Rapid Communications

Anomalous thermalization and transport in disordered interacting Floquet systems

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Local observables in generic periodically driven closed quantum systems are known to relax to values described by periodic infinite temperature ensembles. At the same time, ergodic static systems exhibit anomalous thermalization of local observables and satisfy a modified version of the eigenstate thermalization hypothesis (ETH), when disorder is present. This raises the question, how does the introduction of disorder affect relaxation in periodically driven systems? In this Rapid Communication, we analyze this problem by numerically studying transport and thermalization in an archetypal example. We find that thermalization is anomalous and is accompanied by subdiffusive transport with a disorder-dependent dynamical exponent. Distributions of matrix elements of local operators in the eigenbases of a family of effective time-independent Hamiltonians, which describe the stroboscopic dynamics of such systems, show anomalous departures from predictions of ETH, signaling that only a modified version of ETH is satisfied. The dynamical exponent is shown to be related to the scaling of the variance of these distributions with system size.

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Introduction. Recent advances in cold-atom [1,2] and trapped-ion [3] experiments have stimulated interest in nonequilibrium dynamics and thermalization or lack thereof in *isolated* quantum systems. Thermalization in both classical and quantum systems requires an effective loss of information contained in the initial state of the system. For classical systems, this occurs naturally, since the underlying equations of motion are nonlinear and therefore typically chaotic. For quantum systems, the situation is more delicate, since all the information about the state of the system is encoded in the wave function which evolves under the *linear* Schrödinger equation. While the information about the whole system cannot be lost under unitary evolution, this is not the case for subsystems as the *corresponding reduced* density matrices evolve nonunitarily. Thus, the objects of interest in the context of thermalization are local observables supported on a subsystem while the rest of the system serves as an effective bath. Written in the eigenbasis of the Hamiltonian, the diagonal elements of local observables encode information about the stationary state, while the off-diagonal elements contain dynamical information. The probability distributions of diagonal and off-diagonal matrix elements of local operators were studied already in the 1980's [4–8], but regained interest after the introduction of the "eigenstate thermalization hypothesis" (ETH) by Deutsch [9] and Srednicki [10] in the following decade. ETH was confirmed to hold in a variety of systems [11–14], and is concerned with only the first and the second moments of the distribution of the matrix elements, implicitly assuming that the probability distribution is Gaussian. This assumption was motivated by Berry's conjecture, which states that eigenstates of nonintegrable quantum systems are reminiscent of random states drawn from a Gaussian distribution [15,16]. More specifically, ETH requires that the matrix elements of local operators are described by a smooth functional dependence on the extensive energy and the distance from the diagonal, superimposed by random Gaussian fluctuations. The variance of the fluctuations is assumed to decay exponentially with the system size [10], which was verified for a number of generic quantum systems [17–23].

In a recent article, it was shown that for a class of disordered *ergodic* systems, which for sufficiently strong disorder undergo the many-body localization (MBL) transition [24] (see Refs. [25–27] for recent reviews), ETH has to be modified, since the decay of the fluctuations of the off-diagonal matrix elements acquires a power-law correction to their scaling with system size [28,29]. This is accompanied with anomalous (subdiffusive) relaxation to equilibrium, a situation which was dubbed *anomalous thermalization* [28].

In this Rapid Communication, we show that a similar phenomenology exists also in disordered periodically driven (Floquet) systems, which undergo the Floquet-MBL transition for sufficiently strong disorder [30–33]. The stroboscopic dynamics of these systems is governed by the unitary Floquet operator, which is the time-evolution operator over one period. The Floquet operator can be expressed in terms of a family of effective Hamiltonians, which allows the generalization of the concept of thermalization and ETH to this time-dependent case [34–37]. It was shown that ETH assumes a simplified form since the smooth part of the diagonal matrix elements is

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constant and corresponds to the trace of the local observable [34–37]. This is consistent with the expectation that for any generic initial state, the system heats up to a state which is locally indistinguishable from the infinite temperature state. In previous studies it was numerically shown that disordered Floquet systems exhibit anomalous heat absorption from the periodic drive [38–41], and it was suggested that spin transport in such systems is anomalous [38,39]. Here, using numerically exact methods, we study the nature of transport and thermalization in disordered Floquet systems.

