

Fractional exclusion statistics and thermodynamics of the Hubbard chain in the spin-incoherent Luttinger liquid regime

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Bethe ansatz and bosonization procedures are used to describe the thermodynamics of the strong-coupled Hubbard chain in the *spin-incoherent* Luttinger liquid (LL) regime: $J(\equiv 4t^2/U) \ll k_B T \ll E_F$, where t is the hopping amplitude, $U(\gg t)$ is the repulsive on-site Coulomb interaction, and $k_B T (E_F \sim t)$ is the thermal (Fermi) energy. We introduce a fractional Landau LL approach, whose $U = \infty$ fixed point is exactly mapped onto an ideal gas with two species obeying the Haldane-Wu *exclusion* fractional statistics. This phenomenological approach sheds light on the behavior of several thermodynamic properties in the spin-incoherent LL regime: specific heat, charge compressibility, magnetic susceptibility, and Drude weight. In fact, besides the hopping (mass) renormalization, the fractional Landau LL parameters, due to quasiparticle interaction, are determined and relationships with velocities of holons and spinons are unveiled. The specific heat thus obtained is in very good agreement with previous density matrix renormalization group (DMRG) simulations of the t - J model in the spin-incoherent regime. A phase diagram is provided and two thermodynamic paths to access this regime clarifies both the numerical and analytical procedures. Further, we show that the high- T limit of the fractional Landau LL entropy and chemical potential exhibit the expected results of the t - J model, under the condition $U \gg k_B T$. Lastly, finite-temperature Lanczos simulations of the single-particle distribution function confirm the characteristics of the spin-incoherent regime and the high- T limit observed in previous DMRG studies.

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I. INTRODUCTION

Very recently, experimental realization of one-dimensional (1D) ultracold fermions with a tunable number of spin components has been reported in the crossover regime of temperature between spin-ordered and spin-incoherent Luttinger liquid (LL) [1]. In particular, the subtle bosonic limit [2] is evidenced for strongly repulsive ^{173}Yb atoms with nuclear spin $I = 5/2$. In addition, studies using analytical and numerical methods have shown [3] that the spin-incoherent 1D spin-1 Bose LL in a harmonic trap and in the Tonks-Girardeau limit (infinite repulsion) [4], exhibits the universal $1/p^4$ dependence momentum distribution, which is, however, broader than the spinless case, due to spin-function overlaps. We also remark that the Tonks-Girardeau limit has been experimentally achieved in ultra cold boson atoms [5], and also verified in frustrated quantum spin chains [6].

On the theoretical side, the method of bosonization [7] has provided an efficient means to derive analytical results for low-dimensional interacting fermion systems in condensed matter and field theory, thereby allowing the emergence of new physical concepts. In this context, the LL theory has been proposed [8] as a unified framework to describe the low-energy physics of a large class of 1D quantum many-body systems [9–11]. Emphasis has been given to those systems subjected

to strong quantum fluctuations and exhibiting new features not fully described by the standard Fermi liquid theory [12] governed by the zero coupling-strength fixed point [13]. Notwithstanding, several aspects of a Landau-Luttinger theory were discussed at length [14–16]. Further, generalization of the standard Fermi liquid theory was also proposed with aim in describing the unusual properties of heavy-fermion systems, in particular close to a metal-insulator transition [17].

Following the LL concept we have witnessed a vigorous development in the study of 1D strongly correlated electron systems, particularly in connection with the nature and the role played by charge and spin excitations, and the related phenomenon of spin-charge separation [9]. Comparison of results derived using bosonization with those from other methods, such as the Bethe-ansatz and density matrix renormalization group techniques [11,18], has also proved valuable. More recently, a very interesting regime of the LL, namely the spin-incoherent LL, has received special attention [19]. For both continuous [20,21] and lattice [22–24] versions of the 1D Hubbard model [10], this regime is realized under the condition $J(\equiv 4t^2/U) \ll k_B T \ll E_F (\sim t)$, where t is the nearest-neighbor hopping amplitude, U is the repulsive on-site Coulomb interaction, $\beta = 1/(k_B T)$ is the inverse temperature measured in units of the Boltzmann constant, J is the antiferromagnetic exchange coupling, and E_F is the Fermi energy. Alternatively, for low carrier densities, quantum wires [25–29] are near the 1D Wigner crystal limit at which the electrostatic energy between the particles greatly exceeds

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their kinetic energy leading to $J \ll E_F$, so that for $k_B T \gg J$ the observed conductance is about half the usual LL value $2e^2/h$ due to the spin-incoherent contribution to the resistance, where e is the magnitude of electron charge and h is the Planck constant. Indeed, it has been shown that, despite that features of spin-charge separation persist, the spin part of the correlation function exhibits an exponential spatial decay [20,21] not consistent with the usual LL power-law decay. Moreover, at half filling [22], the effective gapped charged excitations are modified due to the presence of the uncorrelated spin degrees of freedom.

In this paper we shall demonstrate that the thermodynamic properties of the Hubbard chain in the spin-incoherent regime can be described by using arguments from complementary powerful methods in the realm of quantum statistical mechanics and many-body theory, notably the Haldane-Wu *exclusion* fractional statistics [30]. In this context, the fractional character of the excitations of Hubbard models with short-range Coulomb interaction and correlated hopping [31–33] (bond-charge interaction), and infinite-range Coulomb interaction [34] as well, has been invoked to properly describe phase diagrams exhibiting metal-insulator transition, including the unexpected absence of conductivity at half filling due to a topological change in the Fermi surface, and η -pairing [35] induced 1D critical superconductivity [36]. Correlated hopping can also play a relevant role in 2D models of high-temperature superconductors [37]. In addition, particles obeying exclusion fractional statistics have been considered in the context of optical lattices [38,39], including the (1D) Tonks-Girardeau limit [40]. In 2D systems, it was suggested [41] that spectroscopy measurements on ultracold atoms can be used to demonstrate the fractional exclusion statistics of quasiholes in the Laughlin state of bosons. On the other hand, neutral anyonic excitations, which satisfy fractional *exchange* statistics in two dimensions, can be identified [42] through measurements of spectral functions near the threshold. The structure factor follows a universal power-law behavior, whose exponent is the signature of the anyon statistics and the underlying topologically ordered states that should occur in spin liquids and fractional Chern insulators. Moreover, it was proposed [43] that superfluid to Mott insulator quantum phase transitions in an anyon-Hubbard model with three-body interaction can be driven by the statistics or by the interaction.

In Sec. II, we use a strong-coupling perturbative expansion [44] of the Takahashi's Bethe-ansatz grand-canonical free energy [45–47] to calculate the Helmholtz free energy, energy and entropy in the spin-incoherent regime. From these thermodynamic potentials and the Luttinger theory, we present in Sec. III the specific heat, isothermal compressibility, Luttinger liquid parameter, magnetic susceptibility, and the Drude weight, to leading order in J/E_F . In Sec. IV, we show that the thermodynamics of the infinite- U Hubbard chain is exactly mapped onto an ideal exclusion gas of two species obeying the Haldane-Wu *exclusion* fractional statistics [30]. In Sec. V we introduce a fractional Landau LL approach, which provides nontrivial insights and a direct connection with the LL theory in the spin-incoherent regime. Indeed, our results provide strong evidence that the fractional exclusion entropy describes very well the thermodynamics of the spin-incoherent regime. We can thus identify the pertinent fractional

Landau LL parameters, and their relationship with the LL properties, namely, the velocity of holons and spinons. Despite that there have been previous attempts [17,48,49] towards a generalization of the Fermi liquid theory to particles obeying fractional exclusion statistics, a realization of these ideas, as presented here, is apparently missing. In Sec. VI we consider the high- T limit [50] of the particle distribution function, chemical potential, and entropy. Finally, concluding remarks are reserved to Sec. VII.

II. SPIN-INCOHERENT REGIME OF THE HUBBARD CHAIN

The Hamiltonian of the Hubbard chain of L sites in the presence of an external magnetic field along the z direction is given by

$$\mathcal{H} = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu_B H \sum_i (n_{i\uparrow} - n_{i\downarrow}), \quad (1)$$

where $\langle i, j \rangle$ denotes nearest-neighbor sites, $\sigma \in \{\uparrow, \downarrow\}$, $c_{i\sigma}$ ($c_{i\sigma}^\dagger$) is the electron annihilation (creation) operator, $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ is the number operator, $\mu_B H$ is the Zeeman energy, and μ_B is the Bohr magneton. The t - J model, which projects out doubly occupied states in the strong-coupling regime of the Hubbard chain, reads:

$$\begin{aligned} \mathcal{H}_{t-J} = & -t \sum_{\langle i,j \rangle, \sigma} (1 - n_{i\bar{\sigma}}) c_{i\sigma}^\dagger c_{j\sigma} (1 - n_{j\bar{\sigma}}) \\ & + J \sum_{\langle i,j \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right) - 2\mu_B H S^z, \quad (2) \end{aligned}$$

where $\bar{\sigma} = -\sigma$, $S^z = \frac{1}{2} \sum_i (n_{i\uparrow} - n_{i\downarrow})$, with $\hbar \equiv 1$, and

$$J = 4t^2/U.$$

The spin-incoherent LL regime is found at temperatures such that

$$J (\equiv 4t^2/U) \ll k_B T \ll E_F \sim t. \quad (3)$$

This regime is characterized by low-energy collective charge excitations (holons) with a velocity $v_c^{(\text{inch})}$ of interacting spinless fermions, and by the absence of collective spin excitations, since the very small strong-coupling spinon velocity v_s ($\sim J$) implies a very small correlation length $\xi = v_s/\pi k_B T \sim J/2k_B T \ll 1$. In this context, we note that the special point $J = 0$ ($U = \infty$) is also a spin-incoherent LL, since it is a spin-disordered state, with $v_s = J = 0$ and infinite spin degeneracy in the thermodynamic limit; thereby, only holon excitations exist.

