.

[8]The emergent anomaly for $\mathbb{C}P^1$ model argued in Ref. [48] is related to this extra phase factor [see below Eq. (8) of the reference], but we will not discuss it here.

brief comment for clarity. We perform the $\mathbb{Z}_N$ permutation to the left-hand side of (65), then

$$\begin{aligned}&\varepsilon_{f_1 f_2 \dots f_N}\phi_{f_1 1}\phi_{f_2 2}\cdots\phi_{f_N N}\\ &\quad\mapsto \varepsilon_{f_1 f_2\dots f_N}\phi_{f_1 2}\phi_{f_2 3}\cdots\phi_{f_N 1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\frac{N(N+1)}{2}}\\ &\quad= (-1)^{N-1}\mathrm{e}^{\pi\mathrm{i}(N+1)}\varepsilon_{f_1' f_2'\dots f_N'}\phi_{f_1' 1}\phi_{f_2' 2}\cdots\phi_{f_N' N}.\end{aligned} \tag{67}$$

The negative sign coming out of the epsilon tensor for even N is exactly canceled by the additional phase factor, and the transformation is consistent with the constraint (65). This suggests that we do not need such factors for odd N as in the case of $N=3$. Indeed, we can eliminate those factors by $U(1)^{N-1}$ gauge transformations for odd N but it is impossible for even N. The easiest way to understand it is to perform the $\mathbb{Z}_N$ permutation N times, then $\boldsymbol{\phi}_\ell \mapsto (-1)^{N-1}\boldsymbol{\phi}_\ell$. For even N, the $\mathbb{Z}_N$ permutation acts projectively on $\boldsymbol{\phi}$ fields.[9]

A. Permutation symmetry, $\mathbb{Z}_n$ subgroup, and $SU(N)/\mathbb{Z}_N$-$\mathbb{Z}_n$ anomaly

Let us first consider the $\mathbb{Z}_N$ permutation. Further, let n be a divisor of N, so we can consider a subgroup $\mathbb{Z}_n \subset \mathbb{Z}_N$ that maps $a_\ell \mapsto a_{\ell+N/n}$. The change of the topological θ term is given by

$$\Delta S_{\mathrm{top}} = \mathrm{i}\sum_{\ell=1}^{N}\frac{\theta_{\ell-N/n}-\theta_\ell}{2\pi}\int \mathrm{d}a_\ell. \tag{68}$$

In order for $\Delta S_{\mathrm{top}}=0 \mod 2\pi$ for arbitrary topological charges, we find the condition

$$\theta_{\ell+N/n} = \theta_\ell + \alpha, \quad \mod 2\pi, \tag{69}$$

for some constant α because of (66). Repeating this transformation n times, we obtain that $n\alpha = 0 \mod 2\pi$, and thus

$$\alpha = \frac{2\pi p}{n} \tag{70}$$

for some $p = 0,1,\dots,n-1$. In addition, we still have $(n-1)$ free parameters $\theta_1,\dots,\theta_{N/n-1}$, so $\mathbb{Z}_n$-invariant points form $(N/n-1)$-dimensional planes. In particular, the $\mathbb{Z}_N$-symmetric points are given by

$$\theta_\ell = \frac{2\pi p\ell}{N} \quad \mod 2\pi, \tag{71}$$

for some $p=0,1,\dots,N-1$.

We can discuss the mixed 't Hooft anomaly between $SU(N)/\mathbb{Z}_N$ and $\mathbb{Z}_n$ permutation, where n is a divisor of N. By gauging $SU(N)/\mathbb{Z}_N$, we introduce the $SU(N)$ one-form gauge field A and $\mathbb{Z}_N$ two-form gauge field B. The important thing is that the $U(1)$ field strength, $\mathrm{d}a_\ell$, is no longer gauge invariant under a one-form $U(1)$ gauge symmetry, and it must be replaced by $\mathrm{d}a_\ell + B$. As a consequence, the constraint on the field-strength becomes

$$\sum_{\ell=1}^{N}(\mathrm{d}a_\ell + B) = 0. \tag{72}$$

Now, let us compute the effect of $\mathbb{Z}_n$ permutation under these background gauge fields (A,B). As an example of $\mathbb{Z}_n$-invariant plane, we take

$$\theta_{i+jN/n} = \theta_i + \frac{2\pi p}{n}j \tag{73}$$

for $i=1,\dots,N/n$ and $j=0,1,\dots,n-1$, $\theta_{N/n}$ is fixed so that $\theta_N = 0$. Under the $\mathbb{Z}_n$ permutation, the change of the topological action at this point does not vanish mod 2π, but it becomes

$$\Delta S_{\mathrm{top}} = -\mathrm{i}\frac{N}{n}p\int B, \quad \mod 2\pi. \tag{74}$$

Since $\int B \in \frac{2\pi}{N}\mathbb{Z}$, this gives the nontrivial phase to the change of the partition function $Z[(A,B)]$. This is the $SU(N)/\mathbb{Z}_N$-$\mathbb{Z}_n$ 't Hooft anomaly.

B. $SU(N)/\mathbb{Z}_N$-C anomaly and global inconsistency

Next, we consider the charge conjugation symmetry. C_k changes the topological action by

$$\Delta S_{\mathrm{top}} = -\mathrm{i}\sum_{\ell=1}^{N}\frac{\theta_{-\ell-k}+\theta_\ell}{2\pi}\int \mathrm{d}a_i. \tag{75}$$

In order for $\Delta S_{\mathrm{top}} = 0 \mod 2\pi\mathrm{i}$, we get

$$\theta_{-\ell-k}+\theta_\ell = \beta \quad \mod 2\pi, \tag{76}$$

for some constant β.

Gauging $SU(N)/\mathbb{Z}_N$, the above change of the topological term is replaced by

$$\Delta S_{\mathrm{top}} = -\mathrm{i}\sum_{\ell=1}^{N}\frac{\theta_{-\ell-k}+\theta_\ell}{2\pi}\int (\mathrm{d}a_\ell + B). \tag{77}$$

The C_k invariance without background B only requires $\theta_{-\ell-k}+\theta_\ell = \beta \mod 2\pi$ because $\int \mathrm{d}a_\ell \in 2\pi\mathbb{Z}$, but this is not true with B since $\int B \in \frac{2\pi}{N}\mathbb{Z}$. This derives the $SU(N)/\mathbb{Z}_N$-C_k mixed 't Hooft anomaly or global inconsistency depending on whether the anomaly can be canceled by the local counterterm with the discrete level in $\int B$.

C. Anomalies and inconsistency involving $\mathbb{Z}_N \times \mathbb{Z}_N \subset SU(N)/\mathbb{Z}_N$

Recall that much of the discussion of the anomalies in the spin systems having a $SU(3)/\mathbb{Z}_3$ global flavor symmetry remained even if only $\mathbb{Z}_3\times\mathbb{Z}_3 \subset SU(3)/\mathbb{Z}_3$ was preserved. A generalization of this argument to linearized $SU(N)/U(1)^{N-1}$ models is straightforward.

[9] For $N=2$, the model is the familiar $\mathbb{C}P^1$ nonlinear sigma model. There, the charge conjugation is defined as $\boldsymbol{\phi}_1 \mapsto \mathrm{i}\sigma^y\overline{\boldsymbol{\phi}}_1$ for consistency with the spin $SU(2)$ rotation. Doing this charge conjugation twice, we get $\boldsymbol{\phi}_1 \mapsto -\boldsymbol{\phi}_1$. Here, we have argued that the same thing is true for larger even N. We should still call the global symmetry as $\mathbb{Z}_N$, because such phases does not appear on gauge-invariant operators.

The relevant $\mathbb{Z}_N\times\mathbb{Z}_N$ symmetry can be seen being generated by $SU(N)$ matrices[10]

$$M_C=\omega^{\frac{N-1}{2}}\begin{pmatrix}1 & 0 & \dots & \dots & 0\\ 0 & \omega & 0 & \dots & \vdots\\ \vdots & 0 & \omega^2 & & \vdots\\ \vdots & \dots & \dots & \ddots & \vdots\\ 0 & \dots & \dots & \dots & \omega^{N-1}\end{pmatrix},$$

$$M_S=\omega^{\frac{N-1}{2}}\begin{pmatrix}0 & 1 & \dots & \dots & 0\\ \vdots & 0 & 1 & \dots & \vdots\\ \vdots & \vdots & \ddots & & \vdots\\ 0 & \dots & \dots & \ddots & 1\\ 1 & \dots & \dots & \dots & 0\end{pmatrix}, \tag{78}$$

with $\omega=\mathrm{e}^{2\pi\mathrm{i}/N}$. Then, they commute up to a center element, $M_SM_C=\omega M_CM_S$, and $M_C^N=\mathbf{1}$ and $M_S^N=(-1)^{N-1}\mathbf{1}$. Considering the homomorphism $H:SU(N)\to PSU(N)$, which sends the center elements of $SU(N)$ to unity in $PSU(N)$, makes it clear that $H(S)$ and $H(C)$ generate $\mathbb{Z}_N\times\mathbb{Z}_N\subset PSU(N)$.

As before, we have that when we gauge this $\mathbb{Z}_N\times\mathbb{Z}_N$ symmetry by twisting the index f of $\phi_{\ell,a}$ fields by an $M_S\in SU(N)$ matrix in one direction and $M_C\in SU(N)$ in the other direction, we are forced to have fractional fluxes for $U(1)$ gauge fields a_ℓ. In fact, for all $\ell=1,\dots,N$,

$$\int \mathrm{d}a_\ell=\int B \mod 2\pi, \tag{79}$$

where B is the $\mathbb{Z}_N$ two-form gauge field of the $PSU(N)$ gauge bundle.

The change in the action is

$$\Delta S_{\mathrm{top}}=\mathrm{i}p\int B, \tag{80}$$

where we set $\theta_\ell=\frac{2\pi p\ell}{N}$, so that the $\mathbb{Z}_N$ was a symmetry prior to gauging the $\mathbb{Z}_N\times\mathbb{Z}_N\subset PSU(N)$. This gives the mixed 't Hooft anomaly for $\mathbb{Z}_N\times\mathbb{Z}_N\subset PSU(N)$ flavor symmetry and $\mathbb{Z}_N$ permutation. The anomaly and global inconsistency discussed in previous subsections can also be found in the same manner.