Model. We numerically investigate a disordered one-dimensional Heisenberg model subject to a periodic modulation of the staggered magnetization. The driving protocol we consider is generated by two alternating noncommuting Hamiltonians $\hat{H}_\pm$ (each applied for half a period $T/2$),

$$\hat{H}_\pm = \sum_{i=1}^{L-1} J\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1} + \sum_{i=1}^{L}[h_i \pm (-1)^i\Delta]\hat{S}_i^z, \tag{1}$$

where L is the length of the chain, $\hat{S}_i^{x,y,z}$ are spin-1/2 operators, J is the spin-spin interaction (which we set to unity), $h_i \in [-W, W]$ are independent random fields drawn from a uniform distribution, and Δ is the driving amplitude. Both Hamiltonians commute with the total magnetization $\hat{M}_z = \sum_\ell \hat{S}_\ell^z$, and in what follows, we work in the $M_z = 0$ sector for even L and $M_z = 1/2$ for odd L. We set $T = 3$, such that the frequency of the drive is well below the many-body bandwidth (of the undriven system) for all considered system sizes, and choose the driving amplitude to be small enough to still have a Floquet-MBL transition for sufficiently strong disorder (e.g., Refs. [30,31]), yet not much smaller than the single-particle bandwidth ($\Delta = 0.5$).

Distributions of matrix elements. In Floquet systems, the Hamiltonian is time dependent, therefore for the study of thermalization, the quantity of interest is not the Hamiltonian but the unitary Floquet operator $\hat{U}_F(T,0)$, which we take to be $\hat{U}_F(T,0) = e^{-i\hat{H}_+T/4}e^{-i\hat{H}_-T/2}e^{-i\hat{H}_+T/4}$ [42]. Using full diagonalization, we obtained all the eigenstates (denoted by Greek letters) of $\hat{U}_F(T,0)$ for various system sizes and computed the matrix elements of the local magnetization $S_{\alpha\beta}^z \equiv \langle\alpha|\hat{S}_{L/2}^z|\beta\rangle$. For clean Floquet systems, it was shown that the smooth part of the diagonal matrix elements of local operators does not depend on the quasienergy, with fluctuations which decrease with the system size [36]. We verified that this also holds for the *disordered* system we consider here [43]. We note in passing that our finding rules out the existence of a mobility edge in the quasienergy spectrum, since assuming its existence would imply that for the localized states $|\alpha\rangle$, $(S_{\alpha\alpha}^z)^2 \approx 1/4$, while for the delocalized $(S_{\alpha\alpha}^z)^2 \approx 0$, which is not consistent with our numerical observation [43].

While ETH is concerned only with the first and second moments of the distributions of matrix elements, we study the full distribution of both diagonal and off-diagonal matrix elements. The independence of the diagonal elements on the quasienergy allows us to accumulate statistics not only over different disorder realizations but also across all quasienergies. To eliminate correlations, all statistical errors are computed by bootstrap resampling over the disorder realizations only.

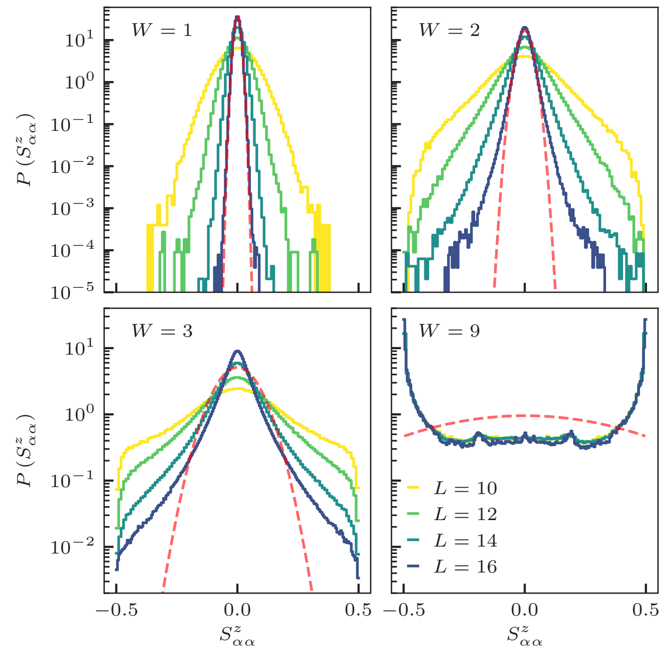

FIG. 1. Probability distribution $P(S_{\alpha\alpha}^z)$ for $W = 1$, 2, 3, and 9 and $L = 10$, 12, 14, and 16. The red dashed line in each panel depicts a Gaussian distribution with the same standard deviation as that of the corresponding $P(S_{\alpha,\alpha}^z)$ for $L = 16$. Statistics is obtained combining all Floquet eigenstates and all disorder realizations (over 500 realizations for $L < 16$ and 50 realizations for $L = 16$). Other parameters are $J = 1$, $\Delta = 0.5$, and $T = 3$.