The thermodynamic Bethe ansatz has been successfully implemented for the Hubbard chain long ago [45]. However, difficulties exist in deriving closed-form expressions for thermodynamic quantities from the infinite coupled integral equations. Notwithstanding, it has been shown [44] that it is possible to solve the set of integral equations perturbatively in the strong coupling limit ($t \ll U$), and consistent high-temperature series expansions have been provided. In particular, in Appendix A the results reported in Ref. [44] for the grand canonical free

energy $\Omega(T, \mu, H)$ can be used in order to obtain corrections of $\mathcal{O}(t^2/U)$ to the $U = \infty$ limit. Most importantly, as we show in this work, these corrections are suitable to describe the t - J limit of the Hubbard chain in the regime $U \gg k_B T$, including the spin-incoherent regime for $k_B T \ll t$. In fact, in Appendix A we find that $\Omega(T, \mu, H)$ in the spin-incoherent regime reads:

$$\begin{aligned} & \frac{\Omega_{\text{inch}}(T, \mu, H)}{L} \\ &= -k_B T \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln[1 + e^{-\beta(\varepsilon_k - \mu - \mu_B H)} + e^{-\beta(\varepsilon_k - \mu + \mu_B H)}] \\ & - \frac{k_B T}{\cosh(\beta \mu_B H)} \left(\frac{t}{U}\right) \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{2}{e^{\beta(\varepsilon_k - \mu)} + 2 \cosh(\beta \mu_B H)} \\ & \times \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos k \ln[1 + e^{-\beta(\varepsilon_k - \mu - \mu_B H)} + e^{-\beta(\varepsilon_k - \mu + \mu_B H)}] \\ & + \dots, \end{aligned} \quad (4)$$

where μ is the chemical potential and $\varepsilon_k = -2t \cos k$ is the dispersion relation of tight-binding fermionic particles, which is the exact dispersion relation for the $U = \infty$ case [51]. In fact, making $U = \infty$ in Eq. (4), we obtain the exact expression of the grand-canonical free energy [45] at this extremal coupling value. The grand-canonical free energy (4) is also suitable to describe the spin-incoherent regime, since using the inequalities in (3): $4t^2/U \ll k_B T \ll t$, we find $U/k_B T \gg 1/(k_B T/t)^2 \gg 1$.

The chemical potential μ is calculated from $n = -\frac{1}{L} \left(\frac{\partial \Omega}{\partial \mu}\right)$:

$$\begin{aligned} & \mu_{\text{inch}}(T, n) = -2t \cos(n\pi) \\ & - \frac{nt^2}{U} \left[1 + 2 \sin^2(n\pi) - \frac{\sin(2n\pi)}{2n\pi}\right] - k_B T \ln 2 \\ & + \frac{\pi^2 (k_B T)^2 \cos(n\pi)}{12t \sin^2(n\pi)} \left\{1 + \left(\frac{2t}{U}\right) \left[\frac{n}{\cos(n\pi)} - \frac{\sin(n\pi)}{\pi}\right]\right\} \\ & + \dots, \end{aligned} \quad (5)$$

The corresponding expansion for the Helmholtz free energy $F(= \mu N + \Omega)$, energy $E(= F - T \partial F / \partial T)$, and entropy $S(= -\partial F / \partial T)$ read:

$$\begin{aligned} & \frac{F_{\text{inch}}(T, n)}{L} = -\frac{2t \sin(n\pi)}{\pi} - \left(\frac{t^2}{U}\right) n^2 \left[1 - \frac{\sin(2n\pi)}{2n\pi}\right] \\ & - nk_B T \ln 2 - \frac{\pi (k_B T)^2}{12t^* \sin(n\pi)} + \dots; \end{aligned} \quad (6)$$

$$\begin{aligned} & \frac{E_{\text{inch}}(T, n)}{L} = -\frac{2t \sin(n\pi)}{\pi} - \left(\frac{t^2}{U}\right) n^2 \left[1 - \frac{\sin(2n\pi)}{2n\pi}\right] \\ & + \frac{\pi (k_B T)^2}{12t^* \sin(n\pi)} + \dots; \end{aligned} \quad (7)$$

$$\frac{S_{\text{inch}}(T, n)}{L} = nk_B \ln 2 + \frac{\pi k_B^2 T}{6t^* \sin(n\pi)} + \dots, \quad (8)$$

where the T -dependent terms have coefficients with a hopping parameter t^* given by, up to $\mathcal{O}(t/U)$,

$$t^* = t \left[1 - \frac{2nt \cos(n\pi)}{U}\right]. \quad (9)$$

We stress that up to $\mathcal{O}(t/U)$ doubly occupied sites are forbidden [52]. In fact,

$$\begin{aligned} & \frac{\langle N_{\uparrow\downarrow} \rangle}{L} = \frac{\partial(E_{\text{inch}}/L)}{\partial U} = n^2 \left(\frac{t}{U}\right)^2 \left[1 - \frac{\sin(2n\pi)}{2n\pi}\right] \\ & - \left(\frac{n\pi}{6}\right) \left(\frac{k_B T}{U}\right)^2 \cot(n\pi) + \dots \end{aligned} \quad (10)$$

The above results show that the charge degrees of freedom in the regime $J \ll k_B T \ll t$ or $J = 0$ and $k_B T \ll t$ are described by a gas of free spinless fermions. Indeed, the first term in $E_{\text{inch}}(T, n)$ is the ground-state energy of a gas of free spinless fermions with dispersion $\varepsilon_k = -2t \cos k$; while T -dependent terms in $E_{\text{inch}}(T, n)$ and $S(T, n)$ are contributions from thermally excited spinless fermions, with a mass $\sim 1/t^*$, above the Fermi surface, which is defined by the wave vectors $k = \pm k_F$, with $k_F = n\pi$. The spin-incoherent regime is identified by noticing that the first term in the entropy $S_{\text{inch}}(T, n)$ indicates that the spin degrees of freedom are fully disordered.

III. RESPONSE FUNCTIONS AND SPIN-INCOHERENT LL PARAMETERS

The Hamiltonian of the system in the spin-incoherent regime and *zero field* can be mapped onto the following charged bosonized LL Hamiltonian [27]:

$$\mathcal{H}_{\text{inch}} = v_c^{(\text{inch})} \int \frac{dx}{2\pi} \left[\frac{1}{g} (\partial_x \theta)^2 + g (\partial_x \phi)^2 \right], \quad (11)$$

where $v_c^{(\text{inch})}$ is the holon velocity, $(1/\pi)(\partial_x \theta)$ is the fluctuation in electron density, and the commutation relation $[\theta(x), \partial_{x'} \phi(x')] = i\pi \delta(x - x')$ holds. The coupling g can be written in terms of the LL parameter K_c , which governs the decay of the correlation functions:

$$K_c = \frac{1}{2g}. \quad (12)$$

The specific heat $C = -\frac{T}{L} (\partial^2 F / \partial T^2)$:

$$C_{\text{inch}}(T, n) = \gamma_{\text{inch}} k_B^2 T + \dots, \quad (13)$$

displays a free spinless Fermi gas form where the specific-heat coefficient γ_{inch} and the holon velocity are, respectively,

$$\gamma_{\text{inch}} = \frac{\pi}{3v_c^{(\text{inch})}}; \quad (14)$$

$$v_c^{(\text{inch})} = 2t^* \sin(n\pi). \quad (15)$$

On the other hand, the charge compressibility $\kappa^{-1} = n^2 (\partial \mu / \partial n)$ reads:

$$\begin{aligned} & \kappa_{\text{inch}}^{-1}(T, n) = 2\pi t n^2 \sin(n\pi) \\ & \times \left\{1 - \left(\frac{2t}{U}\right) \left[\frac{\sin(n\pi)}{\pi} + n \cos(n\pi) + \mathcal{O}\left(\frac{k_B^2 T^2}{t^2}\right)\right]\right\}. \end{aligned} \quad (16)$$

Further, in the spin-incoherent LL regime $g_{\text{inch}}^{-1} = \pi v_c^{(\text{inch})} \kappa_{\text{inch}} n^2$, we find

$$g_{\text{inch}} = 1 - \left(\frac{2t}{U}\right) \frac{\sin(n\pi)}{\pi}, \quad (17)$$

and

$$K_c^{(\text{inch})} = \frac{1}{2g_{\text{inch}}} = \frac{1}{2} + \left(\frac{t}{U}\right) \frac{\sin(n\pi)}{\pi}. \quad (18)$$

Notice that using Eqs. (9) and (15), we can verify that $v_c^{(\text{inch})}$ is not the holon velocity of the standard LL theory at $T = 0$.

Lastly, since [53] $\sigma_0 = 2K_c v_c$, the Drude weight that measures the dc peak in the conductivity, $\sigma(\omega) = \sigma_0 \delta(\omega)$, in the spin-incoherent LL regime is given by

$$\sigma_0^{(\text{inch})} = 2t \sin(n\pi) \left[1 + \frac{2t}{U} \left(\frac{\sin(n\pi)}{\pi} - n \cos(n\pi) \right) \right], \quad (19)$$

where use was made of Eqs. (18) and (21). We also confirm the spin-incoherent regime by probing the spin degrees of freedom through the susceptibility $\chi(T, \mu)$. As shown in Appendix B, the canonical susceptibility and spinon velocity read, respectively:

$$\chi_{\text{inch}}(T, n) = \mu_B^2 \beta n \left[1 - \frac{nv_s}{\pi k_B T} + \mathcal{O}\left(\frac{J}{t}\right) \right]; \quad (20)$$

$$v_s = \frac{2\pi t^2}{U} \left[1 - \frac{\sin(2n\pi)}{2n\pi} \right], \quad (21)$$

where v_s is the strong-coupling spinon velocity [53]. The correction of $\mathcal{O}(v_s/k_B T)$ to the dominant Curie response is the one we expect in view of the highly excited spin degrees of freedom and implies $v_s(n)|_{U=\infty} = 0$, for any value of T . For finite J , we use the fluctuation-dissipation theorem: $\chi = \beta \int G(x) dx$, where $G(x)$ is the spin-correlation function. In order to satisfy Eq. (20), $G(x) = \mu_B^2 n [\delta(x) - n e^{-x/\xi}]$, with a correlation length ξ given by the expected result [8,54,55]: $\xi = v_s/(\pi k_B T) \sim [J/(2k_B T)] \ll 1$, thus confirming the spin-incoherent regime for finite J .