D. C-T′ and $\mathbb{Z}_N$-T′ 't Hooft anomaly for even N

Here, we will discuss anomalies involving time-reversal symmetry, which were discussed in [54] for the $N=2$ case. The time-reversal symmetry T acts on $\boldsymbol{\phi}_i$ as

$$\mathsf{T}:\boldsymbol{\phi}_\ell(x,t)\mapsto\overline{\boldsymbol{\phi}}_\ell(x,-t). \tag{81}$$

When N is even, we can also define

$$\mathsf{T}':\boldsymbol{\phi}_\ell(x,t)\mapsto\mathcal{T}\overline{\boldsymbol{\phi}}_\ell(x,-t), \tag{82}$$

with

$$\mathcal{T}=\underbrace{(\mathrm{i}\tau^2)\otimes\cdots\otimes(\mathrm{i}\tau^2)}_{N/2}\in SU(N). \tag{83}$$

The matrix $\mathcal{T}$ satisfies $\mathcal{T}^\dagger=\mathcal{T}^t=-\mathcal{T}$ and $\mathcal{T}^2=-\mathbf{1}_N$. T' thus satisfies $\mathsf{T}'^2=(-1)^{N_\phi}$, where $N_{\boldsymbol{\phi}}$ counts the number of $\boldsymbol{\phi}_i'$s, and generates $\mathbb{Z}_2$ symmetry on gauge-invariant operators. Using this time-reversal symmetry, we can put the theory on nonorientable manifolds with the structure $\mathrm{Pin}^{\tilde{c}}=\mathrm{Pin}_-\ltimes_{\mathbb{Z}_2}U(1)$. This means that we put the background gauge field for T' symmetry.

After gauging T', the $U(1)$ gauge fields $\mathrm{d}a_\ell$ should obey

$$\int F_\ell=\pi\int w_2(TM_2) \mod 2\pi, \tag{84}$$

where $w_2(TM_2)$ is the second Stiefel-Whitney class of the tangent bundle of M_2. That is, πw_2 plays the role of $(N/2)B$ if we want to make a correspondence to the analysis with $SU(N)/\mathbb{Z}_N$ background gauge fields. (For derivation of this, see Ref. [54].)

Now, we can do the same analysis for $\mathbb{Z}_N$ permutation and C_k symmetries. Considering special θ angles with these additional symmetries, we discuss the change of the partition function under those transformations with $w_2(TM_2)$.

As an example, let us take a $\mathbb{Z}_N$-symmetric point $\theta_\ell=2\pi\ell/N$. The $\mathbb{Z}_N$ permutation changes the topological term as

$$\Delta S_{\mathrm{top}}=-\mathrm{i}\pi\int w_2(TM_2), \tag{85}$$

and thus the partition function on $\mathbb{R}P^2$, $Z(\mathbb{R}P^2)$, changes the sign under $\mathbb{Z}_N$ permutation. This means that there is the mixed 't Hooft anomaly between $\mathbb{Z}_N$ and T' symmetries.

As another example, let us take a C_1-symmetric point, $\theta_1=\pi$ and $\theta_\ell=0$ for $\ell\geqslant 2$. Again, the change of the topological term under C_1 is given by

$$\Delta S_{\mathrm{top}}=-\mathrm{i}\pi\int w_2(TM_2), \tag{86}$$

and we find the mixed anomaly between C_1 and T'. This mixed anomaly for spin systems without spin rotational symmetries was first found in the study [54] of $\mathbb{C}P^1$ nonlinear sigma model at $\theta=\pi$ (see also Refs. [62,75]).

V. $SU(N)$ WESS-ZUMINO-WITTEN MODEL

As we have mentioned briefly in Sec. III E, the $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model at the $\mathbb{Z}_3$-symmetric point is believed to be gapless. It is therefore an important and interesting problem to ask which conformal field theory appears in the low-energy limit. The anomaly matching condition tells us that the conformal field theory must have the same 't Hooft anomaly.

In this section, we compute the 't Hooft anomaly of $(1+1)$-dimensional $SU(N)$ Wess-Zumino-Witten (WZW) model and make connections with the $SU(N)/U(1)^{N-1}$ nonlinear sigma model. The classical action of the level-k $SU(N)$ WZW model is defined as [76–78]

$$S=-\frac{|k|}{8\pi}\int_{M_2}\mathrm{tr}[\mathrm{d}U\wedge\star\mathrm{d}U^\dagger]+k\Gamma_{\mathrm{WZ}}[U], \tag{87}$$

[10]Note that the prefactor was chosen so that the determinant is unity.

where U is the $SU(N)$-valued scalar field on M_2, and the Wess-Zumino term Γ_{WZ} is defined by

$$\Gamma_{\mathrm{WZ}}[U] = \frac{\mathrm{i}}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger \mathrm{d}U)^3]. \tag{88}$$

Here, M_3 is the three-dimensional manifold with $\partial M_3 = M_2$, and U is extended to M_3. By imposing the condition that $\exp(S)$ is independent of this extension, the level k is quantized to integers, $k \in \mathbb{Z}$. Since the parity flips the sign of the level k, it is often considered only for $k > 0$ and the level $-k$ shows the same conformal behavior.

The model has the $[SU(N)_L \times SU(N)_R]/\mathbb{Z}_N$ global symmetry. $SU(N)_L \times SU(N)_R \ni (g_L, g_R)$ acts on U as $U \mapsto g_L U g_R^\dagger$, but the symmetry group must be divided by $\mathbb{Z}_N$ since the diagonal center $\mathbb{Z}_N \subset SU(N)_L \times SU(N)_R$ does not change U. The subgroup $(SU(N)/\mathbb{Z}_N)_V \times (\mathbb{Z}_N)_L$ is of our interest. In previous studies [79–81], it was shown that the modular invariance of CFT and $(\mathbb{Z}_N)_L$ cannot be gauged (or orbifolded) simultaneously. Our interest here is the anomaly matching between the UV theory $[SU(N)/U(1)^{N-1}$ nonlinear sigma model] and the IR conformal behavior $[SU(N)$ WZW], and we are interested in the anomaly of the common global symmetries. $g_V \in SU(N)_V$ acts on U as $U \mapsto g_V U g_V^\dagger$, and $\omega^\ell \in (\mathbb{Z}_N)_L$ acts on U as $U \mapsto \omega^\ell U$ with $\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$ and $\ell = 0, 1, \ldots, N-1$. For connection with the main part of this paper, we would like to identify the $PSU(N)$ flavor symmetry as $SU(N)_V/\mathbb{Z}_N$ and the $\mathbb{Z}_N$ permutation symmetry as $(\mathbb{Z}_N)_L$.

A. ’t Hooft anomaly of WZW model

In order to gauge $(SU(N)/\mathbb{Z}_N)_V$, we introduce the $SU(N)$ one-form gauge field A_V and the $\mathbb{Z}_N$ two-form gauge field B. The $\mathbb{Z}_N$ two-form gauge field is realized as the pair of the $U(1)$ two-form gauge field B and the $U(1)$ one-form gauge field C satisfying the constraint

$$NB = \mathrm{d}C \tag{89}$$

and we construct the $U(N)$ gauge field $\widetilde{A}_V$ by

$$\widetilde{A}_V = A_V + \frac{1}{N} C \mathbf{1}_N. \tag{90}$$

The naive minimal coupling procedure is to replace $U^\dagger \mathrm{d}U$ by

$$U^\dagger D_V U = U^\dagger(\mathrm{d}U + \mathrm{i}\widetilde{A}_V U - \mathrm{i}U\widetilde{A}_V). \tag{91}$$

It is important to notice that the constraint and the covariant derivatives are invariant under the $U(1)$ one-form gauge transformation,

$$\widetilde{A}_V \mapsto \widetilde{A}_V + \xi, \quad C \mapsto C + N\xi, \quad B \mapsto B + \mathrm{d}\xi. \tag{92}$$

If we do this minimal coupling procedure for Γ_{WZ}, however, it becomes dependent on the choice of M_3. We will reconsider this to compute the anomaly soon later, and we will find that $SU(N)_V/\mathbb{Z}_N$ itself does not have the anomaly but has a mixed anomaly with $(\mathbb{Z}_N)_L$.

In order to gauge $(\mathbb{Z}_N)_L$, we introduce the $\mathbb{Z}_N$ one-form gauge field A_L. As we have done above, it is convenient to realize it by the $U(1)$ one-form gauge field satisfying the constraint

$$NA_L = \mathrm{d}\varphi, \tag{93}$$

where φ is the 2π-periodic scalar field. To do it, we first regard $U \in SU(N)$ as $U \in U(N)$ with the constraint

$$\det(U) = 1. \tag{94}$$

In gauging $\mathbb{Z}_N$, we replace the constraint by

$$\mathrm{e}^{\mathrm{i}\varphi}\det(U) = 1, \tag{95}$$

and the derivative $U^\dagger \mathrm{d}U$ is also replaced by the familiar form

$$U^\dagger D_L U = U^\dagger(\mathrm{d} + \mathrm{i}A_L)U. \tag{96}$$

The constraint and the covariant derivative are both invariant under the $U(1)$ zero-form gauge transformation,

$$U \mapsto \mathrm{e}^{-\mathrm{i}\psi}U, \quad \varphi \mapsto \varphi + N\psi, \quad A_L \mapsto A_L + \mathrm{d}\psi, \tag{97}$$

where ψ is the gauge parameter and 2π-periodic scalar.