In Fig. 1 we show the distributions computed for various disorder strengths W and system sizes L. For weak disorder ($W \approx 1$) the distributions are very close to Gaussian with variances which decrease with L, indicating the validity of ETH. For sufficiently strong disorder, $W = 9$, the distributions become bimodal, and almost independent of the system size, signaling the failure of ETH which occurs in the Floquet-MBL phase. The most fascinating situation occurs for intermediate disorder where the variances still decrease with L, however, the tails of the distributions acquire more weight with a clear departure from a Gaussian form. Interestingly, after rescaling the matrix elements by the standard deviation of their distributions, we find that the distributions collapse reasonably well for various L, not only for the Gaussian case, but also when the distributions are non-Gaussian, indicating that the anomalous behavior persists also in the thermodynamic limit [43]. In what follows, we set $0.5 \leqslant W \leqslant 4$ and focus only on the ergodic, albeit anomalous, phase [43].

While the diagonal matrix elements of local operators are related to the stationary state of the system, the off-diagonal matrix elements are directly connected to thermalization, or the relaxation to the stationary state. In Fig. 2 we show the distributions of the off-diagonal matrix elements $P(S_{\alpha\beta}^z)$ for two disorder values and various system sizes. To take statistics over the entire quasienergy spectrum, for every Floquet eigenstate we consider ten Floquet eigenstates closest to it in quasienergy and calculate $S_{\alpha\beta}^z$ for all the pairs in each group. The matrix elements are then accumulated over all the groups as also over different disorder realizations. For weak disorder ($W = 1$) the

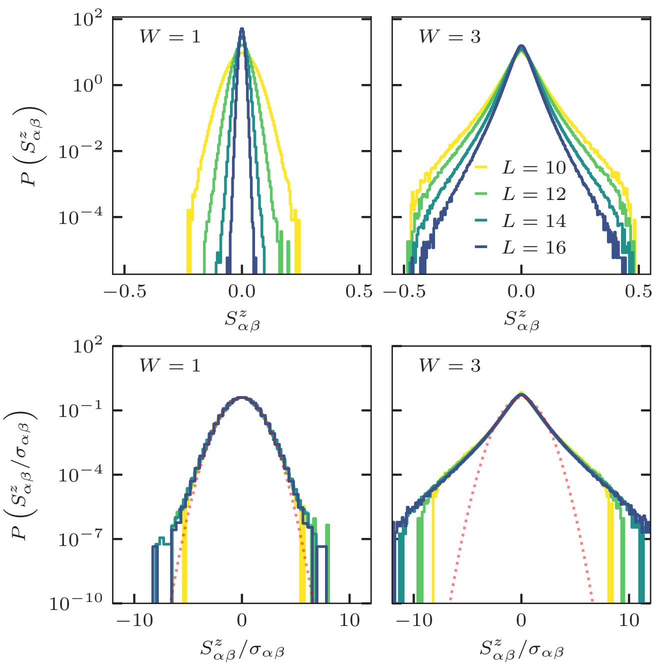

FIG. 2. Top row: Probability distribution $P(S^z_{\alpha\beta})$ for $W = 1$ and 3 and $L = 10$, 12, 14, and 16. Bottom row: Same as top row, but with the distributions scaled with the standard deviation of $P(S^z_{\alpha\beta})$, $\sigma_{\alpha\beta}$. The dashed red lines in the bottom row denote the standard normal distribution. Other parameters are the same as in Fig. 1.

distributions are close to Gaussian, in accordance with what is stipulated by ETH. However, for intermediate disorder, but still in the ergodic phase ($W = 3$), the distributions are clearly non-Gaussian. Rescaling the distributions by their standard deviation collapses all system sizes on top of each other (see the bottom panels of Fig. 2), indicating the convergence of the shape of the distributions to their thermodynamic limit. Similar anomalous behavior of the distributions of the off-diagonal elements was observed by two of us for static disordered systems, where it was established that only a *modified* version of the ETH is satisfied [28].