A. $T \rightarrow 0$ limit: The standard LL regime, with charge and spin collective excitations

Here we show that we can infer the parameters of the standard LL regime, which settles as $T \rightarrow 0$, from the above spin-incoherent results. In doing so, we take advantage of the description of the $U \rightarrow \infty$ limit of the Hubbard chain put forward in Ref. [56]. In particular, by using the Bethe ansatz solution, it has been shown that the ground-state wave function of the system can be constructed as a product of a spinless fermion wave function $|\Psi\rangle$ and a squeezed spin wave function $|\chi\rangle$. The wave function $|\chi\rangle$ are eigenfunctions of the following Heisenberg Hamiltonian:

$$\mathcal{H}_S = \sum_{i=1}^N \sum_{\alpha=x,y,z} \tilde{J}^\alpha \left(S_i^\alpha S_{i+1}^\alpha - \frac{1}{4} \delta_{\alpha,z} \right), \quad (22)$$

where

$$\tilde{J}^\alpha = n \frac{4t^2}{U} \left[1 - \frac{\sin(2n\pi)}{2n\pi} \right] \quad (23)$$

is determined by the ground-state energy wave function of the spinless fermions $|\Psi^{GS}\rangle$. Notice that, at half filling, we have the standard coupling $J = 4t^2/U$. Therefore, the contribution

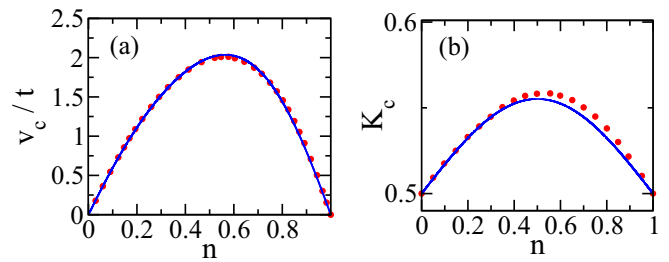


FIG. 1. (a) Charge velocity [Eq. (26)] and (b) correlation exponent K_c [Eq. (27)] at $T = 0$ as a function of n for $U = 16t$. In both figures, the dots displayed were obtained from Ref. [53].

of \mathcal{H}_S to the ground-state energy per site is given by

$$\frac{\langle \chi^{GS} | \mathcal{H}_S | \chi^{GS} \rangle}{L} \equiv \frac{E^{GS}}{L} = -n^2 \left(\frac{4t^2}{U} \right) \frac{[1 - 4\gamma_S(T=0)]}{4} \times \left[1 - \frac{\sin(2n\pi)}{2n\pi} \right], \quad (24)$$

where

$$\gamma_S(T) = \langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle = \begin{cases} 1/4 - \ln 2, & T = 0; \\ 0, & k_B T \gg t^2/U \end{cases} \quad (25)$$

denotes the T -dependent nearest-neighbor spin correlation function of the Heisenberg model [57]. This contribution at $T = 0$, together with that of spinless fermions [first term in Eq. (7)] is the exact ground-state result up to $\mathcal{O}(t/U)$ [56,58–61] of the 1D t - J model. We thus infer that the ground state energy of the Hubbard chain in the spin-incoherent regime obtains through the replacement of $\gamma_S(T=0)$ by $\gamma_S(T \gg J/k_B) = 0$. This correspondence was already noticed in the study of the thermodynamics of the Hubbard chain in the spin-disordered regime at half filling [22]. We have also noted that several expressions valid in the spin-incoherent LL regime differ from the corresponding ones at $T = 0$ by the multiplying factor $[1 - 4\gamma_S(T=0)]$.

Consider first the charge velocity at $T = 0$:

$$v_c(T=0, n) = 2t \sin(n\pi) \times \left\{ 1 - \frac{2[1 - 4\gamma_S(0)]nt \cos(n\pi)}{U} \right\} = 2t \sin(n\pi) \left[1 - \frac{8 \ln 2}{U} nt \cos(n\pi) \right] \quad (26)$$

which is the extension of Eq. (15) to $T = 0$ using Eq. (25), in agreement with Bethe-ansatz analytical results [62] of the strongly coupled Hubbard model at $T = 0$. In Fig. 1(a) we plot $v_c(T=0)$ as a function of n for $U = 16t$. Note the remarkable agreement with early Bethe-ansatz numerical [53] result at $T = 0$.

Now, consider the LL parameter at $T = 0$:

$$K_c(T=0, n) = \frac{1}{2} + [1 - 4\gamma_S(0)] \left(\frac{t}{U} \right) \frac{\sin(n\pi)}{\pi} = \frac{1}{2} + \frac{4 \ln 2}{U\pi} t \sin(n\pi). \quad (27)$$

The validity of this formula is confirmed in Fig. 1(b), where the plot of $K_c(T=0, n)$ as a function of n for $U = 16t$ is

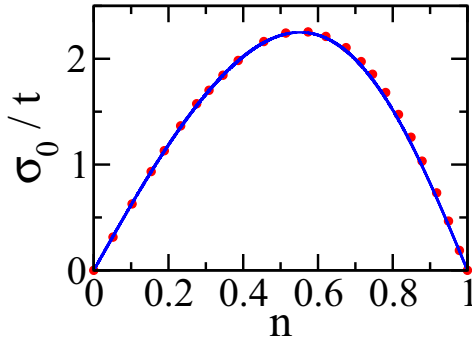


FIG. 2. Drude weight as a function of band filling for $U = 16t$ and $T = 0$. Solid curve is the plot of Eq. (28) and the dots in highlight were obtained from Ref. [53].

exhibited. In addition, we note that for $n \rightarrow 0$: $K_c(T = 0, n) = 1/2 + (4 \ln 2)(nt/U)$, which coincides with the expression for K_c reported in Ref. [63].

The previous results imply that the Drude weight [64] at $T = 0$ is given by

$$\sigma_0(T = 0) = 2K_c v_c = 2t \sin(n\pi) \left\{ 1 + 8 \ln 2 \left(\frac{t}{U} \right) \times \left[\frac{\sin(n\pi)}{\pi} - n \cos(n\pi) \right] \right\}, \quad (28)$$

where use of Eqs. (26) and (27) has been made. As shown in Fig. 2, the agreement between this formula for $U = 16t$ and early numerical results [53] is excellent. Lastly, concerning the specific-heat coefficient, as $T \rightarrow 0$ the spin-spin correlation function displays power-law behavior and the prediction for γ is [10]:

$$\gamma = \frac{\pi}{3} \left(\frac{1}{v_c} + \frac{1}{v_s} \right)_{T=0}. \quad (29)$$

IV. $U = \infty$ AS AN EXACT IDEAL GAS OF EXCLUSIONS OR FREE SPINLESS FERMIONS

The concept of a Luttinger liquid is the paradigm for describing the low-energy physics of interacting electron systems in one dimension. Notwithstanding, it is important to investigate alternative approaches that can shed light on the physics of such systems. In this context, a remarkable result that follows from previous works [31,34,36] by two of the authors is that the properties of $U = \infty$ limit can be viewed as derived from an ideal exclusion gas of two fractional species: $\alpha = 1$ for particles with spin up and $\alpha = 2$ for particles with spin down, coupled by the Haldane statistical matrix

$$[g]_{kk';\alpha\alpha'} = \delta_{kk'} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (30)$$

in which case double occupation is excluded. In fact, the same 3×3 statistical matrix describes the referred Hubbard models [31,34,36], including double occupancy effects. This is confirmed by noting that Eq. (4) with $U = \infty$ can be written

in the form:

$$\Omega_\infty(T, \mu_\infty, H) = -\frac{1}{\beta} \sum_{k,\alpha} \ln(1 + w_{k,\alpha}^{-1}), \quad (31)$$

where $w_{k,\alpha}$'s satisfy the Haldane-Wu distribution [30]:

$$w_{k,1} = e^{\beta(\varepsilon_{k,1} - \mu_\infty)}, \quad (32)$$

$$w_{k,2} = (1 + w_{k,1})e^{\beta(\varepsilon_{k,2} - \varepsilon_{k,1})}. \quad (33)$$

In addition, $\langle n_{k,\alpha} \rangle$ satisfies the exclusion relation:

$$\langle n_{k,\alpha} \rangle w_{k,\alpha} = 1 - \sum_{k',\lambda} g_{kk';\alpha\lambda} \langle n_{k',\lambda} \rangle, \quad (34)$$

where

$$\langle n_{k,\alpha} \rangle = \frac{e^{-\beta(\varepsilon_{k,\alpha} - \mu_\infty)}}{2 + \sum_{\lambda=1} e^{-\beta(\varepsilon_{k,\lambda} - \mu_\infty)}}. \quad (35)$$

More specifically:

$$\langle n_{k,1} \rangle = e^{2\beta\mu_B H} \langle n_{k,2} \rangle, \quad (36)$$

$$= \frac{e^{\beta\mu_B H}}{e^{\beta(\varepsilon_k - \mu_\infty)} + 2 \cosh(\beta\mu_B H)}, \quad (37)$$

in agreement with an independent calculation for the Hubbard model at $U = \infty$ in Ref. [65]. Although the matrix given in Eq. (30) is asymmetric, it should be noted that the spin-up and spin-down symmetry is preserved, as we can see from Eq. (35): $\langle n_{k,1} \rangle_H = \langle n_{k,2} \rangle_{-H}$. Moreover, the entropy reads:

$$S_\infty(T, \mu, H) = -k_B \sum_k [\langle n_{k,1} \rangle \ln \langle n_{k,1} \rangle + \langle n_{k,2} \rangle \ln \langle n_{k,2} \rangle + (1 - \langle n_{k,1} \rangle - \langle n_{k,2} \rangle) \times \ln(1 - \langle n_{k,1} \rangle - \langle n_{k,2} \rangle)], \quad (38)$$

which carries the signature of the statistical matrix in Eq. (30).

In zero field, Eq. (35), or Eqs. (36) and (37), reduces to

$$\langle n_{k,1} \rangle_{H=0} = \langle n_{k,2} \rangle_{H=0} = \frac{1}{e^{\beta(\varepsilon_k - \mu_\infty)} + 2} \equiv \langle n_k \rangle, \quad (39)$$

in agreement with early results [52], so $\langle n_k \rangle$ develops a rigorous step discontinuity at the Fermi surface as $T \rightarrow 0$, with

$$n = \frac{2}{L} \sum_k \langle n_k \rangle_{T=0}. \quad (40)$$

We also mention that the fractional character of $\langle n_k \rangle$, Eq. (39), stems from the fact that, in the exclusion formalism, both charge and spin degrees of freedom are combined to form a single distribution. However, by summing up in the fractional species, we obtain the free spinless fermion distribution:

$$\langle n_k^{(F)} \rangle = \langle n_{k,1} \rangle_{H=0} + \langle n_{k,2} \rangle_{H=0} = \frac{1}{e^{\beta(\varepsilon_k - \mu_\infty^{(F)})} + 1}, \quad (41)$$

where $\mu_\infty^{(F)}$ is the chemical potential of the free spinless Fermi gas:

$$\mu_\infty^{(F)}(T, n) = \mu_\infty + k_B T \ln 2. \quad (42)$$

Lastly, using Eqs. (41) and (42), the zero-field entropy per site in Eq. (38) can be written as

$$\frac{S_\infty(T, n)}{L} = nk_B \ln 2 - \frac{k_B}{L} \sum_k [\langle n_k^{(F)} \rangle \ln \langle n_k^{(F)} \rangle + (1 - \langle n_k^{(F)} \rangle) \ln (1 - \langle n_k^{(F)} \rangle)] \quad (43)$$

$$= nk_B \ln 2 + \frac{S_\infty^{(F)}(T, n)}{L}, \quad (44)$$

where $S_\infty^{(F)}$ is the entropy of the free spinless Fermi gas. We stress that Eqs. (43) and (44) or (38) in zero field reproduce the two low- T leading terms in Eq. (8) in the limit $U = \infty$, i.e., $t^* = t$, after eliminating μ_∞ or $\mu_\infty^{(F)}$ in favor of n . Therefore, the specific heat calculated from either of the referred equations has the same value, since the difference between the two forms of the entropy function is a constant term, $nk_B \ln 2$, associated with the disordered spin degrees of freedom.