Now, let us gauge $(SU(N)/\mathbb{Z}_N)_V \times (\mathbb{Z}_N)_L$. For this procedure, we perform the two gauging procedures explained above simultaneously. We introduce the left and right gauge fields by

$$L := \mathrm{i}(\widetilde{A}_V + A_L), \quad R := \mathrm{i}\widetilde{A}_V, \tag{98}$$

respectively, and the covariant derivative is defined as

$$U^\dagger DU = U^\dagger(\mathrm{d}U + LU - UR). \tag{99}$$

As we have mentioned, the gauged action obtained by this procedure is manifestly gauge-invariant, but the Wess-Zumino term with the covariant derivative is no longer the 2D action mod 2π:

$$\begin{aligned}
&\frac{\mathrm{i}}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger DU)^3] \\
&\quad = \frac{\mathrm{i}}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger \mathrm{d}U)^3] \\
&\quad + \frac{\mathrm{i}}{4\pi}\int_{M_3} \mathrm{tr}\Big[-\mathrm{d}(U\mathrm{d}U^\dagger)L - \mathrm{d}(\mathrm{d}U^\dagger U)R \\
&\quad - U\mathrm{d}U^\dagger L^2 + U^\dagger \mathrm{d}U R^2 + LUR\mathrm{d}U^\dagger - L(\mathrm{d}U)RU^\dagger \\
&\quad - U^\dagger L^2 UR + U^\dagger LUR^2 + \frac{1}{3}(L^3 - R^3)\Big].
\end{aligned} \tag{100}$$

In order to eliminate the three-dimensional mixed term of U and gauge fields without breaking the gauge invariance, we add the following three-dimensional gauge-invariant term:

$$\frac{\mathrm{i}}{4\pi}\int_{M_3} \mathrm{tr}[(UDU^\dagger)F_L - (U^\dagger DU)F_R], \tag{101}$$

where $F_L = \mathrm{d}L + L^2$ and $F_R = \mathrm{d}R + R^2$. Note that this term is invariant under the $U(1)$ one-form gauge symmetry, because one can show that $\mathrm{tr}[U^\dagger DU - UDU^\dagger] = 2\mathrm{tr}[L - R + U^\dagger \mathrm{d}U] = 0$. As a consequence, we obtain

$$\begin{aligned}
&\Gamma_{\mathrm{WZW}}[U, (\widetilde{A}_V, B), A_L] \\
&\quad = \frac{\mathrm{i}}{12\pi}\int_{M_3} \mathrm{tr}[(U^\dagger \mathrm{d}U)^3] + \frac{\mathrm{i}}{4\pi}\int_{M_3} \mathrm{d}\{\mathrm{tr}[LU\mathrm{d}U^\dagger \\
&\quad - RU^\dagger \mathrm{d}U - RU^\dagger LU]\} + \frac{\mathrm{i}}{4\pi}\int_{M_3} \mathrm{tr}\Big[\Big(R\mathrm{d}R + \frac{2}{3}R^3\Big) \\
&\quad - \Big(L\mathrm{d}L + \frac{2}{3}L^3\Big)\Big].
\end{aligned} \tag{102}$$

Let us substitute (98) into this expression to find the anomaly. We obtain

$$\begin{aligned}\Gamma_{\mathrm{WZW}}[U,(\widetilde{A}_V,B),A_L]\\ =\frac{\mathrm{i}}{12\pi}\int_{M_3}\mathrm{tr}[(U^\dagger \mathrm{d}U)^3]+\frac{\mathrm{i}}{4\pi}\int_{M_3}\mathrm{d}\{\mathrm{tr}[(\mathrm{i}\widetilde{A}_V)(U\mathrm{d}U^\dagger\\ -U^\dagger \mathrm{d}U-\mathrm{i}U^\dagger\widetilde{A}_V U)]\}\\ +\frac{\mathrm{i}}{4\pi}\int_{M_3}\mathrm{tr}\big[A_L\big(\mathrm{d}\widetilde{A}_V+\mathrm{i}\widetilde{A}_V^2\big)\big].\end{aligned}\tag{103}$$

The first two terms on the right-hand side of (103) give the well-defined action on M_2 mod 2π, but the last one cannot be written as a local term in two-dimensions. Indeed, it is equal to the three-dimensional topological action,

$$S_{\mathrm{SPT}}=\frac{\mathrm{i}N}{2\pi}\int_{M_3}A_L\wedge B,\tag{104}$$

which describes the (2 + 1)D SPT phase protected by the $\mathbb{Z}_N$ zero-form and $\mathbb{Z}_N$ one-form symmetries. We have shown that the gauged WZW partition function, $Z_{\mathrm{WZW},k}[(A_V,B),A_L]$, is not gauge invariant as a two-dimensional field theory. Adding the three-dimensional SPT phase (104), then the combined system,

$$Z_{\mathrm{WZW},k}[(A_V,B),A_L]\exp\left(kS_{\mathrm{SPT}}[A_L,B]\right),\tag{105}$$

is gauge-invariant. As a coefficient of the topological term, there is the identification $k\sim k+N$. As a consequence, the level-k $SU(N)$ WZW model has a mixed 't Hooft anomaly between $(SU(N)/\mathbb{Z}_N)_V$ and $(\mathbb{Z}_N)_L$ for $k\neq 0$ mod N.

B. Possible renormalization-group flows between $SU(N)$ Wess-Zumino-Witten models

We can discuss the possible renormalization-group (RG) flow of the $SU(N)_k$ WZW model under the perturbations preserving $PSU(N)\times(\mathbb{Z}_N)_L$. Since the 't Hooft anomaly is RG invariant, the low-energy effective theory must also have the anomaly given by $kS_{\mathrm{SPT}}[A_L,B]$. If k is a multiple of N, then $kS_{\mathrm{SPT}}=0$ mod 2π and thus the system can be gapped with the unique ground state. In the following, we assume that $1\leqslant \gcd(N,k)<N$, where $\gcd(N,k)$ is the greatest common divisor of N and k.

To be specific, let us ask if $SU(N)_k$ WZW model can flow into $SU(N)_{k'}$ WZW model in view of anomaly matching. For that discussion, we specify how the symmetry $PSU(N)\times\mathbb{Z}_N$ is realized in UV and IR theories. In both limits, $PSU(N)$ is assumed to be realized as the vector-like symmetry. For $SU(N)_k$ WZW theory, $\mathbb{Z}_N$ is generated by

$$U_{\mathrm{UV}}\mapsto \mathrm{e}^{2\pi\mathrm{i}/N}U_{\mathrm{UV}}.\tag{106}$$

In this setting, as we have discussed, the anomaly is given by

$$kS_{\mathrm{SPT}}[A_L,B]=\frac{kN}{2\pi}\int A_L\wedge B.\tag{107}$$

At low energies, we assume that $\mathbb{Z}_N$ is generated by

$$U_{\mathrm{IR}}\mapsto \mathrm{e}^{2\pi\mathrm{i}q/N}U_{\mathrm{IR}},\tag{108}$$

for some integer q with $\gcd(N,q)=1$. In this case, the anomaly is given by

$$k'S_{\mathrm{SPT}}[qA_L,B]=\frac{k'qN}{2\pi}\int A_L\wedge B.\tag{109}$$

The anomaly matching claims that we have to have

$$k=k'q \mod N.\tag{110}$$

It is important to notice that the above constraints are satisfied especially for

$$k'=\gcd(N,k),\tag{111}$$

by choosing $q=k/\gcd(N,k)$ mod N. The c theorem [82] tells us that the modulus of level-k must decrease along the renormalization group flow since [83] $c=k(N^2-1)/(k+N)$. If $k'<\gcd(N,k)$, then there is no integer q satisfying both $\gcd(N,q)=1$ and $k=k'q$ mod N. Therefore, $k'=\gcd(N,k)$ is the minimal level of the $SU(N)$ WZW model satisfying the constraint of anomaly matching. When preserving the $SU(N)_V/\mathbb{Z}_N\times(\mathbb{Z}_N)_L$ symmetry, we thus conclude that the $SU(N)_k$ WZW model can flow into the $SU(N)_{\gcd(N,k)}$ WZW model.

We point out that our conclusion is consistent with the previous conjecture [84] about the RG flow of WZW models: based on the assumption of adiabatic continuity for certain spin systems, it has been conjectured that $SU(N)_k$ WZW model can be deformed into $SU(N)_1$ WZW model if N and k are coprime, i.e., $\gcd(N,k)=1$. This is consistent with the anomaly matching condition discussed above by redefining the generator of $(\mathbb{Z}_N)_L$ symmetry as $q=k/\gcd(N,k)=k$. We note that the anomaly matching argument gives the constraint also when $\gcd(N,k)>1$.

Let us make several remarks about spin chains. The $SU(2)/U(1)$ nonlinear sigma model at $\theta=\pi$ shows the conformal behavior in the long-range limit, and that behavior is described by the $SU(2)_1$ WZW model. The nonlinear sigma model has a mixed 't Hooft anomaly between $SU(2)/\mathbb{Z}_2$ flavor symmetry and $\mathbb{Z}_2$ charge conjugation symmetry [43]. The anomaly polynomial is exactly given by (104) for $N=2$ and $k=1$, so the anomaly matching is satisfied for the Haldane conjecture [80].

We generalize this Haldane conjecture to the $SU(N)/U(1)^{N-1}$ nonlinear sigma models. If no other unknown protection for the RG flow occurs, we can conjecture for the $SU(N)/U(1)^{N-1}$ nonlinear sigma model at $\theta_\ell=2\pi p\ell/N$; when $p\neq 0$ mod N and the low-energy behavior is conformal, it is given by the $SU(N)_{\gcd(N,p)}$ WZW model.

C. Connection between $SU(N)$ WZW and $SU(N)/U(1)^{N-1}$ models

We here would like to argue in a complementary way the $SU(N)$ WZW model has the same 't Hooft anomaly of $SU(N)/U(1)^{N-1}$ nonlinear sigma model at the specific θ angles. We will make a direct connection between those two models preserving the relevant symmetry, $SU(N)/\mathbb{Z}_N\times\mathbb{Z}_N\subset\frac{SU(N)_L\times SU(N)_R}{\mathbb{Z}_N}$, and therefore provide an independent and intuitive proof for the matching of anomalies. For this purpose, let us consider the potential term breaking $[SU(N)_L\times$

$SU(N)_R]/\mathbb{Z}_N$ to $SU(N)_V/\mathbb{Z}_N\times(\mathbb{Z}_N)_L$, following the idea of Ref. [85]:

$$V=g\int_{M_2}\mathrm{d}^2x(\mathrm{tr}[U]^{2N}+\mathrm{tr}[U^\dagger]^{2N}). \tag{112}$$

Since this perturbation respects $SU(N)_V/\mathbb{Z}_N\times(\mathbb{Z}_N)_L$, the 't Hooft anomaly for this symmetry does not change for any values of the coupling g. For $g<0$, the classical minima are given by $U=\exp(2\pi \mathrm{i}n/N)\mathbf{1}$ with $n=0,1,\dots,N-1$, and thus $\mathbb{Z}_N$ is spontaneously broken and N degenerate vacua ensue. For $g>0$, the classical minimum satisfies $\mathrm{tr}[U]=0$, and $SU(N)_V/\mathbb{Z}_N\times(\mathbb{Z}_N)_L$ is not spontaneously broken. To study this region, we take the limit $g\to+\infty$ and set the strict constraint,

$$\mathrm{tr}[U]=0, \tag{113}$$

on M_2.