Anomalous spin transport. Previous works on many-body localization in static disordered systems identified a regime of anomalously slow dynamics at the ergodic side of the MBL transition. In particular, subdiffusive transport of spin or particles [44–48] as well as subballistic spreading of quantum information [49,50] was observed. It was also shown that these dynamical properties are related to a regime of slow anomalous thermalization described by a *modified* version of the ETH [28]. For the driving in Eq. (1), the total magnetization is conserved, which allows us to study spin transport. For this purpose we examine the infinite temperature spin-spin correlation function

$$C_i(t) = \frac{1}{\mathcal{N}}\mathrm{Tr}\big[\hat{S}^z_i(t)\hat{S}^z_{L/2}\big], \tag{2}$$

where $\mathcal{N}$ is the Hilbert space dimension and $\hat{S}^z_i(t)$ is the spin operator on site i written in the Heisenberg picture and evolved according to the aforementioned driving protocol. This correlation function encodes the spreading of an initial magnetization excitation created at center of the lattice $L/2$

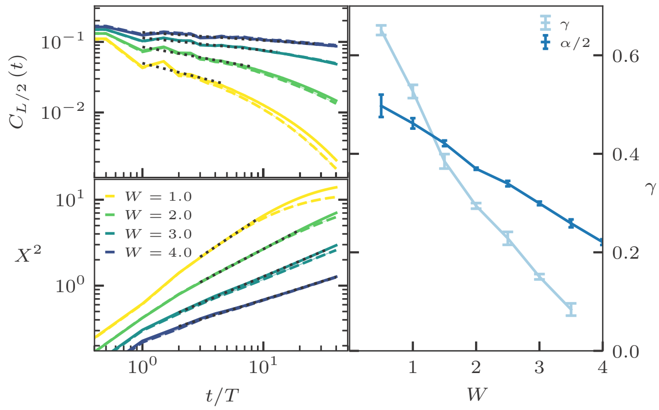

FIG. 3. Left: Autocorrelation function $C_{L/2}(t)$ (top) and mean-square displacement $X^2(t)$ (bottom) as a function of time on a log-log scale. Dashed lines: $L = 23$; solid lines: $L = 27$. Shades indicate statistical uncertainty (in most cases smaller than the linewidth) and best fits of the underlying power law are indicated by dotted black lines. Right: The exponents γ and $\alpha/2$ as a function of the disorder strength W. Error bars indicate only statistical errors and not systematic uncertainty.

at time $t = 0$. To calculate (2) we exploit the concept of dynamical typicality (for details, see Sec 5.1.5 of Ref. [51]). Practically, we approximate the trace in (2) by the expectation value with respect to a random state $|\psi\rangle$ sampled according to the Haar measure. The error of this approximation is inversely proportional to the square root of the Hilbert space dimension. After this substitution, the calculation of $C_i(t) = \langle\psi|\hat{S}^z_i(t)\hat{S}^z_{L/2}|\psi\rangle$ can be reduced to the propagation of two wave functions according the driving protocol in Eq. (1). The propagation is performed using standard Krylov space time-evolution methods (see Sec. 5.1.2 in Ref. [51]) for spin-1/2 chains of up to 27 spins. We characterize transport by the calculation of the spin-spin autocorrelation function $C_{L/2}(t)$ and the mean-square displacement (MSD),

$$X^2(t) = \sum_{i=1}^{L}\left(i - \frac{L}{2}\right)^2[C_i(t) - C_i(0)], \tag{3}$$

which is directly related to the current-current correlation function and therefore to transport (see the Appendix of Ref. [51]). The autocorrelation function of transported quantities decays as $C_{L/2}(t) \sim t^{-\gamma}$ and the MSD grows as $X^2(t) \propto t^{\alpha}$. For diffusive systems, $C_i(t)$ asymptotically assumes a Gaussian form, yielding the connection $\gamma = \alpha/2$, with $\alpha = 1$. For subdiffusive transport, $\alpha < 1$, and typically $\gamma \neq \alpha/2$, since $C_i(t)$ is non-Gaussian (cf. Fig. 3). In the left panels of Fig. 3 we present the autocorrelation function and the MSD for various disorder strengths. Both quantities are calculated by averaging the correlation function $C_i(t)$ over 100–1000 disorder realizations.

For weak disorder, we observe a fast decay of the autocorrelation function and close to linear growth of the MSD, consistent with diffusive transport observed in a similar clean Floquet model [52]. At intermediate disorder, we find a clearly sublinear growth of the MSD with an exponent $\alpha < 1$ which decreases as a function of the disorder strength (see the right panel of Fig. 3). The exponents were obtained by restricting

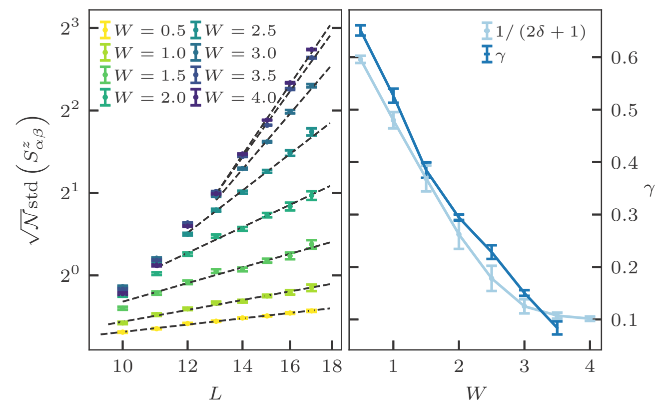

FIG. 4. Left: Extraction of the exponent δ from the scaling of $\sqrt{\mathcal{N}}\,\mathrm{std}(S^z_{\alpha\beta})$ with L. Right: Comparison of the exponent γ to its value extracted from the relation $\gamma = 1/(2\delta + 1)$.

the fits to times for which finite-size effects on the MSD are comparable to the statistical errors.