V. FRACTIONAL LANDAU LUTTINGER LIQUID

In the previous section, we have described the low-energy physics of the Hubbard chain for $J (\equiv 4t^2/U) \ll k_B T \ll E_F (\sim t)$ from the standpoint of a spin-incoherent LL and have determined the parameters g and v_c that govern this class of fluid. In this section, our aim is to show that the system can also be mapped onto a fractional Landau LL [48,66,67]. This phenomenological approach, which is a suitable generalization of the standard Landau Fermi liquid theory, can shed light on the underlying aspects that characterize the crossover behavior from the fixed point associated with $U = \infty$ at $T = 0$ to the spin-incoherent LL regime at a given temperature $k_B T > J \ll t$.

In Fig. 3 we present a schematic phase diagram $k_B T$ versus $J/t = 4t/U \ll 1$ that illustrates two possible thermodynamic paths of the Hubbard model to reach the spin-incoherent LL regime. The first one (Path I) is physically attained by increasing the temperature of the system, initially in the ground state of the strong-coupling regime of the LL. The system undergoes a crossover and ends up at $T \gg J/k_B$, a spin disordered regime characterized by a zero pair spin correlation function: $\langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle = \gamma_S(T) = 0$, as discussed in Sec. III. In the second path (Path II), which helps us to understand the Landau LL approach, the system starts at the fixed point $T = 0$ and $U = \infty$, the temperature increases up to a value at which the interaction is switched on and triggers the system into the spin-incoherent regime.

We thus assume that when corrections of $\mathcal{O}(t^2/U)$ are switched on, the low-energy spectrum can be obtained from the following expansion of the functional $E_L(T) - E_0(T = 0)$:

$$E_L(T) - E_0(T = 0) = \sum_{k,\alpha} \tilde{\epsilon}_{k,\alpha} \delta \langle \hat{n}_{k,\alpha} \rangle + \frac{1}{2} \sum_{k,\alpha,k',\alpha'} f_{k,\alpha;k',\alpha'} \delta \langle \hat{n}_{k,\alpha} \rangle \delta \langle \hat{n}_{k',\alpha'} \rangle, \quad (45)$$

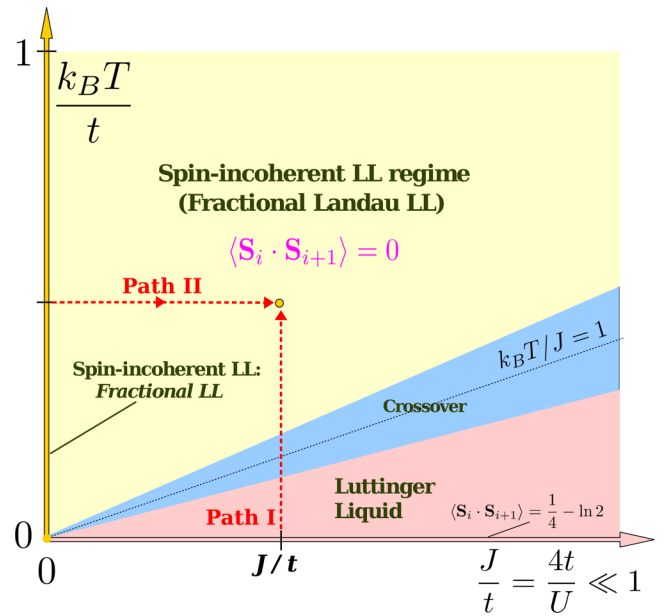


FIG. 3. Schematic phase diagram of the Hubbard chain in the strong-coupling regime, $J/t = 4t/U \ll 1$, and $k_B T \lesssim t$. At the line $J = 0$ the electrons are in a spin-incoherent Luttinger liquid (LL) phase with Curie response (spin correlation length $\xi = 0$). Further, this $U = \infty$ fixed point is exactly mapped onto an ideal gas with two species obeying the Haldane-Wu exclusion fractional statistics, i.e., a fractional LL. At the $T = 0$ line, excluding the point $J = 0$, the system is found in an LL phase with algebraic decay of the charge and spin correlation functions. Increasing T from a point at this line, Path I in the diagram, there is a crossover to a spin-incoherent regime with spin correlation length $\xi = v_s / (\pi k_B T) \sim [J / (2k_B T)] \ll 1$, so $\langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle = 0$. This regime can also be achieved through Path II, associated with both fractional LL and the fractional Landau LL: starting at $T = 0$ and $U = \infty$, the temperature increases up to a value at which the interaction is switched on and triggers the system into the spin-incoherent regime.

where $E_0(T = 0)$ is the ground state energy,

$$\tilde{\epsilon}_{k,\alpha} = -2t^* \cos k, \quad (46)$$

t^* is the renormalized hopping amplitude with no effect of quasiparticle interaction,

$$\delta \langle \hat{n}_{k,\alpha} \rangle = \langle \hat{n}_{k,\alpha}(T) \rangle - \langle \hat{n}_{k,\alpha}(0) \rangle, \quad (47)$$

and $f_{k,\alpha;k',\alpha'}$ represents the interaction energy between quasiparticles. In addition, it is assumed that the entropy has the same fractional functional form of S_∞ , Eq. (38):

$$S(T, \mu, H) = -k_B \sum_k [\langle \hat{n}_{k,1} \rangle \ln \langle \hat{n}_{k,1} \rangle + \langle \hat{n}_{k,2} \rangle \ln \langle \hat{n}_{k,2} \rangle + (1 - \langle \hat{n}_{k,1} \rangle - \langle \hat{n}_{k,2} \rangle) \ln (1 - \langle \hat{n}_{k,1} \rangle - \langle \hat{n}_{k,2} \rangle)]. \quad (48)$$

It means that the statistics of the fractional quasiparticles are also governed by the statistical matrix (30).

The equilibrium distribution of the quasiparticles is obtained by solving the equation $\partial \Omega / \partial \langle \hat{n}_{k,\alpha} \rangle = 0$, where

$\Omega = E - TS - \mu N$ and

$$n = \frac{1}{L} \sum_{k,\alpha} \langle \hat{n}_{k,\alpha} \rangle. \quad (49)$$

After some algebra, one finds a distribution that is formally identical to Eq. (35):

$$\langle \hat{n}_{k,\alpha} \rangle = \frac{e^{-\beta(\hat{\epsilon}_{k,\alpha} - \mu_L)}}{2 + \sum_{\lambda=1} e^{-\beta(\hat{\epsilon}_{k,\lambda} - \mu_L)}}, \quad (50)$$

where

$$\hat{\epsilon}_{k,\alpha} = \tilde{\epsilon}_{k,\alpha} + \sum_{k',\alpha'} f_{k,\alpha;k',\alpha'} \delta \langle \hat{n}_{k',\alpha'} \rangle \quad (51)$$

is the energy of the fractional Landau LL quasiparticle [12]. By symmetry considerations, the interaction energy between quasiparticles satisfies:

$$f_{k,1;k',1} = f_{k,2;k',2} \equiv f_{k,k'}^s + f_{k,k'}^a, \quad (52)$$

$$f_{k,2;k',1} = f_{k,1;k',2} \equiv f_{k,k'}^s - f_{k,k'}^a, \quad (53)$$

which define the spin symmetric $f_{k,k'}^s$ and spin antisymmetric $f_{k,k'}^a$ parts of the fractional quasiparticle interaction [12]. In terms of these quantities, one has in zero field

$$\hat{\epsilon}_{k,1} = \hat{\epsilon}_{k,2} = -2t^* \cos k + 2 \sum_{k'} f_{k,k'}^s \delta \langle \hat{n}_{k'} \rangle \equiv \hat{\epsilon}_k. \quad (54)$$

In the following, it is our task to demonstrate that the above phenomenological approach proves useful in the understanding of the underlying low-energy behavior of the Hubbard chain in the spin-incoherent regime. We emphasize that, regardless of the fact that the quasiparticles effects occur in the neighborhood of the Fermi surface $\{\pm k_F\}$, the final results are shown to be fully compatible with those derived in the previous sections through a proper identification of the fractional Landau LL parameters.

A. Thermodynamic properties

In order to compute the specific heat $C(T, n)$, we make the usual Landau assumption of neglecting corrections to $\hat{\epsilon}_{k,\alpha}$ due to interaction between the quasiparticles, so that only the hopping amplitude is renormalized:

$$\hat{\epsilon}_k \simeq \tilde{\epsilon}_k = -2t^* \cos k. \quad (55)$$

Next, we insert Eq. (55) into Eq. (49) in order to obtain the fractional Landau LL chemical potential, μ_L :

$$\mu_L(T, n) = -2t^* \cos(n\pi) - k_B T \ln 2 + \frac{\pi^2 \cos(n\pi) (k_B T)^2}{12t^* \sin^2(n\pi)} + \dots; \quad (56)$$

therefore, the fractional Landau LL energy per site, and the fractional Landau LL specific heat, thus read:

$$\begin{aligned} \frac{E_L(T, n)}{L} - \frac{E_0(T=0, n)}{L} &= \frac{2}{L} \sum_k \tilde{\epsilon}_k [\langle \tilde{n}_k \rangle - \langle \hat{n}_k(0) \rangle] \\ &= \frac{\pi (k_B T)^2}{12t^* \sin(n\pi)} + \dots, \end{aligned} \quad (57)$$

where

$$\langle \tilde{n}_k \rangle = \frac{1}{e^{\beta(\tilde{\epsilon}_k - \mu_L)} + 2} \quad (58)$$

and

$$C_L(T, n) = \frac{\pi k_B^2 T}{6t^* \sin(n\pi)} + \dots \quad (59)$$

Comparing the above equation with Eqs. (9), (13)–(15) associated with C_{inch} , we confirm our choice of t^* in Eq. (9).

The consistence of the fractional Landau LL approach is confirmed by the prediction for the entropy. In fact, by using Eqs. (55), (56), and (58) into Eq. (48), we obtain

$$\frac{S_L(T, n)}{L} = nk_B \ln 2 + \frac{\pi k_B^2 T}{6t^* \sin(n\pi)} + \dots, \quad (60)$$

in complete agreement with $S_{\text{inch}}(T, n)$ in Eq. (8). Remarkably, the fractional Landau LL quasiparticles carry all the entropy of the system in the spin-incoherent regime $J \ll k_B T \ll E_F$ and correctly describe the fermionic spinless charge degrees of freedom and the background of fully disordered spin degrees of freedom.