We consider the decomposition of $U\in SU(N)$ satisfying (113) on M_2 as

$$U(x_1,x_2)=\mathcal{U}(x_1,x_2)\Omega_0\mathcal{U}(x_1,x_2)^\dagger, \tag{114}$$

where $\Omega_0=\omega^{-(N-1)/2}\mathrm{diag}[1,\omega,\dots,\omega^{N-1}]$ with $\omega=\mathrm{e}^{2\pi\mathrm{i}/N}$. We take its extension to M_3 as[11]

$$U(x_1,x_2,x_3)=\mathcal{U}(x_1,x_2)\Omega(x_3)\mathcal{U}(x_1,x_2)^\dagger \tag{115}$$

with $M_3=M_2\times I_-$, where $I_-=(-\infty,0]$ is the half-line. The boundary condition is

$$\Omega(0)=\Omega_0,\quad \Omega(-\infty)=\mathbf{1}. \tag{116}$$

We have set $\Omega(-\infty)=\mathbf{1}$ so that $U(x_1,x_2,-\infty)=\mathbf{1}$, and we can regard $M_2\times\{-\infty\}$ as a point. This decomposition of U to $\mathcal{U}$ has a redundancy given by the maximal Abelian subgroup $U(1)^{N-1}$ of $SU(N)$. We want to think of the matrix $\mathcal{U}$ as corresponding to the complex scalar fields $\boldsymbol{\phi}_\ell$ of $SU(N)/U(1)^{N-1}$ nonlinear sigma model, i.e.,

$$\mathcal{U}=[\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\dots,\boldsymbol{\phi}_N]. \tag{117}$$

To justify that, let us first make the correspondence of symmetries, and compute the WZW action.

The redundancy of the decomposition is related to the gauge invariance under $\boldsymbol{\phi}_\ell\mapsto \mathrm{e}^{\mathrm{i}\varphi_\ell}\boldsymbol{\phi}_\ell$, with $\sum_{\ell=1}^N\varphi_l=0$. The flavor symmetry corresponds to $SU(N)_V/\mathbb{Z}_N$ because $V\in SU(N)$ acts as $\mathcal{U}\mapsto V\cdot\mathcal{U}$ and the center subgroup $\mathbb{Z}_N\subset SU(N)_V$ can be compensated by the $U(1)^{N-1}$ gauge invariance. The $\mathbb{Z}_N$ permutation symmetry corresponds to $(\mathbb{Z}_N)_L$. Let us introduce the permutation matrix $\mathcal{P}$ by

$$\mathcal{U}=[\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\dots,\boldsymbol{\phi}_N]\mapsto[\boldsymbol{\phi}_2,\dots,\boldsymbol{\phi}_N,\boldsymbol{\phi}_1]=:\mathcal{U}\mathcal{P}, \tag{118}$$

then it satisfies $\mathcal{P}\Omega_0\mathcal{P}^{-1}=\omega\Omega_0$. Applying the $\mathbb{Z}_N$ permutation to U, we get

$$U=\mathcal{U}\Omega_0\mathcal{U}^\dagger\mapsto\mathcal{U}\mathcal{P}\Omega_0\mathcal{P}^{-1}\mathcal{U}^\dagger=\omega U, \tag{119}$$

which is nothing but $(\mathbb{Z}_N)_L$.

Next, let us compute the WZW action using this parametrization. For the $SU(N)/U(1)^{N-1}$ nonlinear sigma model, the origin of 't Hooft anomaly is the topological θ term, so we have to reproduce the correct θ terms from the Wess-Zumino term Γ_{WZ}. Thus the kinetic term is unimportant, but let us write down its result just for completeness:

$$\begin{aligned}\frac{1}{2}\int_{M_2}&\mathrm{tr}[\mathrm{d}U\wedge\star\mathrm{d}U^\dagger]\\&=\sum_\ell\int_{M_2}(\mathrm{d}\overline{\boldsymbol{\phi}}_\ell\wedge\star\mathrm{d}\boldsymbol{\phi}_\ell-(\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_\ell)\wedge\star(\mathrm{d}\overline{\boldsymbol{\phi}}_\ell\cdot\boldsymbol{\phi}_\ell))\\&\quad-\sum_{\ell\neq m}\int_{M_2}\omega^{m-\ell}(\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_m)\wedge\star(\mathrm{d}\overline{\boldsymbol{\phi}}_m\cdot\boldsymbol{\phi}_\ell).\end{aligned} \tag{120}$$

The first term on the right-hand side is the usual kinetic term with the $U(1)$ gauge fields $a_\ell=\mathrm{i}\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_\ell$. The second one did not exist in our $SU(N)/U(1)^{N-1}$ sigma model, but it is gauge invariant and does not break any global symmetries. Therefore, we can add it without any problem and it does not affect the discussion of the 't Hooft anomalies.

In order to compute the Wess-Zumino term conveniently, let us specify our extension to the x_3 direction in more concrete way. We parametrize $\Omega(x_3)$ as

$$\Omega(x_3)=\mathrm{diag}[\mathrm{e}^{\mathrm{i}\theta_1(x_3)},\mathrm{e}^{\mathrm{i}\theta_2(x_3)},\dots,\mathrm{e}^{\mathrm{i}\theta_N(x_3)}]. \tag{121}$$

The boundary condition on $\Omega(x_3)$ can be rephrased as

$$\theta_\ell(-\infty)=0,\quad \theta_\ell(0)=\frac{2\pi\ell}{N}, \tag{122}$$

up to an overall shift of $\theta_\ell(0)'$s, but such overall shift does not change the following argument so we can take this convention. Since $\mathcal{U}$ does not depend on x_3, we obtain that

$$\begin{aligned}&\mathrm{tr}[(U^\dagger\mathrm{d}U)^3]\\&\quad=3\,\mathrm{tr}[(\mathcal{U}^\dagger\mathrm{d}U)^2\mathrm{d}\Omega\Omega^\dagger+(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})^2\Omega^\dagger\mathrm{d}\Omega]\\&\qquad+3\,\mathrm{tr}[(\mathcal{U}^\dagger\mathrm{d}\mathcal{U})\Omega^\dagger(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})\mathrm{d}\Omega-(\mathcal{U}^\dagger\mathrm{d}\mathcal{U})\Omega(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})\mathrm{d}\Omega^\dagger].\end{aligned} \tag{123}$$

As we shall see, the first line on the right-hand side gives the θ term, while the second one gives the generalization of the λ term. The computation of the first term can be done as follows:

$$\begin{aligned}\frac{\mathrm{i}}{12\pi}\int_{M_3}&3\,\mathrm{tr}[(\mathcal{U}^\dagger\mathrm{d}\mathcal{U})^2\mathrm{d}\Omega\Omega^\dagger+(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})^2\Omega^\dagger\mathrm{d}\Omega]\\&=\frac{\mathrm{i}}{2\pi}\int_{M_2}\sum_{\ell,m}(\overline{\boldsymbol{\phi}}_m\cdot\mathrm{d}\boldsymbol{\phi}_\ell)(\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_m)\int_{I_-}\mathrm{i}\mathrm{d}\theta_m(x_3)\\&=\sum_m\frac{\theta_m(0)}{2\pi}\int_{M_2}\mathrm{d}\overline{\boldsymbol{\phi}}_m\cdot\mathrm{d}\boldsymbol{\phi}_m.\end{aligned} \tag{124}$$

This is exactly the θ term, and the θ angles are given by $\theta_\ell=\theta_\ell(0)=2\pi\ell/N$. This is nothing but the $\mathbb{Z}_N$ symmetric point of the $SU(N)/U(1)^{N-1}$ nonlinear sigma model. Doing the similar computation of the second term gives

$$\begin{aligned}\frac{\mathrm{i}}{4\pi}\int_{M_3}&\mathrm{tr}[(\mathcal{U}^\dagger\mathrm{d}\mathcal{U})\Omega^\dagger(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})\mathrm{d}\Omega-(\mathcal{U}^\dagger\mathrm{d}\mathcal{U})\Omega(\mathrm{d}\mathcal{U}^\dagger\mathcal{U})\mathrm{d}\Omega^\dagger]\\&=\frac{1}{4\pi}\sum_{m\neq\ell}\sin(\theta_m(0)-\theta_\ell(0))\int_{M_2}(\overline{\boldsymbol{\phi}}_m\cdot\mathrm{d}\boldsymbol{\phi}_\ell)(\overline{\boldsymbol{\phi}}_\ell\cdot\mathrm{d}\boldsymbol{\phi}_m).\end{aligned} \tag{125}$$

[11]We are allowed to take this specific extension to M_3 since any extensions give the same value of Γ_{WZ} up to $2\pi\mathrm{i}$.

For the case of the λ term, $\sin(\theta_m(0)-\theta_\ell(0))$ should be replaced by $\lambda(\delta_{m,\ell+1}-\delta_{m,\ell-1})$, so this gives its generalization. The generalization does not break any symmetry, so it does not change the argument of the 't Hooft anomaly matching.

VI. LINEAR SIGMA MODELS

In this section, we construct the linear sigma model version of the $SU(N)/U(1)^{N-1}$ nonlinear sigma model, and show that they have the same 't Hooft anomaly and global inconsistency explicitly. In certain limits of linear sigma models, we can perform the analytic computation of the partition function, and we can check the conjecture on the phase diagram given in Sec. III.

A. Linear realization of $SU(N)/U(1)^{N-1}$ nonlinear sigma model

We here consider the case $N=3$ for simplicity of the presentation, and the generalization is straightforward. Instead of the three copies of $\mathbb{C}P^2$ obeying orthogonality conditions, let us take 3 copies of an $SU(3)$ triplet, without orthogonality constraints on them. The kinetic term of the Lagrangian is

$$S_{\rm kin}=-\frac{1}{2g}\int\sum_{\ell=1,2,3}|(\mathrm{d}+\mathrm{i}a_\ell)\mathbf{\Phi}_\ell|^2, \tag{126}$$

where $\mathbf{\Phi}_\ell=(\Phi_{1\ell},\Phi_{2\ell},\Phi_{3\ell})$, and we put the strict constraint on the $U(1)$ gauge fields,

$$a_3=-a_1-a_2. \tag{127}$$

What this means is that the triplet $\mathbf{\Phi}_3$ is charged as $(-1,-1)$ under the gauge charges a_1 and a_2.