The analysis of the power-law decay of the autocorrelation function is more involved, mostly due to the superimposed oscillations occurring for short times. Fast transport (for smaller disorder strengths) results in a very short domain of power-law decay and less reliable exponents γ for $W \leqslant 1$. At stronger disorder, finite-size effects are less pronounced due to slower transport, yielding a longer domain of the power laws and more reliable γ. We have verified that the domain of the power-law decay increases with increasing the system size (see the left top panel of Fig. 3).

Anomalous thermalization. A relation between the scaling of the variance of the off-diagonal matrix elements and the decay of the autocorrelation function was previously derived [28]. While ETH assumes $\sqrt{\mathcal{N}}\,\mathrm{std}(S^z_{\alpha\beta}) = O(1)$, for static systems it was shown that for energies ϵ_α and ϵ_β satisfying $|\epsilon_\alpha - \epsilon_\beta| < L^{-1/\gamma}$, the scaling is modified to $\sqrt{\mathcal{N}}\,\mathrm{std}(S^z_{\alpha\beta}) \sim L^\delta$, namely, the decay of $\mathrm{std}(S^z_{\alpha\beta})$ with system size is not purely exponential, but includes a power-law correction with system size. Moreover. it was shown that $\delta = (1 - \gamma)/(2\gamma)$ [28]. While the derivation of this relation is rather general, it was only tested for static systems. Here, we investigate its validity for the Floquet system (1). The exponent γ is obtained from the decay of the autocorrelation function, as explained above. The exponent δ is extracted from the scaling of $\sqrt{\mathcal{N}}\,\mathrm{std}\,(S^z_{\alpha\beta})$ with system size (see the left panel of Fig. 4). To calculate the standard deviation of $S^z_{\alpha\beta}$ we consider approximately ten nearby (in quasienergy) Floquet eigenstates $|\alpha\rangle$ and $|\beta\rangle$. Since the density of states is constant and exponentially large in L, taking a fixed finite number of nearby states guarantees that $|\epsilon_\alpha - \epsilon_\beta| < L^{-1/\gamma}$ is satisfied for sufficiently large systems. The extracted exponent δ is nonzero, indicating a regime of anomalous (slow) thermalization similar to the situation in static disordered systems [28]. The exponent δ is increasing with the disorder strength, presumably diverging at the Floquet-MBL transition, where both ETH and its generalized version fail. We note that the domain of validity of the power law shifts to larger system sizes at intermediate disorder since the tails of the distributions are only observable for large system sizes. The right panel of Fig. 4 shows an excellent agreement between the exponent γ, and its value calculated from δ using the relation $\delta = (1 - \gamma)/(2\gamma)$.

Discussion. In this Rapid Communication, we have studied spin transport and thermalization in an archetypal disordered and interacting Floquet system, which has a Floquet-MBL transition for sufficiently strong disorder. We have found that, similar to their static counterparts, spin transport for disordered Floquet systems is subdiffusive and is accompanied by anomalous thermalization with a modified form of ETH. The distributions of matrix elements of local operators written in the eigenbasis of the Floquet operator are non-Gaussian, although the variance of these distributions still decays with the system size. We demonstrated that the decay of the variance is directly related to the temporal decay of the spin-spin autocorrelation function. Given the above, we conjecture that the slow dynamical regime and anomalous thermalization is a *generic* feature of the ergodic phase of systems exhibiting MBL, and does not rely on energy conservation. It is interesting to see how the removal of all conservation laws affects the dynamics of the generic correlation functions in the system.

Interestingly, the disordered Floquest system we consider not only has a flat many-body density of states, but also structureless diagonal elements of local operators written in the eigenbasis of the Floquet operator. We argue that this finding is inconsistent with a mobility edge in the quasienergy spectrum, such that the Floquet-MBL transition *occurs at a critical disorder strength and has no additional structure in quasienergy*. We leave a detailed comparison of the MBL transition in static and Floquet systems for future studies.

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