The prediction for κ is obtained as follows. From $n = \sum_k 2 \langle \hat{n}_k \rangle / L$, we get

$$\frac{\partial n}{\partial \mu} = \frac{2}{L} \sum_k \frac{\beta(1 - \partial \hat{\epsilon}_k / \partial \mu) e^{\beta(\hat{\epsilon}_k - \mu)}}{[e^{\beta(\hat{\epsilon}_k - \mu)} + 2]^2}, \quad (61)$$

where

$$\frac{\partial \hat{\epsilon}_k}{\partial \mu} = 2 \sum_{k'} \frac{f_{k,k'}^s \beta(1 - \partial \hat{\epsilon}_{k'} / \partial \mu) e^{\beta(\hat{\epsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\epsilon}_{k'} - \mu)} + 2]^2}. \quad (62)$$

At low T , the above integrands have sharp peaks centered at the k vectors of the Fermi surface $\{\pm k_F\}$; therefore, one obtains (see Appendix C)

$$\begin{aligned} \frac{\partial \hat{\epsilon}_k}{\partial \mu} &= (f_{k,k_F}^s + f_{k,-k_F}^s) \sum_{k'} \frac{\beta(1 - \partial \hat{\epsilon}_{k'} / \partial \mu) e^{\beta(\hat{\epsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\epsilon}_{k'} - \mu)} + 2]^2} \\ &= (f_{k,k_F}^s + f_{k,-k_F}^s) \left(\frac{L}{2} \right) \left(\frac{\partial n}{\partial \mu} \right). \end{aligned} \quad (63)$$

By inserting this back into $\frac{\partial n}{\partial \mu}$ and using $\kappa^{-1} = n^2 (\partial \mu / \partial n)$, we find

$$\kappa_L^{-1}(T, n) = 2\pi t^* n^2 \sin(n\pi) (1 + F_0^s) + \dots, \quad (64)$$

where

$$F_0^s = \frac{L(f_{k_F,k_F}^s + f_{k_F,-k_F}^s)}{4\pi t^* \sin(n\pi)} \quad (65)$$

is the Landau-Luttinger parameter associated with the spin symmetric part of the quasiparticle interaction at the Fermi level ($k_F = n\pi$). A comparison of Eqs. (64) and (16) implies:

$$F_0^s = -\frac{v_{c,\infty}}{\pi U}, \quad (66)$$

with $v_{c,\infty} = v_c^{\text{(inch)}}|_{U=\infty} = 2t \sin(n\pi)$. Notice that F_0^s is in fact the ratio of the total kinetic energy per site for $U = \infty$ at $T = 0$ over the on-site Coulomb repulsion U .

We now calculate the prediction for χ . In the presence of a magnetic field, we replace $\hat{\varepsilon}_k$ by $\hat{\varepsilon}_{k,\alpha} = \tilde{\varepsilon}_k \mp \mu_B H$ in Eq. (50). Thus the spin susceptibility is given by

$$\begin{aligned} \chi_L(T, n) &= \frac{\mu_B^2}{L} \sum_k \frac{\partial}{\partial H} (\langle \hat{n}_{k,1} \rangle - \langle \hat{n}_{k,2} \rangle)_{H=0} \\ &= \frac{\mu_B^2}{L} \sum_k \frac{\beta}{[e^{\beta(\hat{\varepsilon}_k - \mu)} + 2]} \frac{\partial}{\partial H} (\hat{\varepsilon}_{k,2} - \hat{\varepsilon}_{k,1})_{H=0}, \end{aligned} \quad (67)$$

where

$$\begin{aligned} \frac{\partial}{\partial H} (\hat{\varepsilon}_{k,2} - \hat{\varepsilon}_{k,1})_{H=0} &= 2 \\ &+ 2 \sum_{k'} f_{k,k'}^a \frac{\partial}{\partial H} (\langle \hat{n}_{k',2} \rangle - \langle \hat{n}_{k',1} \rangle)_{H=0}. \end{aligned} \quad (68)$$

Since we expect $f_{k,k'}^a = \mathcal{O}(t^2/U)$, we can take

$$\frac{\partial}{\partial H} (\langle \hat{n}_{k',2} \rangle - \langle \hat{n}_{k',1} \rangle)_{H=0} = -\frac{2\beta}{e^{\beta(\hat{\varepsilon}_{k'} - \mu)} + 2} \quad (69)$$

in the last expression. Therefore, the spin susceptibility becomes

$$\chi_L(T, n) = \frac{\mu_B^2 n}{k_B T} (1 - \beta t F_0^a) + \dots, \quad (70)$$

where

$$F_0^a = \frac{4}{tN} \sum_k \sum_{k'} f_{k,k'}^a \langle \hat{n}_k \rangle \langle \hat{n}_{k'} \rangle. \quad (71)$$

In contrast to Eqs. (61) and (62), the absence of sharp peaks at the Fermi surface in Eq. (71) is a clear manifestation of the fact that the spin degrees of freedom are highly thermalized. A comparison between Eqs. (20) and (70), however, allows us to identify

$$F_0^a = \frac{nv_s}{\pi t} \quad (72)$$

without the need of specifying the range of integration. If, in addition, we make the assumption that $f_{k,k'}^a$ is k independent, Eq. (72) implies $Lf^a = v_s/\pi$. Notice also that F_0^a is the ratio between the energy per site of the Heisenberg Hamiltonian in the spin-incoherent regime and nt [see Eqs. (6), (7)–(24), and (25)].

Lastly, we shall digress on the eventual crossover of the magnetic susceptibility as $T \rightarrow 0$. Unlike the crossover associated with the charge response functions, which is governed by the spin-spin correlation function, as discussed in Sec. III, in the magnetic susceptibility case there is a change of paradigm as $T \rightarrow 0$. First, as a guess, we notice that, to $\mathcal{O}(t^2/U)$: $\lim_{T \rightarrow 0} \chi_L(T, n) = \lim_{T \rightarrow 0} \frac{\mu_B^2 \beta n}{1 + \beta t F_0^a} = \pi \mu_B^2 / v_s$. It thus suggests the following *ansatz* for the Landau parametrization: $\lim_{T \rightarrow 0} (t F_0^a) = (-\beta^{-1} + n\pi v_s/2)$, which implies [10] $\lim_{T \rightarrow 0} \chi_L(T, n) = 2\mu_B^2 / \pi v_s$. It entails that, as $T \rightarrow 0$, the strong-coupling exchange enhancement of $\mathcal{O}(t^2/U)$ suppresses the Curie behavior and gives rise to the LL power-law decay of the spin correlation function and the very low- T behavior of $C(T)$ shown in Fig. 4, with dominant spinon contribution, see Eq. (29).

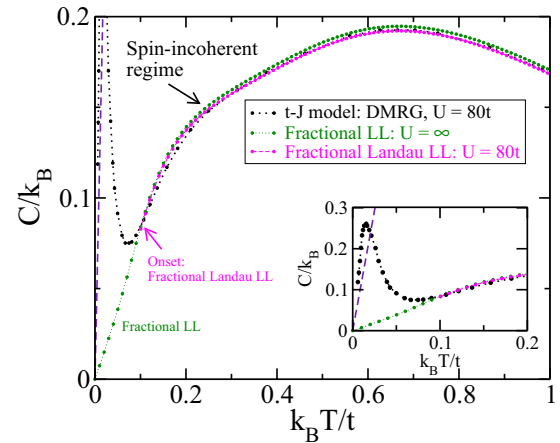


FIG. 4. Specific heat C in units of k_B as a function of the thermal energy $k_B T$ in units of t for chains with $n = 3/4$. The DMRG data for a t - J chain with 32 sites and $J = 0.05t$ ($U = 80t$), from Ref. [23]. Also shown are predictions from the fractional Landau LL in zero field for $U = 80t$ and the fractional LL at $U = \infty$. Notably, the results of the fractional Landau LL are in very good agreement with the DMRG data in the spin-incoherent regime. For completeness, we show the straight line of the $T \rightarrow 0$ limit of C/k_B , whose coefficient is $\gamma k_B T$, with γ in Eq. (29). The inset shows details of the referred estimates for C/k_B in a narrow low T interval. Notice that in Fig. 3 Path I is associated with the DMRG data, while Path II with the fractional LL and the fractional Landau LL.

B. Drude weight

In the presence of an external electric field ϕ , the spectrum E_∞ of the Hubbard chain with $U = \infty$, or $J = 0$ in Eq. (2), is altered according to the well known prescription [68]

$$E_\infty \rightarrow \sum_k \varepsilon_k(\phi) n_k, \quad (73)$$

where

$$\varepsilon_k(\phi) = -2t \cos(k + \phi). \quad (74)$$

Since Eqs. (45) and (51) establish a one-to-one mapping between the eigenstates of the Hamiltonian for $J = 0$ and $J \neq 0$, in the presence of ϕ we have

$$\begin{aligned} E(\phi) - E_0 &= \sum_{k,\alpha} \tilde{\varepsilon}_{k,\alpha}(\phi) \delta \langle \hat{n}_{k,\alpha} \rangle_\phi \\ &+ \frac{1}{2} \sum_{k,\alpha,k',\alpha'} f_{k,\alpha;k',\alpha'} \delta \langle \hat{n}_{k,\alpha} \rangle_\phi \delta \langle \hat{n}_{k',\alpha'} \rangle_\phi, \end{aligned} \quad (75)$$

where

$$\tilde{\varepsilon}_{k,\alpha}(\phi) = -2t^* \cos(k + \phi), \quad (76)$$

$$\hat{\varepsilon}_{k,\alpha}(\phi) = \tilde{\varepsilon}_{k,\alpha}(\phi) + \sum_{k',\alpha'} f_{k,\alpha;k',\alpha'} \delta \langle \hat{n}_{k',\alpha'} \rangle_\phi, \quad (77)$$

$$\delta \langle \hat{n}_{k,\alpha} \rangle_\phi = \frac{1}{e^{\beta[\hat{\varepsilon}_{k,\alpha}(\phi) - \mu]} + 2} - \langle \hat{n}_{k,\alpha} \rangle_{\phi=0}^{T=0}. \quad (78)$$

We are now in a position to obtain the Drude weight [68] (see Appendix D):

$$\begin{aligned}\sigma_0 &= -\frac{\pi}{L} \left[\frac{\partial^2 E(\phi)}{\partial \phi^2} \right]_{\phi=0} \\ &= 2t^* \sin(n\pi) - \left(\frac{L}{\pi} \right) (f_{k_F, k_F}^s - f_{k_F, -k_F}^s).\end{aligned}\quad (79)$$

Now using Eq. (19), one obtains

$$\frac{L(f_{k_F, k_F}^s - f_{k_F, -k_F}^s)}{2\pi t^* \sin(n\pi)} = F_0^s. \quad (80)$$

A combination of Eqs. (65) and (80) determines the spin symmetric part of the interaction energy between quasiparticles:

$$L f_{k_F, k_F}^s = \frac{3\pi v_{c, \infty}}{2} (1 - 1/g), \quad (81)$$

$$L f_{k_F, -k_F}^s = \frac{\pi v_{c, \infty}}{2} (1 - 1/g), \quad (82)$$

with $v_{c, \infty} = v_c^{\text{(inch)}}|_{U=\infty}$. Note in addition that the renormalized hopping can be written as

$$t^* = t \left[\frac{v_c^{\text{(inch)}}}{v_{c, \infty}} \right]. \quad (83)$$

It is now clear that Eq. (72) and Eqs. (81)–(83) establish the connection between the fractional Landau LL parametrization and that of the LL in the spin-incoherent regime.