The analog of the λ term is given as follows: we first define the gauge-covariant one-form,

$$\omega_{i,j}=\overline{\mathbf{\Phi}}_i\cdot(\mathrm{d}+\mathrm{i}a_j)\mathbf{\Phi}_j-\mathbf{\Phi}_j\cdot(\mathrm{d}-\mathrm{i}a_i)\overline{\mathbf{\Phi}}_i, \tag{128}$$

and write down the gauge-invariant term as

$$S_\lambda=\frac{\lambda}{2\pi}\int\sum_{\ell=1}^{3}\omega_{\ell+1,\ell}\wedge\omega_{\ell,\ell+1}. \tag{129}$$

The model above, $S_{\rm kin}+S_\lambda$, has the following symmetries, which are the same with those of $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model: (1) $PSU(3)=SU(3)/\mathbb{Z}_3$ flavor symmetry, acting projectively on $\mathbf{\Phi}_\ell$ as $\mathbf{\Phi}_\ell\mapsto U\mathbf{\Phi}_\ell$ with $U\in SU(3)$, (2) time-reversal T that sends $\mathbf{\Phi}_\ell(x,t)\mapsto\overline{\mathbf{\Phi}}_\ell(x,-t)$, (3) $\mathbb{Z}_3$ exchange symmetry, which cyclically permutes $\mathbf{\Phi}_\ell\mapsto\mathbf{\Phi}_{\ell+1}$ and $a_\ell\mapsto a_{\ell+1}$, and (4) charge conjugations C_k that sends $\mathbf{\Phi}_\ell\mapsto-\overline{\mathbf{\Phi}}_{-\ell-k}$ and $a_\ell\mapsto-a_{-\ell-k}$.

In addition, there is an extra $U(1)/\mathbb{Z}_3$ symmetry, given by

$$\mathbf{\Phi}_\ell\mapsto \mathrm{e}^{\mathrm{i}\varphi}\mathbf{\Phi}_\ell \tag{130}$$

with $\varphi\sim\varphi+2\pi$, but the physical identification on gauge-invariant operators is $\varphi\sim\varphi+2\pi/3$.

In order to match the symmetry with that of nonlinear sigma model, we shall break $U(1)/\mathbb{Z}_3$ symmetry explicitly by the potential term. The invariant tensors of $SU(3)$ are the Kronecker delta and the epsilon tensors. The gauge-invariant quadratic invariants of $SU(3)$ made of $\mathbf{\Phi}_\ell$ are $\overline{\mathbf{\Phi}}_\ell\cdot\mathbf{\Phi}_{\ell'}$. Using the epsilon tensor, we also have the gauge-invariant $SU(3)$ invariant,

$$g=\varepsilon_{abc}\Phi_{a1}\Phi_{b2}\Phi_{c3},\quad \overline{g}=\varepsilon_{abc}\overline{\Phi}_{a1}\overline{\Phi}_{b2}\overline{\Phi}_{c3}. \tag{131}$$

This operator is invariant under $U(1)$ gauge symmetries, and has the unit charge under $U(1)/\mathbb{Z}_3$ global symmetry, and

$$\mathsf{T}:g(x,t)\mapsto\overline{g}(x,-t),\ \ \mathsf{C}_k:g\mapsto\overline{g}. \tag{132}$$

We therefore add the following potential term,

$$S_{\rm pot}=\int\mathrm{d}^2x\left\{\sum_\ell V_1(|\mathbf{\Phi}_\ell|^2)+\sum_{\ell>\ell'}V_2(|\overline{\mathbf{\Phi}}_\ell\cdot\mathbf{\Phi}_{\ell'}|^2)\right\} +\int\mathrm{d}^2x\,V_3(\varepsilon_{abc}(\Phi_{a1}\Phi_{b2}\Phi_{c3}+\overline{\Phi}_{a1}\overline{\Phi}_{b2}\overline{\Phi}_{c3})). \tag{133}$$

By taking a certain limit of V_1 and V_2, we can reproduce the orthonormality constraint (2) of the nonlinear sigma model, and the matrix $[\mathbf{\Phi}_1,\mathbf{\Phi}_2,\mathbf{\Phi}_3]\in U(3)$. The potential V_3 gives the condition on its determinant as in (3), and $[\mathbf{\Phi}_1,\mathbf{\Phi}_2,\mathbf{\Phi}_3]\in SU(3)$. Since $U(1)\times U(1)$ gauge invariance says that this target space is redundant by $U(1)\times U(1)$, we can obtain the nonlinear $SU(3)/[U(1)\times U(1)]$ sigma model as a low-energy effective theory of $S_{\rm kin+pot+\lambda}$ in this limit. As we shall see, the anomaly discussed in Sec. III exists for the generic potential V_1, V_2, and V_3.

As in the case of nonlinear sigma model, we introduce the topological term that breaks $\mathbb{Z}_3$ and C_k for general values:

$$S_{\rm top}=\sum_{\ell=1}^{3}\frac{\mathrm{i}\theta_\ell}{2\pi}\int\mathrm{d}a_\ell. \tag{134}$$

Because of the constraint on the gauge charge, we can choose one of the θ angles to be zero, and we set $\theta_3=0$ in this section.

The model above has all the same symmetry, and, save for the more parameter freedom, largely the same structure as the $SU(3)/U(1)^2$ nonlinear sigma model. It therefore has the same anomalies that we have been discussing so far.

Such models, which can be supplemented with arbitrary local terms in the Lagrangian, are of interest as they better capture all the possible phases of relevant spin chains. The $SU(3)/U(1)^2$ nonlinear sigma model on the other hand is supposed to describe only a Heisenberg spin chain, and even that one can be reliably related to it only via the limit of large dimension of the $SU(3)$ representations (i.e., large spin limit). The statement that the $SU(3)/U(1)^2$ nonlinear sigma model is the effective model of the spin chains is therefore imprecise. Rather the more precise statement is that the effective theory of general spin chains is described by a linear sigma model, with *a priori* unknown couplings. Still anomalies and inconsistencies give constraint on possible vacuum realizations of such models so the phase diagram is guaranteed to be interesting. They also allow for more semiclassical regimes, because they have more tunable parameters. In particular, we can add a mass to the $\mathbf{\Phi}_\ell$ fields, preserving all the symmetries and, therefore, all the anomalies. Upon taking this mass to large values, a free photon ensues. We discuss this next.

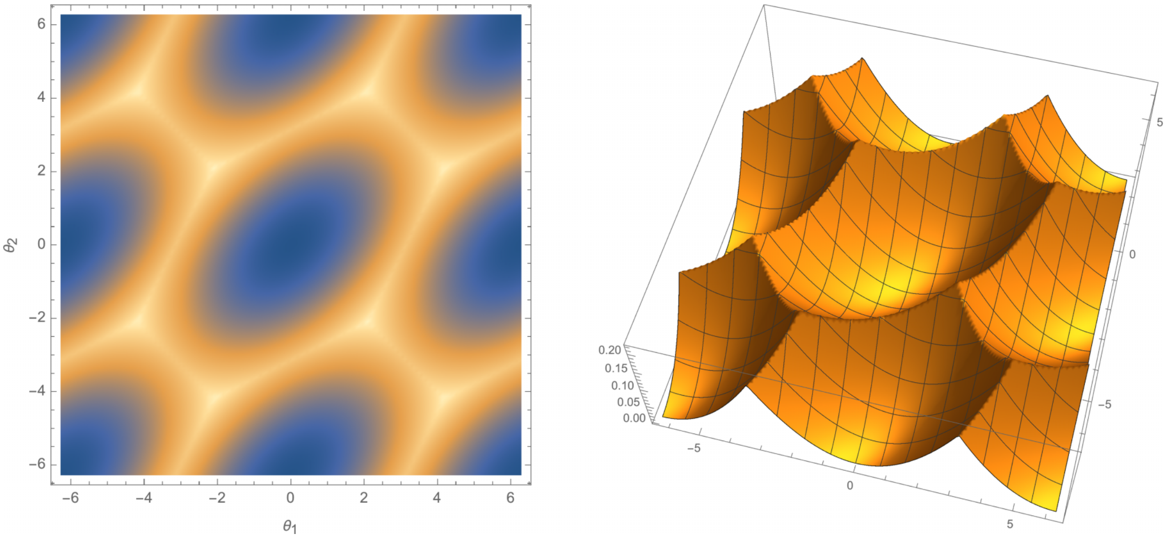

FIG. 3. The energy density of the ground state as a function of the two θ parameters. Notice that the same picture emerges as discussed in Sec. III. On the 3D plot on the right, it is clear that level crossings occur at the C-symmetric lines, which meet at $\mathbb{Z}_3$-cyclic permutation symmetric points which carry a 't Hooft anomaly.

B. Free photon limit

Using the linear sigma model description, let us take the limit to compute the free energy analytically. This will provide a check and deepen the understanding of how the anomaly and global inconsistency matching discussed in Sec. III E is realized. The easiest thing we can do is to send the mass of the $\Phi_{\ell,f}$ fields to be large, then the matter fields can be integrated out, and we obtain the local field theory of photons.

The effective theory is a free $U(1)^{N-1}$ gauge theory, with the (real-time) Lagrangian given by

$$\mathcal{L} = \sum_{\ell=1}^{N} \frac{1}{2e^2} F_\ell^2 + \sum_{\ell=1}^{N} \frac{\theta_\ell}{2\pi} F_\ell, \tag{135}$$

where $F_\ell = \partial_t a_{x\ell} - \partial_x a_{t\ell}$ and e is the effective coupling constant. The gauge fields satisfy the constraint $a_1 + \cdots + a_N = 0$, and we can set $\theta_N = 0$.