C. Specific heat and numerical data

We shall now demonstrate that in the spin-incoherent regime the fractional Landau LL approach provides a very good description of the T behavior of the zero-field specific heat of the system derived from the entropy defined in Eq. (48). We stress that this procedure will prove rewarding in establishing an exact connection between the fractional Landau LL and an *interacting* spinless Fermi gas, similarly to the one that we have discussed between the fractional LL and the *free* spinless Fermi gas in Sec. IV. However, the fractional Landau LL is valid only under the condition $J(=4t^2/U) \lesssim k_B T \lesssim E_F(\sim t)$, while the fractional LL is an exact description at $U = \infty$ and any T .

In zero field, using the Landau assumption in the calculation of the specific heat, Eqs. (55) and (58), and summing up the two fractional species we can obtain a direct relation between $\langle \tilde{n}_k \rangle$ and the *interacting* spinless Fermi gas distribution function:

$$\langle \tilde{n}_k^{(F)} \rangle = 2\langle \tilde{n}_k \rangle = \frac{1}{e^{\beta(\tilde{\epsilon}_k - \mu_L^{(F)})} + 1}, \quad (84)$$

with

$$\mu_L^{(F)} = \mu_L + k_B T \ln 2, \quad (85)$$

where μ_L is the fractional Landau LL chemical potential and $\mu_L^{(F)}$ is the chemical potential of the related interacting spinless Fermi gas. Lastly, by replacing $\langle \hat{n}_k \rangle \rightarrow \langle \tilde{n}_k \rangle$ in Eq. (48), and using Eqs. (84) and (85), we can obtain a relation between the fractional Landau LL entropy, S_L , and the related interacting

spinless Fermi gas entropy, $S_L^{(F)}$:

$$\begin{aligned}\frac{S_L(T, n)}{L} &= nk_B \ln 2 - \frac{k_B}{L} \sum_k [\langle \tilde{n}_k^{(F)} \rangle \ln \langle \tilde{n}_k^{(F)} \rangle \\ &\quad + (1 - \langle \tilde{n}_k^{(F)} \rangle) \ln (1 - \langle \tilde{n}_k^{(F)} \rangle)]\end{aligned}\quad (86)$$

$$= nk_B \ln 2 + \frac{S_L^{(F)}(T, n)}{L}, \quad (87)$$

which is formally identical to Eq. (44) at $U = \infty$. The function $\mu_L(T, n)$, to order $(k_B T/t^*)^2$, is given by Eq. (56); however, in order to attain a good description for a wide range of temperatures we have calculated $\mu_L(T, n)$ numerically using the constraint equation

$$\frac{2}{L} \sum_k \langle \tilde{n}_k \rangle = n, \quad (88)$$

where n is the average density of spinless fermions.

From either entropy above, we can numerically calculate the specific heat of the fractional Landau LL gas using $C = T(\partial S/\partial T)$. In Fig. 4 we show $C(T)/k_B$ for the fractional LL ($U = \infty$) and the fractional Landau LL for $U = 80t$ ($J = 0.05t$) for chains with $n = 3/4$. The specific heat of the fractional Landau LL in zero field is derived using Eqs. (48), (55), and (9), for $U = 80t$, whereas for the fractional LL, $U = \infty$, use is made of Eq. (38). Remarkably, the fractional Landau LL prediction quantitatively agrees with the DMRG data in the temperature range of the spin-incoherent regime up to $k_B T \sim t$. Despite the tiny value of $\frac{t}{t^*} = 0.987$, the fractional Landau LL approach adequately quantifies the first order correction, (t/U) , to the $U = \infty$ curve in the spin-incoherent regime.

The two paths to the spin-incoherent LL regime shown in Fig. 3 can be discussed with the aid of Fig. 4. Path I of Fig. 3 is associated with the DMRG data of Ref. [23] showed in Fig. 4, in which case we witness the linear behavior of the specific heat, with spin and charge contributions at very low temperature, and the crossover to the spin-incoherent regime. Further, Path II of Fig. 3 is associated with the analytical results plotted in Fig. 4. Indeed, in this figure we indicate the onset of the spin-incoherent regime, in which case we can notice that the specific heat data of the fractional LL and that of the Landau fractional LL, both due to charge contribution only, practically meet at the onset of the spin-incoherent regime, since they differ by the small correction term of order t/U .

VI. HIGH-TEMPERATURE LIMIT

In previous sections we have studied the Hubbard chain in the spin-incoherent regime: $J \ll k_B T \ll t$, using a perturbative Bethe ansatz procedure, valid for $U/k_B T \gg 1$, combined with a phenomenological approach. In this section, we find it instructive to study the high- T limit, so we can provide direct contact with well established results for the t - J models derived using quantum transfer matrix techniques [50]. The high- T limit is accessed under the conditions: $e^{-\beta \tilde{\epsilon}_k} \rightarrow 1$, with $\frac{\mu_L}{k_B T}$ a function of n . Indeed, from either Eqs. (84) and (85) or Eq. (88), we find that $\langle \tilde{n}_k \rangle = n/2$ and

$$\lim_{T \rightarrow \infty} \frac{\mu_L}{k_B T} = \ln \left(\frac{n/2}{1-n} \right), \quad (89)$$

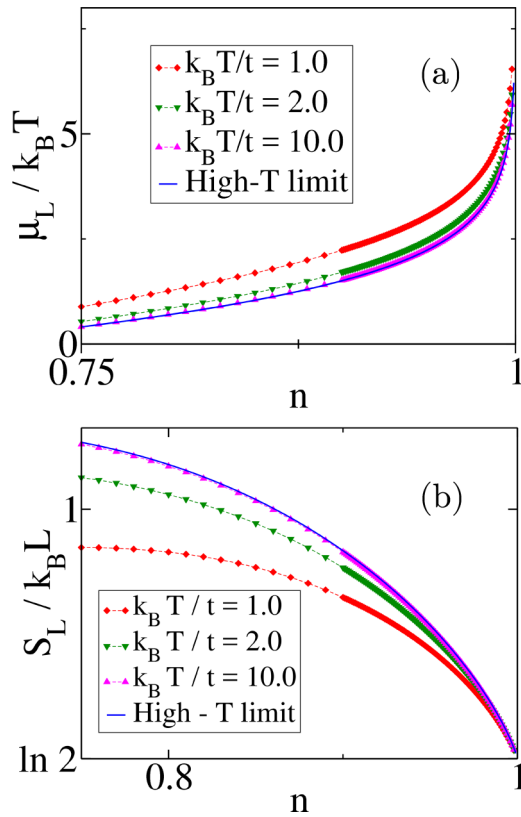


FIG. 5. (a) Fractional Landau LL chemical potential, μ_L , in units of $k_B T$ as a function of particle density n , for $U = 80t$. The high- T limit, given by $\frac{\mu_L}{k_B T} = \ln\left(\frac{n}{2-2n}\right)$, is indicated. (b) Fractional Landau LL entropy per site, S_L/L , in units of k_B , as a function of n , for $U = 80t$. The high- T limit, given by $\frac{S_L}{k_B L} = n \ln 2 - n \ln n - (1-n) \ln(1-n)$, is also shown.

which exhibits a Van-Hove singularity as $n \rightarrow 1$, as illustrated in Fig. 5(a). These results imply that S_L in Eq. (86) reads:

$$\lim_{T \rightarrow \infty} \frac{S_L(T, n)}{k_B L} = n \ln 2 - n \ln n - (1-n) \ln(1-n), \quad (90)$$

which is exactly the result expected by counting the total number of states of the t - J model at a density n , with $N_\uparrow = N_\downarrow$, in the thermodynamic limit. In Fig. 5(b) we present $S_L(T, n)$ with a density n for $U = 80t$. It is also interesting to notice that $S_L(T, n)/k_B L$ approaches $\ln 2$ at half filling due to the Van-Hove singularity. In addition, we stress that the high- T limit is taken under the proviso that $U/k_B T \gg 1$, as is the case in Figs. 5(a) and 5(b), in which case $U = 80t$. It is worth mentioning that as $T \rightarrow \infty$, U increases accordingly, so that Eq. (90) is the $T \rightarrow \infty$ entropy of the $U = \infty$ Hubbard chain, Eq. (44).

Lastly, in order to confirm the high- T limit of the particle occupation number, $\langle n_k \rangle$, of the t - J model, Eq. (2), we use the Lanczos exact diagonalization and finite temperature Lanczos method (FTLM) [69] to calculate $\langle n_k \rangle$ in finite chains under periodic boundary conditions (PBC). In fact, our analysis provides strong evidence in favor of our analytical results and, most importantly, verifies the consistency of the fractional Landau LL phenomenological approach.

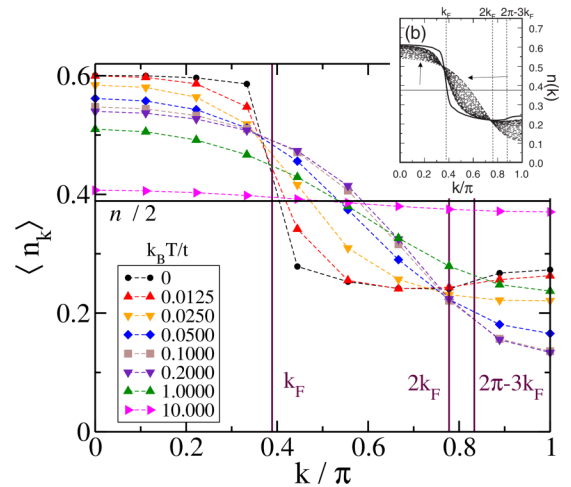


FIG. 6. Momentum distribution function $\langle n_k \rangle$ calculated through FTLM for a t - J chain with 18 sites under PBC, particle density $n = 7/9$, $J = 0.05t$, and $k_F = n\pi/2$ for the indicated values of temperature $k_B T/t$. Notice that the limit $\langle n_k \rangle = n/2$ as $(k_B T/t) \rightarrow \infty$ is nearly attained for $(k_B T/t) = 10$. The inset is a copy of Fig. 3(b) of Ref. [23]: DMRG data for $n = 0.75$, $J = 0.05t$, for a chain with 32 sites. Arrows indicate increasing $t/k_B T$ in steps of 4. The horizontal line at $\langle n_k \rangle = n/2 = 0.375$ indicates the value of $\langle n_k \rangle$ for $(k_B T/t) \rightarrow \infty$.