Now, we canonically quantize the system in order to find energy eigenstates, and we take the temporal gauge $a_{t\ell} = 0$ for this purpose. The coordinates $a_{x\ell}$ ($\ell = 1, \ldots, N-1$) have the canonical momentum

$$\pi_x^\ell = \frac{\partial \mathcal{L}}{\partial(\partial_t a_{x\ell})} = \frac{1}{e^2} F_\ell + \frac{1}{e^2} \sum_{\tilde{\ell}=1}^{N-1} F_{\tilde{\ell}} + \frac{\theta_\ell}{2\pi}. \tag{136}$$

Solving it for F_ℓ, we have

$$F_\ell = e^2 \Pi_x^\ell - \frac{e^2}{N} \sum_{\tilde{\ell}=1}^{N-1} \Pi_x^{\tilde{\ell}}, \tag{137}$$

where

$$\Pi_x^\ell = \pi_x^\ell - \frac{\theta_\ell}{2\pi}, \tag{138}$$

for $\ell = 1, \ldots, N-1$. The Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= \sum_{\ell=1}^{N-1} \pi_x^\ell (\partial_t a_{x\ell}) - \mathcal{L} \\ &= \sum_{\ell=1}^{N-1} \Pi_x^\ell F_\ell - \frac{1}{2e^2} \sum_{\ell=1}^{N-1} F_\ell^2 - \frac{1}{2e^2} \left(\sum_{\ell=1}^{N-1} F_\ell \right)^2 + \pi_x^\ell \partial_x a_{t\ell} \\ &= \frac{e^2}{2} \left[\sum_{\ell=1}^{N-1} (\Pi_x^\ell)^2 - \frac{1}{N} \left(\sum_{\ell=1}^{N-1} \Pi_x^\ell \right)^2 \right] + \pi_x^\ell \partial_x a_{t\ell}. \end{aligned} \tag{139}$$

When demanding that $[H, \pi_0^\ell]$, the last term causes the secondary constraint $\partial_x \pi_x^\ell = 0$ — the Gauss law. Further the spectrum is simply given by the eigenvalues of π_x^ℓ, which are integers m_ℓ, i.e.,

$$\begin{aligned}&\mathcal{E}_{\{m_k\}}(\theta_\ell) \\ &= \frac{e^2}{2} \left[\sum_{\ell=1}^{N-1} \left(m_\ell - \frac{\theta_\ell}{2\pi} \right)^2 - \frac{1}{N} \left(\sum_{\ell=1}^{N-1} \left(m_\ell - \frac{\theta_\ell}{2\pi} \right) \right)^2 \right]. \end{aligned} \tag{140}$$

The ground-state energy is given by the minimum among those sectors:

$$\mathcal{E}(\theta_\ell) = \min_{\{m_k\} \in \mathbb{Z}^{N-1}} \mathcal{E}_{\{m_k\}}(\theta_\ell). \tag{141}$$

Using this expression for $N = 3$, we can confirm the phase diagram of Fig. 1 (or the left one of Fig. 2 in Sec. III E). We plot the $N = 3$ energy density given by (140) for the ground state (Fig. 3), and we can clearly see the pattern that emerged from our general discussion.

Now consider the $\mathbb{Z}_N$ permutation symmetry sends $F_\ell \mapsto F_{\ell+1}$, where $F_N = -F_1 - F_2 \cdots - F_{N-1}$. This symmetry acts on the canonical momentum as

$$\pi_x^\ell \mapsto \pi_x^{\ell+1} - \pi_x^1 + \frac{-\theta_{\ell+1}+\theta_\ell+\theta_1}{2\pi}, \quad \ell = 1,2\dots,N-2; \tag{142}$$

$$\pi_x^{N-1} \mapsto -\pi_x^1 + \frac{\theta_{N-1}+\theta_1}{2\pi}. \tag{143}$$

If we replace $\theta_\ell = 2\pi p\ell/N$, we find the action of the $\mathbb{Z}_N$ permutation on the eigenvalues $\{m_\ell\}_{\ell=1,\dots,N-1}$:

$$m_\ell \mapsto m_{\ell+1} - m_1, \quad \ell = 1,2\dots,N-2; \tag{144}$$

$$m_{N-1} \mapsto -m_1 + p. \tag{145}$$

We now look for a fixed point of the transformation, i.e., that

$$m_\ell = m_{\ell+1} - m_1, \quad \ell = 1,2\dots,N-2; \tag{146}$$

$$m_{N-1} = -m_1 + p. \tag{147}$$

The first equation implies that

$$m_\ell = \ell m_1, \quad \ell = 1,\dots,N-1 \tag{148}$$

and, in particular, that

$$m_{N-1} = (N-1)m_1, \tag{149}$$

while the second one implies

$$m_{N-1} = -m_1 + p. \tag{150}$$

Consistency of the two equations demands that

$$Nm_1 = p, \tag{151}$$

which is only possible if $p = 0$ mod N, so that $m_1 = p/N \in \mathbb{Z}$. This is precisely the case where there is no anomaly in the full theory, so we get consistency. When $p \neq 0$ mod N, there is no fixed point of the $\mathbb{Z}_N$ transformation acting on integers m_ℓ, so all states (and in particular the ground state) are degenerate, and the anomaly is saturated by breaking the $\mathbb{Z}_N$ global symmetry.

Let us discuss this a bit more from the point of view of anomalies. Originally, the theory has the $PSU(N)$ flavor symmetry, but it is gone in the low-energy effective theory since matter fields are very massive. Instead, the theory acquires the emergent $\mathbb{Z}_N$ one-form symmetry, which is further enhanced to $U(1)^{N-1}$ one-form symmetry in the free-photon Lagrangian (135). We can understand the above energy spectrum (140) by gauging this $U(1)^{N-1}$ one-form symmetry. To see it, we introduce the $U(1)$ two-form gauge fields, B_ℓ, for $\ell = 1,\dots,N-1$ and impose the invariance under the $U(1)$ one-form gauge transformations,

$$B_\ell \mapsto B_\ell + \mathrm{d}\lambda_\ell, \quad a_\ell \mapsto a_\ell + \lambda_\ell. \tag{152}$$

We have to replace the field strength $\mathrm{d}a_\ell$ by $\mathrm{d}a_\ell - B_\ell$ in order to satisfy this invariance. We can further add the local gauge-invariant terms of B_ℓ in the gauging procedure, so we add

$$\mathrm{i}\sum_\ell m_\ell \int B_\ell \tag{153}$$

for integers $\{m_\ell\} \in \mathbb{Z}^{N-1}$. We can easily find that

$$\begin{aligned}\exp(-V\mathcal{E}_{\{m_k\}}) &= \int \mathcal{D}B_\ell \int \mathcal{D}a_\ell \exp\sum_\ell\left[-\frac{1}{4e^2}\int_{M_2}|\mathrm{d}a_\ell - B_\ell|^2\right.\\ &\left.+\frac{\mathrm{i}\theta_\ell}{2\pi}\int_{M_2}(\mathrm{d}a_\ell - B_\ell) + \mathrm{i}m_\ell\int_{M_2}B_\ell\right],\end{aligned} \tag{154}$$

where the path integral is done with the constraint $a_1+\dots+a_N = 0$ and $B_1+\dots+B_N = 0$, and V is the volume of M_2. The labels $\{m_\ell\}$ of the energy eigenstate are now understood as the coefficient of the counterterm for gauging $U(1)^{N-1}$ one-form symmetry [42,49]. Since the original $PSU(N)$ symmetry corresponds to the diagonal subgroup $\mathbb{Z}_N \subset U(1)^{N-1}$, we would like to set

$$B \equiv B_1 = B_2 = \dots = B_{N-1}, \quad NB = \mathrm{d}C, \tag{155}$$

with some $U(1)$ gauge field C. The corresponding local counterterm becomes

$$\mathrm{i}(m_1+\dots+m_{N-1})\int B, \tag{156}$$

and the coefficient $m_{\mathrm{tot}} = (m_1+\dots+m_N)$ is meaningful only in $\mathbb{Z}_N$. This means that the interaction term coming out of the matter fields $\mathbf{\Phi}_\ell$ can mix the states with the same m_{tot} mod N, but the different ones cannot be mixed by such interactions. In this sense, we can regard m_{tot} as the $\mathbb{Z}_N$ charge of the $\mathbb{Z}_N$ one-form symmetry, or $PSU(N)$ symmetry of the original theory.

Now notice that the $\mathbb{Z}_N$ permutation acts on the $\mathbb{Z}_N$ charge of the one-form symmetry, $m_{\mathrm{tot}} = (m_1+\dots+m_{N-1})$, as

$$m_{\mathrm{tot}} \mapsto m_{\mathrm{tot}} + p \mod N. \tag{157}$$

This means that we cannot find the simultaneous eigenstate of the $\mathbb{Z}_N$ one-form symmetry and the $\mathbb{Z}_N$ permutation symmetry for $p \neq 0$ mod N. Furthermore, if p and N are relatively prime, then all of the states must be N-fold degenerate. Since the $\mathbb{Z}_N$ one-form symmetry emerges from $PSU(N)$ symmetry, (157) should be regarded as the consequence of $PSU(N)$-$\mathbb{Z}_N$ 't Hooft anomaly. We can repeat a similar discussion for $PSU(N)$-C global inconsistency.[12]

VII. CIRCLE COMPACTIFICATION WITH PERSISTENT 'T HOOFT ANOMALY

In the previous section, Sec. VI B, we give the linear sigma model description of $SU(3)/[U(1)\times U(1)]$ model, and it is computable in the limit where the matter fields $\mathbf{\Phi}$ are very massive. The original interest of the model is the case where the matter fields are would-be Nambu-Goldstone bosons, and thus it is very appealing if we can consider a setup to study

[12]Note, however, that on the C-invariant lines of pure $U(1)^{N-1}$ gauge theory, there is a genuine anomaly between the C symmetry and the $U(1)^{N-1}$ center symmetry. It therefore guarantees that the phase transition lines are as in Fig. 3. Only if the center symmetry is reduced down to $\mathbb{Z}_N$ with odd N, does the anomaly turn to a global inconsistency.

that regime analytically. In this section, we provide a setup for reliable semiclassical computations of $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model.

The nonlinear sigma models in two dimension shows the asymptotic freedom in general when the target space has positive curvature [86,87], which means that they become strongly coupled in the infrared regime. It is therefore quite difficult to extract the low-energy behavior of the theory analytically. For example, the semiclassical analysis using instantons suffers from the severe IR divergences, and gives the wrong results even qualitatively [88]. One possible way to evade IR divergences is to put the theory on a small circle $\mathbb{R}\times S^1$. The size of the circle L provides an energy scale $1/L \gg \Lambda$, which can be arbitrarily high, and hence has a potential to render the asymptotically free theories weakly coupled. However, if compactification is done naively, then typically the phase transition along the circle size L takes place, and the wanted low-energy behavior cannot be found in a semiclassical way [89].

The idea of semiclassical analysis with S^1 compactification can be revived by compactifying the theory with twisted boundary conditions or, equivalently with a nontrivial holonomy background [90–93] (see also Refs. [94–118]). Such systems often exhibit a weakly coupled regimes for small L and thus can be treated semiclassically without IR divergences. Their properties look remarkably similar to the low-energy behavior expected for uncompactified theories. It is therefore conjectured that the large and small circles are adiabatically connected thanks to the nontrivial holonomy, but the role of the nontrivial holonomy was not so clear when it was proposed. One of the authors (T. S.) has shown that such holonomies can lead to a vast cancellation in the spectrum preventing a would-be thermal phase transition [115]. In Ref. [115], it was explicitly shown how such cancellations lead to a large N volume independence for $\mathbb{C}P^{N-1}$ and $O(N)$ sigma models, and the conjecture acquired solid ground for certain models. The problem was revisited by the another author (Y. T.) from the viewpoint of 't Hooft anomaly matching, and it is shown that the nontrivial holonomy is essential for persistence of 't Hooft anomaly under S^1 compactification [51,52,119].

In this section, we discuss the S^1 compactification, under which the $SU(3)/\mathbb{Z}_3$-$\mathbb{Z}_3$ mixed anomaly and $SU(3)/\mathbb{Z}_3$-C global inconsistency survives following Ref. [51]. This provides an opportunity for future works to study the low-energy behavior of the $SU(3)/[U(1)\times U(1)]$ nonlinear sigma model by an analytic semiclassical computations.

We take $M_2 = M_1\times S^1$, and the circumference of S^1 is L. Using the clock matrix $C=\mathrm{diag}[1,\omega,\omega^2]$ with $\omega=\mathrm{e}^{2\pi\mathrm{i}/3}$, we define the boundary condition

$$\boldsymbol{\phi}_\ell(x,t+L)=C\cdot\boldsymbol{\phi}_\ell(x,t). \tag{158}$$

We take the periodic boundary condition for the gauge field, $a_\ell(x,t+L)=a_\ell(x,t)$. This defines our S^1-compactified theory.

The above boundary condition is equivalent to introducing the background $SU(3)$ holonomy along the compactified direction. To see this, let us define $\widetilde{\boldsymbol{\phi}}_\ell(x,t)$ obeying the periodic boundary condition by

$$\phi_{f,\ell}(x,t)=\mathrm{e}^{2\pi\mathrm{i}ft/3L}\widetilde{\phi}_{f,\ell}(x,t) \tag{159}$$

for $f=1,2,3$. Then, the covariant time derivative is given as

$$|D_t\phi_{f,\ell}|^2=\left|\left(\partial_0+a_{\ell,0}+\frac{2\pi\mathrm{i}f}{3L}\right)\widetilde{\phi}_{f,\ell}\right|^2, \tag{160}$$

and we can see that the $SU(3)$-flavor background gauge field is introduced in addition to the $U(1)$ gauge field.

Because of the flavor-dependent boundary condition, the $SU(3)/\mathbb{Z}_3$ flavor symmetry is explicitly broken down to its maximal Abelian subgroup $[U(1)\times U(1)]/\mathbb{Z}_3$. In addition, the system has the symmetry involving the shift matrix and one-form transformation [51] (see also Ref. [120]). Since $SC=\mathrm{e}^{2\pi\mathrm{i}/3}CS$, the shift matrix itself does not generate the symmetry of the S^1-compactified theory. Indeed, the kinetic term is changed as

$$\sum_f\left|\left(\partial_0+a_{\ell,0}+\frac{2\pi\mathrm{i}f}{3L}\right)\widetilde{\phi}_{f,\ell}\right|^2 \mapsto \sum_f\left|\left(\partial_0+a_{\ell,0}+\frac{2\pi\mathrm{i}(f+1)}{3L}\right)\widetilde{\phi}_{f,\ell}\right|^2. \tag{161}$$

In order to compensate the difference, we have to perform

$$a_{\ell,0}\mapsto a_{\ell,0}-\frac{2\pi\mathrm{i}}{3L}, \tag{162}$$

which is nothing but the $\mathbb{Z}_3$ one-form transformation on $U(1)$ Polyakov loops. Let us call this $\mathbb{Z}_3$ symmetry as the intertwined shift symmetry, $(\mathbb{Z}_3)_{\mathrm{shift}}$.

Now, we want to gauge the $\mathbb{Z}_3$ intertwined shift symmetry, and we denote the $\mathbb{Z}_3$ one-form gauge field $B^{(1)}$. Since it acts on the one-form gauge field a_ℓ, it should be related to the $\mathbb{Z}_3$ two-form gauge field B for $SU(3)/\mathbb{Z}_3$ flavor symmetry. Indeed, B and $B^{(1)}$ is related by

$$B=B^{(1)}\wedge L^{-1}\mathrm{d}t. \tag{163}$$

We thus denote the partition function with $B^{(1)}$ as $Z_{M_1\times S^1}[B^{(1)}]$. Using this result, we can obtain the mixed 't Hooft anomaly and global inconsistency of S^1-compactified theory just by substituting this correspondence into the 't Hooft anomaly and global inconsistency in two dimensions [51]. Under $\mathbb{Z}_3$ permutation, $\boldsymbol{\phi}_\ell\mapsto\boldsymbol{\phi}_{\ell+1}$, the $SU(3)/\mathbb{Z}_3$-$\mathbb{Z}_3$ anomaly at $(\theta_1,\theta_3)=(2\pi/3,-2\pi/3)$ implies

$$Z_{M_1\times S^1}[B^{(1)}]\mapsto Z_{M_1\times S^1}[B^1]\exp\left(-\mathrm{i}\int_{M_1}B^{(1)}\right), \tag{164}$$

and the $(\mathbb{Z}_3)_{\mathrm{shift}}$-$(\mathbb{Z}_3)_{\mathrm{permutation}}$ anomaly is found for the S^1-compactified theory. Similarly, the $(\mathbb{Z}_3)_{\mathrm{shift}}$-$\mathsf{C}$ global inconsistency can be found from the $SU(3)/\mathbb{Z}_3$-C global inconsistency.

We have shown that the phase diagram of the S^1-compactified theory with the boundary condition (158) is constrained by the same 't Hooft anomaly and global inconsistency with that of two-dimensional $SU(3)/[U(1)\times U(1)]$

sigma model. Since the phase diagrams in Figs. 1 and 2 are found by the $SU(3)/\mathbb{Z}_3$-$\mathbb{Z}_3$ anomaly and the $SU(3)/\mathbb{Z}_3$-C global inconsistency matching arguments, we claim that the circle-compactified model will have the same structure of the phase diagram. It is thus an interesting future study to consider the analytic semiclassical computation of this model in order to get more physical insights on the $SU(3)/[U(1)\times U(1)]$ sigma model.

VIII. THE 2+1D SYSTEMS AND DOMAIN WALLS

We here briefly discuss the anomalies of the QFT system when it is lifted to 2+1D. Our discussion will be cursory, leaving a more detailed discussion for the future. We will restrict ourselves to the case of $N=3$.

In 2+1D, we can no longer have θ terms. Instead the two $U(1)$ gauge fields now have a $[U(1)\times U(1)]_T$ topological symmetry, generated by the charges

$$Q_{1,2}=\frac{1}{2\pi}\int \mathrm{d}a_{1,2}. \tag{165}$$

The relevant $U(1)$ Noether currents are just

$$j_{1,2}=\frac{1}{2\pi}\star \mathrm{d}a_{1,2}. \tag{166}$$

If we now couple the currents to a background gauge fields via the minimal coupling, we have to add a term

$$\mathcal{S}=\frac{\mathrm{i}}{2\pi}\int_{M_3} A_1\wedge \mathrm{d}a_1+\frac{\mathrm{i}}{2\pi}\int_{M_3} A_2\wedge \mathrm{d}a_2. \tag{167}$$

The above action is invariant under the $U(1)\times U(1)$ gauge transformation sending $A_{1,2}\to A_{1,2}+\mathrm{d}\varphi_{1,2}$ because of the quantization of the fluxes $\int \mathrm{d}a_1$ and $\int \mathrm{d}a_2\in 2\pi\mathbb{Z}$. However, if we now gauge the $PSU(3)$ symmetry, the fluxes will fail to be quantized in multiples of 2π by the amount $\int B\in \frac{2\pi}{3}\mathbb{Z}$, where $B\in H^2(M_3,\pi_1(PSU(3)))$ is the $\mathbb{Z}_3$ two-form gauge field, indicating an anomaly. However, if we gauge transform with the choice $\varphi_1=-\varphi_2=\varphi$, the action is still invariant. This indicates that while there is an anomaly between the diagonal part $U(1)_V\subset[U(1)\times U(1)]_T$, which we will refer to as the "vector" part of the global topological symmetry, there is no anomaly involving only the $U(1)_A$ (A is for "*axial*") symmetry, which is generated by the conserved charge Q_1-Q_2, and the $PSU(3)$ spin symmetry only. So there is a mixed 't Hooft anomaly between the $U(1)_V$ and the $PSU(3)$ global symmetries.

Let us therefore gauge the $U(1)_V$ symmetry, by setting the vectorlike gauge field $A_1=A_2=V$: $\mathcal{S}=\frac{\mathrm{i}}{2\pi}\int V\wedge\{(\mathrm{d}a_1+B)+(\mathrm{d}a_2+B)\}$. The gauge transformation $V\to V+\mathrm{d}\varphi$ causes a change in the action

$$\Delta\mathcal{S}=\frac{\mathrm{i}}{2\pi}\int \mathrm{d}\varphi\wedge(2B)\mod 2\pi\mathrm{i}. \tag{168}$$

To fix this, we may consider adding a term $\frac{i}{2\pi}\int V\wedge B$, which would make the action invariant under the $V\to V+\mathrm{d}\varphi$, but term is not gauge invariant under the transformation $B\to B+\mathrm{d}\xi$, where ξ is a $U(1)$ gauge field. In order to achieve the invariance under both gauge transformations, we must put the $(3+1)$D SPT action, $S_{4\mathrm{D}}=\frac{\mathrm{i}}{2\pi}\int V\wedge \mathrm{d}B$.