The FTLM uses the states from R independent Lanczos exact diagonalization procedures to estimate thermodynamic functions of finite systems. For each Lanczos run, a maximum of M Lanczos basis states is generated. The MR approximate eigenenergies and eigenstates are used to calculate the thermodynamic functions of interest. We take $R = 12000$ and $M = 50$ in our calculations and have exploited translational symmetry and rotational symmetry in spin space.

The distribution function of spin \uparrow electrons of momentum k is calculated through:

$$\langle n_k \rangle = \frac{1}{L} \sum_{l=1, m=1}^L \langle c_{l\uparrow}^\dagger c_{m\uparrow} \rangle e^{ik(l-m)}, \quad (91)$$

where $\langle \dots \rangle$ indicates thermal and quantum averages. In Fig. 6 we present $\langle n_k \rangle$ for $J = 0.05t$ and $n = 7/9$, calculated with the Lanczos method ($T = 0$) and FTLM ($T \neq 0$), as well as DMRG data from Ref. [23] for $n = 0.75$ and $J = 0.05t$, shown in the inset. At $T = 0$, the singularities [23,59] at k_F and $3k_F$ (shown at $2\pi - 3k_F$) are evident in our FTLM results for $(k_B T/t) = 0$ and 0.0125, with $k_F = \pi n/2$. The spin-incoherent regime, $k_B T \gtrsim J$, is signaled [23] by the presence of an inflection point at $2k_F$, as observed in Fig. 6 for $(k_B T/t) = 0.05, 0.10$, and 0.20. We thus conclude that both the FTLM and DMRG methods grasp the main features of the crossover between the low- T LL to its spin-incoherent regime.

VII. CONCLUDING REMARKS

We have studied the Hubbard chain in the spin-incoherent Luttinger liquid regime, both for $J = 0$ and finite $J (\ll k_B T)$. In the former case, we have shown that its thermodynamic

properties are exactly those of an ideal gas of two species of noninteracting particles obeying fractional statistics. It implies that the charge degrees of freedom are governed by the free spinless Fermi gas, while the spin degrees of freedom are fully disordered (Curie response). On the other hand, the latter case was investigated using an expression for the grand-canonical free energy derived perturbatively by Ha from Takahashi's integral equations. Based on this result, and using $U \gg k_B T$, we were able to obtain an expression for the Helmholtz free energy suitable to describe the system in the spin-incoherent regime: $J (\equiv 4t^2/U) \ll k_B T \ll E_F$, from which several thermodynamic quantities were derived. In particular, we have reported on the specific heat, charge compressibility, magnetic susceptibility, Drude weight, charge and spin velocities, and the Luttinger liquid (LL) parameter.

We have also discussed the interesting possibility of looking at the system with finite J as a fractional Landau LL. In this framework, the low-energy physics of the system is also described in terms of fractional quasiparticles obeying the Haldane-Wu fractional entropy. At the same time, it enables us to interpret corrections of $\mathcal{O}(t^2/U)$ as coming from (i) renormalization of the hopping t only, which is the case of the specific heat and charge velocity; (ii) hopping renormalization and the interaction of fractional Landau quasiparticles, as found for the charge compressibility, and the Drude weight; (iii) interaction of fractional Landau LL quasiparticles only, as for the magnetic susceptibility. In addition, we have calculated the fractional Landau LL parameters and showed that they are fixed by the ones of the incoherent LL derived from pure thermodynamic grounds and arguments of bosonization. In particular, a phase diagram was provided and two thermodynamic paths to access the spin-incoherent LL regime shed light on the numerical and analytical procedures. Lastly, through a numerical analysis of the exclusion fractional Landau LL entropy and the use of finite temperature Lanczos method, we have calculated the temperature behavior of the specific heat and the particle momentum distribution, respectively, both in very good agreement with previous density matrix renormalization group calculations in the spin-incoherent and the high- T limit.

In conclusion, we believe that our reported results using complementary approaches have provided interesting insights on several features of the thermodynamics of the spin-incoherent Luttinger liquid regime of the Hubbard chain. They might stimulate further theoretical and experimental work, since this special LL regime has been of interest in the context of several physical systems mentioned in our paper, particularly quantum wires at low temperature. In addition, the crossover [67,70] from the (1D) spin-incoherent LL regime (fractional Landau LL) to a higher dimensional phenomenology (due to 2D or 3D coupling between chains), e.g., standard Landau Fermi liquid theory, also deserves further investigation.

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APPENDIX A: THE GRAND-CANONICAL FREE ENERGY FOR $U \gg k_B T$, EQ. (4)

In Ref. [44], Ha derived a strong coupling ($U \gg t$) perturbative λ expansion of the grand-canonical free energy $\frac{\Omega(T, \mu, H)}{L}$:

$$\frac{\Omega(T, \mu, H)}{L} = \omega^{(0)} + \omega^{(1)} + \dots, \quad (\text{A1})$$

where

$$\omega^{(0)} = \frac{U}{2} - \mu - \frac{1}{\beta} \ln 2a - \frac{1}{\beta} I_1 \quad (\text{A2})$$

and

$$\omega^{(1)} = \frac{t}{U} I_2 \left[\left(\frac{1}{a^2} - \frac{1}{b^2} \right) \frac{1}{\beta} I_3 - \frac{1}{a^2} \right]; \quad (\text{A3})$$

with

$$a = \cosh \left[\beta \left(\frac{U}{2} - \mu \right) \right], \quad (\text{A4})$$

$$b = \cosh(\beta \mu_B H), \quad (\text{A5})$$

$$I_1 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln \left[1 + \frac{b}{a} e^{-\beta(\varepsilon_k - \frac{U}{2})} \right], \quad (\text{A6})$$

$$I_2 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[1 + \frac{a}{b} e^{\beta(\varepsilon_k - \frac{U}{2})} \right]^{-1}, \quad (\text{A7})$$

and

$$I_3 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos k \ln \left[1 + \frac{b}{a} e^{-\beta(\varepsilon_k - \frac{U}{2})} \right], \quad (\text{A8})$$

in which $\varepsilon_k = -2t \cos k$.

The above expansion was used to obtain two expansions in distinct limits: (i) $U, k_B T \gg t$ with $U/k_B T$ fixed, which was shown to be in very good agreement with previous high- T expansions [71]; (ii) $U \gg t$ at fixed $k_B T$. We shall use the latter alternative in the limit $U \gg k_B T$, in which case we have $e^{\beta U} \gg 1$ and $a \sim \frac{1}{2} e^{\beta(\frac{U}{2} - \mu)} \gg 1$, such that

$$\omega^{(0)} = -k_B T I_1 \quad (\text{A9})$$

and

$$\omega^{(1)} = \frac{-t}{U} I_2 \frac{1}{b^2} \frac{1}{\beta} I_3 = \frac{-k_B T}{\cosh^2(\beta \mu_B H)} \left(\frac{t}{U} \right) I_2 I_3; \quad (\text{A10})$$

with

$$I_1 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \ln [1 + e^{-\beta(\varepsilon_k - \mu - \mu_B H)} + e^{-\beta(\varepsilon_k - \mu + \mu_B H)}], \quad (\text{A11})$$

$$I_2 = \cosh(\beta \mu_B H) \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{2}{e^{\beta(\varepsilon_k - \mu)} + 2 \cosh(\beta \mu_B H)}, \quad (\text{A12})$$

and

$$I_3 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos k \ln [1 + e^{-\beta(\varepsilon_k - \mu - \mu_B H)} + e^{-\beta(\varepsilon_k - \mu + \mu_B H)}]. \quad (\text{A13})$$

Therefore, Eq. (A1) with $\omega^{(0)}$ and $\omega^{(1)}$ given by Eqs. (A9) and (A10), with I_1 , I_2 , and I_3 defined through Eqs. (A11), (A12), and (A13), respectively, leads to Eq. (4).

APPENDIX B: SUSCEPTIBILITY AT $H = 0$, EQ. (20)

The magnetic susceptibility per site for $H = 0$ is given by

$$\begin{aligned}\chi(T, \mu) &= \left. \frac{\partial M(T, H, \mu)}{\partial H} \right|_{H=0} \\ &= - \left. \frac{\mu_B}{L} \frac{\partial^2 \Omega(T, H, \mu)}{\partial H^2} \right|_{H=0},\end{aligned}\quad (\text{B1})$$

with Ω/L as written in Eq. (4). Using Eq. (B1), and taking the limits $\sinh(\beta\mu_B H) \rightarrow \beta\mu_B H$, $\tanh(\beta\mu_B H) \rightarrow \beta\mu_B H$, and $\cosh(\beta\mu_B H) \rightarrow 1$, we obtain the following expression for the susceptibility:

$$\chi = \frac{\mu_B^2}{k_B T} \left(I_{2,0} - \frac{t}{U} I_{2,0} I_{3,0} - \frac{4t}{U} I_{4,0} I_{3,0} + \frac{4t}{U} \frac{I_{2,0}}{2} I_{5,0} \right),\quad (\text{B2})$$

where $I_{1,0}$, $I_{2,0}$, and $I_{3,0}$ are given by Eqs. (A11), (A12), and (A13), respectively, with $H = 0$, while

$$I_{4,0} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{[e^{\beta(\varepsilon_k - \mu)} + 2]^2},\quad (\text{B3})$$

$$I_{5,0} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{\cos k}{e^{\beta(\varepsilon_k - \mu)} + 2}.\quad (\text{B4})$$

Now, by performing a Sommerfeld-like expansion around $\mu_{\text{inch}}(T = 0) \sim t$, Eq. (5), in the above integrals, we obtain

$$I_{2,0} = n + \dots,\quad (\text{B5})$$

$$I_{3,0} = \frac{nt}{k_B T} \left(1 - \frac{\sin 2\pi n}{2\pi n} \right) + \dots,\quad (\text{B6})$$

$$I_{4,0} = \frac{n}{4} + \dots,\quad (\text{B7})$$

$$I_{5,0} = \frac{\sin(n\pi)}{\pi} + \dots.\quad (\text{B8})$$

Lastly, by substituting Eqs. (B5) and (B6) into Eq. (B2), we obtain

$$\chi = \frac{\mu_B^2}{k_B T} \left[n - \frac{2n^2 t^2}{U k_B T} \left(1 - \frac{\sin 2\pi n}{2\pi n} \right) + \frac{2t n \sin(n\pi)}{U \pi} \right].\quad (\text{B9})$$

From the above equation, we readily reproduce Eq. (20), where we noticed that the last term in Eq. (83) is of order (J/t) and will be neglected.

APPENDIX C: DERIVATION OF EQ. (63)

Equation (62) can be written as

$$\begin{aligned}\frac{\partial \hat{\varepsilon}_k}{\partial \mu} &= 2 \sum_{k' > 0} \frac{f_{k,k'}^s \beta (1 - \partial \hat{\varepsilon}_{k'} / \partial \mu) e^{\beta(\hat{\varepsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\varepsilon}_{k'} - \mu)} + 2]^2} \\ &+ 2 \sum_{k' < 0} \frac{f_{k,k'}^s \beta (1 - \partial \hat{\varepsilon}_{k'} / \partial \mu) e^{\beta(\hat{\varepsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\varepsilon}_{k'} - \mu)} + 2]^2}.\end{aligned}\quad (\text{C1})$$

Next we explore the presence of sharp peaks at the Fermi surface:

$$\begin{aligned}\frac{\partial \hat{\varepsilon}_k}{\partial \mu} &\simeq 2 f_{k,k_F}^s \sum_{k' > 0} \frac{\beta (1 - \partial \hat{\varepsilon}_{k'} / \partial \mu) e^{\beta(\hat{\varepsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\varepsilon}_{k'} - \mu)} + 2]^2} \\ &+ 2 f_{k,-k_F}^s \sum_{k' < 0} \frac{\beta (1 - \partial \hat{\varepsilon}_{k'} / \partial \mu) e^{\beta(\hat{\varepsilon}_{k'} - \mu)}}{[e^{\beta(\hat{\varepsilon}_{k'} - \mu)} + 2]^2}.\end{aligned}\quad (\text{C2})$$

Both integrands are now even, thus after using $2 \sum_{k' > 0} (\dots) = 2 \sum_{k' < 0} (\dots) = \sum_{k'} (\dots)$, one gets Eq. (63) with the help of Eq. (61).

APPENDIX D: DERIVATION OF EQ. (79)

Deriving $E(\phi)$ with respect to ϕ , one gets

$$\begin{aligned}\frac{\partial E(\phi)}{\partial \phi} &= \sum_{k,\alpha} 2t^* \sin(k + \phi) \delta \langle \hat{n}_{k,\alpha} \rangle_\phi \\ &+ \sum_{k,\alpha;k',\alpha'} f_{k,\alpha;k',\alpha'} \delta \langle \hat{n}_{k,\alpha} \rangle_\phi \frac{\partial}{\partial \phi} \delta \langle \hat{n}_{k',\alpha'} \rangle_\phi,\end{aligned}\quad (\text{D1})$$

where we have explored the symmetry $f_{k,\alpha;k',\alpha'} = f_{k',\alpha';k,\alpha}$ and neglected the exponentially small term

$$W \equiv \sum_{k,\alpha} \left[\tilde{\varepsilon}_{k,\alpha}(\phi) \frac{\partial}{\partial \phi} \delta \langle \hat{n}_{k,\alpha} \rangle_\phi \right].\quad (\text{D2})$$

In order to demonstrate this point, we derive Eq. (77) with respect to ϕ :

$$\frac{\partial \hat{\varepsilon}_{k,\alpha}(\phi)}{\partial \phi} = 2t^* \sin(k + \phi) + \sum_{k',\alpha'} f_{k,\alpha;k',\alpha'} \frac{\partial}{\partial \phi} \delta \langle \hat{n}_{k',\alpha'} \rangle_\phi.\quad (\text{D3})$$

Since $f_{k,\alpha;k',\alpha'} = \mathcal{O}(t^2/U)$, we can use in (D3) the approximation

$$\delta \langle \hat{n}_{k',\alpha'} \rangle_\phi = \frac{1}{e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} + 2} - \left. \langle \hat{n}_{k',\alpha'} \rangle \right|_{\substack{\phi=0 \\ T=0 \\ U=\infty}},\quad (\text{D4})$$

where $\varepsilon_{k',\alpha'}(\phi) = -2t \cos(k' + \phi)$. We note that $\sum_{k',\alpha'} \frac{1}{e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} + 2} = N$ implies $\mu = -2t \cos(n\pi) - k_B T \ln 2 + \dots$, which is ϕ independent. Therefore,

$$\frac{\partial}{\partial \phi} \delta \langle \hat{n}_{k',\alpha'} \rangle_\phi = - \frac{\beta e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} 2t \sin(k' + \phi)}{[e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} + 2]^2}.\quad (\text{D5})$$

After inserting this derivative back into Eq. (D3), one has

$$\begin{aligned}\frac{\partial \hat{\varepsilon}_{k,\alpha}(\phi)}{\partial \phi} &= 2t^* \sin(k + \phi) \\ &- \sum_{k',\alpha'} f_{k,\alpha;k',\alpha'} \frac{\beta e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} 2t \sin(k' + \phi)}{[e^{\beta[\varepsilon_{k',\alpha'}(\phi) - \mu]} + 2]^2}.\end{aligned}\quad (\text{D6})$$

We now sum over all values of α' to get an expression that is α independent:

$$\begin{aligned}\frac{\partial \hat{\varepsilon}_{k,\alpha}(\phi)}{\partial \phi} &= 2t^* \sin(k + \phi) \\ &- 2 \sum_{k'} f_{k,k'}^s \frac{\beta e^{\beta[\varepsilon_{k'}(\phi) - \mu]} 2t \sin(k' + \phi)}{[e^{\beta[\varepsilon_{k'}(\phi) - \mu]} + 2]^2},\end{aligned}\quad (\text{D7})$$

with omission of the subscript α' in $\varepsilon_{k',\alpha'}(\phi)$. We shall now demonstrate that the above sum is weakly dependent on ϕ . In the thermodynamic limit, it reads

$$I(k) \equiv -\frac{L}{\pi} \int_{-\pi}^{\pi} dk' f_{k,k'}^s \frac{\beta e^{\beta[\varepsilon_{k'}(\phi)-\mu]} 2t \sin(k'+\phi)}{[e^{\beta[\varepsilon_{k'}(\phi)-\mu]} + 2]^2}, \quad (\text{D8})$$

where it is to be noted that the integrand exhibits sharp peaks at $k'+\phi = \pm k_F$. After the transformation $k'+\phi = q$ one obtains

$$I(k) = -\frac{L}{\pi} \int dq f_{k,q-\phi}^s \frac{\beta e^{\beta(\varepsilon_q-\mu)} 2t \sin q}{[e^{\beta(\varepsilon_q-\mu)} + 2]^2}, \quad (\text{D9})$$

so that the dependence on ϕ is removed from the integrand (except for the very small dependence of $f_{k,q-\phi}^s$) and we can take $\phi = 0$ with negligible error [$\mu_0 = -2t \cos(n\pi)$]:

$$I(k) = -\frac{L}{2\pi} \int_{-\pi}^{\pi} dq f_{k,q}^s \frac{\beta e^{\beta(\varepsilon_q-\mu_0)} 2t \sin q}{[e^{\beta(\varepsilon_q-\mu_0)} + 1]^2}, \quad (\text{D10})$$

with limit of integrations restituted to their original values. Exploring again the presence of sharp peaks at the Fermi surface (see Appendix C), we obtain

$$I(k) = -\frac{L}{2\pi} (f_{k,k_F}^s - f_{k,-k_F}^s), \quad (\text{D11})$$

with the use of $|\sin q| = \sqrt{1 - \cos^2 q}$.

We now return to Eq. (D6):

$$\frac{\partial \hat{\varepsilon}_{k,\alpha}(\phi)}{\partial \phi} = 2t^* \sin(k+\phi) - \frac{L}{2\pi} (f_{k,k_F}^s - f_{k,-k_F}^s). \quad (\text{D12})$$

The derivative of $\delta\langle \hat{n}_{k,\alpha} \rangle_\phi$, Eq. (78), with respect to ϕ can be now calculated:

$$\begin{aligned} \frac{\partial}{\partial \phi} \delta\langle \hat{n}_{k,\alpha} \rangle_\phi &= -\frac{\beta e^{\beta[\varepsilon_{k,\alpha}(\phi)-\mu]}}{[e^{\beta[\varepsilon_{k,\alpha}(\phi)-\mu]} + 2]^2} \\ &\times \left[2t^* \sin(k+\phi) - \frac{L}{2\pi} (f_{k,k_F}^s - f_{k,-k_F}^s) \right]. \end{aligned} \quad (\text{D13})$$

We are now in a position to show that W is exponentially small. After inserting Eq. (D13) into (D2), one gets

$$\begin{aligned} W &= \sum_{k,\alpha} \left[4(t^*)^2 \sin(k+\phi) \cos(k+\phi) \right. \\ &\quad \left. - 2t^* \cos(k+\phi) \left(\frac{L}{2\pi} \right) (f_{k,k_F}^s - f_{k,-k_F}^s) \right] \\ &\quad \times \frac{\beta e^{\beta[\varepsilon_k(\phi)-\mu]}}{[e^{\beta[\varepsilon_k(\phi)-\mu]} + 2]^2}. \end{aligned} \quad (\text{D14})$$

Once again, we call attention to the fact that the integrand displays sharp peaks at $k+\phi = \pm k_F$. Thus, after making the transformation $q = k+\phi$, the resulting integrand becomes odd in q . Using the same arguments that we have applied to go from Eq. (D8) to (D11), we thus conclude that W is exponentially small. We can now return to Eq. (D1) and derive it one more time with respect to ϕ :

$$\begin{aligned} \left(\frac{\partial^2 E(\phi)}{\partial \phi^2} \right)_{\phi=0} &= \sum_{k,\alpha} 2t^* \sin k \left(\frac{\partial}{\partial \phi} \delta\langle \hat{n}_{k,\alpha} \rangle_\phi \right)_{\phi=0} \\ &\quad + \sum_{k,\alpha} 2t^* \cos k \delta\langle \hat{n}_{k,\alpha} \rangle_{\phi=0} \\ &\quad + \sum_{k,\alpha;k',\alpha'} f_{k,\alpha;k',\alpha'} \left(\frac{\partial}{\partial \phi} \delta\langle \hat{n}_{k,\alpha} \rangle_\phi \right)_{\phi=0} \left(\frac{\partial}{\partial \phi} \delta\langle \hat{n}_{k',\alpha'} \rangle_\phi \right)_{\phi=0} \\ &\quad + \sum_{k,\alpha;k',\alpha'} f_{k,\alpha;k',\alpha'} \delta\langle \hat{n}_{k,\alpha} \rangle_{\phi=0} \left(\frac{\partial^2}{\partial \phi^2} \delta\langle \hat{n}_{k',\alpha'} \rangle_\phi \right)_{\phi=0}. \end{aligned} \quad (\text{D15})$$

At low temperatures, we neglect terms containing $\delta\langle \hat{n}_{k,\alpha} \rangle_{\phi=0}$ in (D15), and make use of Eq. (D13) and of the procedure that led to Eq. (D11) to obtain the final result:

$$\begin{aligned} \left(\frac{\partial^2 E(\phi)}{\partial \phi^2} \right)_{\phi=0} &= -\frac{2t^* L}{\pi} \sin(n\pi) \\ &\quad + \left(\frac{L}{\pi} \right)^2 (f_{k_F,k_F}^s - f_{k_F,-k_F}^s), \end{aligned} \quad (\text{D16})$$

which implies Eq. (79).

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