The background fields A_1,A_2 generally break explicitly the $\mathbb{Z}_3$ exchange symmetry, which takes $a_1\to a_2$ and $a_2\to -a_1-a_2$. Note that this breaking of $\mathbb{Z}_3$ permutation is not subject to 't Hooft anomaly matching, although it is the breaking of symmetry due to the background gauge field. As an example, let us again gauge the vector part, $A_1=A_2=V$, then the $\mathbb{Z}_3$ permutation changes the action as

$$\mathcal{S}\mapsto\mathcal{S}-\frac{\mathrm{i}}{2\pi}\int V\wedge\{2(\mathrm{d}a_1+B)+(\mathrm{d}a_2+B)\}. \tag{169}$$

Since the breaking term of the symmetry contains the dynamical gauge fields, we cannot prepare the $(3+1)$D SPT phase canceling this anomaly, and thus this is not a 't Hooft anomaly.[13] However, if we define $\mathbb{Z}_3: A_1\to A_2-A_1$, $A_2\to -A_1$, the action is $\mathbb{Z}_3$ invariant. Still a generic fixed background of the A_1,A_2 fields will break the $\mathbb{Z}_3$ cyclic permutation symmetry.

Consider a $\mathbb{Z}_3$ preserving background, given by the axial-vector-like gauge field $A_1=-A_2=A$, where A is a $\mathbb{Z}_3$ gauge field now, i.e., it can be written as $3A=\mathrm{d}\alpha$, where $\alpha\in[0,2\pi)$ is an angle-valued field. We then have the action

$$\mathcal{S}=\frac{\mathrm{i}}{2\pi}\int A\wedge(\mathrm{d}a_1-\mathrm{d}a_2). \tag{170}$$

The above action corresponds to gauging a $\mathbb{Z}_3^A\subset U(1)_A$. Indeed, it can be checked that because of the 2π quantization of the fluxes $F_{1,2}$, such a background preserves the $\mathbb{Z}_3$ cyclic permutation symmetry. However if we now gauge the $PSU(3)$ symmetry, the cyclic permutation symmetry will induce a change in the action

$$\Delta\mathcal{S}=\frac{\mathrm{i}}{2\pi}\int 3A\wedge(\mathrm{d}a_2+B)=\frac{3\mathrm{i}}{2\pi}\int A\wedge B\mod 2\pi\mathrm{i}, \tag{171}$$

which indicates an anomaly among three symmetries; the $\mathbb{Z}_3^A$, the $\mathbb{Z}_3$ cyclic permutation symmetry and $PSU(3)$.

So we have found two anomalies, which both must be saturated. Both of them include the $PSU(3)$ spin symmetry, and so both can be saturated by breaking the $PSU(3)$. This is the Néel phase which, unsurprisingly, saturates both anomalies. If $PSU(3)$ symmetry is restored, barring topologically ordered phase, the vector topological symmetry must be broken. At the same time, either the $\mathbb{Z}_3$ axial symmetry or the $\mathbb{Z}_3$ cyclic permutation symmetry must be spontaneously broken.

In a realistic spin system, the $U(1)\times U(1)$ topological symmetry will be explicitly broken to some discrete subgroup. Let us assume that the only $\mathbb{Z}_n^V\subset U(1)_V$ survives. Then we have that the $A_1=A_2=V$, where now V is a $\mathbb{Z}_n$ gauge field (i.e., $nV=\mathrm{d}\alpha$). However, now we are allowed a local counterterm of the form

$$\mathcal{S}_{\text{counter}}=\frac{inp}{2\pi}\int V\wedge B,\ p\in\mathbb{Z}, \tag{172}$$

[13] In Ref. [32], this is called the 't Hooft anomaly *not* of Dijkgraaf-Witten type. The usual 't Hooft anomaly corresponds to the 't Hooft anomaly of the Dijkgraaf-Witten type.

which is invariant under the $PSU(3)$ gauge transformation, $B \mapsto B + \mathrm{d}\xi$. If we can satisfy the condition

$$pn = 2 \bmod 3, \tag{173}$$

the anomaly (168) can be canceled. When $n = 3k$ for $k \in \mathbb{Z}$, the above condition can never be satisfied. So we conclude that as long as the symmetry $\mathbb{Z}_{3k} \subset U(1)_V$ is preserved, the anomaly between $U(1)_V$ and $PSU(3)$ persists. If $n = 3k + l$, where $l = 1, 2$, we can choose $p = 2, 1$, respectively, and so there is no anomaly. On the other hand, we have already seen that there is an anomaly between the $\mathbb{Z}_3^A \subset U(1)_A$, the $\mathbb{Z}_3$ cyclic permutation and the $PSU(3)$ symmetry.

We expect that this $(2+1)$D model corresponds to an effective theory of some $SU(3)$ quantum magnet. The anomaly in such systems is saturated either by breaking $PSU(3)$ symmetry (the Néel order) or by breaking the topological or $\mathbb{Z}_3$ cyclic permutation symmetry, which is related to the breaking of lattice symmetries and onset of the valence-bond-solid (VBS) order. It is interesting to explore its phase diagram and relation to the microscopic theory, as well as whether the Néel to VBS transition supports quantum criticality which was proposed for the $SU(2)$ spin systems [121]. Furthermore, when discrete global symmetries are spontaneously broken, there exist domain walls connecting different vacua. Under the setup with the domain walls, we can perform the anomaly inflow argument to uncover the property of domain walls. We leave these interesting subjects for future works.

IX. CONCLUSION

In this work, we discussed a number of particular quantum field theories in 1+1D, which are related to the antiferromagnetic $SU(3)$ chains in the p-box symmetric representations. In the large p limit, the effective model is an $SU(3)/[U(1) \times U(1)]$ nonlinear sigma model [24]. To consider a general spin chain, our discussion covers a linearized version, given by a particular $U(1)^2$ Abelian-Higgs model, with a $PSU(3)$ global (spin) symmetry.

These models have two θ angles, and we studied the phase diagram of those θ angles by using not only the 't Hooft anomaly matching but also the global inconsistency matching. These findings are generalized to $SU(N)/U(1)^{N-1}$ nonlinear sigma models and their linearized cousins.

We first found the $SU(3)/\mathbb{Z}_3$-$\mathbb{Z}_3$ mixed 't Hooft anomaly for special θ angles, which provides the field-theoretic description of the LSM theorem for $SU(N)$ spin chains [4,24]. The anomaly matching tells us that the ground states must be threefold degenerate or there must exist gapless excitations, so the symmetric gapped vacuum is ruled out from possible low-energy behaviors. We also found that distinct regions of the phase diagram are globally inconsistent $SU(3)/\mathbb{Z}_3$-C, indicating the presence of the phase transition lines in the phase diagram. We discussed possible scenarios which satisfy the global inconsistency and the anomaly matching. A minimal scenario is consistent with the proposal of Ref. [24] as well as the calculation of the pure-gauge limit of the linear sigma model, with phase-transition lying along the charge-conjugation-invariant lines; the charge-conjugation symmetry is spontaneously broken on the phase transition lines. However, the global inconsistency matching also leaves open another possibility; along the charge-conjugation-invariant lines the ground state could be a nontrivial SPT phase protected by $SU(3)/\mathbb{Z}_3$, which means that it must be separated from the origin of the phase-diagram by a phase transition line. These two conditions, combined with the 2π periodicity of θ angles, restrict the possible phase diagrams strongly to these two scenarios.

At the nontrivial $\mathbb{Z}_N$ symmetric point, the $SU(N)/U(1)^{N-1}$ nonlinear sigma model is believed to show conformal behavior. We therefore study the $SU(N)$ WZW model, and have shown that the level k $SU(N)$ WZW model and the $SU(N)/U(1)^{N-1}$ sigma model at $\theta_\ell = 2\pi p\ell/N$ have the same 't Hooft anomaly if $kq = p \bmod N$ for some q with $\gcd(N, q) = 1$. Combining the constraint from the c theorem, we conjecture that if $SU(N)/U(1)^{N-1}$ sigma model is conformal then it is generically described by $SU(N)_{\gcd(N,p)}$ WZW model.

We constructed the linear sigma model corresponding to the $SU(N)/U(1)^{N-1}$ nonlinear sigma model, and showed that they have the same 't Hooft anomaly and global inconsistency explicitly. In certain limits of linear sigma models, we can perform the analytic computation of the partition function. It therefore provides an intuitive and concrete understandings on how the anomaly and global inconsistency matching is realized, and we have checked the conjecture on the phase diagram. Study of nonlinear sigma models is usually tough because of the asymptotic freedom, so we also consider the adiabatic circle compactification of the model. We have shown that the 't Hooft anomaly and global inconsistency persist under this circle compactification, and thus it is an interesting future study to analyze this circle-compactified model using reliable semiclassical analysis.

We also briefly discussed the $(2+1)$-dimensional version of the model. It is expected to describe the $SU(3)$ quantum spin magnet in two spatial dimension. We show that it has various 't Hooft anomalies involving topological symmetries, generated by the conserved Abelian fluxes. Such setups leave open the possibilities of nontrivial domain walls, like the ones discussed in Refs. [44,73,74].

Note added. Finalizing the draft, the authors noticed that Ref. [122] appears on arXiv, which partially overlaps with Sec. V.

ACKNOWLEDGMENTS

The authors thank Ian Affleck, Gerald Dunne, Yuta Kikuchi, Zohar Komargodski, Miklós Lajkó, and Frédéric Mila for their comments. The authors also thank Philippe Lecheminant for telling us the conjecture about the $SU(N)$ Wess-Zumino-Witten model in Ref. [84]. The authors would like to thank the hospitality at the KITP institute and the organizers of the "Resurgent Asymptotics in Physics and Mathematics" workshop where a part of this was started. T.S. would like to thank Hartmut Wittig and the Institute for Nuclear Physics at Mainz University for their hospitality during the time that this work was completed. The work at KITP is supported by the National Science Foundation under Grant No. NSF PHY17-48958. The work of Y.T. is supported by RIKEN Special Postdoctoral Program.

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